ENTRANCE EXAMINATION - MSC

Duration: 360 minutes

During your work:

(1) you MAY use textbooks or course notes;
(2) you SHOULD NOT use external help;
(3) you SHOULD NOT use electronic devices for symbolic computations;
(4) you SHOULD NOT use the internet for search or communication.

BY WRITING THIS TEST,
YOU AGREE TO THE TERMS LISTED ABOVE.

This examination contains 8 pages, including this cover page.

Each page contains a test with problems on one of the following topics:

0. **BASIC MATHEMATICS**
1. **ALGEBRA**
2. **ANALYSIS**
3. **COMBINATORICS**
4. **GEOMETRY**
5. **PROBABILITY THEORY**

You have to do the test from BASIC MATHEMATICS.

You also have to choose 3 further topics from the subjects 1–5 and attempt all problems from the chosen topics. Each topic is worth 100 marks.

Please, pay attention to the presentation. Send us your work even if you have not solved each problem or have only partial results in a certain problem. Unless otherwise stated, you SHOULD JUSTIFY your answers.

Send the scanned copy of your solutions WITHIN 6 HOURS of receiving the problem sheet to:

`agoston@cs.elte.hu`

A confirmation letter will be sent within the next 24 hours.

GOOD LUCK!

0. BASIC MATHEMATICS: 10 multiple choice questions

Each problem has exactly one correct answer.
Please, write your solutions into the boxes at the bottom of each page.

0/1. How can one *disprove* a statement of the form "If A then B." ?
- a) We prove that if A holds then B is false.
- b) We prove that if B holds then A is false.
- c) We show an example when A holds but B is false.
- d) We show an example when B holds but A is false.

0/2. What is the negation of the following statement? *There exists a town in Dreamland in which every child has a dragon.*
- a) There exists a town in Dreamland, in which no child has a dragon.
- b) There exists a town in Dreamland, in which there exists a child with no dragon.
- c) In every town in Dreamland there exists a child with no dragon.
- d) In every town in Dreamland there exists a child who has a dragon.

0/3. Which set equals to $\{x : x \in A \Longrightarrow x \in B\}$?
- a) $A \cup B$
- b) $A \cap B$
- c) $A \cap \overline{B}$
- d) $\overline{A} \cup B$

0/4. Which set equals to $\{y : |x| < y \Longrightarrow x < 4\}$?
- a) $(-\infty, 4)$
- b) $(-\infty, 4]$
- c) $(0, 4)$
- d) $[0, 4]$

0/5. Which of the following statements is *false*?
- a) Every real number with finite decimal representation is rational.
- b) Every nonzero real number with finite decimal representation has two infinite decimal representations.
- c) Every rational number has two infinite decimal representations.
- d) If a real number has two infinite decimal representations then it is rational.

Solutions to questions 1-5:

1	2	3	4	5

THIS TEST CONTINUES ON THE NEXT PAGE!

0/6. Which of the following statements is true?
a) If a set of real numbers has a finite supremum then it also has a maximum.
b) The sumpremum of a set of real numbers is always an element of the set.
c) If the supremum of a set of real numbers is an element of the set then the set has a maximum.
d) Every bounded set of real numbers has a maximum.

0/7. Let A and B subsets of the reals. What is the connection between the following statements?

P: $(\forall x \in A)\ (\exists y \in B)\ x < y$ Q: $(\forall y \in B)\ (\exists x \in A)\ x < y$

a) $P \Rightarrow Q$ but $Q \not\Rightarrow P$
b) $Q \Rightarrow P$ but $P \not\Rightarrow Q$
c) $P \Leftrightarrow Q$
d) $P \not\Rightarrow Q$ and $Q \not\Rightarrow P$

0/8. Let (a_n) be a sequence of real numbers. Which of the following statements is *false*?
a) If (a_n) is bounded and nondecreasing then it is convergent.
b) If (a_n) tends to infinity then it is nondecreasing.
c) If (a_n) is nondecreasing then it has a (finite or infinite) limit.
d) If (a_n) tends to infinity then it is unbounded.

0/9. Which of the following statements is true?
a) The function f is strictly increasing on A if there exist $x_1, x_2 \in A$ such that $x_1 < x_2$ and $f(x_1) < f(x_2)$.
b) The function f is **not** strictly increasing on A if there exist $x_1, x_2 \in A$ such that $x_1 < x_2$ and $f(x_1) \geq f(x_2)$.
c) The function f is strictly increasing on A if for every $x_1, x_2 \in A$ we have $x_1 < x_2$ and $f(x_1) < f(x_2)$.
d) If the function f is **not** strictly increasing on A then

$$x_1, x_2 \in A, x_1 < x_2 \Longrightarrow f(x_1) \geq f(x_2).$$

0/10. Suppose that $\lim_{x \to 5} f(x) = 7$. What happens if f is increased at 5 by 1 (and remains the same everywhere else)?
a) The new function will tend to 7 at 5.
b) The new function will tend to 8 at 5.
c) The new function will have no limit at 5.
d) We cannot claim for sure any of the above three statements.

Solutions to questions 6-10:

6	7	8	9	10

1. ALGEBRA: 4 problems

1/1. Find the polar form of the complex number $z = 2 - 2\sqrt{3}i$ and give the fourth roots of z in polar form. What is the algebraic (canonical) form of the product of the four fourth roots of z?

1/2. Find the value of the following $n \times n$ determinant:

$$\begin{vmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & \vdots & & \ddots & \vdots \\ b & b & b & \dots & a \end{vmatrix}$$

1/3. Consider the following matrix:

$$A = \begin{pmatrix} 0 & 2 & -2 \\ 2 & 0 & 2 \\ 2 & -2 & 4 \end{pmatrix}$$

Find the characteristic polynomial, eigenvalues and eigenvectors of A and determine whether the matrix is diagonalizable. If it is, find a matrix T such that $T^{-1}AT$ is a diagonal matrix.

1/4. For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ the infinite sequence $(r_1, r_2, \dots, r_k, \dots)$ will be called be the *rank sequence* of T, if $r_i = \dim \operatorname{Im} T^i$ for every $i = 1, 2, \dots$. For each of the following sequences determine whether the sequence is the rank sequence of a linear transformation $T : \mathbb{R}^{10} \to \mathbb{R}^{10}$. Justify your answer.

a) $(9, 8, 7, \dots, 0, 0 \dots)$;
b) $(10, 9, 8, \dots, 0, 0 \dots)$;
c) $(8, 6, 4, 4, \dots)$;
d) $(8, 6, 4, 4, 2, \dots)$,

2. ANALYSIS: 5 problems

2/1. Prove that $G_1 \cap G_2$ is open if G_1 and G_2 are open sets in $\mathbb{R}$. Is the statement true for the intersection of infinitely many open sets?

2/2. Assume that $\sum a_n$ is convergent. Does it imply the convergence of $\sum a_n^2$?

2/3. Determine the value of $\sqrt{2}$ with accuracy $1/16$ by using some terms of the Taylor series of $f(x) = \sqrt{1+x}$ and Lagrange's remainder.

2/4. Is the function

$$f(x,y) = \frac{x^2 y}{x^2+y^2}, \quad f(0,0) = 0$$

continuous? Is this function differentiable at $(0,0)$?

2/5. Determine the local maxima and minima for $f(x,y) = x^3 + 3y^3 - 12x - 81y + 8$.

3. COMBINATORICS: 5 problems

3/1. Let $\alpha(G)$ denote the size of a largest independent vertex set of a graph $G = (V; E)$, and let $\chi(G)$ denote the chromatic number of G. Show that $\alpha(G)\chi(G) \geq |V|$.

3/2. Prove that

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

3/3. The degrees of the vertices of a tree admit two different values; 92 vertices admit the first value, 9 vertices admit the second one. What are the degrees in question?

3/4. Find the number of eight digit numbers that contain exactly two '1's.

3/5. Consider a 10×10 table filled with positive real numbers. The *neighborhood* of an entry is the set of (at most eight) entries that are tightly around it. An entry a of the table is called *quasi dominant* if there is at most one entry in its neighborhood that is at least as large as a. Find the maximum possible number of quasi dominant entries.

4. GEOMETRY: 4 problems

4/1. Assume that O, A, A', B, and B' are points in the Euclidean plane, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OA'} = \lambda\mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OB'} = \mu\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are linearly independent vectors, λ and μ are real numbers such that $\lambda \cdot \mu \neq 1$. Let I be the intersection point of the lines AB' and $A'B$. Express the vector $\overrightarrow{OI}$ as a linear combination of the vectors $\mathbf{a}$ and $\mathbf{b}$.

4/2. Let M be the orthocenter of the triangle ABC. Show that

$$AB^2 + MC^2 = BC^2 + MA^2 = CA^2 + MB^2.$$

4/3. Suppose that the vertices of a cube have integer coordinates with respect to a Cartesian coordinate system. Show that both the volume of the cube, the area of its facet are integers. Derive from this that the edge length of the cube is also an integer.

4/4. Assume that a Cartesian coordinate system is fixed in the 3-dimensional Euclidean space. Compute the reflection of the point $(1, 2, 3)$ in the line defined by the system of equations

$$\begin{array}{rcrcrcr} 3x & + & 4y & - & 4z & = & 2, \\ 2x & + & 3y & - & 5z & = & -7. \end{array}$$

5. PROBABILITY THEORY: 4 problems

5/1. Let X and Z be independent random variables, both having Poisson distribution with parameter 3. What is the probability that $X + Z$ is less than 5?

5/2. Let X and Y be independent random variables with standard normal distribution. Calculate $Var(2X - Y)$.

5/3. We roll a die, till we get a value, which was previously already observed (e.g. in case of a sequence 5325, 4 rolls were needed). What is the probability, that we had to roll the die exactly 3 times?

5/4. Let X have a uniform distribution over the interval $[0, 1]$. Give the density function of $(X - 0.5)^2$.