Selected topics in graph theory

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Homework 2.
(Corrected version)

Submission deadline: Friday, November 25. Solutions can be submitted on paper or by email, in English or Hungarian.

1 Let G be a simple 3-connected planar map. A *corner* is determined by a face and one of its nodes. Prove that we can assign each corner either to its country or to its node so that each country is assigned to 2 corners, two nodes are not assigned to any corner, and each of the remaining nodes is assigned to 2 corners.

2 Prove that every graph on $n \geq 3$ nodes contains two nodes i and j with $|N(i)\triangle N(j)| \leq (n+1)/2$. Show that for infinitely many n, this bound cannot be improved to $(n-3)/2$.

3 Construct a graph on 2^r nodes ($r \geq 2$, even) with no twin nodes, whose adjacency matrix has rank r over the 2-element field $GF(2)$.

4 Let M be a real $n \times n$ matrix. Recall that we have defined

$$\|M\|_1 = \frac{1}{n^2}\sum_{i,j}|M_{ij}|, \quad \text{and} \quad \|M\|_\square = \frac{1}{n^2}\max_{S,T\subseteq\{1,\dots,n\}}\Big|\sum_{\substack{i\in S\\ j\in T}} M_{ij}\Big|.$$

(a) Prove that $\|M\|_1 \leq 2n\|M\|_\square$. (b)[Bonus problem] Prove that $\|M\|_1 \leq 2\sqrt{n}\|M\|_\square$.

5 Prove that if a graph G on n nodes has a partition $\mathcal{P}$ into k classes such that $d_\square(G, G_\mathcal{P}) = \varepsilon$, then it has a partition $\mathcal{Q}$ into k^2 classes with either $\lfloor n/k^2 \rfloor$ or $\lceil n/k^2 \rceil$ elements such that $d_\square(G, G_\mathcal{Q}) \leq \varepsilon + O(1/k)$.