Chapter 3

Planar graphs and rubber bands

3.1 Preliminaries

We start with collecting results on planar graphs. Most of these will be familiar, and in most cases we don't give proofs.

3.1.1 Maps

A graph G is *planar*, if it can be drawn in the plane so that its edges are Jordan curves and they intersect only at their endnodes. (We use the word *node* for the node of a graph, the word *vertex* for the vertex of a polytope, and the word *point* for points in the plane or in other spaces.) A *planar map* is a planar graph with a fixed embedding. We also use this phrase to denote the image of this embedding, i.e., the subset of the plane which is the union of the set of points representing the nodes and the Jordan curves representing the edges.

The complement of a planar map G decomposes into a finite number of arcwise connected pieces, which we call the *countries* of the map. (Often the countries are called "faces", but we reserve the word *face* for the faces of polyhedra, and the word *facet* for maximum dimensional proper faces.) The set of its countries will be denoted V^*, and often we use the notation $f = |V^*|$.

If a planar map is connected (which we are going to assume most of the time), then every country is homeomorphic to an open disc, except for one, which is unbounded, and homeomorphic to an open ring. Each country has a *boundary* consisting of a cyclic sequence of edges. An edge can occur twice in the boundary sequence; this happens if and only if it is a cut-edge (isthmus) of the graph. If the graph is 2-edge-connected, then no boundary sequence contains a repetition, and it is 2-node-connected, then every boundary is a (simple) cycle. The country also defines a cyclic sequence of nodes; a node my occur many times in this sequence. Each occurrence of a node in this sequence is called a *corner*. Each corner is incident with two edges of the country, called the *edges of the corner*; these two edges are different except if the corner is a node of degree 1.

Every planar map G has a *dual map* $G^* = (V^*, E^*)$ (Figure 2.1) As an abstract graph, this can be defined as the graph whose nodes are the countries of G, and if two countries share k edges, then we connect them in G^* by k edges. So each edge $e \in E$ will correspond to an edge e^* of G^*, and $|E^*| = |E| = m$. (If the same country is incident with e from both sides, then e^* is a loop.)

This dual map has a natural drawing in the plane: in the interior of each country F of G we select a point v_F (which can be called its *capital*), and on each edge $e \in E$ we select a point u_e (this will not be a node of G^*, just an auxiliary point). We connect v_F to the points u_e for each edge on the boundary of F by nonintersecting Jordan curves inside F. If the boundary of F goes through e twice (i.e., both sides of e belong to F), then we connect v_F to u_e by two curves, entering e from two sides. The two curves entering u_e form a single Jordan curve representing the edge e^*. It is not hard to see that each country of G^* will contain a unique node of G, and so $(G^*)^* = G$.

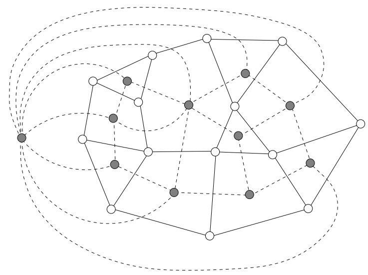

Figure 3.1: A planar map and its dual.

Instead of maps in the plane, we could speak about maps on the sphere. Often this leads to simpler statements, since one does not need to distinguish an unbounded face. (On the other hand, it is easier to follow arguments in the plane.)

3.1.2 Euler's Formula

We often need the following basic fact about planar graphs.

Theorem 3.1.1 (Euler's Formula) *For every connected planar map, $|V| - |E| + |V^*| = 2$ holds.* □

Some important consequences of Euler's Formula are the following.

Corollary 3.1.2 (a) *A simple planar graph with n nodes has at most $3n - 6$ edges.*

(b) *A simple bipartite planar graph with n nodes has at most $2n - 4$ edges.*

(c) *Every simple planar graph has a node with degree at most 5.*

(d) *Every simple bipartite planar graph has a node with degree at most 3.* □

From (a) and (b) it follows immediately that the "Kuratowski graphs" K_5 and $K_{3,3}$ (see Figure 2.2) are not planar. This observation leads to the following characterization of planar graphs.

Theorem 3.1.3 (Kuratowski's Theorem) *A graph G is embedable in the plane if and only if it does not contain a subgraph homeomorphic to the complete graph K_5 or the complete bipartite graph $K_{3,3}$.* □

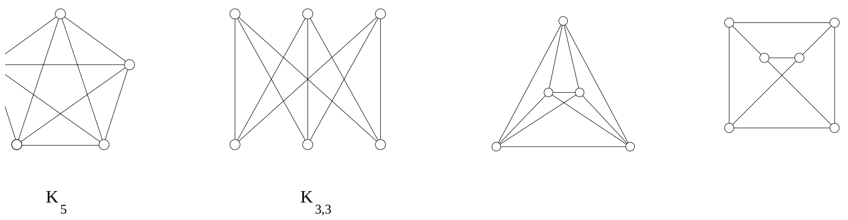

Figure 3.2: The two Kuratowski graphs. The two drawings on the right hand side show that both graphs can be drawn in the plane with a single crossing.

Among planar graphs, 3-connected planar graphs are especially important. A cycle C in a graph G is called *separating*, if $G \setminus V(C)$ has at least two connected components, where any chord of C is counted as a connected component here.

Proposition 3.1.4 *In a 3-connected planar graph a cycle bounds a country if and only if it is non-separating.* □

Corollary 3.1.5 *Every simple 3-connected planar graph has an essentially unique embedding in the plane in the sense that the set of cycles that bound countries is uniquely determined.* □

3.1.3 Planarity and polytopes

Let P be a 3-polytope. The vertices and edges of P form a graph G_P, which we call the *skeleton* of P.

Proposition 3.1.6 *The skeleton of every 3-polytope is a 3-connected planar graph.*

We describe the simple proof, because this is an example of how a geometric representation implies a purely graph-theoretic property, namely 3-connectivity.

Proof. Planarity of G_P can be proved by constructing an embedding called the *Schlegel diagram* of the polytope. Let F be any facet of P, and let x be a point that is outside P but very close to an interior point of F; more precisely, assume that the plane Σ of F separates x from P, but for every other facet F', x is on the same side of the plane of F' as P. Let us

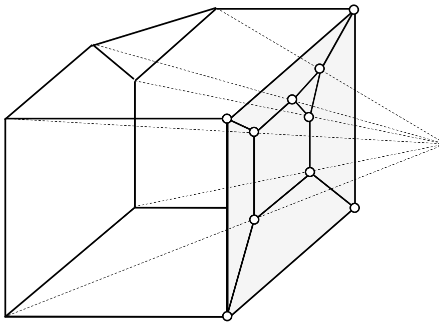

Figure 3.3: Projecting the skeleton of a polytope into one of the facets.

project the skeleton of P from x to the plane Σ. Then we get an embedding of G_P in the plane (Figure 2.3).

To see that G_P is 3-connected, it suffices to show that for any four nodes a, b, c, d there is a path from a to b which avoids c and d.

If a, b, c, d are not coplanar, then let Π be a plane that separates $\{a, b\}$ from $\{c, d\}$; then we can connect a and b by a polygon consisting of edges of P that stays on the same side of Π as a and b, and so avoids c and d.

If a, b, c, d are coplanar, let Π be a plane that contains them. One of the open halfspaces bounded by Π contains at least one vertex of P. We can then connect a and b by a polygon consisting of edges of P that stays on this side of Π (except for its endpoints a and b), and so avoids c and d. $\square$

The converse of this last proposition is an important and much more difficult theorem, whose proof is a main goal of this chapter.

Theorem 3.1.7 (Steinitz's Theorem) *A simple graph is isomorphic with the skeleton of a 3-polytope if and only if it is 3-connected and planar.*

Figure 2.4 illustrates the theorem. A bijection between the nodes of a simple graph G and the vertices of a convex polytope P in $\mathbb{R}^3$ that gives an isomorphism between G and the skeleton of P is called a *Steinitz representation* of the graph G.

Duality and polarity. Let K be a convex body containing the origin as an interior point. The *polar* of K is defined by

$$K^* = \{\mathbf{x} \in \mathbb{R}^d : \ \mathbf{x}^\mathsf{T}\mathbf{y} \le 1 \ \forall \mathbf{y} \in P\}.$$

It is not hard to prove that K^* is a convex body as well, containing the origin in its interior. For every convex body K we have $(K^*)^* = K$.

The polar of a polytope (containing the origin in its interior) is a polytope. For every vertex $\mathbf{v}$ of P, the inequality $\mathbf{v}^\mathsf{T}\mathbf{x} \le 1$ defines a facet of P^*, and vice versa. The vector $\mathbf{v}$

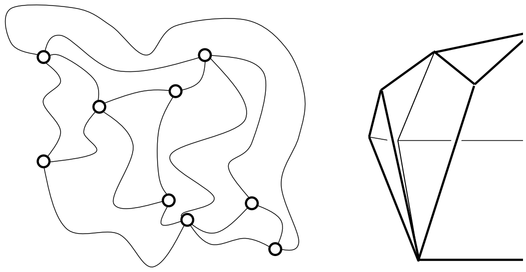

Figure 3.4: Representing a 3-connected planar map by a polytope.

is a normal vector of the facet $\mathbf{v}^\mathsf{T}\mathbf{x} \le 1$. More generally, if $\mathbf{v}_0, \dots, \mathbf{v}_m$ are the vertices of a k-dimensional face F of P, then

$$F^\perp = \{\mathbf{x} \in P^* : \ \mathbf{v}_0^\mathsf{T}\mathbf{x} = 1, \dots, \mathbf{v}_m^\mathsf{T}\mathbf{x} = 1\}$$

defines a $d-k-1$-dimensional face of P^*. Furthermore, $(F^\perp)^\perp = F$.

In dimension 3, the above means that there is a bijection between the edges of a polytope and its polar. In fact it is not hard to see that the skeleton of a convex body (containing the origin in its interior) is isomorphic to the dual map of the skeleton of its polar.

3.2 Geometric representation, a.k.a. vector labeling

A *vector-labeling* in d dimensions of a graph G is a map $\mathbf{u} : \ V \to \mathbb{R}^d$. We will also write $(\mathbf{u}_i : \ i \in V)$ for such a representation. We can also think of the vectors $\mathbf{u}_i$ as the positions of the nodes in $\mathbb{R}^d$. The mapping $i \mapsto \mathbf{u}_i$ can be thought of as a "drawing", or "embedding", or "geometric representation" of the node set of the graph in a euclidean space. (We think of the edges as straight line segments.) These phrases are useful as a visual help for following certain arguments, but all three are ambiguous, and we are going to use "vector labeling" in formal statements. Nevertheless, the vector-label of a node will be sometimes called the *position* of the node, if this use of words suggests a better intuitive picture. At this time, we don't assume that the mapping $i \mapsto \mathbf{u}_i$ is injective; this will be a pleasant property to have, but not always achievable. The pair $(G, \mathbf{u})$ will be called a *structure* in R^d.

For every vector labeling $\mathbf{u} : \ V \to \mathbb{R}^d$, it is often useful to consider the matrix U whose columns are the vectors $\mathbf{u}_i$. This is a $d \times V$ matrix (the rows are indexed by $1, \dots, d$, the columns are indexed by the nodes). We denote the space of such matrices by $\mathbb{R}^{d\times V}$. Of course, this matrix U contains the same information as the labeling itself.

Most of the time we will assume that the vector labeling $\mathbf{u}$ of the graph is not contained in a lower dimensional linear subspace of $\mathbb{R}^d$. This means that the corresponding matrix U has rank d, or (equivalently) its rows are linearly independent. Often we make a stronger assumption, namely that the node positions $\mathbf{u}_i$ are not all contained in a lower dimensional *affine* subspace. We call such structures *full-dimensional* (linearly or affinely). If we have to consider vector labelings that do live in a lower dimensional space, we will call the dimension of this space the *effective (linear or affine) dimension* of the representation.

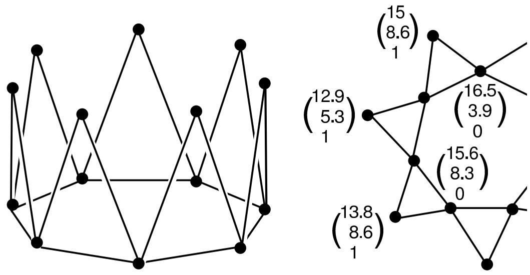

Figure 3.5: Two ways of looking at a graph with vector labeling (not all vector labels are shown).

Remark 3.2.1 While "geometric representation" and "vector labeling" (when defined appropriately) mean the same thing, they do suggest two different ways of visualizing the graph. The latter is the computer science view: we have a graph and store additional information for each node. The former considers the graph as a structure in euclidean space. The main point in this book is to relate geometric and graph-theoretic properties, so this way of visualizing is often very useful (see Figure 2.5).

3.3 Rubber band representation

Let G be a connected graph and $\emptyset \neq S \subseteq V$. Fix an integer $d \geq 1$ and a map $\overline{\mathbf{u}}: S \to \mathbb{R}^d$. We extend this to a representation $\mathbf{u}: V \to \mathbb{R}^d$ (a vector-labeling of G) as follows.

First, let's give an informal description. Replace the edges by ideal rubber bands (satisfying Hooke's Law). Think of the nodes in S as nailed to their given position (node $i \in S$ to $\overline{\mathbf{u}}_i \in \mathbb{R}^d$), but let the other nodes settle in equilibrium. (We are going to see that this equilibrium position is uniquely determined.) We call this equilibrium position of the nodes the *rubber band representation* of G in $\mathbb{R}^d$ extending $\overline{\mathbf{u}}$. The nodes in S will be called *nailed*, and the other nodes, *free* (Figure 2.6).

To be precise, let $\mathbf{u}_i = (u_{i1}, \dots, u_{id})^\mathsf{T} \in \mathbb{R}^d$ be the position of node $i \in V$. By definition, $\mathbf{u}_i = \overline{\mathbf{u}}_i$ for $i \in S$. The *energy* of this representation is defined as

$$\mathcal{E}(\mathbf{u}) = \sum_{ij \in E} |\mathbf{u}_i - \mathbf{u}_j|^2 = \sum_{ij \in E} \sum_{k=1}^{d} (u_{ik} - u_{jk})^2. \tag{3.1}$$

We want to find the representation with minimum energy, subject to the boundary conditions:

$$\begin{aligned} &\text{minimize } \mathcal{E}(\mathbf{u}) \\ &\text{subject to } \mathbf{u}_i = \overline{\mathbf{u}}_i \text{ for all } i \in S. \end{aligned} \tag{3.2}$$

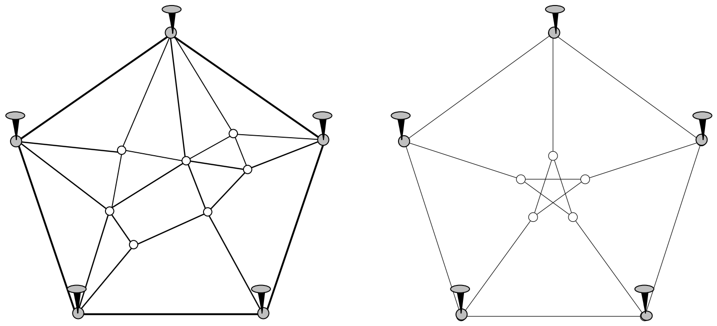

Figure 3.6: Rubber band representation of a planar graph and of the Petersen graph.

Lemma 3.3.1 *The function* $\mathcal{E}: \mathbb{R}^{d\times(V\setminus S)} \to \mathbb{R}$ *is strictly convex.*

Proof. In (2.1), every function $(u_{ik} - u_{jk})^2$ is convex, so $\mathcal{E}$ is convex. Furthermore, moving along an (affine) line in $\mathbb{R}^{d\times(V\setminus S)}$, this function is strictly convex unless $u_{ik} - u_{jk}$ remains constant along this line. If this applies to each coordinate of each edge, then moving along the line means parallel translation of the vectors $\mathbf{u}_i$, which is impossible if at least one node is nailed. □

It is trivial that if any of the vectors $\mathbf{u}_i$ tends to infinity, then $\mathcal{E}(\mathbf{u})$ tends to infinity (still assuming the boundary conditions 2.2 hold, where S is nonempty). Together with Lemma 2.3.1, this implies that the representation with minimum energy is uniquely determined. If $i \in V \setminus S$, then for the representation minimizing the energy, the partial derivative of $\mathcal{E}(\mathbf{u})$ with respect to any coordinate of $\mathbf{u}_i$ must be 0:

$$\sum_{j\in N(i)} (\mathbf{u}_i - \mathbf{u}_j) = 0 \qquad (i \in V \setminus S). \tag{3.3}$$

We can rewrite this as

$$\mathbf{u}_i = \frac{1}{\deg(i)} \sum_{j\in N(i)} \mathbf{u}_j. \tag{3.4}$$

This equation means that *every free node is placed in the center of gravity of its neighbors.* Equation (2.3) has a nice physical meaning: the rubber band connecting i and j pulls i with force $\mathbf{u}_j - \mathbf{u}_i$, so (2.3) states that the forces acting on i sum to 0 (as they should at the equilibrium). It is easy to see that (2.3) characterizes the equilibrium position.

It will be useful to extend the rubber band construction to the case when the edges of G have arbitrary positive weights (or "strengths"). Let $S_{ij} > 0$ denote the strength of the edge ij. We define the energy function of a representation $\mathbf{u}$ by

$$\mathcal{E}_S(\mathbf{u}) = \sum_{ij\in E} S_{ij}|\mathbf{u}_i - \mathbf{u}_j|^2. \tag{3.5}$$

The simple arguments above remain valid: $\mathcal{E}_S$ is strictly convex if at least one node is nailed, there is a unique optimum, and for the optimal representation every $i \in V \setminus S$ satisfies

$$\sum_{j\in N(i)} S_{ij}(\mathbf{u}_i - \mathbf{u}_j) = 0. \tag{3.6}$$

This we can rewrite as

$$\mathbf{u}_i = \frac{1}{\sum_{j\in N(i)} S_{ij}} \sum_{j\in N(i)} S_{ij}\mathbf{u}_j. \tag{3.7}$$

Thus $\mathbf{u}_i$ is no longer at the center of gravity of its neighbors, but it is still a convex combination of them with positive coefficients. In particular, it is in the relative interior of the convex hull of its neighbors.

3.4 Rubber bands, planarity and polytopes

3.4.1 How to draw a graph?

The rubber band method was first analyzed By W.T. Tutte. He described how to use "rubber bands" to draw a 3-connected planar graph with straight edges and convex countries.

Let G be a 3-connected planar graph, and let p_0 be any country of it. Let C_0 be the cycle bounding p_0. Let us nail the nodes of C_0 to the vertices of a convex polygon P_0 in the plane, in the appropriate cyclic order, and let the rest find its equilibrium. We draw the edges of G as straight line segments connecting the appropriate endpoints. Figure 2.7 shows the rubber band representation of the skeletons of the five platonic bodies.

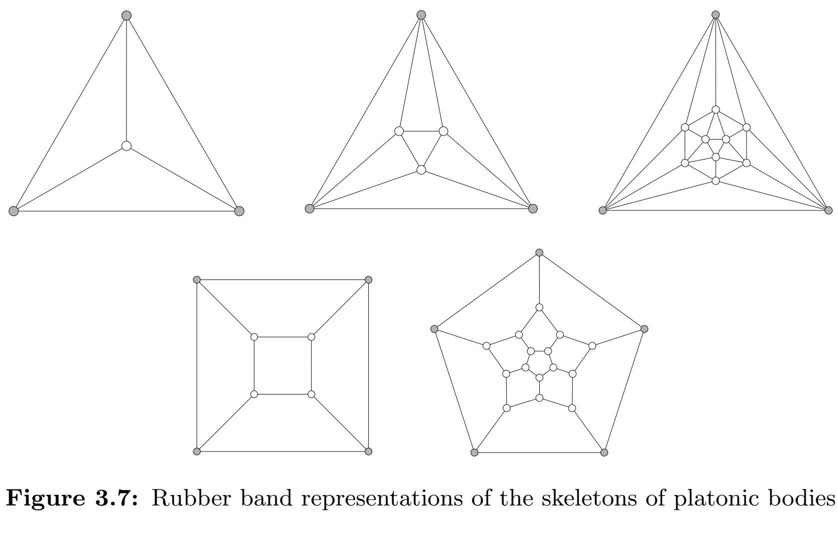

Figure 3.7: Rubber band representations of the skeletons of platonic bodies

By the above, we know that each node not on C_0 is positioned at the center of gravity of its neighbors. Tutte's main result about this representation is the following:

Theorem 3.4.1 *If G is a simple 3-connected planar graph, then every rubber band representation of G (with the nodes of a particular country p_0 nailed to a convex polygon) gives an embedding of G in the plane, such that each country is a convex polygon.*

Proof. Let $\mathbf{u} : V \to \mathbb{R}^2$ be this rubber band representation of G. Let ℓ be a line intersecting the interior of the polygon P_0, and let U_0, U_1 and U_2 denote the sets of nodes of G mapped on ℓ and on the two (open) sides of ℓ, respectively.

The key to the proof is the following claim.

Claim 1. *The sets U_1 and U_2 induce connected subgraphs of G.*

Let us prove this for U_1. Clearly the nodes of the outer cycle p_0 in U_1 form a (nonempty) path P_1. We may assume that ℓ does not through any node (by shifting it very little in the direction of U_1) and that it is not parallel to any line connecting two distinct positions (by rotating it with a small angle). Let $a \in U_1 \setminus V(C_0)$, we show that it is connected to P_1 by a path in U_1. Since $\mathbf{u}_a$ is a convex combination of the positions of its neighbors, it must have a neighbor a_1 such that $\mathbf{u}_{a_1}$ is in U_1 and at least as far away from ℓ as $\mathbf{u}_a$. By our choice of ℓ, either $\mathbf{u}_{a_1}$ is strictly farther from ℓ than $\mathbf{u}_a$, or $\mathbf{u}_{a_1} = \mathbf{u}_a$.

At this point, we have to deal with an annoying degeneracy. There may be several nodes represented by the same vector $\mathbf{u}_a$ (later it will be shown that this does not occur). Consider all nodes represented by $\mathbf{u}_a$, and the connected component H containing a of the subgraph of G induced by these nodes. If H contains a nailed node, then it contains a path from a to P_1, all in U_1. Else, there must be an edge connecting a node $a' \in V(H)$ to a node outside H (since G is connected). Since the system is in equilibrium, a' must have a neighbor a_1 such that $\mathbf{u}_{a_1}$ is farther away from ℓ than $\mathbf{u}_a = \mathbf{u}_{a'}$ (here we use that no edge is parallel to ℓ). Hence $a_1 \in U_1$, and thus a is connected to a_1 by a path in U_1.

Either a_1 is nailed (and we are done), or we can find a node $a_2 \in U_1$ such that a_1 is connected to a_2 by a path in U_1, and $\mathbf{u}_{a_2}$ is farther from ℓ than $\mathbf{u}_{a_1}$, etc. This way we get a path Q in G that starts at a, stays in U_1, and eventually must hit P_1. This proves the claim (Figure 2.8).

Next, we exclude some possible degeneracies. (Note that we are not assuming any more that no edge is parallel to ℓ: this assumption could be made for the proof of Claim 1 only.)

Claim 2. *Every node $u \in U_0$ has neighbors in both U_1 and U_2.*

This is trivial if $u \in V(C_0)$, so suppose that u is a free node. If u has a neighbor in U_1, then it must also have a neighbor in U_2; this follows from the fact that $\mathbf{u}_u$ is the center of gravity of the points $\mathbf{u}_v$, $v \in N(u)$. So it suffices to prove that not all neighbors of u are contained in U_0.

Let T be the set of nodes $u \in U_0$ with $N(u) \subseteq U_0$, and suppose that this set is nonempty. Consider a connected component H of $G[T]$ (H may be a single node), and let S be the set

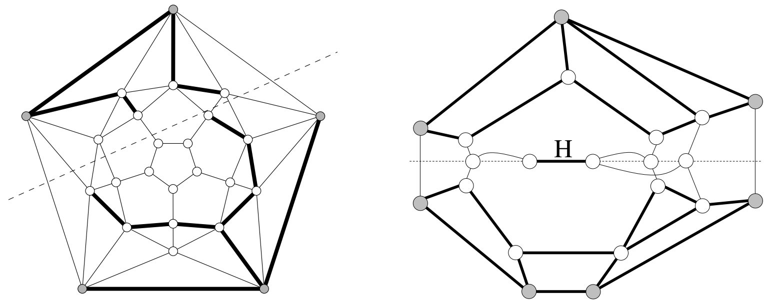

Figure 3.8: Left: every line cuts a rubber band representation into connected parts. Right: Each node on a line must have neighbors on both sides of the line.

of neighbors of H outside H. Since $V(H) \cup S \subseteq U_0$, the set $V(H) \cup S$ cannot contain all nodes, and hence S is a cutset. Thus $|S| \geq 3$ by 3-connectivity.

If $a \in S$, then $a \in U_0$ by the definition of S, but a has a neighbor not in U_0, and so it has neighbors in both U_1 and U_2 by the argument above (see Figure 2.8). The set $V(H)$ induces a connected graph by definition, and U_1 and U_2 induce connected subgraphs by Claim 1. So we can contract these sets to single nodes. These three nodes will be adjacent to all nodes in S. So G can be contracted to $K_{3,3}$, which is a contradiction since it is planar. This proves Claim 2.

Claim 3. *Every country has at most two nodes in U_0.*

Suppose that $a, b, c \in U_0$ are nodes of a country p. Clearly $p \neq p_0$. Let us create a new node d and connect it to a, b and c; the resulting graph G' is still planar. On the other hand, the same argument as in the proof of Claim 2 (with $V(H) = d$ and $S = \{a, b, c\}$) shows that G' has a $K_{3,3}$ minor, which is a contradiction.

Claim 4. *Let p and q be the two countries sharing an edge ab, where $a, b \in U_0$. Then $V(p_1) \setminus \{a, b\} \subseteq U_1$ and $V(p_2) \setminus \{a, b\} \subseteq U_2$ (or the other way around).*

Suppose not, then p has a node $c \neq a, b$ and q has a node $d \neq a, b$ such that (say) $c, d \in U_1$. (Note that $c, d \notin U_0$ by Claim 3.) By Claim 1, there is a path P in U_1 connecting c and d (Figure 2.9). Claim 2 implies that both a and b have neighbors in U_2, and again Claim 1, these can be connected by a path in U_2. This yields a path P' connecting a and b whose inner nodes are in U_2. By their definition, P and P' are node-disjoint. But look at any planar embedding of G: the edge ab, together with the path P', forms a Jordan curve that does not go through c and d, but separates them, so P cannot exist.

Claim 5. *The boundary of every country q is mapped onto a convex polygon P_q.*

This is immediate from Claim 4, since no edge of a country, extended to a line, can intersect its interior.

Claim 6. *The interiors of the polygons P_q ($q \neq p_0$) are disjoint.*

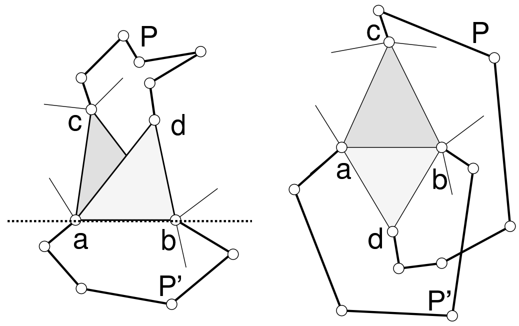

Figure 3.9: Two adjacent countries having nodes on the same side of ℓ in the rubber band representation (left), and the supposedly disjoint paths in the planar embedding (right).

Let $\mathbf{x}$ be a point inside P_{p_0}, we want to show that it is covered by one P_q only. Clearly we may assume that $\mathbf{x}$ is not on the image of any edge. Draw a line through $\mathbf{x}$ that does not go through the position of any node, and see how many times its points are covered by interiors of such polygons. As we enter P_{p_0}, this number is clearly 1. Claim 4 says that as the line crosses an edge, this number does not change. So $\mathbf{x}$ is covered exactly once.

Now the proof is essentially finished. Suppose that the images of two edges have a common point (other than their common endpoints). Then two of the countries incident with them would have a common interior point, which is a contradiction except if these countries are the same, and the two edges are consecutive edges of this country. □

Before going on, let's analyze this proof a little. The key step, namely Claim 1, is very similar to a basic fact concerning convex polytopes, namely that every hyperplane intersecting the interior of the polytope cuts the skeleton into connected parts. Let us call a vector-labeling of a graph *section-connected*, if for every open halfspace, the subgraph induced by those nodes that are mapped into this halfspace is either connected or empty. The skeleton of a polytope is section-connected; and so is the rubber-band representation of a planar graph. Note that the proof of Claim 1 did not make use of the planarity of G (see Exercise 2.3).

Remark 3.4.2 Tutte's method, as described above, is a very efficient procedure to find straight-line embeddings of 3-connected planar graphs in the plane. These embeddings look nice for many graphs (as our figures show), but they may have bad parts, like points getting too close. Figure 2.10 shows a simple situation in which positions of nodes get exponentially close to each other and to the midpoint of an edge. You may play with the edge weights, but finding a good weighting adds substantially to the algorithmic cost.

A reasonable way to exclude nodes being positioned too close is to require that their coordinates are integers. Then the question is, of course, how to minimize these coordinates. In which rectangles $[0,a]\times[0,b]$ can every planar graph on n nodes be squeezed in so that we still get a straight line embedding? It turns out that this can be achieved with $a,b=O(n)$.

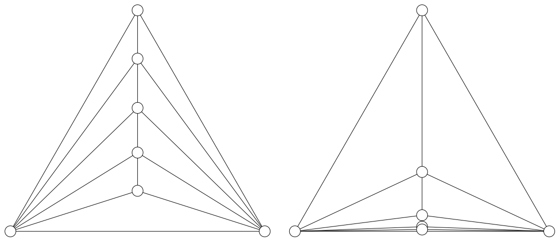

Figure 3.10: The rubber band representation can lead to crowding of the nodes. Each node on the middle line will be placed at or below the center of gravity of the triangle formed by the lower edge and the node immediately above it. So the distance between the k-th node from the top and the lower edge decreases faster than 3^{-k}.

Representing edges by straight lines is not always the most important format of useful drawings. In some very important applications of planar embedding, most notably the design of integrated circuits (chips), the goal is to embed the graph in a grid, so that the nodes are drawn on gridpoints, and the edges are drawn as zig-zagging grid paths. (Of course, we must assume that no degree is larger than 4.) This way all nonzero distances between nodes and/or edges are automatically at least 1 (the edge-length of the grid). Besides trying to minimize the size or area of the grid in which the embedding lies, it is very natural (and practically important) to minimize the number of bends. The good news is that this can be achieved within a very reasonable area ($O(n^2)$) and with $O(n)$ bends See the Handbook of Graph Drawing and Visualization for details.

3.4.2 How to lift a graph?

We are now ready to prove Steinitz's Theorem. An old construction going back to Cremona and Maxwell in the 19-th century can be used to "lift" Tutte's rubber band representation to a Steinitz representation. We will begin with analyzing the reverse procedure: projecting a convex polytope on a face.

Let P be a convex 3-polytope, let F be one of its faces, and let Σ be the plane containing H. Suppose that for every vertex $\mathbf{v}$ of P, its orthogonal projection onto H is an interior point of F; we say that the polytope P is *straight over the face F*.

Theorem 3.4.3 (a) *Let P be a 3-polytope that is straight over its face F, and let G be the orthogonal projection of the skeleton of P onto the plane of F. Then we can assign positive strengths to the edges of G so that G will be the rubber band representation of the skeleton with the vertices of F nailed.*

(b) *Let G be a 3-connected planar graph, and let T be a triangular country of G, and let Δ be a triangle in a plane Σ. Then there is a convex polytope P in 3-space such that T is a face of P, and the orthogonal projection of P onto the plane Σ gives the rubber band representation of G obtained by nailing T to Δ.*

In other words (b) says that we can assign a number $\eta_i \in \mathbb{R}$ to each node $i \in V$ (the height above the plane Σ) such that $\eta_i = 0$ for $i \in V(T)$, $\eta_i > 0$ for $i \in V \setminus V(T)$, and the mapping

$$i \mapsto \mathbf{v}_i = \begin{pmatrix} \mathbf{u}_i \\ \eta_i \end{pmatrix} = \begin{pmatrix} u_{i1} \\ u_{i2} \\ \eta_i \end{pmatrix} \in \mathbb{R}^3$$

is a Steinitz representation of G.

Example 3.4.4 (Triangular Prism) Consider the rubber band representation of a triangular prism in Figure 2.11. If this is an orthogonal projection of a convex polyhedron, then the lines of the three edges pass through one point: the point of intersection of the planes of the three quadrangular faces. It is easy to see that this condition is necessary and sufficient for the picture to be a projection of a prism. To see that it is satisfied by a rubber band representation, it suffices to note that the inner triangle is in equilibrium, and this implies that the lines of action of the forces acting on it must pass through one point. ♦

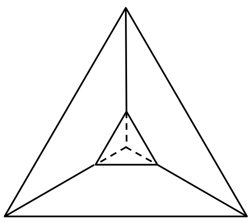

Figure 3.11: The rubber band representation of a triangular prism is the projection of a polytope.

Proof. (a) Let's call the plane of the face F "horizontal", spanned by the first two coordinate axes, and the third coordinate direction "vertical", so that the polytope is "above" the plane of F. For each face p, let $\mathbf{g}_p$ be a normal vector. Since no face is vertical, no normal vector $\mathbf{g}_p$ is horizontal, and hence we can normalize $\mathbf{g}_p$ so that its third coordinate is 1. Clearly for each face p, $\mathbf{g}_p$ will be an outer normal, except for $p = F$, when $\mathbf{g}_p$ is an inner normal.

Write $\mathbf{g}_p = \binom{\mathbf{h}_p}{1}$. Let ij be any edge of G, and let p and q be the two countries on the left and right of ij. Then

$$(\mathbf{h}_p - \mathbf{h}_q)^\mathsf{T}(\mathbf{u}_i - \mathbf{u}_j) = 0. \tag{3.8}$$

Indeed, both $\mathbf{g}_p$ and $\mathbf{g}_q$ are orthogonal to the edge $\mathbf{v}_i\mathbf{v}_j$ of the polytope, and therefore so is their difference, and

$$(\mathbf{h}_p - \mathbf{h}_q)^\mathsf{T}(\mathbf{u}_i - \mathbf{u}_j) = \begin{pmatrix} \mathbf{h}_p - \mathbf{h}_q \\ 0 \end{pmatrix}^\mathsf{T} \begin{pmatrix} \mathbf{u}_i - \mathbf{u}_j \\ \eta_i - \eta_j \end{pmatrix} = (\mathbf{g}_p - \mathbf{g}_q)^\mathsf{T}(\mathbf{v}_i - \mathbf{v}_j) = 0.$$

We have $\mathbf{h}_T = 0$, since the face T is horizontal.

Let R denote the counterclockwise rotation in the plane by 90°, then it follows that $\mathbf{h}_p - \mathbf{h}_q$ is parallel to $R(\mathbf{u}_j - \mathbf{u}_i)$, and so there are real numbers c_{ij} such that

$$\mathbf{h}_p - \mathbf{h}_q = c_{ij} R(\mathbf{u}_j - \mathbf{u}_i). \tag{3.9}$$

We claim that $c_{ij} > 0$. Let k be any node on the boundary of p different from i and j. Then $\mathbf{u}_k$ is to the left from the edge ij, and hence

$$(\mathbf{u}_k - \mathbf{u}_i)^{\mathsf{T}} R(\mathbf{u}_j - \mathbf{u}_i) > 0. \tag{3.10}$$

Going up to the space, convexity implies that the point $\mathbf{v}_k$ is below the plane of the face q, and hence $\mathbf{g}_q^{\mathsf{T}} \mathbf{v}_k < \mathbf{g}_q^{\mathsf{T}} \mathbf{v}_i$. Since $\mathbf{g}_p^{\mathsf{T}} \mathbf{v}_k = \mathbf{g}_p^{\mathsf{T}} \mathbf{v}_i$, this implies that

$$c_{ij}(R(\mathbf{u}_j - \mathbf{u}_i))^{\mathsf{T}}(\mathbf{u}_k - \mathbf{u}_i) = (\mathbf{h}_p - \mathbf{h}_q)^{\mathsf{T}}(\mathbf{u}_k - \mathbf{u}_i) = (\mathbf{g}_p - \mathbf{g}_q)^{\mathsf{T}}(\mathbf{v}_k - \mathbf{v}_i) > 0. \tag{3.11}$$

Comparing with (2.10), we see that $c_{ij} > 0$.

Let us make a remark here that will be needed later. Using that not only $\mathbf{g}_p - \mathbf{g}_q$, but also $\mathbf{g}_p$ is orthogonal to $\mathbf{v}_j - \mathbf{v}_i$, we get that

$$0 = \mathbf{g}_p^{\mathsf{T}}(\mathbf{v}_j - \mathbf{v}_i) = \mathbf{h}_p^{\mathsf{T}}(\mathbf{u}_j - \mathbf{u}_i) + \eta_j - \eta_i,$$

and hence

$$\eta_j - \eta_i = -\mathbf{h}_p^{\mathsf{T}}(\mathbf{u}_j - \mathbf{u}_i). \tag{3.12}$$

To complete the proof of (a), we argue that the projection of the skeleton is indeed a rubber band embedding with strengths c_{ij}, with F nailed. We want to prove that every free node i is in equilibrium, i.e.,

$$\sum_{j \in N(i)} c_{ij}(\mathbf{u}_j - \mathbf{u}_i) = 0. \tag{3.13}$$

Using the definition of c_{ij}, it suffices to prove that

$$\sum_{j \in N(i)} c_{ij} R(\mathbf{u}_j - \mathbf{u}_i) = \sum_{j \in N(i)} (\mathbf{h}_{p_j} - \mathbf{h}_{q_j}) = 0,$$

where p_j is the face to the left and q_j is the face to the right of the edge ij. But this is clear, since every term occurs once with positive and once with negative sign.

(b) The proof consists of going through the steps of the proof of part (a) in reverse order: given the Tutte representation, we first reconstruct the vectors $\mathbf{h}_p$ so that all equations (2.8) are satisfied, then using these, we reconstruct the numbers η_i so that equations (2.12) are satisfied. It will not be hard to verify then that we get a Steinitz representation.

We need a little preparation to deal with edges on the boundary triangle. Recall that we can think of $\mathbf{F}_{ij} = c_{ij}(\mathbf{u}_j - \mathbf{u}_i)$ as the force with which the edge ij pulls its endpoint i. Equilibrium means that for every free node i,

$$\sum_{j\in N(i)} \mathbf{F}_{ij} = 0. \tag{3.14}$$

This does not hold for the nailed nodes, but we can modify the definition of $\mathbf{F}_{ij}$ along the three boundary edges so that $\mathbf{F}_{ij}$ remains parallel to the edge $\mathbf{u}_j - \mathbf{u}_i$ and (2.14) will hold for all nodes (this is the only point where we use that the outer country is a triangle). This is natural by a physical argument: let us replace the outer edges by rigid bars, and remove the nails. The whole structure will remain in equilibrium, so appropriate forces must act in the edges ab, bc and ac to keep balance. To translate this to mathematics, one has to work a little; this is left to the reader as Exercise 2.4.

We claim that we can choose vectors $\mathbf{h}_p$ for all countries p so that

$$\mathbf{h}_p - \mathbf{h}_q = R\mathbf{F}_{ij} \tag{3.15}$$

if ij is any edge and p and q are the two countries on its left and right. This follows by the "potential argument" perhaps familiar from physics. Starting with $\mathbf{h}_T = 0$, and moving from country to adjacent country, this equation will determine the value of $\mathbf{h}_p$ for every country p. What we have to show is that we don't run into contradiction, i.e., if we get to the same country p in two different ways, then we get the same vector $\mathbf{h}_p$. This is equivalent to saying that if we walk around a closed cycle of countries, then the total change in the vector $\mathbf{h}_p$ is zero. It suffices to verify this when we move around countries incident with a single node. In this case, the condition boils down to

$$\sum_{i\in N(j)} R\mathbf{F}_{ij} = 0,$$

which follows by (2.14). This proves that the vectors $\mathbf{h}_p$ are well defined.

Second, we construct numbers η_i satisfying (2.12) by a similar argument (just working on the dual graph). We set $\eta_i = 0$ if i is a node of the unbounded country. Equation (2.12) tells us what the value at one endpoint of an edge must be, if we have it for the other endpoint.

One complication is that (2.12) gives two conditions for each difference $\eta_i - \eta_j$, depending on which country incident with it we choose. But if p and q are the two countries incident with the edge ij, then

$$\mathbf{h}_p^{\mathsf{T}}(\mathbf{u}_j - \mathbf{u}_i) - \mathbf{h}_q^{\mathsf{T}}(\mathbf{u}_j - \mathbf{u}_i) = (\mathbf{h}_p - \mathbf{h}_q)^{\mathsf{T}}(\mathbf{u}_j - \mathbf{u}_i) = (R\mathbf{F}_{ij})^{\mathsf{T}}(\mathbf{u}_j - \mathbf{u}_i) = 0,$$

since $\mathbf{F}_{ij}$ is parallel to $\mathbf{u}_i - \mathbf{u}_j$ and so $R\mathbf{F}_{ij}$ is orthogonal to it. Thus the two conditions on the difference $\eta_i - \eta_j$ are the same.

As before, equation (2.12) determines the values η_i, starting with $\eta_a = 0$. To prove that it does not lead to a contradiction, it suffices to prove that the sum of changes is 0 if we walk around a country p. In other words, if C is the cycle bounding a country p (oriented, say, clockwise), then

$$\sum_{\overrightarrow{ij}\in E(C)} \mathbf{h}_p^{\mathsf{T}}(\mathbf{u}_j - \mathbf{u}_i) = 0,$$

which is clear. It is also clear that $\eta_b = \eta_c = 0$.

Now define $\mathbf{v}_i = \binom{\mathbf{u}_i}{\eta_i}$ for every node i, and $\mathbf{g}_p = \binom{\mathbf{h}_p}{1}$ for every country p. It remains to prove that $i \mapsto \mathbf{v}_i$ maps the nodes of G onto the vertices of a convex polytope, so that edges go to edges and countries go to facets. We start with observing that if p is a country and ij is an edge of p, then

$$\mathbf{g}_p^{\mathsf{T}}\mathbf{v}_i - \mathbf{g}_p^{\mathsf{T}}\mathbf{v}_j = \mathbf{h}_p^{\mathsf{T}}(\mathbf{u}_i - \mathbf{u}_j) + (\eta_i - \eta_j) = 0,$$

and hence there is a scalar α_p so that all nodes of p are mapped onto the hyperplane $\mathbf{g}_p^{\mathsf{T}}\mathbf{x} = \alpha_p$. We know that the image of p under $i \mapsto \mathbf{u}_i$ is a convex polygon, and so the same follows for the map $i \mapsto \mathbf{v}_i$.

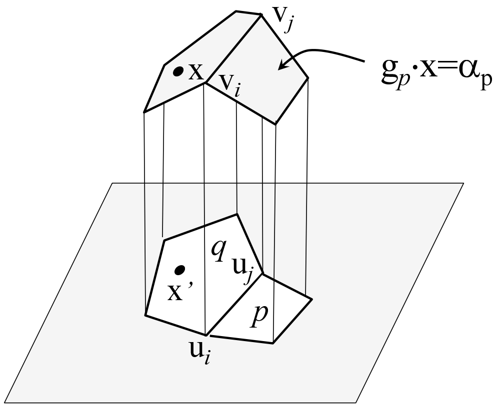

Figure 3.12: Lifting a rubber band representation to a polytope.

To conclude, it suffices to prove that if ij is any edge, then the two convex polygons obtained as images of countries incident with ij "bend" in the right way; more exactly, let p and q be the two countries incident with ij, and let Q_p and Q_q be two corresponding convex polygons (see Figure 2.12). We claim that Q_p lies on the same side of the plane $\mathbf{g}_p^{\mathsf{T}}\mathbf{x} = \alpha_p$ as the bottom face. Let $\mathbf{v}_k$ be any vertex of the polygon Q_q different from $\mathbf{v}_i$ and $\mathbf{v}_j$. We want to show that $\mathbf{g}_p^{\mathsf{T}}\mathbf{v}_k < \alpha_p$. Indeed,

$$\mathbf{g}_p^{\mathsf{T}}\mathbf{v}_k - \alpha_p = \mathbf{g}_p^{\mathsf{T}}\mathbf{v}_k - \mathbf{g}_p^{\mathsf{T}}\mathbf{v}_i = \mathbf{g}_p^{\mathsf{T}}(\mathbf{v}_k - \mathbf{v}_i) = (\mathbf{g}_p - \mathbf{g}_q)^{\mathsf{T}}(\mathbf{v}_k - \mathbf{v}_i)$$

(since both $\mathbf{v}_k$ and $\mathbf{v}_i$ lie on the plane $\mathbf{g}_q^\top \mathbf{x} = \alpha_q$),

$$= \begin{pmatrix} \mathbf{h}_p - \mathbf{h}_q \\ 0 \end{pmatrix}^\top (\mathbf{v}_k - \mathbf{v}_i) = (\mathbf{h}_p - \mathbf{h}_q)^\top (\mathbf{u}_k - \mathbf{u}_i) = (R\mathbf{F}_{ij})^\top (\mathbf{u}_k - \mathbf{u}_i) < 0$$

(since $\mathbf{u}_k$ lies to the right from the edge $\mathbf{u}_i\mathbf{u}_j$). □

3.4.3 How to find a triangular country?

To complete the construction of a Steinitz representation for 3-connected planar graphs, we need one further, rather easy consideration. Theorem 2.4.3 proves Steinitz's theorem in the case when the graph has a triangular country. We are also home if the dual graph has a triangular country; then we can represent the dual graph as the skeleton of a 3-polytope, choose the origin in the interior of this polytope, and consider its polar; this will represent the original graph.

So the proof of Steinitz's theorem is complete, if we prove the following simple fact:

Lemma 3.4.5 *Let G be a 3-connected simple planar graph. Then either G or its dual has a triangular country.*

Proof. If G^* has no triangular country, then every node in G has degree at least 4, and so

$$|E(G)| \geq 2|V(G)|.$$

If G has no triangular country, then similarly

$$|E(G^*)| \geq 2|V(G^*)|.$$

Adding up these two inequalities and using that $|E(G)| = |E(G^*)|$ and $|V(G)| + |V(G^*)| = |E(G)| + 2$ by Euler's theorem, we get

$$2|E(G)| \geq 2|V(G)| + 2|V(G^*)| = 2|E(G)| + 4,$$

a contradiction. □

Exercises to hand in:

Exercise 3.1 Prove that $\min_{\mathbf{u}} \mathcal{E}_S(\mathbf{u})$ (where the minimum is taken over all positions $\mathbf{u}$ with some nodes nailed) is a concave function of the edge weights $S \in \mathbb{R}^E$.

Exercise 3.2 In a rubber band representation, increase the strength of an edge between two non-nailed nodes (while leaving the other edges invariant). Prove that the length of this edge decreases.

Exercise 3.3 Let G be a connected graph, and let $\mathbf{u}$ be a vector-labeling of an induced subgraph H of G (in any dimension). If $(H, \mathbf{u})$ is section-connected, then its rubber-band extension to G is section-connected as well.

Exercise 3.4 Let $\mathbf{u}$ be a rubber band representation of a planar map G in the plane with the nodes of a country T nailed to a convex polygon. Define $\mathbf{F}_{ij} = \mathbf{u}_i - \mathbf{u}_j$ for all edges in $E \setminus E(T)$. (a) If T is a triangle, then we can define $\mathbf{F}_{ij} \in \mathbb{R}^2$ for $ij \in E(T)$ so that $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, $\mathbf{F}_{ij}$ is parallel to $\mathbf{u}_j - \mathbf{u}_i$, and $\sum_{i \in N(j)} \mathbf{F}_{ij} = 0$ for every node i. (b) Show by an example that (a) does not remain true if we drop the condition that T is a triangle.

Exercise 3.5 Prove that every Schlegel diagram with respect to a face F can be obtained as a rubber band representation of the skeleton with the vertices of F nailed (the strengths of the rubber bands must be chosen appropriately).

More exercises for practice:

Exercise 3.6 A *convex representation* of a graph G (in dimension d, with boundary $S \subseteq V$) is a mapping of $V \to \mathbb{R}^d$ such that every node in $V \setminus S$ is in the relative interior of the convex hull of its neighbors.(a) The rubber band representation extending any map from $S \subseteq V$ to $\mathbb{R}^d$ is convex with boundary S. (b) Not every convex representation is constructible this way.

Exercise 3.7 Let G be a 3-connected simple planar graph with a triangular country $p = \overline{abc}$. Let q, r, s be the countries adjacent to p. Let G^* be the dual graph. Consider a rubber band representation $\mathbf{u} : V(G) \to \mathbb{R}^2$ of G with a, b, c nailed down (both with unit rubber band strengths). Prove that the segments representing the edges can be translated so that they form a rubber band representation of $G^* - p$ with q, r, s nailed down (Figure 2.13).

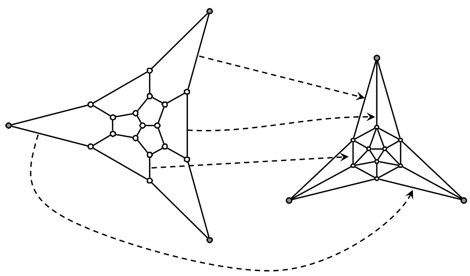

Figure 3.13: Rubber band representation of a dodecahedron with one node deleted, and of an icosahedron with the edges of a triangle deleted. Corresponding edges are parallel and have the same length.

Exercise 3.8 Let G be a connected graph, $\emptyset \neq S \subseteq V$, and $\overline{\mathbf{u}} : S \to \mathbb{R}^d$. Extend $\overline{\mathbf{u}}$ to $\mathbf{u} : V \setminus S \to \mathbb{R}^d$ as follows: starting a random walk at j, let i be the (random) node where S is first hit, and let $\mathbf{u}_j$ denote the expectation of the vector $\overline{\mathbf{u}}_i$. Prove that $\mathbf{u}$ is the same as the rubber band extension of $\overline{\mathbf{u}}$.

Exercise 3.9 Prove that a 1-dimensional rubber band representation of a 2-connected graph, with boundary nodes s and t, nondegenerate in the sense that the nodes are all different, is s-t-numbering (as defined in the Introduction). Show that instead of the 2-connectivity of G, it suffices to assume that deleting any node, the rest is either connected or has two components, one containing s and one containing t.