4.4 Counting Lemma

The *homomorphism density* $t(F,G)$ of a simple graph F into a simple graph G, is defined as the probability that a uniform random map of $V(F)$ into $V(G)$ preserves the edges. In formula

$$t(F,G) = \frac{\hom(F,G)}{|V(G)|^{|V(F)|}},$$

where $\hom(F,G)$ is the number of homomorphisms (edge-preserving maps) $V(F) \to V(G)$. As an example, $t(K_2,G) = 2|E|/|V|^2$.

Lemma 4.4.1 *For any three simple graphs F, G_1 and G_2,*

$$|t(F,G_1) - t(F,G_2)| \leq |E(F)|\, \delta_\Box(G_1,G_2).$$

Proof. Let ϕ be a uniform random map $V(F) \to V(G)$. Let $E(F) = E_1 \cup E_2$ be a partition of $E(F)$ into to (possibly empty) sets. For $X \subseteq E$, consider the probability

$$p(X) = \mathsf{P}(\phi(X) \subseteq E(G_1), \phi(E \setminus X) \subseteq E(G_2)).$$

Clearly $p(E) = t(F,G_1)$ and $p(\emptyset) = t(F,G_2)$. It suffices to prove that if $X' = X \cup \{ij\}$ $(ij \in E \setminus X)$, then $|p(X') - p(X)| \leq \delta_\Box(G_1,G_2)$.

Let $V' = V \setminus \{i,j\}$ and fix the map $\phi' = \phi|_{V'}$. Node i can be mapped onto a set $S \subseteq V$ so that the edges between i and V' are mapped properly (this is the same for X and X'), and similarly j can be mapped onto a set $T \subseteq V$. The edge ij must be mapped into $E(G_1)$ to contribute to $p(X')$, but into $E(G_2)$ to be contribute to $p(X)$. So

$$\mathsf{P}(\phi(X) \subseteq E(G_1), \phi(E \setminus X) \subseteq E(G_2)) \mid \phi') = \frac{e_{G_2}(S,T)}{n^2},$$

and

$$\mathsf{P}(\phi(X') \subseteq E(G_1), \phi(E \setminus X') \subseteq E(G_2)) \mid \phi') = \frac{e_{G_1}(S,T)}{n^2}.$$

It follows that

$$|\mathsf{P}(\phi(X) \subseteq E(G_1), \phi(E\setminus X) \subseteq E(G_2)) \mid \phi') - \mathsf{P}(\phi(X') \subseteq E(G_1), \phi(E\setminus X') \subseteq E(G_2)) \mid \phi')| = \left|\frac{e_{G_2}(S,T)}{n^2} - \frac{e_{G_2}(S,T)}{n^2}\right| \leq$$

Since this holds for every ϕ', the lemma follows. $\Box$