Selected topics in graph theory

László Lovász
Fall 2016

Homework 3.

Submission deadline: Monday, December 19. Solutions can be submitted on paper (to Lilla Tóthmérész) or by email, in English or Hungarian.

1 If $\vartheta(\overline{G}) = 2$, then G is bipartite.

2 (a) If H is an induced subgraph of G, then $\vartheta(H) \le \vartheta(G)$. (b) If H is a spanning subgraph of G, then $\vartheta(H) \ge \vartheta(G)$.

3 Prove that the minimum dimension in which a graph G has an orthonormal representation is at least $\vartheta(G)$.

4 Prove that the vertices of the polytope $\mathrm{FSTAB}(G)$ (see lecture notes) are half-integral, and show by an example that they are not always integral.

5 The *fractional chromatic number* $\chi^*(G)$ is defined as the least real number t for which there exists a family $(A_j : \ j = 1, \dots, p)$ of stable sets in G, and nonnegative weights $(\tau_j : \ j = 1, \dots, p)$ such that $\sum_j \tau_j = t$ and $\sum_j \tau_j \mathbf{1}_{A_j} \ge \mathbf{1}_V$. The *fractional clique number* $\omega^*(G)$ is the largest real number s for which there exist nonnegative weights $(\sigma_i : \ i \in V)$ such that $\sum_i \sigma_i = s$ and $\sum_{i \in A} \sigma_i \le 1$ for every stable set A.

(a) Prove that $\omega(G) \le \omega^*(G)$ and $\chi(G) \ge \chi^*(G)$.

(b) Prove that $\chi^*(G) = \omega^*(G)$.

(b) Prove that $\vartheta(\overline{G}) \le \chi^*(G)$.