Summer School in Mathematics

Introduction to graph limits

Eötvös Loránd University, Budapest

25-29 June, 2018

Special thanks go to Wiebke Höhn and Max Klimm for providing the original template of this guide book. The lecturers were supported by the European Union, co-financed by the European Social Fund (EFOP-3.6.3-VEKOP-16-2017-00002). Supporters:

Contents

Basic information **5**

Sightseeing, restaurants, bars, cafés and more **9**

Ágnes Backhausz - Limits of random graphs **19**

1 Introduction . . . 19
2 Inhomogeneous random graphs . . . 19
3 Dense graph limits . . . 20
4 Convergence of random graphs . . . 21
5 Random regular graphs . . . 24

Péter Csikvári - Statistical physics and graph limits **29**

1 Introduction . . . 29
2 Statistical physical models . . . 29
3 Graph limits and examples . . . 31
4 Monomer-dimer and dimer model . . . 31
5 A proof strategy . . . 38

Péter Frenkel - Homomorphism numbers and graph convergence **47**

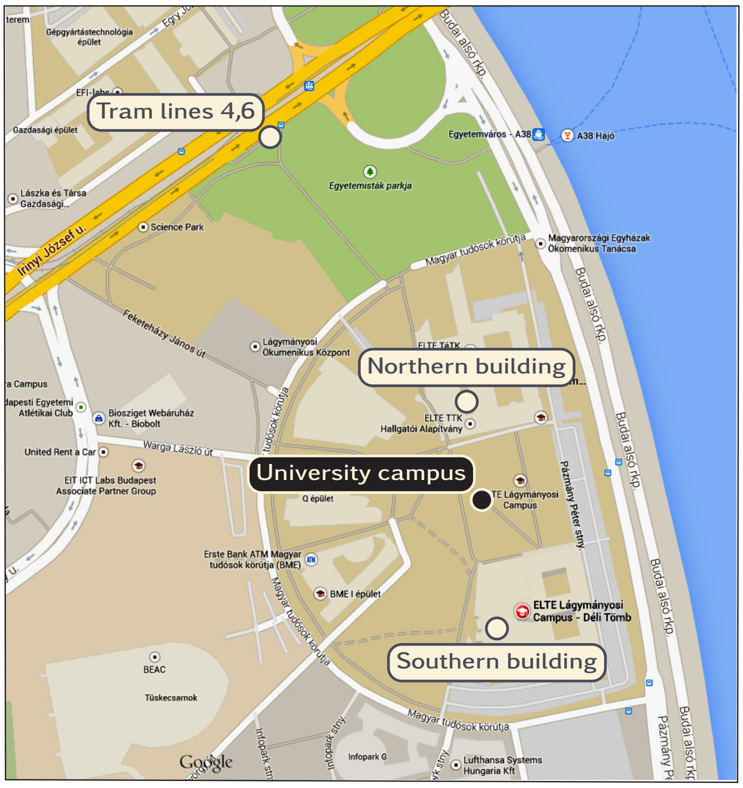
1 Limits of dense graphs . . . 47
2 Local limits of graphs of bounded degree . . . 49

Dávid Kunszenti-Kovács - From combinatorics to analysis **51**

1 A heuristic approach to the limit objects . . . 51
2 From counting homomorphisms to topologies on function spaces . . . 51
3 Cut norm versus L^1, and other norms on kernels . . . 54

László Miklós Lovász - Extremal graph theory: flag algebras and finite forcibility **55**

1 Introduction . . . 55
2 Flag algebras . . . 55
3 Triangle edge count . . . 59
4 Other applications . . . 59
5 Finite forcibility . . . 59



Welcome to Summer School in Mathematics 2018!

In this short guide we would like to provide you with some basic information about practical issues as well as a rather incomplete list of sights, museums, restaurants, bars and pubs.

If you have any question concerning the summer school or your stay in Budapest, do not hesitate to ask us!

We wish you a pleasant stay in Budapest!

The Organizers

Venue

The summer school takes place in the Southern building ("Déli tömb") at the Lágymányosi Campus of the Eötvös Loránd University. The lectures are given in room 4-710, refreshments are provided in room 4-713.

Internet access

Wireless internet connection is available in the lecture room. If you need a scanner or a printer, please contact the organizers.

Lunch

There are several places around the university where you can have lunch. A cafeteria with a limited menu (including vegetarian choices) is in the northern building of the university. For a more complete list of restaurants, see the section "Around the campus".

Contact

If you have any questions or problems, please contact the organizers or the lecturers.

Organizers:

István Ágoston
Email:
agoston@cs.elte.hu
Tel.: +36 1 372 2500 / 8422
Room: 3.708

Kristóf Bérczi
Email:
berkri@cs.elte.hu
Tel.: +36 1 372 2500 / 8582
Room: 3.519

Attila Joó
Email:
attila.joo@uni-hamburg.de
Tel.: -
Room: -

Lecturers:

Ágnes Backhausz
Email:
agnes@cs.elte.hu
Tel.: +36 1 372 2500 / 8529
Room: 3.415

Péter Csikvári
Email:
peter.csikvari@gmail.com
Tel.: +36 1 372 2500 / 8604
Room: 3.616

Péter Frenkel
Email:
frenkelp@renyi.hu
Tel.: +36 1 372 2500 / 8424
Room: 3.710a

Dávid Kunszenti-Kovács
Email:
daku@fa.uni-tuebingen.de
Tel.: -
Room: 3.707

Getting around by taxi

The taxi fares are uniformly calculated as follows: the base fee is 450 HUF with an additional 280 HUF/km or 70 HUF/min. Altogether, getting around by taxi is rather expensive and you might want to consider using public transport instead.

Getting around by bike

You can rent bikes for 2500 - 3500 HUF per day at many places. Here are some possibilities:

- **Budapestbike** - `http://www.budapestbike.hu/`
- **Yellow Zebra Bikes and Segways** - `http://www.yellowzebrabikes.com/`
- **Bikebase** - `http://bikebase.hu/home`
- **Bestbikerental** - `http://www.bestbikerental.hu/`

Recently a large bicycle rental network (so called "Bubi") with many rental and return locations was introduced as part of the public transport system. You will find many automated rental places mostly in the downtown areas. If you want to rent a bicycle, you will have to pay a one week access fee of 2000 HUF plus the usage fee which is free for the first 30 minutes. A deposit is also required. Please, note that the fares are designed to encourage short term use. For the fares and further details see their website:

- **MOL Bubi** - `http://molbubi.bkk.hu/`

Getting around by public transport

The public transportation system in Budapest is a favourite internal travel option for a number of Budapest visitors. The system is efficient, inexpensive and runs throughout all of the major tourist areas of Budapest. The system consists of a combination of the bus, trolley-bus, tram, metro, and train lines and is streamlined so that tickets for all of them can generally be purchased at the same locations.

All regular transportation services stop around midnight (varies by route). However, night buses (blue coloured buses, marked with black in the schedule, numbered 900-999 and tram line 6) replace the metro lines, major tram and bus routes and run through the night until normal service resumes in the morning. Separate schedules for night and day buses are posted at every stop. In the inner areas buses run very frequently (appr. 10-15 min.) Please note: there's front door boarding-only on most lines, except tram 6 and articulated buses.

Budapest currently has four metro lines - M1 (Yellow), M2 (Red), M3 (Blue) and M4 (Green). The Yellow line is the oldest underground transportation line in continental Europe and retains much of its old-fashioned charm. Lines M1, M2 and M3 meet at Deák Ferenc Tér in central Pest. Line M4 opened on 28 March 2014 and connects the Kelenföld railway station located in Buda, and the eastern Keleti station in Pest. Trains run frequently (2-5 minutes on weekdays, 5 minutes on weekends, 10 minutes late night).

Budapest has an extensive system of above-ground trams. The most useful lines for tourists are the famous 4 and 6, which follow the large ring road that encircles the Budapest city center and crosses Margaret bridge before terminating at Széll Kálmán Tér on the Buda side on the North - and Petőfi bridge before it terminates at Móricz Zsigmond Körtér, also on the Buda side; no 47 and 49, which runs through central Pest and across the river all the way to Kelenföldi railway station; no. 2, which follows the Danube River on the Pest side; and no. 19, which follows the Danube River on the Buda Side.

Bus lines of use to most tourists are the 7 and 107 which connect the busy Keleti railway station and the area around the Kelenföld railway station on the Buda side. Some other notable places that they stop along is Blaha Lujza tér (connection to the red M2 metro line, also trams 4 and 6), Ferenciek tere (connection to metro line M3, also very near Váci utca), and in front of the Rudas bath and the Gellért Hotel both on the Buda side. Bus 27 takes you to the top of Gellért Hill from Móricz Zsigmond körtér while bus 26 connects Nyugati station (Metro line 3) and Árpád bridge (on the same metro line) via Margaret Island.

All public transport in Budapest is run by the company BKK. Connections can be easily checked at `http://www.bkk.hu/en/timetables/` or by using the convenient smart phone apps available for **Android** or **iPhone**. Online information - based on onboard GPS devices - about position of buses and timetable of bus, tram and trolley stops is available at `http://futar.bkk.hu`. The map also contains information about the number of available bicycles at the Bubi stations.

On the metro lines, **tickets need to be bought and validated before boarding** while on buses and trams you have to validate your ticket on the spot. For a complete list of tickets and conditions see `http://www.bkk.hu/en/prices/`.

Single ticket 350 HUF
Valid for one uninterrupted trip without change. On the metro lines the ticket must be validated before the start of the trip; on other vehicles immediately after boarding or after the vehicle has departed. Validity period is 80 minutes after stamping; it is 120 minutes on night services.

Block of 10 tickets 3000 HUF
You can buy 10 tickets in a block with some discount compared to buying 10 single tickets separately.

Transfer ticket 530 HUF
Valid for one trip with one change. Trip interruptions - other than changes - and return trips are not permitted. The ticket must be validated at the printed number grids at either end: first when starting a trip at one end and at the other end when changing, with the exception of changes between metro lines.

Short section metro ticket for up to 3 stops 300 HUF
Valid on the metro for one short trip of up to 3 stops for 30 minutes after validation. Trip interruptions and return trips are not permitted.

Single ticket for public transport boat 750 HUF
24-hour, 72-hour, 7-day travelcards and Monthly Budapest pass is valid on weekdays.

Budapest 24-hour travelcard 1650 HUF
Valid for 24 hours from the indicated date and time (month, day, hour, minute) for an unlimited number of trips.

5/30 BKK 24-hour travelcard 4550 HUF
The 5/30 BKK travelcard consists of 5 slips, each with a validity period of 24 hours. The block can be purchased with any starting day with a validity period of 30 days from the starting day.

Budapest 72-hour travelcard 4150 HUF
Valid for 72 hours from the indicated date and time (month, day, hour, minute) for unlimited number of trips on the public transport services ordered by BKK on tram, trolleybus, underground, metro, cogwheel railway on the whole length of the lines on all days; on the whole length of boat services but only on working

days.

Budapest 7-day travelcard 4950 HUF

Valid from 00:00 on the indicated starting day until 02:00 on the following 7th day for an unlimited number of trips. The travelcard is to be used only by one person; it is non-transferable as it is issued specifically for the holder.

Monthly Budapest pass for students 3450 HUF

Valid from 0:00 of the indicated optional starting day to 2:00 of the same day of the following month. Valid for students in higher education together with a Hungarian or EU or ISIC student ID.

Giraffe Hop On Hop Off

Giraffe Hop On Hop Off tours offer 2 bus, 1 boat and 1 walking tour during the day and 1 bus tour in the evening in Budapest for tourists. They pass several sights on their way; the RED and YELLOW lines are audio guided in 20 languages and the BLUE boat line is audio guided in English and German. The ticket is valid on the day of the first departure while the next day is free.

Things to do in Budapest

- Walk along the Danube on the Pest side between Elisabeth Bridge and Chain Bridge. Then cross the Danube and continue towards Margareth Bridge to see the Parliament Building from Batthyány tér.
- Take Metro 1 from Vörösmarty tér to Hősök tere and see the monument there. You may take a walk in the City Park (Városliget) or go to the Zoo or the Museum of Fine Arts.
- After 7:00pm take a short walk along Ráday utca from Kálvin tér. You may want to enter one of the cafés or restaurants.
- Go to the Great Market Hall on Vámház körút, close to Liberty Bridge (Szabadság híd). After all, this is one of the few things Margaret Thatcher did when she visited Budapest in the 1980's (she bought garlic :)).
- Go to Gellért Hill to get a glimpse of the city from above.
- Go to a concert in one of the major or smaller concert halls, churches or open air locations. Some of them are free.
- Go to Margaret Island and see the fountain on the southern end or the music tower on the northern end of the island. In between you will find a garden of roses and a small zoo.
- Go at night to the Palace of Arts (across Eötvös University, close to Rákóczi Bridge on the Pest side) and enjoy the view of the National Theatre or of the glass walls of the Palace of Arts.
- See the bridges at night. You get a good view from Castle Hill.
- Go to some of the baths in Budapest (Széchenyi bath in the City park, Rudas bath at the Buda side of Elisabeth Bridge or Gellért bath in Hotel Gellért at the Buda side of Liberty Bridge).

Words of caution

- Don't go to a restaurant or café without checking the price list first. A reasonable dinner should not cost you more than 20 Euros (6000 HUF). (Of course you may be willing to pay more but you should know in advance.)
- Don't leave your valuables unattended, especially not in places frequently visited by tourists. Be aware of pick-pockets on crowded buses or trams.
- Budapest is relatively safe even at night, nevertheless if possible, try to avoid being alone on empty streets at night. Some pubs should also be avoided.
- Don't carry too much cash with you: direct payment banking cards and most credit cards are widely accepted. If you withdraw money from a banking machine, be careful and try to do it in a public place.
- If you get on a bus, tram, trolley or metro, usually you have to have a pass or a prepaid ticket which you have to validate upon boarding (or when entering the metro station). Most of the tickets are valid for a single trip only (even if it is only for a short distance). If you get a pass for a week, you have to enter on the ticket the number of a photo id (passport, id card) which you have to carry with you when using the pass. - On some buses you may get a single ticket from the driver but be prepared to have change with you. Even tickets bought from the driver have to be validated.

Shopping centres

Mammut. 1024 Budapest, II. district, Lövőház utca 2-6, +36 1 345 8020 `www.mammut.hu`

Westend City Center. 1062 Budapest, VI. district, Váci út 1-3, +36 1 238 7777 `www.westend.hu`

Corvin Plaza. 1083 Budapest, VIII. district, Futó utca 37, +36 1 799 2440 `www.corvinplaza.hu`

Arena Plaza. 1087 Budapest, VIII. district, Kerepesi út 9, +36 1 880 7010 `www.arenaplaza.hu`

Allee. 1117 Budapest, XI. district, Október huszonharmadika utca 8-10, +36 1 372 7208 `www.allee.hu`

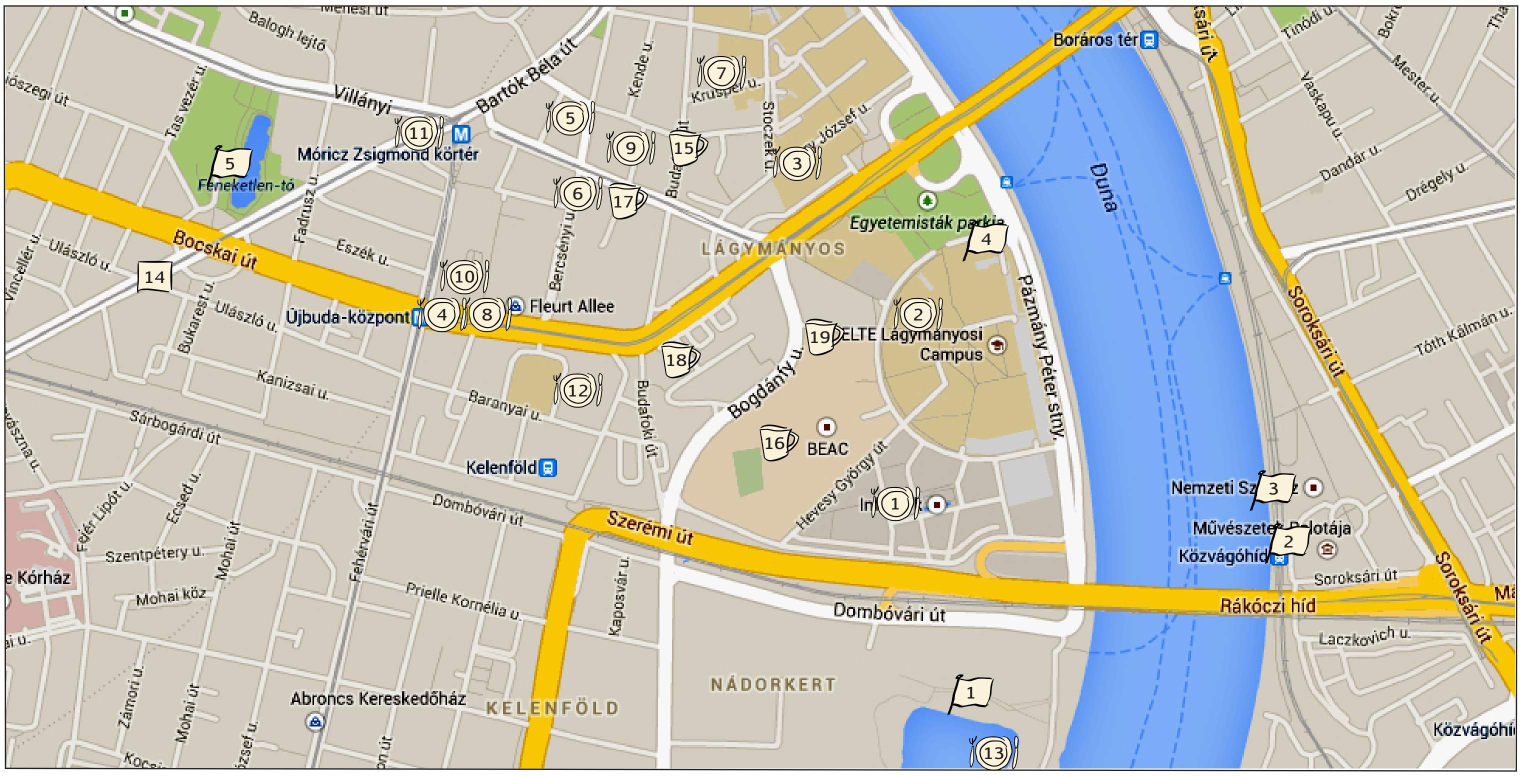
Market halls

Batthyány téri Vásárcsarnok. 1011 Budapest, I. district, Batthyány tér 5

Rákóczi téri Vásárcsarnok. 1084 Budapest, VIII. district, Rákóczi tér 7-9, Mon 6:00am–4:00 pm, Tue–Fri 6:00am–6:00pm, Sat 6:00am–1:00pm

Vámház körúti Vásárcsarnok. 1093 Budapest, IX. district, Vámház körút 1-3, Mon 6:00am–5:00 pm, Tue–Fri 6:00am–6:00pm, Sat 6:00am–3:00pm

Fehérvári úti Vásárcsarnok. 1117 Budapest, XI. district, Kőrösi J. utca 7-9, Mon 6:30am–5:00 pm, Tue–Fri 6:30am–6:00pm, Sat 6:30am–3:00pm



Sights

1 Kopaszi-gát. *1117 Budapest, Kopaszi gát 5, Bus 153, 154, 154B, 6:00am–2:00am.*
Kopaszi-gát is a beautifully landscaped narrow peninsula in south Buda, next to Rákóczi Bridge. Nested in between the Danube on one side and a protected bay, it has a lovely beach feel. Kopaszi-gát is also a favourite picnic spot and the park offers lots of outdoor activities from biking to ball games. The sign in the park says it all: "Fűre lépni szabad!", which means "Stepping on the grass is permitted!"

2 Palace of Arts. *1095 Budapest, Komor Marcell utca 1, Suburban railway 7, Tram 1, 2, 24, Mon-Sun 10:00am–10:00pm (varies).*
The Palace of Arts in Budapest, also known as MÜPA for short (Művészetek Palotája), is located within the Millennium Quarter of the city, between Petőfi and Lágymányosi bridges. It is one of the most buzzing cultural and musical centres in Budapest, and as such one of the liveliest Budapest attractions.

3 National Theatre. *1095 Budapest, Bajor Gizi park 1, Suburban railway 7.*
The building lies on the bank of the Danube, and is a five-minute walk from the Csepel HÉV (Suburban railway 7). The area of the theatre can be functionally separated into three parts. The central part is the nearly round building of the auditorium and stage, surrounded by corridors and public areas. The second is the U-shaped industrial section around the main stage. The third section is the park that surrounds the area, containing numerous memorials commemorating the Hungarian drama and film industry.

4 A38 Ship. *1117 Budapest, a little South from Petőfi bridge, Buda side, Trams 4 and 6 (Petőfi híd, budai hídfő), Buses 153, 154, 154B, Mon–Sat 11:00am–11:00pm.*
The world's most famous repurposed Ukrainian cargo ship, A38 is a concert hall, cultural center and restaurant floating on the Danube near the abutment of Petőfi Bridge on the Buda-side with a beautiful panorama. Since its opening it has become one of Budapest's most important venues, and according to artists' feedback, one of Europe's coolest clubs.

5 Feneketlen-tó. *1114 Budapest, Bus 7, 107, Tram 19, 41, 47, 48, 49.*
Feneketlen-tó, which means bottomless lake, is surrounded by a beautiful park filled with paths, statues and children's playgrounds. The lake is not as deep as its name suggests. In the 19th century there was a brickyard in its place and the large hole dug by the workers filled with water when they accidentally hit a spring. Ever since, locals cherish the park and they come to feed the ducks, relax on the benches or take a stroll around the lake. The lake's water quality in the 1980s began to deteriorate, until a water circulation device was built. The lake today is a popular urban place for fishing.

Restaurants & Eateries

1 Infopark. *Next to the university campus, Mon–Fri 8:00am–6:00pm.*
Infopark is the first innovation and technology park of Central and Eastern Europe. There are several cafeterias and smaller sandwich bars hidden in the buildings, most of them are really crowded between 12:00am-2:00pm.

2 University Cafeteria. *University campus, Northern building, Mon–Fri 8:00am–4:00pm.*
The university has a cafeteria on the ground floor of the Northern building. You can also buy sandwiches, bakeries, etc here.

3 Goldmann restaurant. *1111 Budapest, Goldmann György tér 1, Mon–Fri 11:00am–3:00pm.*
Goldmann is a cafeteria of the Technical University, popular among students for its reasonable offers. Soups are usually quite good.

4 Fehérvári úti vásárcsarnok. *1117 Budapest, Kőrösi József utca 7–9, Mon 6:30am–5:00pm, Tue–Fri 6:30am–6:00pm, Sat 6:30am–2:00pm.*
A farmers market with lots of cheap and fairly good native canteens (e.g. Marika Étkezde) on the upper floors. You can also find cheese, cakes, fruits, vegetables etc.

5 Anyu. *1111 Budapest, Bercsényi utca 8, Mon–Fri 11:00am–4:00pm.*
Tiny bistro selling home-made soup, sandwiches and cakes.

6 Turkish restaurant. *1111 Budapest, Karinthy Frigyes út 26, Mon–Sun 10:00am–0:00am.*

This tiny Turkish restaurant offers gyros, baklava and salads at a reasonable price.

7 **Stoczek.** *1111 Budapest, Stoczek utca 1-3, Mon–Fri 11:00am–3:30pm.*
Stoczek is a cafeteria of the Technical University. It offers decent portions for good price. There are two floors, a café can be found downstairs.

8 **Allee.** *1117 Budapest, Október huszonharmadika utca 8-10, Mon–Sat 9:00am–10:00pm, Sun 9:00am–8:00pm.*
A nearby mall with several restaurants on its 2nd floor.

9 **Íz-lelő étkezde.** *1111 Budapest, Lágymányosi utca 17, Mon–Fri 10:00am–3:00pm.*
Decent lunch for low price, and student friendly atmosphere. Only open from Monday to Friday!

10 **Cserpes Milk Bar.** *1117 Budapest, Október Huszonharmadika utca 8-10, Mon–Fri 7:30am–10:00pm, Sat 9:00am–8:00pm, Sun 9:00am–6:00pm.*
A milk bar just next to the shopping center Allee. Great place for having a breakfast or a quick lunch.

11 **Wikinger Bistro.** *1114 Budapest, Móricz Zsigmond körtér 4, Mon–Sun 11:00am–10:00pm.*
If you are up for hamburgers, Wikinger Bistro offers a huge selection of different burgers.

12 **Hai Nam Bistro.** *1117 Budapest, Október huszonharmadika utca 27, Mon–Sat 10:00am–9:00pm, Sun 10:00am–3:00pm.*
If you like Vietnamese cuisine and Pho, this may be the best place in the city. It is a small place, so be careful, it is totally full around 1:00pm.

13 **Vakvarjú.** *1117 Budapest, Kopaszi gát 2, Mon–Sun 11:30am–10:00pm.*
Vakvarjú can be found on the Kopaszi gát. It is a nice open-air restaurant where you can have lunch and relax next to the Danube for a reasonable price.

Others

14 **Gondola.** *1115 Budapest, Bartók Béla út 69-71, Mon–Sun 10:00am–8:00pm.*
This is a nice little ice cream shop right next to the Feneketlen-tó (Bottomless Lake).

Pubs

15 **Bölcső.** *1111 Budapest, Lágymányosi utca 19, Mon–Fri 11:30am–11:00pm, Sat–Sun 12:00am–11:00pm.*
Bölcső has a nice selection of Hungarian and Czech craft beers and one of the best all-organic homemade burger of the city. Other than burgers, the menu contains homemade beer snacks such as pickled cheese, hermelin (a typical Czech bar snack), and breadsticks. Bölcső also boasts a weekly menu that makes a perfect lunch or dinner.

16 **Szertár.** *1117 Budapest, Bogdánfy utca 10, Mon–Fri 8:30am–4:30am.*
Szertár is a small pub close to the university campus. It is located at the BEAC Sports Center and offers sandwiches and hamburgers as well. A perfect place to relax after a long day at the university where you can also play kicker.

17 **Pinyó.** *1111 Budapest, Karinty Frigyes út 26, Mon 10:00am–0:00am, Tue–Sat 10:00am–1:00am, Sun 4:00pm–0:00am.*
Squeezed to a basement, Pinyó looks like being after a tornado: old armchairs, kicker table, tennis racket on the wall, ugly chairs and tables. It does not promise a lot, but from the bright side, it is completely foolproof. Popular meeting place among students.

18 **Lusta Macska.** *1117 Budapest, Irinyi József utca 38, Mon–Wed 4:00pm–0:45am, Thu–Sat 4:00pm–2:00am.*
Lusta Macska is a cheap pub for students close to the Schönherz dormitory of the Technical University. It is a tiny place with very simple furniture.

19 **Kocka.** *1111 Budapest, Warga László út 1, Mon–Fri 6:30am–7:50pm.*
Nearby the campus, the Kocka Pub is a rather cheap place mainly for students.

Sights

Great Market Hall. *1093 Budapest, Vámház körút 1-3, Metro 4, Tram 47, 48, 49, Mon 6:00am–5:00 pm, Tue–Fri 6:00am–6:00pm, Sat 6:00am–3:00pm.*

1

Central Market Hall is the largest and oldest indoor market in Budapest. Though the building is a sight in itself with its huge interior and its colourful Zsolnay tiling, it is also a perfect place for shopping. Most of the stalls sell fruits and vegetables but you can also find bakery products, meat, dairy products and souvenir shops. In the basement there is a supermarket.

2

Károlyi Garden. *1053 Budapest, Károlyi Mihály utca 16, Metro 2, Bus 5, 7, 8, 107, 133, 233, Tram 47, 48, 49.*

Károlyi Garden is maybe the most beautiful park in the center of Budapest. It was once the garden of the castle next to it (Károlyi Castle, now houses the Petőfi Literature Museum). In 1932 it was opened as a public garden. In the nearby Ferenczy utca you can see a fragment of Budapest's old town wall (if you walk in the direction of Múzeum körút).

3

Gellért Hill and the Citadel. *1118 Budapest, Metro 4, Bus 5, 7, 8, 107, 133, 233.*

The Gellért Hill is a 235 m high hill overlooking the Danube. It received its name after St. Gellért who came to Hungary as a missionary bishop upon the invitation of King St. Stephen I. around 1000 a.d. If you approach the hill from Gellért square, you can visit the Gellért Hill Cave, which is a little chapel. The fortress of the Citadel was built by the Habsburgs in 1851 to demonstrate their control over the Hungarians. Though it was equipped with 60 cannons, it was used as threat rather than a working fortification.
From the panorama terraces one can have a stunning view of the city, especially at night. By a short walk one can reach the Liberation Monument.

Sziklatemplom (Cave Church). *1111 Budapest, Szent Gellért tér, Metro 4, Buses 7, 107, 133, 233, Trams 19, 41, 47, 48, 49, 56, 56A, Mon–Sat 9:30am–7:30pm, 500 HUF.*

4

The Cave Church, located inside Gellért Hill, isn't your typical church with high ceilings and gilded interior. It has a unique setting inside a natural cave system formed by thermal springs.

Rudas Gyógyfürdő és Uszoda. *1013 Budapest, Döbrentei tér 9 (a little South from Erzsébet bridge, Buda side), Buses 5, 7, 8, 107, 109, 110, 112, 233, 239, Trams 19, 41, 56, 56A, Mon–Sun 6:00am–10:00pm; Night bath: Fri–Sat 10:00pm–4:00am, 1350–4200 HUF.*

5

Centered around the famous Turkish bath built in the 16th century, Rudas Spa offers you several thermal baths and swimming pools with water temperatures varying from 16 to 42 Celsius degrees.

Gellért Gyógyfürdő és Uszoda. *1118 Budapest, Kelenhegyi út 4 (at Gellért tér), Metro 4, Buses 7, 107, 133, 233, Trams 19, 41, 47, 48, 49, 56, 56A, Mon–Sun 6:00am–8:00pm, 5100–5500 HUF.*

6

Gellért Thermal Bath and Swimming Pool is a nice spa in the center of the city.

7

National Museum. *1088 Budapest, Múzeum körút 14-16, Metro 3, 4, Bus, 9, 15, 115, Tram 47, 48, 49, Tue–Sun 10:00am–6:00pm, 800 HUF.*

The Hungarian National Museum (Hungarian: Magyar Nemzeti Múzeum) was founded in 1802 and is the national museum for the history, art and archaeology of Hungary, including areas not within Hungary's modern borders such as Transylvania; it is not to be confused with the collection of international art of the Hungarian National Gallery. The museum is in Budapest VIII in a purpose-built Neoclassical building from 1837-47 by the architect Mihály Pollack.

Restaurants & Eateries

1

Pagony. *1114 Budapest, Kemenes utca 10, Mon–Sun 11:00am–10:00pm.*

If you are looking for a cool spot in the blazing summer heat of Budapest, look no further. This joint was created by its resourceful proprietor by converting an unused toddler's pool section of the Gellért bath into a trendy pub. While there is no water (yet) in the pools, you can definitely find a table with comfy chairs which are actually in a wading pool.

2

Hummus Bar. *1225 Budapest, Bartók Béla út 6, Mon–Fri 10:00am–10:00pm, Sat–Sun 12:00am–10:00pm.*

The famous homemade Hummus can be enjoyed in variety of different dishes. The menu offers everything from a wide variety of quality salads, soups, desserts, meats and vegetarian dishes. The food is prepared with great care using only high quality products, and focusing on the simplicity of preparation - thus allowing affordable pricing.

3 **Főzelékfaló Ételbár.** *1114 Budapest, Bartók Béla út 43–47, Mon–Fri 10:00am–9:30pm, Sat 12:00–8:00pm.*

Főzelékfaló Ételbár boasts a selection centered on főzelék, a Hungarian vegetable dish that is the transition between a soup and a stew, but you can get fried meats, several side dishes, and desserts as well.

4 **Főzelékfaló Ételbár.** *1053 Budapest, Petőfi Sándor utca 1 (Ferenciek tere), Mon–Fri 10:00am–8:00pm, Sat 12:00–8:00pm.*

Főzelékfaló Ételbár boasts a selection centered on főzelék, a Hungarian vegetable dish that is the transition between a soup and a stew, but you can get fried meats, several side dishes, and desserts as well.

5 **Púder.** *1091 Budapest, Ráday utca 8, Mon-Sun 12:00am–1:00am.*

Restaurant and bar with a progressive, eclectic interior that was created by Hungarian wizards of visual arts. Its back room gives home to a studio theatre. Many indoor and outdoor cafés, bars, restaurants and galleries are located in the same street, the bustling neighborhood of Ráday Street is often referred to as "Budapest Soho".

Cafés

6 **CD-fű.** *1053 Budapest, Fejér György utca 1, Mon–Thu 6:00pm–12:00pm, Fri-Sat 4:00pm–12:00pm.*

As the third teahouse of Budapest, CD-fű is located in a slightly labyrinth-like basement. With its five rooms it is a bit larger than usual, and also gives place for several cultural events.

7 **Hadik kávéház.** *1118 Budapest, Bartók Béla út 36, Mon–Sat 9:00am–11:00pm.*

A lovely place to relax and soak up the atmosphere of pre-war years in Budapest. Hadik is a pleasant, old-fashioned café serving excellent food.

8 **Sirius Teaház.** *1088 Budapest, Bródy Sándor utca 13, Mon–Sun 12:00am–10:00pm.*

Sirius teahouse has the perfect atmosphere to have a cup of tea with your friends, but it is better to pay attention to the street numbers, this teahouse is very hard to find, there is no banner above the entrance. Customers can choose from 80 different types of tea.

Pubs

9 **Mélypont Pub.** *1053 Budapest, Magyar utca 23, Mon–Tue 6:00pm–1:00am, Wed–Sat 6:00pm–2:00am.*

Basement pub in the old city center. Homey atmosphere with old furniture.

10 **Trapéz.** *1093 Budapest, Imre utca 2, Mon–Tue 10:00am–0:00am, Wed–Fri 10:00am–2:00am, Sat 12:00am–2:00am.*

Nice ruin pub in an old house behind the Great Market Hall which also has an open-air area. You can watch sports events and play kicker on the upper floor.

11 **Élesztő.** *1094 Budapest, Tűzoltó utca 22, Mon–Sun 3:00pm–3:00am.*

Élesztő is the Gettysburg battlefield of the Hungarian craft beer revolution; it's a like a mixture of a pilgrimage site for beer lovers, and a ruin-pub with 17 beer taps, a home brew bar, a theater, a hostel, a craft pálinka bar, a restaurant and a café.

12 **Mr. & Mrs. Columbo.** *1013 Budapest, Szarvas tér 1, Mon–Sat 4:00pm–11:00pm.*

A nice pub with excellent food and czech beers. Their hermelin is really good.

13 **Aréna Corner.** *1114, Budapest, Bartók Béla út 76, Sun–Fri 12:00am–11:00pm, Sat 12:00am-12:00pm.*

A nice place to watch World Cup matches while drinking Czech beer.

Others

14 **Mikszáth square.** *1088 Budapest, Mikszáth Kálmán tér.*

Mikszáth tér and the surrounding streets are home to many cafés, pubs and restaurants usually with nice outdoor terraces. Many places there provide big screens to watch World Cup matches.

Sights

St. Stephen's Basilica. *1051 Budapest, Szent István tér 1, Metro 1, 2, 3, Bus 9, 16, 105, Guided tours Mon–Fri 10:00am–3:00pm, 1200 HUF.*
This Roman Chatolic Basilica is the most important church building in Hungary, one of the most significant tourist attractions and the third highest building in Hungary. Equal with the Hungarian Parliament Building, it is one of the two tallest buildings in Budapest at 96 metres (315 ft) - this equation symbolises that worldly and spiritual thinking have the same importance. According to current regulations there cannot be taller building in Budapest than 96 metres (315 ft). Visitors may access the dome by elevators or by climbing 364 stairs for a 360° view overlooking Budapest.

Opera. *1061 Budapest, Andrássy út 22, Metro 1, Bus 105, Trolleybus 70, 78, Tours start at 2:00pm, 3:00pm and 4:00pm, 1990 HUF.*
The Opera House was opened in 1884 among great splendour in the presence of King Franz Joseph. The building was planned and constructed by Miklós Ybl, who won the tender among other famous contemporary architects.

Kossuth Lajos Square. *1055 Budapest, Metro 2, Bus 15, 115, Tram 2.*
The history of Kossuth Lajos Square goes back to the first half of the 19th century. Besides the Parliament, other attractions in the square refer to the Museum of Ethnography (which borders the square on the side facing the Parliament) and to several monuments and statues. The square is easily accessible, since the namesake metro station is located on the south side of the square.

Fishermen's Bastion. *1014 Budapest, Hess Andras Square, Bus 16, 16A, 116, all day, tower: daily 9:00am–11:00pm;, free, tower: 350 HUF.*
On the top of the old fortress walls, the Fishermen's Bastion was only constructed between 1895-1902. It is named after the fishermen's guild because according to customs in the middle ages this guild was in charge of defending this part of the castle wall. As a matter of fact it has never had a defending function. The architect was Frigyes Schulek, who planned the building in neo-gothic style.

Parliament. *1055 Budapest, Kossuth Lajos tér 1-3, Metro 2, Bus 15, 115, Tram 2, Mon–Suni 8:00am–6:00pm, 3500 HUF, EU citizens and students 1750 HUF, EU students 875 HUF.*
The commanding building of Budapest Parliament stretches between Chain Bridge and Margaret Bridge on the Pest bank of the Danube. It draws your attention from almost every riverside point. The Gellért Hill and the Castle Hill on the opposite bank offer the best panorama of this huge edifice. The Hungarian Parliament building is splendid from the inside too. You can visit it on organised tours. Same-day tickets can be purchased in limited numbers at our ticket office in the Museum of Ethnography. Advance tickets are available online at `www.jegymester.hu/parlament`.

Buda Castle and the National Gallery. *1014 Budapest, Szent György tér 2, Bus 16, Funicular, Tue–Sun 10:00am–6:00pm, 900 HUF.*
Buda Castle is the old royal castle of Hungary, which was damaged and rebuilt several times, last time after World War II. Now it houses the Széchényi Library and the National Gallery, which exhibits Hungarian paintings from the middle ages up to now. The entrance to the castle court is free (except if there is some festival event inside). One of the highlights of the court is the Matthias fountain which shows a group of hunters, and the monument of Prince Eugene Savoy. From the terrace of the monument you have a very nice view of the city.

Matthias Church. *1014 Budapest, Szentháromság tér 2, Bus 16, 16A, 116, Mon–Fri 9:00am–5:00pm, Sat 9:00am–12:15pm, Sun 1:00pm–5:00pm, 1000 HUF.*
Matthias Church (Mátyás-templom) is a Roman Catholic church located in front of the Fisherman's Bastion at the heart of Buda's Castle District. According to church tradition, it was originally built in Romanesque style in 1015. The current building was constructed in the florid late Gothic style in the second half of the 14th century and was extensively restored in the late 19th century. It was the second largest church of medieval Buda and the seventh largest church of medieval Hungarian Kingdom. Currently it also regularly houses various concerts.

Heroes Square. *1146 Budapest, Hősök tere, Metro 1, Bus 20E, 30, 105, Trolleybus 72, 75, 79.*

The monumental square at the end of Andrássy Avenue sums up the history of Hungary. The millennium memorial commemorates the 1000th anniversary of the arrival of the Hungarians in the Carpathian Basin.

9 **Városliget.** *1146 Budapest, Városliget, Metro 1, Bus 20E, 30, 105, Tram 1, Trolleybus 70, 72, 74, 75, 79.*

Városliget (City Park) is a public park close to the centre of Budapest. It is the largest park in the city, the first trees and walkways were established here in 1751. Its main entrance is at Heroes Square, one of Hungary's World Heritage sites.

10 **Vajdahunyad vára.** *1146 Budapest, Városliget, Metro 1, Bus 20E, 30, 105, Trolleybus 70, 72, 75, 79, Courtyard always open, Castle Tue–Sun 10:00am-5:00pm, Courtyard free, Castle 1100 HUF.*

Vajdahunyad Castle is one of the romantic castles in Budapest, Hungary, located in the City Park by the boating lake / skating rink. The castle, despite all appearances, was built in 1896, and is in fact a fantasy pastiche showcasing the architectural evolution through centuries and styles in Hungary. The castle is the home of several festivals, concerts and the exhibitions of the Hungarian Agricultural Museum.

11 **Museum of Fine Arts (Szépművészeti Múzeum).** *1146 Budapest, Dózsa György út 41, Metro 1, Trolleybus 72, 75, 79, Temporarily closed till 2018 due to renovation.*

The Museum of Fine Arts is a museum in Heroes' Square, Budapest, Hungary. The museum's collection is made up of international art (other than Hungarian), including all periods of European art, and comprises more than 100,000 pieces. The Museum's collection is made up of six departments: Egyptian, Antique, Old sculpture gallery, Old painter gallery, Modern collection, Graphics collection.

12 **Zoo Budapest (Fővárosi Állat és Növénykert).** *1146 Budapest, Állatkerti körút 6-12, +36 1 273 4900, Metro 1, Trolleybus 72, 75, 79, Mon–Thu 9:00am–6:00pm, Fri–Sun 9:00am–7:00pm, 1900 HUF.*

The Budapest Zoo and Botanical Garden is one of the oldest in the world with its almost 150 years of history. Some of its old animal houses were designed by famous Hungarian architects. Nowadays it houses more than 1000 different species. Currently the greatest attraction is Asha, the child elephant.

13 **Great Synagogue.** *1072 Budapest, Akácfa utca 47., Metro 2, Bus 5, 7, 8, 9, 107, 133, 178, 233, Tram 47, 48, 49, Trolleybus 74, Sun–Thu 10:00am–6:00pm, Fri 10:00am–4:30pm, 2950 HUF.*

The Great Synagogue in Dohány Street is the largest Synagogue in Europe and the second largest in the world. It can accommodate close to 3,000 worshipers. It was built between 1854 and 1859 in Neo-Moorish style. During World War II, the Great Synagogue was used as a stable and as a radio communication center by the Germans. Today, it's the main center for the Jewish community.

14 **Millenáris.** *1024 Budapest, Kis Rókus utca 16-20, Tram 4, 6, 17 (Széna tér), Bus 6, 11, 111, Mon–Sun 6:00am–11:00pm.*

Located next to the Mammut mall, at the site of the one-time Ganz Electric Works, Millenáris is a nice park and venue for exhibitions, concerts, performances. You can also see a huge hyperbolic quadric and its two reguli.

15 **Batthyány tér.** *1011 Budapest, Metro M2, Tram 19, 41, Bus 11, 39, 111, Suburban railway 5.*

Batthyány square has a great view on the beautiful Hungarian Parliament, one of Europe's oldest legislative buildings, a notable landmark of Hungary.

16 **Erzsébet tér.** *1051 Budapest, Metro 1, 2, 3, Buses 15, 16, 105, Trams 47, 48, 49, Metro 1, 2, 3.*

Erzsébet Square is the largest green area in Budapest's inner city. The square was named after Elisabeth, 'Sisi', wife of Habsburg Emperor Franz Joseph, in 1858. The square's main attraction is the Danubius Fountain, located in the middle of the square, symbolizing Hungary's rivers. One of the world's largest mobile Ferris wheels can be also found on the square. The giant wheel offers fantastic views over Budapest day and night. Standing 65 meters tall, the wheel with its 42 cars is Europe's largest mobile Ferris wheel.

17 **Hungarian Academy of Sciences.** *1051 Budapest, Széchenyi István tér 9, Tram 2, Bus 15,16, 105, 115.*

The Hungarian Academy of Sciences is the most important and prestigious learned society of Hungary. Its seat is at the bank of the Danube in Budapest.

18 **Playground for adults.** *1124 Budapest, Vérmező.*

A playground for adults? Yes, this indeed exists and can be found in a nice park on the Buda side, close to the castle.

Restaurants & Eateries

1 **Onyx restaurant.** *1051 Budapest, Vörösmarty tér 7-8, Tue–Fri 12:00am–2:30pm, 6:30pm–11pm; Sat 6:30pm–11:00pm.*

Exclusive atmosphere, excellent and expensive food – Onyx is a highly elegant restaurant with one Michlein Star. Do not forget to reserve a table.

2 **Pizza King.** *1072 Budapest, Akácfa utca 9, Mon–Fri 10:00am–0:00am, Sat–Sun 10:00am–3:00am.*

During lunchtime on weekdays offers nice menus for 900 HUF, and you can buy cheap pizza there at any time of the day. Also runs pizza takeaways at many locations in the city.

Pubs

3 **Snaps Galéria.** *1077 Budapest, Király utca 95, Mon–Fri 2:00pm–0:00am, Sat 6:00pm–0:00am.*

Snaps is a tiny two-floor pub located in the sixth district. From the outside it is nothing special, but entering it has a calm atmosphere. The beers selection - Belgian and Czech beers- is quite extraordinary compared to other same level pubs.

4 **Noiret Pool and Darts Hall, Cocktail Bar and Pub.** *1066 Budapest, Dessewffy utca 8-10, Mon–Sun 10:00am–4:00am.*

A good place to have a drink and play pool, darts, snooker, or watch soccer.

5 **Szimpla Kert.** *1075 Budapest, Kazinczy utca 14, Mon–Thu 12:00am–4:00am, Fri 10:00am–4:00am, Sat 12:00am–4:00am, Sun 9:00am–5:00am.*

Szimpla Kert (Simple Garden) is the pioneer of Hungarian ruin pubs. It is really a cult place giving new trends. Undoubtedly the best known ruin pub among the locals and the tourists, as well.

Others

6 **Mammut Shopping and Entertainment Centre.** *1024 Budapest, Lövőház utca 2-6, Mon–Sat 6:30am–10:00pm, Sun 9:00am–10:00pm.*

A twin mall in the heart of Buda.

7 **WestEnd City Center.** *1062 Budapest, Váci út 1-3, Mon–Sun 8:00am–11:00pm.*

A big mall with stores, restaurants etc. and a roof garden.

8 **Corvintető.** *1085 Budapest, Blaha Lujza tér 1-2, Mon–Sun 6:00pm–6:00am.*

Situated on the rooftop of once-glorious Corvin Department Store, Corvintető offers world-class DJs and concerts every day of the week. Recommended by The New York Times. Do not mess it up with Corvin Negyed, another stop of trams 4 and 6.

Margaret Island

Margaret Island (Margitsziget) is the green heart of Budapest. It lies in the middle of the Danube between Margaret Bridge and Árpád Bridge. Apart from a couple of hotels and sport facilities, there are no buildings on the Island, it is a huge green park with promenades and benches, great for a date or a picnic. Everyone can find their own cup of tea here: there is the Hajós Alfréd National Sports Swimming Pool, the Palatinus and the running track for the sporty, the petting zoo, the music fountain and the Water Tower for families, and we recommend the Japanese Garden or a ride on a 4-wheel bike car for couples. If you're hungry for culture, check out the open-air stages and the medieval ruins of the Island.

Sights

1 **Entrance of Margit-sziget.** *Budapest, Margit híd, Trams 4 and 6 (Margit híd, Margit-sziget).*
Here you can enter the beautiful Margit-sziget (Margaret Island) on foot. However, you may also take bus 26 to get to the Island.

2 **Kiscelli Múzeum.** *1037 Budapest, Kiscelli utca 108, Tram 17, 19, 41, Bus 165, Tue–Sun 10:00am–6:00pm, 800 HUF.*
Kiscelli Múzeum is located in a beautiful baroque monastery in Old-Buda. It offers exhibitions on the history of Budapest between the 18–21. centuries.

3 **Görzenál.** *1036 Árpád fejedelem útja 125, Bus 29, Suburban railway 5, Mon–Fri 2:00pm–8:00pm, Sat–Sun 9:30am–8:00pm, 900 HUF.*
Görzenál currently is the biggest outdoor roller skating rink in Europe. The skating surface of the Gorzenal Roller Skate and Recreational Park is 14,000 square meters. This rink, which is located in picturesque surroundings along the Danube and Margaret Island, has a skating track as well as park structures for aggressive roller sports and BMX.

4 **Pál-völgyi Cave.** *1025 Budapest, Szépvölgyi út 162, Bus 65, Tue–Sun 10:00am–4:00pm, 1100 HUF.*
An 500-metre long route in a cave with narrow, canyon-like corridors, large level differences, astonishing stone formations, drip stones, glittering calcium-crystals and prints of primeval shells. Even with the 120 steps and the ladder that have to be mounted, the whole tour can easily be fulfilled in normal clothes and comfortable shoes.

5 **Gül baba's türbe.** *1023 Budapest, Mecset utca 14 (entrance: Türbe tér 1), Tram 4, 6, Mon–Sun 10:00am–6:00pm, free.*
The tomb of Gül Baba, "the father of roses", who was a Turkish poet and companion of Sultan Suleiman the Magnificent. He died shortly after the Turkish occupation of Buda in 1541 and his tomb is said to be the northernmost pilgrimage site of the muslims in the world. It is located on a hilltop, surrounded by a beautiful garden which offers a nice view of the city.

Bars

1 **Holdudvar Courtyard.** *1138 Budapest, Margitsziget, Mon–Wed 11:00am–2:00am, Thu 11:00am–4:00am, Fri–Sat 11:00–5:00, Sun 11:00–2:00am.*
A great entertainment spot in Budapest where everybody finds something to do: an open-air cinema, café, bar. The gallery exhibits works of contemporary fine art. Holdudvar hosts fashion shows and various cultural events.

SUMMER SCHOOL IN MATHEMATICS 2018
EÖTVÖS LORÁND UNIVERSITY, BUDAPEST, HUNGARY

	Monday, June 25	Tuesday, June 26	Wednesday, June 27	Thursday, June 28	Friday, June 29
9.00-9.30 9.30-10.00 10.00-10.30	**Ágnes Backhausz:** *Limits of random graphs*	**Péter Frenkel:** *Homomorphism numbers and graph convergence*	**Péter Csikvári:** *Statistical physics and graph limits*	**Péter Frenkel:** *Homomorphism numbers and graph convergence*	**Péter Csikvári:** *Statistical physics and graph limits*
10.30-11.00	Coffee break	Coffee break	Coffee break	Coffee break	Coffee break
11.00-11.30 11.30-12.00 12.00-12.30	**Péter Frenkel:** *Homomorphism numbers and graph convergence*	**Ágnes Backhausz:** *Limits of random graphs*	**Dávid Kunszenti-Kovács:** *From combinatorics to analysis*	**Dávid Kunszenti-Kovács:** *From combinatorics to analysis*	**László Miklós Lovász**
12.30-13.00 13.00-13.30 13.30-14.00	Lunch	Lunch	Lunch	Lunch	Lunch
14.00-14.30 14.30-15.00 15.00-15.30	**Miklós Abért**	**Balázs Szegedy**	**Excursion**	**Péter Csikvári:** *Statistical physics and graph limits*	**Dávid Kunszenti-Kovács:** *From combinatorics to analysis*
15.30-16.00					
16.00-16.30					
16.30-17.00					
17.00-17.30					
17.30-18.00					

1 Introduction

Random graph models have been extensively studied in the last twenty years due to their flexibility for modelling large real-world networks like the internet, online social networks, biological networks etc. In these lecture notes we have a look at this topic from a theoretical point of view, and present how certain questions about random graphs can lead to different notions of graph limits. First we consider sequences of random graphs in which the density of edges converges to a positive number, and have a look at densities of triangles and other subgraphs. In the second part we examine the local properties and convergence of random graphs where the degrees are fixed (which implies that the edge density goes to zero).

2 Inhomogeneous random graphs

In this section we introduce a general family of random graph models (in fact, a general model and its specializations). The starting point is the following well-known homogeneous random graph model from 1959 [12, Gilbert], [9, Erdős–Rényi]. Notice that in this version loops can occur.

Definition 1 (Erdős–Rényi random graph). Let $n \geq 2$ be an integer and $0 \leq p \leq 1$. The Erdős–Rényi random graph $\mathrm{ER}_n(p)$ is a graph on vertices $[n] = \{1, \dots, n\}$. As for the edges, for every pair $1 \leq s \leq t \leq n$ we connect vertices s and t with probability p, independently.

See e.g. [10] for various results on this model.

We are interested in inhomogeneous graphs, where (by definition) there are certain parts of the graph that are dense, and others that are sparse.

Definition 2 (Stochastic block model). Let n, m be positive integers, $q = (q_1, \dots, q_m)$ be a probability distribution on $[m] = \{1, \dots, m\}$, and P be an $m \times m$ symmetric matrix with entries in $[0, 1]$.

The random graph $\mathrm{SBM}(n, P, q)$ is defined as follows. Let $X_1, \dots, X_n$ be independent random variables with distribution q. Then vertices i and j are connected with probability P_{X_i, X_j} independently for each pair of vertices (for $i = j$ as well).

This model has a clustering property: we can find groups of vertices among which there are many edges, or just a few edges. These groups are often called clusters. Regarding this model, one of the recently studied question is the following: given the graph (edges and vertices, but no information on the Xs), how can one find out which vertices belong to the same cluster? See e.g. [1] for a recent survey on this topic.

The following can be considered as a further generalization of the stochastic block model.

Definition 3 (W-random graph, [15], 2006). The W-random graph model $G(n, W)$ produces a graph on vertices $v_1, \dots, v_n$ by using a symmetric measurable function $W : [0, 1]^2 \to [0, 1]$. Let $X_1, \dots, X_n$ be independent random variables with uniform distribution on $[0, 1]$. Then connect vertices v_i and v_j independently with probability $W(X_i, X_j)$ for every $1 \leq i \leq j \leq n$.

A symmetric measurable function $W : [0, 1]^2 \to [0, 1]$ is called a **graphon**.

Notice that the stochastic block model can be considered as a special case of the W-random graph model with an appropriate step function W, defined as follows. Let $[0, 1]$ be the disjoint union of intervals $I_1, \dots, I_m$ such that I_j has length q_j (for $j = 1, 2, \dots, m$), and let W be the following graphon:

$$W(x, y) = P_{j_x, j_y} \quad \text{if} \quad x \in I_{j_x}, y \in I_{j_y}.$$

We will be interested in the limit of a sequence of W-random graphs. Clearly, the limit object will be W as the number of vertices tends to infinity. The question is the following: what does it mean for a (dense) graph sequence to be convergent.

3 Dense graph limits

3.1 Density of edges

Given a finite graph $G = (V(G), E(G))$, we consider the following question: by choosing two vertices uniformly at random (with replacement), what is the probability that the two vertices are connected to each other? This is clearly twice the number of edges divided by $|V(G)|^2$, where $|V(G)|$ is the number of vertices.

In a random graph, the number of edges is random itself. Still, we can ask about the probability that the two vertices are connected to each other. This is the same as the expectation of the edge density, and in the above random graph models it can be calculated as follows, by the law of total expectation.

- Erdős–Rényi random graph on n vertices with edge probability p: p.
- $\mathrm{SBM}(n, P, q)$ model: $\sum_{i=1}^m \sum_{j=1}^m P_{i,j} q_i q_j$.
- W-random graph model: $\int_{[0,1]^2} W(x_1, x_2) dx_1 dx_2$.

3.2 Triangle density

Going further, we can ask the probability that three randomly chosen vertices form a triangle (vertices are chosen with replacement, and loops can occur in the triangle). In a fixed graph, this is six times the number of triangles divided by $|V(G)|^3$. As for the random graph models, the limit of the probability of this event is given as follows.

- Erdős–Rényi random graph on n vertices with edge probability p: p^3.
- $\mathrm{SBM}(n, P, q)$ model: $\sum_{i,j,k=1}^m P_{i,j} P_{j,k} P_{k,i} \cdot q_i q_j q_k$.
- W-random graph model: $\int_{[0,1]^3} W(x_1, x_2) W(x_2, x_3) W(x_3, x_1) dx_1 dx_2 dx_3$.

3.3 Homomorphism density

To understand more of the structure of the graph and define a convergence notion, a more general version of subgraph densities is needed. For more on homomorphisms and graph limits, see e.g. [14].

Definition 4 (Homomorphism density). Let F and G be finite simple graphs. We say that $\varphi : V(F) \to V(G)$ is a *homomorphism* if it is adjacency-preserving; that is, for every $(v, v') \in E(F)$ we have $(\varphi(v), \varphi(v')) \in E(G)$. We denote by $\hom(F, G)$ the number of homomorphisms from F to G. The *homomorphism density* of F in G is defined as follows:

$$t(F, G) = \frac{\mathrm{hom(F,G)}}{|V(G)|^{|V(F)|}}.$$

In other words, $t(F, G)$ is the probability of the following event. We choose random vertices $u_1, \ldots, u_m$ from $V(G)$, uniformly with replacement, and see if for every edge (v_i, v_j) of F the vertices u_i and u_j are connected in G (the vertex set of F is $v_1, \ldots, v_m$). That is, $t(F, G)$ is the probability that if we map the vertices of F to the vertices of G, edges go to edges, and we get a homomorphism. (Notice that there is no condition on the pairs of vertices that are not connected to each other in F.)

Similarly to the previous cases, we can calculate the limit of the expectation of the homomorphism density $t(F, G_n)$, when $F = (V(F), E(F))$ is a fixed finite simple graph on k vertices, and G_n is a random graph on n vertices. Notice that using the previous representation, conditionally on the event that the vertices u_i are different, the equations are true for all n (without taking the limit).

- If G_n is an Erdős–Rényi random graph on n vertices with edge probability p:

$$\lim_{n\to\infty} \mathbb{E}(t(F, G_n)) = p^{|E(F)|}.$$

- If G_n is from the $\mathrm{SBM}(n,P,q)$ model:

$$\lim_{n\to\infty} \mathbb{E}(t(F,G_n)) = \sum_{s_1,\dots,s_k=1}^{m} \prod_{ij\in E(F)} P_{s_is_j} \cdot q_1 \dots q_k.$$

- W-random graph model: $\int_{[0,1]^k} \prod_{ij\in E(F)} W(x_i,x_j)dx_1 \dots dx_k$.

Again, notice that in the definition of homomorphisms there is no condition on the pairs of vertices that are not connected to each other. This is why the product goes only for the edges, and not for all pairs of vertices.

3.4 Dense graph convergence

As we have seen before, in expectation, the homomorphism density of a W-random graph model does not depend on the number of vertices. That is, as the number of vertices tends to infinity, the expectation of the homomorphism density converges. As we will see, based on the following definition, much more can be proved about the convergence of the homomorphism densities in random graphs.

Definition 5 (Graph convergence, [7], 2008). We say that a sequence of finite simple graphs (G_n) is convergent if the sequence $\big(t(F,G_n)\big)$ converges for all finite simple graphs F.

The goal is not just to understand the convergence of graph sequences, but to find objects that can be the limit of the sequence. In order to do this, first we define the homomorphism density of a graph in a graphon, as the expected homomorphism density in a W-random graph (which we have already determined). After that, graphons can be interpreted as limits of dense graph sequences.

Definition 6. The homomorphism density of a finite simple graph $F = ([k], E(F))$ in a graphon $W : [0,1]^2 \to [0,1]$ is given by

$$t(F,W) = \int_{[0,1]^k} \prod_{ij\in E(F)} W(x_i,x_j)dx_1 \dots dx_k.$$

A sequence of finite simple graphs (G_n) converges to a graphon W if for every finite simple graph F the sequence of homomorphism densities $t(F,G_n)$ converges to $t(F,W)$ as $n \to \infty$.

Now we can state one of the fundamental theorems of the theory of dense graph limits, which can be proved by using the martingale convergence theorem.

Theorem 7 (Lovász–Szegedy, 2006, [15]). *Let (G_n) be a convergent sequence of finite simple graphs. Then there exists a graphon W such that G_n converges to W.*

4 Convergence of random graphs

4.1 W-random graphs

As for W-random graphs, more is true than the calculation of the expectation shows. To prove this, we will need Azuma's inequality for martingales (for more details on martingales we refer to the textbook [17]).

Theorem 8 (Azuma inequality for martingales). *Let $(X_n)_{n\ge 0}$ be a martingale with respect to the filtration $(\mathcal{F}_n)_{n\ge 0}$. Suppose that $|X_m - X_{m-1}| \le c_m$ holds for some $c_m > 0$ for every $m \ge 1$. Then for every $t > 0$ and $n \ge 1$ we have*

$$\mathbb{P}(|X_n - X_0| > t) \le 2\exp\left(-\frac{t^2}{2\sum_{m=1}^{n} c_m^2}\right).$$

Theorem 9 (Lovász–Szegedy, 2006, [15]). *For every symmetric measurable function $W : [0,1]^2 \to [0,1]$, the sequence $G(n,W)$ converges to W with probability 1. That is,*

$$\lim_{n\to\infty} t(F,G(n,W)) = t(F,W) = \int_{[0,1]^k} \prod_{ij\in E(F)} W(x_i,x_j)dx_1 \dots dx_k$$

holds for every finite simple graph F (on the vertex set $[k]$) as $n \to \infty$ almost surely.

Proof for a slightly modified version [15]. In the definition of homomorphism densities, one can consider only the maps that are injective. Since the map is from the vertex set of F to the vertex set of G_n, and the first one is fixed, while the latter tends to infinity, most of the maps are injective anyway. In the sequel we present the proof of the statement for injective homomorphisms. The proof can be completed by using Lemma 2.1 from [15].

Let F be a finite simple graph on vertices $\{1,\dots,k\}$, and fix n (until the very end of the proof). We denote by G_n a $G(n,W)$-random graph. Let H be the set of injective maps from $\{1,\dots,k\}$ to the vertex set of G_n. There are $n(n-1)\dots(n-k+1)$ elements in H. For $\varphi\in H$, let A_φ be the event that φ is a homomorphism from F to G_n. The probability of this event is the same for all φ (by the symmetry of the random graph model). In addition, as we discussed after Definition 4, the homomorphism density is the probability that a randomly chosen map is a homomorphism, and hence this probability is equal to the integral $t(F,W)$. Therefore we have

$$t(F,W)=\frac{1}{|H|}\sum_{\varphi\in H}\mathbb{P}(A_\varphi).$$

Given n, for $1\le m\le n$, let G^*_m be the induced subgraph of G_n corresponding to the vertices $v_1,\dots,v_m$. The following sequence becomes a martingale for $1\le m\le n$ with respect to $\mathcal{F}_m=\sigma(G^*_m)$:

$$B_m=\frac{1}{|H|}\sum_{\varphi\in H}\mathbb{P}(A_\varphi|G^*_m).$$

Indeed, we have

$$\mathbb{E}(B_m|\mathcal{F}_{m-1})=\frac{1}{|H|}\sum_{\varphi\in H}\mathbb{E}\Big(\mathbb{P}(A_\varphi|G^*_m)\Big|\mathcal{F}_{m-1}\Big)=B_{m-1},$$

as the σ-algebra generated by G^*_{m-1} is contained by the σ-algebra generated by G^*_m.

Since $\mathcal{F}_0$ is the trivial σ-algebra, $B_0=t(F,W)$. On the other hand, the event A_φ is measurable with respect to $\sigma(G_n)$, hence

$$B_n=\frac{1}{|H|}\sum_{\varphi\in H}\mathbb{P}(A_\varphi|G_n)=\frac{1}{|H|}\sum_{\varphi\in H}\mathbb{I}(A_\varphi)=t(F,G_n),$$

by the definition of homomorphism density.

Notice that

$$\begin{aligned}|B_m-B_{m-1}|&=\frac{1}{|H|}\Big|\sum_{\varphi\in H}(\mathbb{P}(A_\varphi|G_m)-\mathbb{P}(A_\varphi|G_{m-1}))\Big|\\&\le\frac{1}{|H|}\sum_{\varphi\in H}\Big|\mathbb{P}(A_\varphi|G_m)-\mathbb{P}(A_\varphi|G_{m-1})\Big|.\end{aligned}$$

The graphs G_{m-1} and G_m differ only in v_m and the edges connected to it. The event A_φ (whether φ is a homomorphism or not) depends on the subgraph corresponding to $v_{\varphi(1)},\dots,v_{\varphi(k)}$. Hence if m is not in the range of φ, then the conditional expectation of A_φ is the same with respect to G_{m-1} and G_m. The number of maps φ whose range contains k is $k(n-1)(n-2)\dots(n-k+1)$. Therefore

$$|B_m-B_{m-1}|\le\frac{k(n-1)\dots(n-k+1)}{|H|}=\frac{k(n-1)\dots(n-k+1)}{n(n-1)\dots(n-k+1)}=\frac{k}{n}.$$

By the Azuma inequality on martingales with bounded differences (with $c_m=k/m$ for every m, see Theorem 8) we obtain

$$\mathbb{P}(|B_n-B_0|\ge\varepsilon)\le 2\exp\left(-\frac{\varepsilon^2}{2k^2}n\right).$$

This means

$$\mathbb{P}(|t(F,G_n)-t(F,W)|\ge\varepsilon)\le 2\exp\left(-\frac{\varepsilon^2}{2k^2}n\right).$$

For fixed k (which is the number of vertices of F), the right-hand side is finite when we sum it up for n. Hence by the Borel–Cantelli lemma, with probability 1, the event on the left-hand side occurs only finitely many times. Since this holds for every fixed F and ε, we get that $t(F,G_n)$ converges with probability 1 to $t(F,W)$ for every fixed F. □

4.2 Speed of convergence

Given a general convergence theorem, it is natural to ask how fast the convergence is in some appropriate distance. As we will see, for some particular models, more can be proved, but first we summarize some general results on this question.

First we need a notion for distance of graphs, which induces the convergence notion of Definition 5. As in [8], we define this for graphons (based on the cut-norm defined in [11]), and then for finite graphs.

Definition 10 (Cut distance, [8]). The *cut norm* of an integrable function $F : [0,1]^2 \to \mathbb{R}$ is defined by

$$\|F\|_\square = \sup_{S,T\subset[0,1]} \left| \int_{S\times T} F(x,y)\,dx\,dy \right|,$$

where the supremum goes over measurable subsets of $[0,1]$.

The *cut distance* of two graphons U, W is given by

$$\delta_\square(U,W) = \inf_{\phi:[0,1]\to[0,1]} \|U - W^\phi\|_\square,$$

where the infimum goes over all invertible maps $\phi : [0,1] \to [0,1]$ such that both ϕ and its inverse are measure preserving, and the graphon W^ϕ is defined by $W^\phi(x,y) = W(\phi(x),\phi(y))$.

We assign a graphon W_G to every finite simple graph $G = ([n], E(G))$. For $1 \le k < n$, let J_k be the interval $[(k-1)/n, k/n)$, and let $J_n = [(n-1)/n, 1]$.) Then

$$W_G(x,y) = \begin{cases} 1, & \text{if } x \in J_i, y \in J_j \text{ and } ij \in E(G); \\ 0, & \text{otherwise.} \end{cases}$$

Finally, the cut distance of two finite simple graphs is the following:

$$\delta_\square(G,G') = \delta_\square(W_G, W_{G'}).$$

In the sequel, we will often write G instead of W_G when we consider distances of graphs, or a graph and a graphon.

In general, the following is known for the connection between cut distance and convergence.

Theorem 11 ([7], Borgs–Chayes–Lovász–T. Sós–Vesztergombi, 2008). *A sequence of finite simple graphs (G_n) is convergent if and only if it is a Cauchy sequence in the cut distance, i.e. for every $\varepsilon > 0$ there exists n_0 such that for every $n, m \ge n_0$ we have $\delta_\square(G_n, G_m) \le \varepsilon$.*

A sequence of finite simple graphs (G_n) converges to W if and only if $\delta_\square(W_{G_n}, W)$ tends to 0 as $n \to \infty$. Moreover, the vertices of the graphs can be labeled such that $\|W_{G_n} - W\|_\square \to 0$.

Corollary 12 (Uniqueness of the limit). *For a sequence of finite simple graphs (G_n), if (G_n) converges to W and also to W^*, then $\delta_\square(W, W^*) = 0$, that is,*

$$\inf_{\phi:[0,1]\to[0,1]} \|W^\phi - W^*\|_\square = 0.$$

We are interested in the cut-distance of a W-random graph on n vertices and W as a random variable. There are some general results on this. In general, without any assumption on the structure of the graphon, the following can be proved.

Proposition 13 ([14], 2012, Lemma 10.16). *Let W be a graphon, and $G(n,W)$ is a W-random graph on n vertices. Then the following holds:*

$$\mathbb{P}\left(\delta_\square\Big(G(n,W), W\Big) \le \frac{22}{\sqrt{\log n}}\right) \ge 1 - e^{-\frac{n}{2\log n}}.$$

However, one expects that for graphons with simpler structures, the W-random graph on n vertices is closer to the original graphon. Recently, Klopp and Verzelen [13] have proved a result in this direction, by understanding the speed of convergence of graphons with finitely many values. The following theorem shows that faster convergence can be proved if the range of the graphon has much smaller size than the number of vertices. It also implies that the uniform result of Proposition 13 is the best possible.

Theorem 14 ([13], 2017+). *Let W be a graphon that has k possible values ($2 \le k \le n$). Then for the W-random graph on n vertices we have*

$$\mathbb{E}\Big(\delta_\square(G(n,W),W)\Big) \le C\sqrt{\frac{k}{n\log k}},$$

where C is a universal constant (depending neither on n, nor or k).

This is formulated for dense graphs, but in [13], a more general version is proved for sparse random graphs generated from graphons. A related result on L^p-graphons (symmetric L^p-functions on $[0,1]^2$) can be found in [5] and [6].

4.3 Preferential attachment graphs

Various random graph models with preferential attachment dynamics have been examined since the seminal paper of Barabási and Albert [2]. This property means that vertices with larger degree have more chance to get new edges. Now we present briefly a dense random graph model with this property, which converges to a certain graphon.

Definition 15 (Simplified dense preferential attachment graph). Given n, the vertices will be $\{v_1,\dots,v_n\}$. Fix $0 < c < 1$. For $j = 1,2,\dots,2\lceil cn^2\rceil$, choose one of the vertices according to an n-color Pólya urn process: at the beginning, each vertex has a single ball. Then at each step we choose one of the vertices with probabilities proportional to the number of balls, and assign a new ball to the urn of this chosen vertex. We repeat this $2\lceil cn^2\rceil$ times. Finally, we connect the pairs of vertices chosen in steps $2j-1$ and $2j$ for $j = 1,\dots,\lceil cn^2\rceil$ but only if they are different, and once if a pair appears several times (loops and multiple edges are not allowed).

This graph model is called "dense" because the edge density converges to a positive number. As for the current simplified version, the following holds (see also [8] for more on this model).

Theorem 16 ([16], 2012). *The sequence of simplified dense preferential attachment graphs converges to the graphon $W(x,y) = 1-\exp(-c\ln x\ln y)$ with probability 1 as the number of vertices tends to infinity.*

5 Random regular graphs

In this section we examine sparse random graphs, where the degrees of the vertices are fixed numbers, and the set of edges is chosen randomly among the graphs satisfying this condition. This family of random graph models leads to a different notion of graph limits.

5.1 Configuration model

Let $n, d_1,\dots,d_n$ be positive integers such that $d_1+\ldots+d_n$ is even. Our goal is to construct a random graph on vertex set $\{v_1,\dots,v_n\}$ such that the degree of v_j is d_j for every $j = 1,\dots,n$. A possibility is to choose uniformly at random among all graphs satisfying this condition (since they form a finite set). Or, we can exclude the graph containing loops or multiple edges, and choose uniformly among simple graphs. Since both of these approaches are not easy to handle, this gives rise to the following definition.

Definition 17 (Configuration model). Let n be $d_1,\dots,d_n$ positive integers such that the sum of the latter is even. We consider vertices $v_1, v_2,\dots,v_n$, and assign d_j half-edges to v_j (for every $j = 1,2,\dots,n$). Then we choose a perfect matching of the $D = \sum_{j=1}^n d_j$ half-edges uniformly at random. We form edges from the half-edges paired together. The random graph we get (with possible loops and multiple edges) is called the configuration model with $d_1,\dots,d_n$.

This model is easier to handle or to simulate with a computer than the previous definition. Namely, we can choose one of the half-edges arbitrarily, and then choose its pair uniformly at random among the other $D-1$ half-edges. Then again, we take another vertex and choose its pair uniformly at random among the other $D-3$ half-edges. We repeat this until there are no more unpaired half-edges. It is easy to see that all perfect matchings of the D half-edges appear with the same probability.

5.2 Regular graphs

Definition 18 (Random regular graph). A graph is d-regular if all its vertices have degree d. A random d-regular graph on n vertices (with nd even) is the random graph produced by the configuration model with $d_j = d$ for all $j = 1, 2, \dots, n$.

As it was mentioned before, loops may occur in the configuration model. However, according to the following statement, the proportion of vertices with loops tends to 0 in probability as the number of vertices tends to infinity.

Proposition 19. *Let $d \geq 3$ be a positive integer, and S_n be the number of loops in a random d-regular graph on n vertices. Then we have*

$$\lim_{n\to\infty} \frac{\mathbb{E}(S_n)}{n} = 0 \quad \Rightarrow \quad \frac{S_n}{n} \to 0 \text{ in probability.}$$

Proof. *[10]* For $1 \leq j \leq n$ and $1 \leq s < t \leq d$ let $\mathbb{I}_{st,j}$ be the indicator of the event that the sth and tth half-edges of vertex j are paired to each other. Then

$$\mathbb{E}(S_n) = \mathbb{E}\Bigg(\sum_{j=1}^{n} \sum_{1\leq s<t\leq d} \mathbb{I}_{st,j} \Bigg) = \sum_{j=1}^{n} \sum_{1\leq s<t\leq d} \mathbb{E}(\mathbb{I}_{st,j}).$$

The expectation $\mathbb{E}(\mathbb{I}_{st,j})$ is the probability that the sth and tth half-edges of vertex j are attached to each other. Since half-edge s is attached to all other half-edges with the same probability (by symmetry), and there are $nd-1$ half-edges different from this one, this expectation is equal to $\frac{1}{nd-1}$. On the other hand, the number of pairs s,t is $d(d-1)/2$ for each j. Therefore we obtain

$$\mathbb{E}(S_n) = \sum_{j=1}^{n} \sum_{1\leq s<t\leq d} \mathbb{E}(\mathbb{I}_{st,j}) = n \cdot \frac{d(d-1)}{2} \cdot \frac{1}{nd-1} \leq \frac{d}{2}.$$

This implies the first part of the statement. The second part is an application of Markov's inequality for $\varepsilon > 0$ and the random variable $S_n/n \geq 0$, as follows:

$$\mathbb{P}\left(\left| \frac{S_n}{n} \right| > \varepsilon \right) = \mathbb{P}\left(\frac{S_n}{n} > \varepsilon \right) \leq \frac{\mathbb{E}(S_n)}{n\varepsilon} \leq \frac{d}{2n\varepsilon} \to 0 \qquad (n \to \infty). \qquad \square$$

With a similar but more technical proof it can be shown that the expected number of cycles of length $k = 2$ is bounded from above by a constant, and hence the proportion of vertices which are contained by such a cycle converges to 0 in probability. A similar argument works for the cycles of length k with arbitrary k, which leads to the following proposition. (In a finite graph the r-neighborhood of a vertex v is the set of vertices whose distance from v is at most r.)

Proposition 20. *Let $d \geq 3$ and $r \geq 1$ be fixed. In a random d-regular graph on n vertices, let $V_n(r)$ be the number of vertices whose r-neighborhood is a tree (i.e. there are neither loops, nor cycles in the subgraph spanned by the vertices at distance at most r from the vertex). Then we have*

$$\lim_{n\to\infty} \frac{\mathbb{E}(V_n(r))}{n} = 1 \quad \Rightarrow \quad \frac{V_n(r)}{n} \to 1 \text{ in probability as } n \to \infty.$$

To put it in another way, let G_n be a random d-regular graph on n vertices, and choose a vertex v uniformly at random from G_n. Then for every $r \geq 1$ the probability that the r-neighborhood of v is a tree tends to 1 as n goes to infinity.

Exercise. Let $k \geq 2, d \geq 3$ be fixed positive integers, and let $S_n(k)$ be the number of vertices in a random d-regular graph on n vertices which are contained in a cycle of length at most k. Show that $\mathbb{E}(S_n(k)/n) \to 0$ as $n \to \infty$.

Definition 21. Let $B_d(r)$ be the set of finite connected graphs which are rooted (they contain a distinguished vertex o, called the root), in which the maximum of the distance of the vertices from o is at most r, and in which all vertices have degree at most d.

We say that the finite rooted graphs F_1 and F_2 are isomorphic if there exists a bijection $\varphi : V(F_1) \to V(F_2)$ mapping the root of F_1 to the root of F_2 such that (v_1, v_2) are connected to each other in F_1 if and only if $\varphi(v_1)$ and $\varphi(v_2)$ are connected to each other in F_2.

Definition 22 ([3]). Let $d \geq 2$ be fixed, and (G_n) be a sequence of finite graphs such that every vertex in every graph has degree at most d. We say that (G_n) is convergent in the Benjamini–Schramm sense (or locally), if for every $r \geq 1$ and $F \in B_d(r)$ the following holds: the probability that the r-neighborhood of a uniformly randomly chosen vertex of G_n (considered to be the root) is isomorphic to F converges as $n \to \infty$.

Example. Let P_n be the path of length n, and C_n be the cycle of length n. Both sequences are convergent in the Benjamini–Schramm sense. We can also say that their limit is the graph with vertex set $\mathbb{Z}$, where numbers with difference 1 are connected to each other.

Let G_n be the hypercube of side length n in the graph $\mathbb{Z}^d$. Then (G_n) is convergent, and its limit is $\mathbb{Z}^d$.

Proposition 23. *Let $d \geq 3$ be fixed, and for $n \geq 1$ let G_n be a random d-regular graph on n vertices. Then the sequence (G_n) is convergent in Benjamini–Schramm sense with probability 1, and its limit is the infinite d-regular tree T_d.*

5.3 Eigenvalues of random d-regular graphs

In this section we mention a result about the spectral properties of random d-regular graphs. This is related to the previous proposition; it turns out that the limit of the spectral measure of the finite graphs is the spectral measure of the infinite d-regular tree.

Let $G = (V, E)$ be a finite graph (with possible loops and multiple edges) on the vertex set $[n] = \{1, 2, \dots, n\}$. Let A be its adjacency matrix. This is a matrix of size $n \times n$ such that the entry in row i and column j is the number of edges going between vertices i and j. The eigenvalues of A are called the eigenvalues of graph G. Since A is symmetric, all eigenvalues are real numbers. This leads to the following definition.

Definition 24 (Eigenvalues of graphs). Let $G = (V, E)$ be a finite graph, where $V = \{1, 2, \dots, n\}$. Vector $v \in \mathbb{R}^n \setminus \{0\}$ is an eigenvector of graph G with eigenvalue $\lambda \in \mathbb{R}$ if and only if for every $j = 1, 2, \dots, n$ we have

$$\sum_{i: ij \in E} v_i = \lambda v_j.$$

To put it in another way, an eigenvector is a function on the vertices such that the following holds for every vertex v: if we sum up the values at the neighbors of v (weighted by edge multiplicities), then we get the value at v multiplied by λ.

It is clear that for a d-regular graph the constant 1 vector is always an eigenvector with eigenvalue $\lambda = d$. In addition, in this case for every eigenvalue we have $|\lambda| \leq d$ (in fact, $\lambda = -d$ is an eigenvalue if and only if the graph is bipartite). As for random d-regular graphs, the following holds.

Proposition 25. *Let $d \geq 3$ be fixed, and for every $n \geq 1$ (with nd even) let G_n be a random d-regular graph on n vertices. Let $T_n(x)$ be the proportion of eigenvalues of G_n that are smaller than x. Then for every $x \in [-2\sqrt{d-1}, 2\sqrt{d-1}]$ we have*

$$\lim_{n \to \infty} T_n(x) = \int_{-2\sqrt{d-1}}^{x} \frac{d}{2\pi} \cdot \frac{\sqrt{4(d-1) - y^2}}{d^2 - y^2} dy$$

in probability.

The probability distribution with the density function in this proposition is called the Kesten–McKay measure. In addition, this measure is also the spectral measure of the infinite d-regular tree. We refer to the lecture notes [4] for more details on the spectrum of random graphs.

Exercise. Show that a finite simple d-regular graph is bipartite if and only if its smallest eigenvalue is $\lambda = -d$.

[1] E. Abbe, Community detection and stochastic block models: recent developments. arXiv:1703.10146 math.PR

[2] A.-L. Barabási and R. Albert, Emergence of scaling in random networks, Science **286** (1999), no. 5439, 509–512.

[3] I. Benjamini and O. Schramm, Recurrence of distributional limits of finite planar graphs, Electron. J. Probab. **6** (2001), no. 23, 13 pp.

[4] C. Bordenave, Spectrum of random graphs. Preprint, 2016. `https://www.math.univ-toulouse.fr/ bordenave/coursSRG.pdf`

[5] C. Borgs, J. T. Chayes, H. Cohn and S. Ganguly, Consistent nonparametric estimation for heavy-tailed sparse graphs. Preprint. ArXiv: 1508.06675

[6] C. Borgs, J. T. Chayes, H. Cohn and Y. Zhao, An L^p-theory of sparse graph convergence I: limits, sparse random graph models, and power law distributions. Preprint. ArXiv: 1401.2906.

[7] C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós and K. Vesztergombi, Convergent sequences of dense graphs. I. Subgraph frequencies, metric properties and testing, Adv. Math. **219** (2008), no. 6, 1801–1851.

[8] C. Borgs, J. Chayes, L. Lovász, V. Sós, K. Vesztergombi, Limits of randomly grown graph sequences. *Eur. J. Combin.* **32(7)** (2011), pp. 985–999.

[9] P. Erdős and A. Rényi, On the evolution of random graphs, Publications of the Mathematical Institute of the Hungarian Academy of Sciences. **5** (1960), 17–61.

[10] R. van der Hofstad, *Random graphs and complex networks. Vol. 1*, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, Cambridge, 2017.

[11] A. Frieze and R. Kannan, Quick approximation to matrices and applications, Combinatorica **19** (1999), no. 2, 175–220.

[12] E. N. Gilbert, Random graphs, Ann. Math. Statist. **30** (1959), 1141–1144.

[13] O. Klopp, N. Verzelen, Optimal graphon estimation in cut distance. Preprint. ArXiv: 1703.05101.

[14] L. Lovász, *Large networks and graph limits*, American Mathematical Society Colloquium Publications, 60, Amer. Math. Soc., Providence, RI, 2012.

[15] L. Lovász and B. Szegedy, Limits of dense graph sequences, J. Combin. Theory Ser. B **96** (2006), no. 6, 933–957.

[16] B. Ráth, L. Szakács, Multigraph limit of the dense configuration model and the preferential attachment graph, *Acta Math. Hung.* **136** (2012), pp. 196–221.

[17] D. Williams, *Probability with martingales*, Cambridge Mathematical Textbooks, Cambridge University Press, Cambridge, 1991.

1 Introduction

This lecture note concerns with some very basic problems in statistical physics. These particular problems are coming from lattice gas models, and have some combinatorial flavor. To describe an exemplary problem let us consider the d-dimensional grid, $\mathbb{Z}^d$, and let us consider some configuration on it, for instance, an independent set of it. To understand the behavior of this configuration, it is worth first considering a large box of $\mathbb{Z}^d$, say box B_n of size $n\times\cdots\times n$, and understand the behavior of the configuration there. For instance, we may wish to understand what is the expected fraction of the box that is covered by a random independent set chosen uniformly at random from all independent sets of the box B_n. As n goes to infinity we might (or might not) expect that these fractions will converge to some number. If we try to translate this problem to a more general setting, then it is natural to look at the sequence of boxes as a converging graph sequence and $\mathbb{Z}^d$ as a limit of this graph sequence. The proper understanding of this convergence will lead to the definition of the Benjamini–Schramm convergence. The problem of the convergence of the independence ratio, the fraction of vertices covered by a random independent set, is a special case of the more general question of describing those graph parameters that are continuous with respect to the topology determined by the Benjamini–Schramm convergence.

At this point it might look that the only goal of this graph convergence is to extend some graph parameter from finite graphs to infinite ones. Surprisingly, sometimes the other direction is also fruitful. Here the understanding of the infinite object is easy, and then this leads to a better understanding of the finite version of the problem.

This lecture note is organized as follows. In the next section we describe some statistical physical models. In the third section we introduce the Benjamini–Schramm convergence. In Section 4 we give various results about dimer and monomer-dimer models. In particular, we study the number of dimer configurations of grid graphs, and we introduce the so-called matching polynomial. In Section 5 we apply graph limit theory to the monomer-dimer model.

2 Statistical physical models

In this section we collected some very basic statistical physical models.

2.1 Ising-model

Ising-model was introduced to model the phenomenon that if we heat up a magnetic metal, then it eventually lose its magnetism. In this model the vertices of the graph G represent particles. These particles have a spin that can be up $(+1)$ or down (-1). Two adjacent particles have an interaction e^{β} if they have the same spins, and $e^{-\beta}$ if they have different spins (explanation of this sentence comes soon). In the statistical physics literature β is called the inverse temperature. If the temperature is very small, then β is very large and adjacent vertices have the same spin with high probability as we will see soon. On the other hand, if the temperature is very large, then β is very small so the spins of adjacent vertices will be more or less uncorrelated. Suppose also that there is an external magnetic field that breaks the symmetry between $+1$ and -1. This defines a probability distribution on the possible configurations as follows: for a random spin configuration S:

$$\mathbb{P}(\mathsf{S}=\sigma)=\frac{1}{Z}\exp\left(\sum_{(u,v)\in E(G)}\beta\sigma(u)\sigma(v)+B\sum_{u\in V(G)}\sigma(u)\right),$$

where Z is the normalizing constant:

$$Z_{\mathrm{Is}}(G,B,\beta)=\sum_{\sigma:V(G)\to\{-1,1\}}\exp\left(\sum_{(u,v)\in E(G)}\beta\sigma(u)\sigma(v)+B\sum_{u\in V(G)}\sigma(u)\right).$$

Here Z is the so-called partition function of the Ising-model. Now we can see that if β is large, then according to this distribution it is much more likely that adjacent vertices will have the same spin, and so the sum of the spins will have large absolute value which measures the magnetism of the metal.

A priori we assumed so far that $\beta>0$, because this case provides the proper physical intuition. On the other hand, the model makes perfect sense mathematically even if $\beta<0$. When β is a negative number with large absolute value, then the system favors configurations where most pairs of adjacent vertices have different spins. When $\beta > 0$ we say that it is a ferromagnetic Ising-model, and when $\beta < 0$, then we say that it is an antiferromagnetic model. As one might expect it, the model behaves very differently in the two regimes.

2.2 Monomer-dimer and dimer model

For a graph G let $\mathcal{M}(G)$ be the set of all matchings. Recall that a matching M of G is simply a set of edges such that no two edges in the set intersect each other. When this set has k edges, then we say that it is a k-matching or alternatively, the matching M is of size k (and it covers $2k$ vertices). For a $\lambda > 0$ we can associate a probability space on $\mathcal{M}(G)$ by choosing a random matching $\mathbf{M}$ as follows:

$$\mathbb{P}(\mathbf{M} = M) = \frac{\lambda^{|M|}}{M(G,\lambda)},$$

where $M(G,\lambda)$ is the normalizing constant:

$$M(G,\lambda) = \sum_{M \in \mathcal{M}(G)} \lambda^{|M|}.$$

This model is the monomer-dimer model. The name has the following origin. In statistical physics the vertices of the graph represent particles, and edges represent some sort of interaction between certain pair of particles. A dimer is a pair of particles where the interaction is active. Supposing that one particle can be active with at most one other particle, we get that the dimers form a matching. The uncovered vertices are called monomers. We say that $M(G,\lambda)$ is the partition function of the monomer-dimer model. In mathematics it is called the matching generating function. Let $m_k(G)$ denote the number of k-matchings. Then

$$M(G,\lambda) = \sum_k m_k(G)\lambda^k.$$

Note that the sum runs from $k = 0$ as the empty set is a matching by definition. Naturally, once we introduced a probability distribution we can ask various natural questions like what is $\mathbb{E}|\mathbf{M}|$. It is not hard to see that

$$\mathbb{E}|\mathbf{M}| = \sum_{M \in \mathcal{M}(G)} \frac{|M|\lambda^{|M|}}{M(G,\lambda)} = \frac{\lambda M'(G,\lambda)}{M(G,\lambda)}.$$

If G has $2n$ vertices, then we call an n-matching a perfect matching as it covers all vertices. The dimer model is the model where we consider a uniform distribution on the perfect matchings. Clearly, a dimer model is a monomer-dimer model without monomers. The number of perfect matchings is denoted by $\mathrm{pm}(G)$. With our previous notation we have $\mathrm{pm}(G) = m_n(G)$.

2.3 Hard-core model

For a graph G let $\mathcal{I}(G)$ be the set of all independent sets. Recall that an independent set is a subset I of the vertices such that no two elements of I is adjacent in G. Here the vertices of G represent possible places for particles that repulse each other so that no two adjacent vertex can be occupied by particles. For a $\lambda > 0$ we can associate a probability space on $\mathcal{I}(G)$ by choosing a random independent set $\mathbf{I}$ as follows:

$$\mathbb{P}(\mathbf{I} = I) = \frac{\lambda^{|I|}}{I(G,\lambda)},$$

where $I(G,\lambda)$ is the normalizing constant:

$$I(G,\lambda) = \sum_{I \in \mathcal{I}(G)} \lambda^{|M|}.$$

Then $I(G,\lambda)$ is the partition function of the hard-core model. In mathematics it is called the independence polynomial of the graph G. Let $i_k(G)$ denote the number of independent sets of size k. Then

$$I(G,\lambda) = \sum_{k=0}^{n} i_k(G)\lambda^k.$$

Note that the sum runs from $k = 0$ as the empty set is an independent set by definition. Similarly to the case of the matchings we have

$$\mathbb{E}|\mathbf{I}| = \sum_{I \in \mathcal{I}(G)} \frac{|I|\lambda^{|I|}}{I(G,\lambda)} = \frac{\lambda I'(G,\lambda)}{I(G,\lambda)}.$$

Hard-core model describes models with repulsive interactions. For instance, if we pack spheres of radius 1 into some space, then it is a hard-core model where the vertices of the graph are all possible points that can be a center of a sphere and two vertices are adjacent if they have distance at least 2. If we discretize this problem, then maybe we can allow only the points of a lattice to be the center of a sphere.

3 Graph limits and examples

In the introduction we have seen that it would be useful if we can consider a lattice as a limit object of its large boxes. This establishes a claim to handle infinite graphs and connect them to the theory of finite graphs. This is exactly the goal of this section. In what follows we introduce the concept of Benjamini–Schramm convergence with some examples. We will see that this concept will be much more flexible than just considering lattices and its subgraphs.

Before we define this concept one more remark is in order: in this lecture note we will always assume that there is some Δ such that the largest degree of any graph G_i in a given sequence of graphs is at most Δ. In such a case we say that the graph sequence (G_i) is a *bounded-degree* graph sequence. This assumption simplifies our task significantly.

Definition 1. For a finite graph G, a finite connected rooted graph α and a positive integer r, let $\mathbb{P}(G,\alpha,r)$ be the probability that the r-ball centered at a uniform random vertex of G is isomorphic to α.

Let L be a probability distribution on (infinite) connected rooted graphs; we will call L a *random rooted graph*. For a finite connected rooted graph α and a positive integer r, let $\mathbb{P}(L,\alpha,r)$ be the probability that the r-ball centered at the root vertex is isomorphic to α, where the root is chosen from the distribution L.

We say that a bounded-degree graph sequence (G_i) is *Benjamini–Schramm convergent* if for all finite rooted graphs α and $r > 0$, the probabilities $\mathbb{P}(G_i,\alpha,r)$ converge. Furthermore, we say that (G_i) *Benjamini-Schramm converges to* L, if for all positive integers r and finite rooted graphs α, $\mathbb{P}(G_i,\alpha,r) \to \mathbb{P}(L,\alpha,r)$.

The Benjamini–Schramm convergence is also called *local convergence* as it primarily grasps the local structure of the graphs (G_i).

If we take larger and larger boxes in the d-dimensional grid $\mathbb{Z}^d$, then it will converge to the rooted $\mathbb{Z}^d$, that is, the corresponding random rooted graph L is simply the distribution which takes a rooted $\mathbb{Z}^d$ with probability 1. When L is a certain rooted infinite graph with probability 1 then we simply say that this rooted infinite graph is the limit without any further reference on the distribution.

There are other very natural graph sequences which are Benjamini–Schramm convergent, for instance, if (G_i) is a sequence of d–regular graphs such that the girth $g(G_i) \to \infty$ (length of the shortest cycle), then it is Benjamini–Schramm convergent and we can even see its limit object: the rooted infinite d-regular tree $\mathbb{T}_d$

The following problem is one of the main problems in the area, and will be especially crucial for us.

Problem. For which graph parameters $p(G)$ is it true that the sequence $(p(G_i))_{i=1}^{\infty}$ converges whenever the graph sequence $(G_i)_{i=1}^{\infty}$ is Benjamini–Schramm convergent?

The problem in such a generality is intractable, but there are various tools to attack it in special cases. One of the most popular tools is the so-called belief propagation. For matchings we will use another way to attack this problem using certain empirical measures called matching measures.

Concerning the general problem the reader might wish to consult with the papers [4,6,7,8,21,22], the book [16] and the references therein.

4 Monomer-dimer and dimer model

In this section we study the monomer-dimer and dimer models. In the first part we prove a result of Kasteleyn [13], and independently Fisher and Temperley [23], about the number of perfect matchings of grid graphs. Then we give a lower bound on the number of perfect matchings of regular bipartite graphs (though the proof will be given in Section 5). Finally, we introduce the concept of matching polynomial and review its basic properties.

4.1 Dimer model on the lattice

In this section we study the number of perfect matchings of grid graphs. It turns out that if G is a bipartite graph with classes of size n, then the problem of counting the number of perfect matchings of G is equivalent to computing the permanent of a $0-1$ matrix of size n by n. Recall that the permanent of a matrix A is defined as follows:

$$\mathrm{per}(A) = \sum_{\pi \in S_n} a_{1,\pi(1)} a_{2,\pi(2)} \dots a_{n,\pi(n)}.$$

Let us suppose for a moment that all $a_{ij} \in \{0,1\}$, and define a graph G on the vertex set $R \cup C$, where $R = \{r_1, r_2, \dots, r_n\}$ and $C = \{c_1, c_2, \dots, c_n\}$ correspond to the rows and columns of the matrix, respectively. If $a_{ij} = 1$, then put an edge between the vertices r_i and c_j. Now it is clear that $\mathrm{per}(A) = \mathrm{pm}(G)$, the number of perfect matchings of G. Unfortunately, permanents are hard to compute in spite of the fact that their siblings, determinants can be computed in polynomial time. Still we can use their similarity as the proof of the following theorem shows.

Theorem 2 (Kasteleyn [13] and independently Fisher and Temperley [23]). Let $Z_{m,n}$ be the number of perfect matchings of the grid of size $m \times n$. Then

$$Z_{m,n} = \left(\prod_{j=1}^{m} \prod_{k=1}^{n} \left(4\cos^2\left(\frac{\pi j}{m+1}\right) + 4\cos^2\left(\frac{\pi k}{n+1}\right)\right)\right)^{1/4}.$$

Proof. Note that the grid is a bipartite graph, so we can color the vertices of the grid by black and white such that only vertices of different colors are adjacent. Let S be the incidence matrix (also called bipartitie adjacency matrix) of the bipartite graph: $S_{ij} = 1$ if black and white vertices b_i and w_j are adjacent, and 0 otherwise. Then the number of perfect matchings is exactly $\mathrm{per}(S)$, the permanent of S. We will give a "signing" σ of S such that $\mathrm{per}(S) = |\det(S^\sigma)|$.

Let $S^\sigma_{(x,y),(x,y\pm 1)} = i$ and $S^\sigma_{(x,y),(x\pm 1,y)} = 1$, and 0 otherwise. We claim that $\mathrm{per}(S) = |\det(S^\sigma)|$. One way to see it is the following: from any perfect matching M_1 we can arrive to any other perfect matching M_2 by a sequence of moves of the following type: choose two edges of the form $e = ((x,y),(x+1,y)), f = ((x,y+1),(x+1,y+1)$ and replace them by $e' = ((x,y),(x,y+1)), f' = ((x+1,y),(x+1,y+1)$, or do the reverse of this operation (why?). In $\det(S^\sigma)$ this operation does the following thing: the sign of the corresponding permutation changes because we did a transposition, but also the weight of the perfect matching changes since we changed two edges of weight 1 to two edges of weight i or vice versa. So every expansion term of $\det(S^\sigma)$ corresponding to a perfect matching will give the same quantity, hence $\mathrm{per}(S) = |\det(S^\sigma)|$.

Next we will compute $\det(S^\sigma)$. It will be more convenient to work with the matrix

$$A = \begin{pmatrix} 0 & S^\sigma \\ (S^\sigma)^T & 0 \end{pmatrix}.$$

Clearly, $\det(A) = \det(S^\sigma)^2$. It turns out that we can give all eigenvectors and eigenvalues explicitly. Note that the vector consisting of the values $f(x,y)$ is an eigenvector of A belonging to the eigenvalue λ if

$$\lambda f(x,y) = f(x+1,y) + f(x-1,y) + i f(x,y+1) + i f(x,y-1),$$

where $f(r,t) = 0$ if $r \in \{0, m+1\}$ or $t \in \{0, n+1\}$. Let $1 \le j \le m$, $1 \le k \le n$, and $z = e^{2\pi i \frac{j}{m+1}}$ and $w = e^{2\pi i \frac{k}{n+1}}$. Let us consider the vector $f_{j,k}$ defined as follows:

$$f_{j,k}(x,y) = (z^x - z^{-x})(w^y - w^{-y}) = -4\sin\left(\frac{\pi j x}{m+1}\right)\sin\left(\frac{\pi k y}{n+1}\right).$$

Then with $\lambda_{j,k} = z + \frac{1}{z} + i\left(w + \frac{1}{w}\right)$ we have

$$\lambda_{j,k} f_{j,k}(x,y) = f_{j,k}(x+1,y) + f_{j,k}(x-1,y) + i f_{j,k}(x,y+1) + i f_{j,k}(x,y-1).$$

Indeed,

$$f_{j,k}(x+1,y) + f_{j,k}(x-1,y) = (z^{x+1} - z^{-x-1})(w^y - w^{-y}) + (z^{x-1} - z^{-x+1})(w^y - w^{-y}) =$$
$$= (z + z^{-1})(z^x - z^{-x})(w^y - w^{-y}) = (z + z^{-1}) f_{j,k}(x,y),$$

and

$$i f_{j,k}(x,y+1) + i f_{j,k}(x,y-1) = i((z^x - z^{-x})(w^{y+1} - w^{-y-1}) + (z^x - z^{-x})(w^{y-1} - w^{-y+1}) =$$

$$= i(w+w^{-1})(z^x - z^{-x})(w^y - w^{-y}) = i(w+w^{-1}) f_{j,k}(x,y),$$

It is easy to see that the vectors $f_{j,k}$ are pairwise orthogonal to each other, consequently they are linearly independent. Since there are nm eigenvalues of A, we have found all of them. Note that

$$\lambda_{j,k} = 2\cos\left(\frac{\pi j}{m+1}\right) + 2\cos\left(\frac{\pi k}{n+1}\right) i.$$

Hence

$$Z_{m,n} = \left(\prod_{j=1}^{m}\prod_{j=1}^{n} \lambda_{j,k}\right)^{1/2} = \left(\prod_{j=1}^{m}\prod_{j=1}^{n} |\lambda_{j,k}|^2\right)^{1/4} =$$

$$= \left(\prod_{j=1}^{m}\prod_{k=1}^{n}\left(4\cos^2\left(\frac{\pi j}{m+1}\right) + 4\cos^2\left(\frac{\pi k}{n+1}\right)\right)\right)^{1/4}.$$

Corollary 3. We have

$$\lim_{\substack{m,n\to\infty \\ 2\,\mid\, mn}} \frac{1}{mn} \log Z_{m,n} = \frac{4}{\pi^2} \int_0^{\pi/2}\int_0^{\pi/2} \log(4\cos^2(x) + 4\cos^2(y))\, dx\, dy.$$

Remark 4. Surprisingly, there is a nice expression for the above integral:

$$\frac{4}{\pi^2} \int_0^{\pi/2}\int_0^{\pi/2} \log(4\cos^2(x) + 4\cos^2(y))\, dx\, dy = \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}.$$

4.2 Lower bounds on the number of perfect matchings

In the previous section we considered the number of perfect matchings of large grids. These graphs were almost 4–regular bipartite graphs, that is, almost all vertices had exactly 4 neighbors. It is a natural question to ask for a lower bound for the number of perfect matchings of d–regular bipartite graphs. This problem was solved in some approximate sense by Voorhoeve and Schrijver.

Theorem 5 (A. Schrijver [19], for $d = 3$ M. Voorhoeve [25]). Let G be a d–regular bipartite graph on $v(G) = 2n$ vertices, and let $\mathrm{pm}(G)$ denote the number of perfect matchings of G. Then

$$\mathrm{pm}(G) \geq \left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^n.$$

In other words, for every d–regular bipartite graph G we have

$$\frac{\ln \mathrm{pm}(G)}{v(G)} \geq \frac{1}{2}\ln\left(\frac{(d-1)^{d-1}}{d^{d-2}}\right).$$

It turns out that the constant $\frac{1}{2}\ln\left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)$ is the best possible constant as it was shown by Wilf [26], see also [3, 20]. They showed it by computing the expected value of $\mathrm{pm}(G)$ for d–regular random bipartite graphs. There was no explicit construction for regular bipartite graphs with small number of perfect matchings for a long time. Very recently it turned out that if a d–regular bipartite graph has small number of short cycles, then it has asymptotically the same number of perfect matchings as a random d–regular graph, the more precise formulation is the following.

Theorem 6 (M. Abért, P. Csikvári, P. E. Frenkel, G. Kun [1]). Let (G_i) be a sequence of d–regular graphs such that $g(G_i) \to \infty$, where g denotes the girth, that is, the length of the shortest cycle.

(a) For the number of perfect matchings $\mathrm{pm}(G_i)$, we have

$$\limsup_{i\to\infty} \frac{\ln \mathrm{pm}(G_i)}{v(G_i)} \leq \frac{1}{2}\ln\left(\frac{(d-1)^{d-1}}{d^{d-2}}\right).$$

(b) If, in addition, the graphs (G_i) are bipartite, then

$$\lim_{i\to\infty} \frac{\ln \mathrm{pm}(G_i)}{v(G_i)} = \frac{1}{2}\ln\left(\frac{(d-1)^{d-1}}{d^{d-2}}\right).$$

This theorem, more precisely, the condition $g(G_i) \to \infty$ together with our knowledge on Benjamini–Schramm convergence suggests that the quantity $\frac{\ln \mathrm{pm}(G)}{v(G)}$ is minimized by the infinite d–regular tree $\mathbb{T}_d$. Indeed, we will give a proof of Theorem 5 based on this intuition in the next section. In fact, we will prove the following more general theorem.

Theorem 7 ([5]). Let G be a d–regular bipartite graph on $v(G) = 2n$ vertices, and let $m_k(G)$ denote the number of matchings of size k. Let $0 \le p \le 1$, then

$$\sum_{k=0}^{n} m_k(G)\left(\frac{p}{d}\left(1-\frac{p}{d}\right)\right)^k (1-p)^{2(n-k)} \ge \left(1-\frac{p}{d}\right)^{nd}.$$

Observe that in case of $p = 1$ this theorem directly reduces to Theorem 5. To prove this theorem, we will need some preparation, and so the rest of this section is devoted to the study of the so-called matching polynomial.

4.3 Matching polynomial

Recall that if $G = (V,E)$ is a finite graph, then $v(G)$ denotes the number of vertices, and $m_k(G)$ denotes the number of k-matchings ($m_0(G) = 1$). Let

$$\mu(G,x) = \sum_{k=0}^{\lfloor v(G)/2 \rfloor} (-1)^k m_k(G) x^{v(G)-2k}.$$

We call $\mu(G,x)$ the matching polynomial. Clearly, the matching generating function $M(G,\lambda)$ introduced in Section 2 and the matching polynomial encode the same information.

Proposition 8 ([10,12]). (a) Let $u \in V(G)$. Then

$$\mu(G,x) = x\mu(G-u,x) - \sum_{v \in N(u)} \mu(G-\{u,v\},x).$$

(b) For $e = (u,v) \in E(G)$ we have

$$\mu(G,x) = \mu(G-e,x) - \mu(G-\{u,v\},x).$$

(c) For $G = G_1 \cup G_2 \cup \cdots \cup G_k$ we have

$$\mu(G,x) = \prod_{i=1}^{k} \mu(G_i,x).$$

(d) We have

$$\mu'(G,x) = \sum_{u \in V(G)} \mu(G-u,x).$$

Proof. (a) By comparing the coefficients of x^{n-2k} we need to prove that

$$m_k(G) = m_k(G-u) + \sum_{v \in N(u)} m_{k-1}(G-\{u,v\}).$$

This is indeed true since we can count the number of k-matchings of G as follows: there are $m_k(G-u)$ k-matchings which do not contain u, and if a k-matching contains u, then there is a unique $v \in N(u)$ such that the edge (u,v) is in the matching, and the remaining $k-1$ edges are chosen from $G-\{u,v\}$.

(b) By comparing the coefficient of x^{n-2k} we need to prove that

$$m_k(G) = m_k(G-e) + m_{k-1}(G-\{u,v\}).$$

This is indeed true since the number of k-matchings not containing e is $m_k(G-e)$, and the number of k-matchings containing $e = (u,v)$ is $m_{k-1}(G-\{u,v\})$.

(c) It is enough to prove the claim when $G = G_1 \cup G_2$, for more components the claim follows by induction. By comparing the coefficient of x^{n-2k} we need to prove that

$$m_k(G) = \sum_{r=0}^{k} m_r(G_1) m_{k-r}(G_2).$$

This is indeed true since a k-matching fof G uniquely determine an r-matching of G_1 and a $(k-r)$-matching of G_2 for some $0 \leq r \leq k$.

(d) This follows from the fact that

$$(m_k(G)x^{n-2k})' = (n-2k)m_k(G)x^{n-1-2k} = \sum_{u \in V(G)} m_k(G-u)x^{n-1-2k}$$

since we can compute the cardinality of the set

$$\{(M,u) \mid u \notin V(M), |M| = k\}$$

in two different ways.

Theorem 9 (Heilmann and Lieb [12]). All zeros of the matching polynomial $\mu(G,x)$ are real.

Proof. We will prove the following two statements by induction on the number of vertices.

(i) All zeros of $\mu(G,x)$ are real.

(ii) For an x with $\mathrm{Im}(x) > 0$ we have

$$\mathrm{Im}\frac{\mu(G,x)}{\mu(G-u,x)} > 0$$

for all $u \in V(G)$.

Note that in (ii) we already use the claim (i) inductively, namely that $\mu(G-u,x)$ does not vanish for an x with $\mathrm{Im}(x) > 0$. On the other hand, claim (ii) for G implies claim (i). So we need to check claim (i).

By the recursion formula we have

$$\frac{\mu(G,x)}{\mu(G-u,x)} = \frac{x\mu(G-u,x) - \sum_{v \in N(u)} \mu(G-\{u,v\},x)}{\mu(G-u,x)} = x - \sum_{v \in N(u)} \frac{\mu(G-\{u,v\},x)}{\mu(G-u,x)}.$$

By induction we have

$$\mathrm{Im}\frac{\mu(G-u,x)}{\mu(G-\{u,v\},x)} > 0$$

for $\mathrm{Im}(x) > 0$. Hence

$$-\mathrm{Im}\frac{\mu(G-\{u,v\},x)}{\mu(G-u,x)} > 0$$

which gives that

$$\mathrm{Im}\frac{\mu(G,x)}{\mu(G-u,x)} > 0.$$

Remark 10. One can also show that the zeros of $\mu(G,x)$ and $\mu(G-u,x)$ interlace each other just like the zeros of a real-rooted polynomial and its derivative.

Theorem 11 (Heilmann and Lieb [12]). If the largest degree Δ is at least 2, then all zeros of the matching polynomial lie in the interval $(-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1})$.

Proof. First we show that if u is a vertex of degree at most $\Delta - 1$, then for any $x \geq 2\sqrt{\Delta-1}$ we have

$$\frac{\mu(G,x)}{\mu(G-u,x)} \geq \sqrt{\Delta-1}.$$

We prove this statement by induction on the number of vertices. This is true if $G = K_1$, so we can assume that $v(G) \geq 2$. Then

$$\begin{aligned}\frac{\mu(G,x)}{\mu(G-u,x)} &= \frac{x\mu(G-u,x) - \sum_{v \in N_G(u)} \mu(G-\{u,v\},x)}{\mu(G-u,x)} \\ &= x - \sum_{v \in N_G(u)} \frac{\mu(G-\{u,v\},x)}{\mu(G-u,x)} \geq x - (\Delta-1)\frac{1}{\sqrt{\Delta-1}} \geq \sqrt{\Delta-1}.\end{aligned}$$

We used the fact that $v \in N_G(u)$ has degree at most $\Delta - 1$ in the graph $G - u$.

Then for any vertex u we have

$$\frac{\mu(G,x)}{\mu(G-u,x)} = \frac{x\mu(G-u,x) - \sum_{v\in N_G(u)} \mu(G-\{u,v\},x)}{\mu(G-u,x)}$$
$$= x - \sum_{v\in N_G(u)} \frac{\mu(G-\{u,v\},x)}{\mu(G-u,x)} \geq x - \Delta\frac{1}{\sqrt{\Delta-1}} > 0$$

since $v \in N_G(u)$ has degree at most $\Delta - 1$ in the graph $G - u$. This shows $\mu(G,x) \neq 0$ if $x \geq 2\sqrt{\Delta-1}$. Since the zeros of the matching polynomial are symmetric to 0 we get that all zeros lie in the interval $(-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1})$.

Suppose that $\mu(G,x) = \prod_{i=1}^{v(G)}(x-\alpha_i)$. Our next goal is to understand the quantity $\sum_{i=1}^{v(G)} \alpha_i^k$ for some fixed k. It turns out that this quantity is a non-negative integer, and it has some combinatorial interpretation counting certain special walks. Next we will make it more explicit by the use of the so-called path-tree.

Before we proceed it is worth motivate and compare our results with the corresponding result for the characteristic polynomial of a graph G. For a graph G, the adjacency matrix of G denoted by A_G is defined as follows: A_G has size $v(G) \times v(G)$ and its rows and columns are labeled by the vertices of G, the element $(A_G)_{uv} = 1$ if the vertices u and v are adjacent, and 0 otherwise. The characteristic polynomial of G is the characteristic polynomial of the adjacency matrix, $\phi(G,x) = \det(xI - A_G)$. Then $\phi(G,x) = \prod_{i=1}^{v(G)}(x-\lambda_i)$. The numbers λ_i are the eigenvalues of the matrix A_G, they are real since A_G is symmetric. Then

$$\sum_{i=1}^{v(G)} \lambda_i^k = \mathrm{Tr}(A_G^k)$$

since λ_i^k are the eigenvalues of A_G^k and the sum of the eigenvalues is the trace of the matrix, that is, the some of the diagonal elements. The quantity

$$\mathrm{Tr}(A_G^k) = \sum_{v_0\in V(G)} (A_G)_{v_0v_1}(A_G)_{v_1v_2}\dots(A_G)_{v_{k-1}v_0}$$

by the definition of matrix multiplication. The quantity $(A_G)_{v_0v_1}(A_G)_{v_1v_2}\dots(A_G)_{v_{k-1}v_0}$ is 1 if and only if $(v_j, v_{j+1}) \in E(G)$ for $j = 0,\dots,k-1$ and $v_k = v_0$, this is called a closed walk. So $\mathrm{Tr}(A_G^k)$ counts the number of closed walks of length k. This is determined by the statistics of the k-neighborhoods.

Definition 12. Let G be graph with a given vertex u. The *path-tree* $T(G,u)$ is defined as follows. The vertices of $T(G,u)$ are the paths[1] in G which start at the vertex u and two paths joined by an edge if one of them is a one-step extension of the other.

Proposition 13. Let G be a graph with a root vertex u. Let $T(G,u)$ be the corresponding path-tree in which the root is again denoted by u for sake of convenience. Then

$$\frac{\mu(G-u,x)}{\mu(G,x)} = \frac{\mu(T(G,u)-u,x)}{\mu(T(G,u),x)},$$

and $\mu(G,x)$ divides $\mu(T(G,u),x)$.

Proof. The proof of this proposition is again by induction using part (a) of Proposition 8. Indeed,

$$\frac{\mu(G,x)}{\mu(G-u,x)} = \frac{x\mu(G-u,x) - \sum_{v\in N(u)} \mu(G-\{u,v\},x)}{\mu(G-u,x)} =$$
$$= x - \sum_{v\in N(u)} \frac{\mu(G-\{u,v\},x)}{\mu(G-u,x)} = x - \sum_{v\in N(u)} \frac{\mu(T(G-u,v)-v,x)}{\mu(T(G-u,v),x)}$$
$$= x\frac{\prod_{v\in N(u)} \mu(T(G-u,v),x) - \sum_{v\in N(u)} \mu(T(G-u,v)-v,x)\prod_{v'\in N(u)\setminus\{v\}} \mu(T(G-u,v'),x)}{\prod_{v\in N(u)} \mu(T(G-u,v),x)}$$
$$= \frac{x\mu(T(G,u)-u,x) - \sum_{v\in N(u)} \mu(T(G,u)-\{u,v\},x)}{\mu(T(G,u)-u,x)} = \frac{\mu(T(G,u),x)}{\mu(T(G,u)-u,x)}.$$

In the first step we used the recursion formula, and in the third step we used the induction step to the graph $G - u$ and root vertex v. Here it is an important observation that $T(G-u,v)$ is exactly the branch of the tree $T(G,u)$ that we get if we delete the vertex u from $T(G,u)$ and consider the subtree rooted at the path uv.

[1] In statistical physics, paths are called self-avoiding walks.

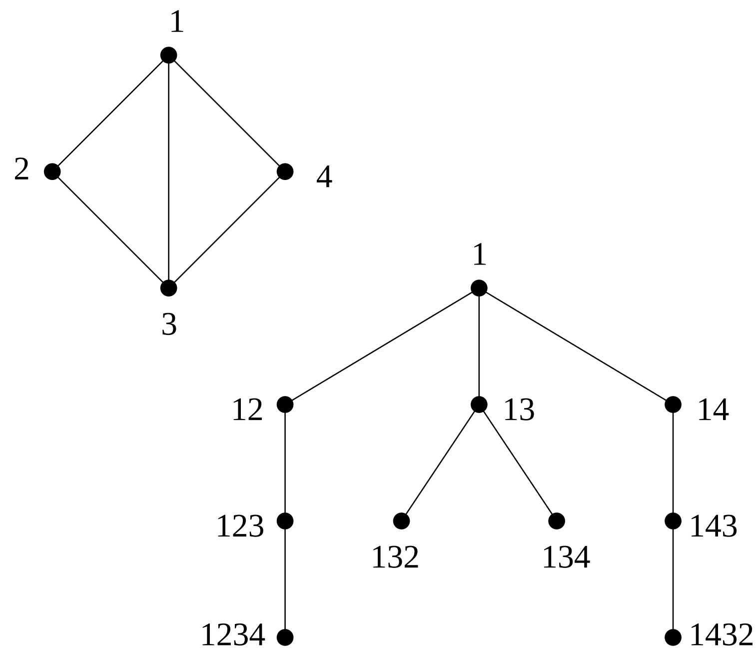


Figure 1: A path-tree from the vertex 1.

Proposition 14 ([10,12]). For a forest T, the matching polynomial $\mu(T,x)$ coincides with the characteristic polynomial $\phi(T,x) = \det(xI - A_T)$.

Proof. Indeed, when we expand the $\det(xI - A)$ we only get non-zero terms when the cycle decomposition of the permutation consists of cycles of length at most 2. These terms correspond to the terms of the matching polynomial.

Remark 15. Clearly, Propositions 13 and 14 together give a new proof of the Heilmann-Lieb theorem since $\mu(G,x)$ divides $\mu(T(G,u),x) = \phi(T(G,u),x)$ whose zeros are real since they are the eigenvalues of a symmetric matrix.

Proposition 16. Let

$$\frac{\mu(G-u,x)}{\mu(G,x)} = \sum_k a_k(G,u)x^{-(k+1)}.$$

Then $a_k(G,u)$ counts the number of closed walks of length k in the tree $T(G,u)$ from u to u.

Proof. This proposition follows from Proposition 13 and 14 and the fact that

$$\frac{\phi(H-u,x)}{\phi(H,x)} = \sum_k W_k(H,u)x^{-(k+1)},$$

where $W_k(H,u)$ counts the number of closed walks of length k from u to u in a graph H. Indeed,

$$\frac{\mu(G-u,x)}{\mu(G,x)} = \frac{\mu(T(G,u)-u,x)}{\mu(T(G,u),x)} = \frac{\phi(T(G,u)-u,x)}{\phi(T(G,u),x)} = \sum_k W_k(T(G,u),u)x^{-k}.$$

Here $W_k(T(G,u),u) = a_k(G,u)$ by definition.

Remark 17. A walk in the tree $T(G,u)$ from u can be imagined as follows. Suppose that in the graph G a worm is sitting at the vertex u at the beginning. Then at each step the worm can either grow or pull back its head. When it grows it can move its head to a neighboring unoccupied vertex while keeping its tail at vertex u. At each step the worm occupies a path in the graph G. A closed walk in the tree $T(G,u)$ from u to u corresponds to the case when at the final step the worm occupies only vertex u. C. Godsil calls these walks tree-like walks in the graph G.

Proposition 18. (a) Let

$$\frac{\mu'(G,x)}{\mu(G,x)} = \sum_k a_k(G)x^{-(k+1)}.$$

Then $a_k(G)$ counts the number of closed tree-like walks of length k.

(b) If $\mu(G,x) = \prod_{i=1}^{v(G)}(x-\alpha_i)$, then for all $k \geq 1$ we have

$$a_k(G) = \sum_{i=1}^{v(G)} \alpha_i^k.$$

Remark 19. The quantity $a_k(G,u)$ makes perfect sense for a random rooted graph L as it only depends on the k neighborhood of the root vertex. So we can define:

$$a_k(L) = \sum_{\alpha} \mathbb{P}(L,\alpha,k)\mathrm{TW}_k(\alpha),$$

where $\mathrm{TW}_k(\alpha)$ is the number of closed tree-like walks of length k from the root vertex u of α in the rooted graph α.

5 A proof strategy

The goal of this section is to present an almost complete proof of Theorem 5 and 7. As we have discussed in section 4.2 certain intuition suggests that the extremal graph for (perfect) matchings is not finite, but the infinite d–regular tree $\mathbb{T}_d$. This raises the question: how can we attack a problem if we conjecture that the d–regular tree is the extremal graph for a given graph parameter:

Problem: Given a graph parameter $p(G)$. We would like to prove that among d–regular graphs we have

$$p(G) \geq p(\mathbb{T}_d).$$

A possible two-step solution..

- Find a graph transformation φ for which $p(G) \geq p(\varphi(G))$, and for every graph G there exists a sequence of graphs (G_i) such that $G = G_0$ and $G_i = \varphi(G_{i-1})$, and $G_i \to \mathbb{T}_d$.
- Show that if (G_i) converges in Benjamini–Schramm sense, then $p(G_i)$ is convergent, and compute $p(\mathbb{T}_d)$. (Or at least, show it in the case of $G_i \to \mathbb{T}_d$.) Then

$$p(G) = p(G_0) \geq p(G_1) \geq p(G_2) \geq \dots \geq p(\mathbb{T}_d).$$

Concerning the first step we will be more explicit: it seems that the 2-lift transformation can be used in a wide range of problems. Experience shows that the second step can be the most difficult, but the first step can also be tricky. Nevertheless, in the special case when we only consider a graph sequence converging to $\mathbb{T}_d$, there are many available tools: see for instance the paper of D. Gamarnik and D. Katz [9]. If $p(G) = \ln I(G,\lambda)/v(G)$, where $I(G,\lambda)$ denotes the partition function of the hard-core model, then the first step is very easy for d–regular bipartite graphs, while the second step concerning the limit theorem was established by A. Sly and N. Sun [21]. If $p(G) = \ln Z_{\mathrm{Is}}(G,B,\beta)/v(G)$, then both steps are somewhat tricky, but the first step is still easier.

In this section we demonstrate this plan by sketching the proof of Theorem 7. In the following sections we study each step separately.

5.1 First step: graph transformation

In this section we introduce the concept of 2-lift.

Definition 20. A k-cover (or k-lift) H of a graph G is defined as follows. The vertex set of H is $V(H) = V(G) \times \{0,1,\dots,k-1\}$, and if $(u,v) \in E(G)$, then we choose a perfect matching between the vertices (u,i) and (v,j) for $0 \leq i,j \leq k-1$. If $(u,v) \notin E(G)$, then there are no edges between (u,i) and (v,j) for $0 \leq i,j \leq k-1$.

When $k=2$ one can encode the 2-lift H by putting signs on the edges of graph G: the $+$ sign means that we use the matching $((u,0),(v,0)),((u,1),(v,1))$ at the edge (u,v), the $-$ sign means that we use the matching $((u,0),(v,1)),((u,1),(v,0))$ at the edge (u,v). For instance, if we put $+$ signs to every edge, then we simply get $G \cup G$ as H, and if we put $-$ signs everywhere, then the obtained 2-cover H is simply $G \times K_2$.

The following result will be crucial for our argument.

Lemma 21 (N. Linial [15]). For any graph G, there exists a graph sequence $(G_i)_{i=0}^{\infty}$ such that $G_0 = G$, G_i is a 2-lift of G_{i-1} for $i \geq 1$, and $g(G_i) \to \infty$, where $g(H)$ is the girth of the graph H, that is, the length of the shortest cycle. In particular, if G_0 is d–regular, then $G_i \to \mathbb{T}_d$.

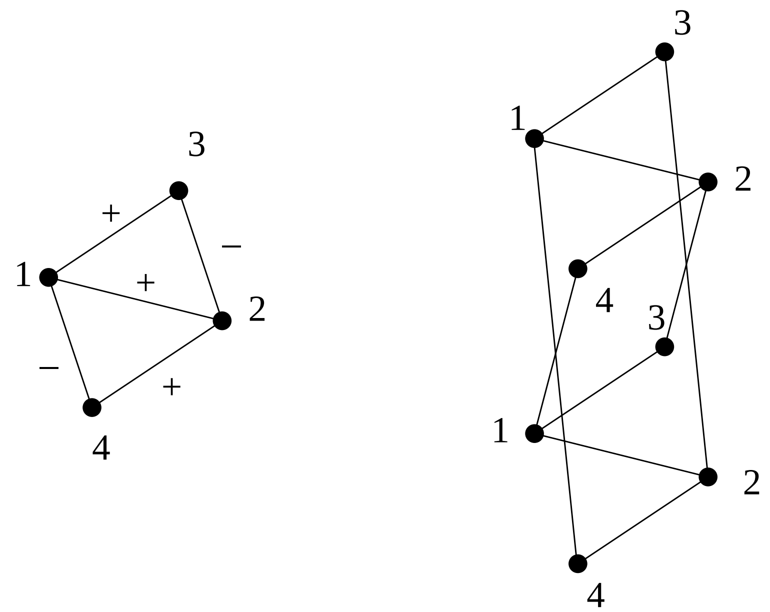


Figure 2: A 2-lift.

Proof. It is clear that if H' is a 2-lift of H, then $g(H') \geq g(H)$. Hence it is enough to show that for every H there exists an H'' obtained from H by a sequence of 2-lifts such that $g(H'') > g(H)$. We show that if the girth $g(H) = k$, then there exists a lift of H with fewer k-cycles than H. Let X be the random variable counting the number of k-cycles in a random 2-lift of H. Every k-cycle of H lifts to two k-cycles or a $2k$-cycle with probability $1/2$ each, so $\mathbb{E}X$ is exactly the number of k-cycles of H. But $H \cup H$ has two times as many k-cycles than H, so there must be a lift with strictly fewer k-cycles than H has. Choose this 2-lift and iterate this step to obtain an H'' with girth at least $k+1$.

Note that if G is a bipartite d–regular graph, and H is a 2-lift of G, then H is again a d–regular bipartite graph.

The following theorem shows that the first step of the plan works for matchings of bipartite graphs.

Theorem 22. Let G be a graph, and let H be an arbitrary 2-lift of G. Then

$$m_k(H) \leq m_k(G \times K_2),$$

where $m_k(\cdot)$ denotes the number of matchings of size k.

In particular, if $H = G \cup G$, then $m_k(G \cup G) \leq m_k(G \times K_2)$ for every k. It follows that we have $\mathrm{pm}(G)^2 \leq \mathrm{pm}(G \times K_2)$.

Furthermore, if G is a bipartite graph and H is a 2-lift of G, then

$$\frac{\ln M(G,\lambda)}{v(G)} = \frac{\ln M(G \cup G, t)}{v(G \cup G)} \geq \frac{\ln M(H,\lambda)}{v(H)},$$

where $M(G,\lambda) = \sum_k m_k(G)\lambda^k$. (Note that $M(G \cup G, \lambda) = M(G,\lambda)^2$.)

Proof. Let M be any matching of a 2-lift of G. Let us consider the projection of M to G, then it will consist of cycles, paths and "double-edges" (i.e, when two edges project to the same edge). Let $\mathcal{R}$ be the set of these configurations. Then

$$m_k(H) = \sum_{R \in \mathcal{R}} |\phi_H^{-1}(R)|$$

and

$$m_k(G \times K_2) = \sum_{R \in \mathcal{R}} |\phi_{G \times K_2}^{-1}(R)|,$$

where ϕ_H and $\phi_{G \times K_2}$ are the projections from H and $G \times K_2$ to G. Note that

$$|\phi_{G \times K_2}^{-1}(R)| = 2^{k(R)},$$

where $k(R)$ is the number of cycles and paths of R. Indeed, in each cycle or path we can lift the edges in two different ways. The projection of a double-edge is naturally unique. On the other hand,

$$|\phi_H^{-1}(R)| \leq 2^{k(R)},$$

since in each cycle or path if we know the inverse image of one edge, then we immediately know the inverse images of all other edges. Clearly, there is no equality in general for cycles. Hence

$$|\phi_H^{-1}(R)| \le |\phi_{G\times K_2}^{-1}(R)|$$

and consequently,

$$m_k(H) \le m_k(G\times K_2).$$

Note that if G is bipartite, then $G\times K_2 = G\cup G$, and so

$$\frac{1}{v(H)}\ln M(H,\lambda) \le \frac{1}{v(G\cup G)}\ln M(G\cup G,\lambda) = \frac{1}{2v(G)}\ln M(G,\lambda)^2 = \frac{1}{v(G)}\ln M(G,\lambda)$$

Remark 23. Sometimes it is also possible to prove that for a certain graph parameter $p(\cdot)$ one has $p(G)\ge p(H)$ for all k-cover H of G. Such a result was given by N. Ruozzi in [18] for attractive graphical models. The advantage of using k-covers is that one can spare the graph limit step in the above plan, and replace it with a much simpler averaging argument over all k-covers of G with k converging to infinity. For homomorphisms this averaging argument was given by P. Vontobel [24]. For matchings such a result was established by C. Greenhill, S. Janson and A. Ruciński [11].

Exercise 24. Let G be a graph, and let H be an arbitrary 2-lift of G. Show that

$$i_k(H) \le i_k(G\times K_2),$$

where $i_k(\cdot)$ denotes the number of independent sets of size k. Furthermore, show that if G is a bipartite graph and H is a 2-lift of G, then

$$\frac{\ln I(G,\lambda)}{v(G)} = \frac{\ln I(G\cup G,\lambda)}{v(G\cup G)} \ge \frac{\ln I(H,\lambda)}{v(H)},$$

where $I(G,\lambda)=\sum_k i_k(G)\lambda^k$, that is, the partition function of the hard-core model. (Note that $I(G\cup G,\lambda)=I(G,\lambda)^2$.)

Exercise 25. (hard) Let G be a graph, and let H be an arbitrary 2-lift of G. Show that if $\beta>0$ then

$$\frac{\ln Z_{\mathrm{Is}}(G,B,\beta)}{v(G)} \ge \frac{\ln Z_{\mathrm{Is}}(H,B,\beta)}{v(H)},$$

where $Z_{\mathrm{Is}}(G,B,\beta)$ is the partition function of the Ising-model.

5.2 Second step: graph limit theory

In this subsection we carry out the second step of our plan. First we develop the necessary terminology.

Definition 26 (M. Abért, P. Csikvári, P. E. Frenkel, G. Kun [1]). The matching measure of a finite graph G is defined as

$$\rho_G = \frac{1}{v(G)}\sum_{z_i:\ \mu(G,z_i)=0}\delta(z_i),$$

where $\delta(s)$ is the Dirac-delta measure on s, and we take every z_i into account with its multiplicity. In other words, it is the uniform distribution on the zeros of $\mu(G,x)$.

Example: Let us consider the matching measure of C_6.

$$\mu(C_6,x)=x^6-6x^4+9x^2-2=$$

$$=(x-\sqrt{2})(x+\sqrt{2})(x-\sqrt{2+\sqrt{3}})(x+\sqrt{2+\sqrt{3}})(x-\sqrt{2-\sqrt{3}})(x+\sqrt{2-\sqrt{3}})$$

Hence

$$\int f(z)\,d\rho_{C_6}(z)=$$

$$=\frac{1}{6}\left(f(\sqrt{2})+f(-\sqrt{2})+f(\sqrt{2+\sqrt{3}})+f(-\sqrt{2+\sqrt{3}})+f(\sqrt{2-\sqrt{3}})+f(-\sqrt{2-\sqrt{3}})\right).$$

The following theorem enables us to consider the matching measure of a unimodular random graph which can be obtained as a Benjamini–Schramm limit of finite graphs. In particular, it provides an important tool to establish the second step of our plan.

Theorem 27 (M. Abért, P. Csikvári, P. E. Frenkel, G. Kun [1]). Let (G_i) be a Benjamini–Schramm convergent bounded degree graph sequence. Let ρ_{G_i} be the matching measure of the graph G_i. Then the sequence (ρ_{G_i}) is weakly convergent, i. e., there exists some measure ρ_L such that for every bounded continuous function f, we have

$$\lim_{i\to\infty}\int f(z)\,d\rho_{G_i}(z)=\int f(z)\,d\rho_L(z).$$

Proof. For $k\geq 0$ let

$$\mu_k(G)=\int z^k\,d\rho_G(z)$$

be the k-th moment of ρ_G. By Proposition 18 we have

$$\mu_k(G)=\mathbb{E}_v a_k(G,v)$$

where $a_k(G,v)$ denotes the number of closed walks of length k of the tree $T(G,v)$ starting and ending at the vertex v.

Clearly, the value of $a_k(G,v)$ only depends on the k-ball centered at the vertex v. Let $\mathrm{TW}_k(\alpha)=a_k(G,v)$ where the k-ball centered at v is isomorphic to α. Note that the value of $\mathrm{TW}_k(\alpha)$ depends only on the rooted graph α and k and does not depend on G.

Let $\mathcal{N}_k$ denote the set of possible k-balls in G. The size of $\mathcal{N}_k$ and $\mathrm{TW}_k(\alpha)$ are bounded by a function of k and the largest degree of G. By the above, we have

$$\mu_k(G)=\mathbb{E}_v a_k(G,v)=\sum_{\alpha\in\mathcal{N}_k}\mathbb{P}(G,\alpha,k)\cdot\mathrm{TW}_k(\alpha).$$

Since (G_i) is Benjamini–Schramm convergent, we get that for every fixed k, the sequence of k-th moments $\mu_k(G_i)$ converges. The same holds for $\int q(z)\,d\rho_{G_i}(z)$ where q is any polynomial. By the Heilmann–Lieb theorem, ρ_{G_i} is supported on $[-2\sqrt{\Delta-1},2\sqrt{\Delta-1}]$ where Δ is the absolute degree bound for G_i. Since every continuous function can be uniformly approximated by a polynomial on $[-2\sqrt{\Delta-1},2\sqrt{\Delta-1}]$, we get that the sequence (ρ_{G_i}) is weakly convergent.

Assume that (G_i) Benjamini–Schramm converges to L. Then for all $k\geq 0$ we have $\mathbb{P}(G_n,\alpha_k,k)\to 1$ where α_k is the k-ball in L, which implies

$$\lim_{n\to\infty}\mu_k(G_n)=\lim_{n\to\infty}\sum_{\alpha\in\mathcal{N}_k}\mathbb{P}(G_n,\alpha,k)\cdot\mathrm{TW}_k(\alpha)=\sum_{\alpha}\mathbb{P}(L,\alpha,k)\cdot\mathrm{TW}_k(\alpha)=a_k(L)$$

where v is any vertex in L. This means that all the moments of ρ_L and $\lim\rho_{G_n}$ are equal, so $\lim\rho_{G_n}=\rho_L$.

Remark 28. It is clear from the proof that the crucial point of the proof is that $\int z^k\,d\rho_G(z)$ can be determined by knowing the statistics of the k-balls. This phenomenon is not restricted to the matching polynomial. In fact, it is a very general phenomenon. A better-known example is that for the spectral measure, that is, for the uniform distribution on the eigenvalues of the adjacency matrix of the graph, this integral is determined by the number of closed walks of length k as we discussed before Definition 12.

To illustrate the power of Theorem 27, let us consider an application that also provides us the second step of our plan.

Theorem 29 (M. Abért, P. Csikvári, T. Hubai [2]). Let (G_i) be a Benjamini–Schramm convergent graph sequence of bounded degree graphs. Then the sequences of functions

$$\frac{\ln M(G_i,\lambda)}{v(G_i)}$$

is pointwise convergent.

Proof. If G is a graph on $v(G)=2n$ vertices and

$$M(G,\lambda)=\prod_{i=1}^{v(G)/2}(1+\gamma_i\lambda),$$

then

$$\mu(G,x)=\prod_{i=1}^{v(G)/2}(x-\sqrt{\gamma_i})(x+\sqrt{\gamma_i}),$$

and therefore

$$\frac{\ln M(G,\lambda)}{v(G)}=\frac{1}{v(G)}\sum_{i=1}^{v(G)/2}\ln(1+\gamma_i\lambda)=\int\frac{1}{2}\ln(1+\lambda z^2)\,d\rho_G(z).$$

Since $\frac{1}{2}\ln(1+\lambda z^2)$ is a continuous function for every fixed positive λ, the theorem immediately follows from Theorem 27.

Exercise 30. When we introduced the monomer-dimer model we have seen that the expected size of a random matching is

$$\mathbb{E}_G|\mathbf{M}| = \frac{\lambda M'(G,\lambda)}{M(G,\lambda)}.$$

Show that if (G_i) is a Benjamini–Schramm convergent graph sequence of bounded degree graphs, then $\mathbb{E}_{G_i}|\mathbf{M}|/v(G_i)$ is convergent.

It is worth introducing the notation

$$p_\lambda(G) = \frac{\ln M(G,\lambda)}{v(G)}.$$

By Theorem 29 we can also introduce $p_\lambda(L)$ if L is a Benjamini–Schramm-limit of a sequence of finite graphs (G_i). (In fact, from the proof it is clear that it is possible to define the function $p_\lambda(L)$ if L is not the Benjamini–Schramm-limit of finite graphs.) In particular, we can speak about $p_\lambda(\mathbb{T}_d)$.

If we know the matching measure of a random unimodular graph, then it is just a matter of computation to derive various results on matchings.

In the particular case when the sequence (G_i) converges to the infinite d–regular tree $\mathbb{T}_d$, the limit measure $\rho_{\mathbb{T}_d}$ turns out to be the so-called Kesten-McKay measure. It is true in general that for any finite tree or infinite random rooted tree the matching measure coincides with the so-called spectral measure, for details see [1]. In particular, this is true for the infinite d–regular tree $\mathbb{T}_d$. Its spectral measure is computed explicitly in the papers [14] and [17]. The Kesten-McKay measure is given by the density function

$$f_d(x) = \frac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)}\chi_{[-\omega,\omega]},$$

where $\omega = 2\sqrt{d-1}$. Hence for any continuous function $h(z)$ we have

$$\int h(z)\,d\rho_{\mathbb{T}_d}(z) = \int_{-\omega}^{\omega} h(z) f_d(z)\,dz.$$

In particular,

$$p_\lambda(\mathbb{T}_d) = \int \frac{1}{2}\ln(1+\lambda z^2)\,d\rho_{\mathbb{T}_d}(z) = \frac{1}{2}\ln S_d(\lambda),$$

where

$$S_d(\lambda) = \frac{1}{\eta_\lambda^2}\left(\frac{d-1}{d-\eta_\lambda}\right)^{d-2} \quad \text{and} \quad \eta_\lambda = \frac{\sqrt{1+4(d-1)\lambda}-1}{2(d-1)\lambda}.$$

It is worth introducing the following substitution:

$$\lambda = \frac{\frac{p}{d}\left(1-\frac{p}{d}\right)}{(1-p)^2}.$$

As p runs through the interval $[0,1)$, λ runs through the interval $[0,\infty)$ and we have

$$\eta_\lambda = \frac{1-p}{1-\frac{p}{d}} \quad \text{and} \quad S_d(\lambda) = \frac{\left(1-\frac{p}{d}\right)^d}{(1-p)^2}.$$

5.3 The end of the proof of Theorem 7

For every sequence of 2-covers we know from Theorem 22 that

$$p_\lambda(G_0) \geq p_\lambda(G_1) \geq p_\lambda(G_2) \geq p_\lambda(G_3) \geq \dots$$

Furthermore, from Theorem 21 and 29 we know that we can choose the sequence of 2-covers such that the sequence $p_\lambda(G_i)$ converges to $p_\lambda(\mathbb{T}_d)$, hence $p_\lambda(G) \geq p_\lambda(\mathbb{T}_d)$ for any d–regular bipartite graph G. In other words,

$$\frac{1}{2n}\ln M(G,\lambda) \geq \frac{1}{2}\ln S_d(\lambda).$$

With the substitution $\lambda = \frac{\frac{p}{d}\left(1-\frac{p}{d}\right)}{(1-p)^2}$ we arrive to the inequality

$$M\left(G, \frac{\frac{p}{d}\left(1-\frac{p}{d}\right)}{(1-p)^2}\right) \geq \frac{1}{(1-p)^{2n}}\left(1-\frac{p}{d}\right)^n.$$

After multiplying by $(1-p)^{2n}$, we get that

$$\sum_{k=0}^{n} m_k(G)\left(\frac{p}{d}\left(1-\frac{p}{d}\right)\right)^k (1-p)^{2(n-k)} \geq \left(1-\frac{p}{d}\right)^{nd}.$$

This is true for all $p \in [0,1)$ and so by continuity it is also true for $p = 1$, where it directly reduces to Schrijver's theorem since all but the last term vanish on the left hand side.

[1] M. Abért, P. Csikvári, P. Frenkel, and G. Kun, *Matchings in Benjamini–Schramm convergent graph sequences*, Transactions of the American Mathematical Society, **368** (2016), pp. 4197–4218.

[2] M. Abért, P. Csikvári, and T. Hubai, *Matching measure, Benjamini–Schramm convergence and the monomer–dimer free energy*, Journal of Statistical Physics, **161** (2015), pp. 16–34.

[3] B. Bollobás and B. D. McKay, *The number of matchings in random regular graphs and bipartite graphs*, Journal of Combinatorial Theory, Series B, **41** (1986), pp. 80–91.

[4] C. Borgs, J. Chayes, J. Kahn, and L. Lovász, *Left and right convergence of graphs with bounded degree*, Random Structures & Algorithms, **42** (2013), pp. 1–28.

[5] P. Csikvári, *Lower matching conjecture, and a new proof of Schrijver's and Gurvits's theorems*, Journal of European Mathematical Society, **19** (2017), pp. 1811–1844.

[6] A. Dembo and A. Montanari, *Gibbs measures and phase transitions on sparse random graphs*, Brazilian Journal of Probability and Statistics, (2010), pp. 137–211.

[7] A. Dembo, A. Montanari, A. Sly, and N. Sun, *The replica symmetric solution for Potts models on d-regular graphs*, Communications in Mathematical Physics, **327** (2014), pp. 551–575.

[8] A. Dembo, A. Montanari, and N. Sun, *Factor models on locally tree-like graphs*, The Annals of Probability, **41** (2013), pp. 4162–4213.

[9] D. Gamarnik and D. Katz, *Sequential cavity method for computing free energy and surface pressure*, Journal of Statistical Physics, **137** (2009), pp. 205–232.

[10] C. D. Godsil, *Algebraic combinatorics*, vol. 6, CRC Press, 1993.

[11] C. Greenhill, S. Janson, and A. Ruciński, *On the number of perfect matchings in random lifts*, Combinatorics, Probability and Computing, **19** (2010), pp. 791–817.

[12] O. J. Heilmann and E. H. Lieb, *Theory of monomer-dimer systems*, Communications in Mathematical Physics, (1972), pp. 190–232.

[13] P. W. Kasteleyn, *The statistics of dimers on a lattice: I. The number of dimer arrangements on a quadratic lattice*, Physica, **27** (1961), pp. 1209–1225.

[14] H. Kesten, *Symmetric random walks on groups*, Transactions of the American Mathematical Society, **92** (1959), pp. 336–354.

[15] N. Linial, *Lifts of graphs (talk slides)*, `http://www.cs.huji.ac.il/~nati/PAPERS/lifts_talk.pdf`.

[16] L. Lovász, *Large networks and graph limits*, vol. 60, American Mathematical Soc., 2012.

[17] B. D. McKay, *The expected eigenvalue distribution of a large regular graph*, Linear Algebra and its Applications, **40** (1981), pp. 203–216.

[18] N. Ruozzi, *The Bethe partition function of log-supermodular graphical models*, in Advances in Neural Information Processing Systems, 2012, pp. 117–125.

[19] A. Schrijver, *Counting 1-factors in regular bipartite graphs*, Journal of Combinatorial Theory, Series B, **72** (1998), pp. 122–135.

[20] A. Schrijver and W. G. Valiant, *On lower bounds for permanents*, in Indagationes Mathematicae (Proceedings), vol. 83, Elsevier, 1980, pp. 425–427.

[21] A. Sly and N. Sun, *The computational hardness of counting in two-spin models on d-regular graphs*, in Foundations of Computer Science (FOCS), 2012 IEEE 53rd Annual Symposium on, IEEE, 2012, pp. 361–369.

[22] A. Sly and N. Sun, *Counting in two-spin models on d–regular graphs*, The Annals of Probability, **42** (2014), pp. 2383–2416.

[23] H. N. V. Temperley and M. E. Fisher, *Dimer problem in statistical mechanics-an exact result*, Philosophical Magazine, **6** (1961), pp. 1061–1063.

[24] P. O. Vontobel, *Counting in graph covers: A combinatorial characterization of the Bethe entropy function*, IEEE Transactions on Information Theory, **59** (2013), pp. 6018–6048.

[25] M. Voorhoeve, *A lower bound for the permanents of certain $(0,1)$-matrices*, in Indagationes Mathematicae (Proceedings), vol. 82, Elsevier, 1979, pp. 83–86.

[26] H. S. Wilf, *On the permanent of a doubly stochastic matrix*, Canadian Journal of Mathematics, **18** (1966), pp. 758–761.

1 Limits of dense graphs

1.1 Convergence of dense graphs

Imagine a sequence (G_n) of finite, simple, undirected graphs having more and more nodes. When should we say that this graph sequence is convergent, i.e., the graphs look more and more alike (apart from the growth in size)? The idea is to describe any graph by certain numerical parameters and then require all of these parameters to converge along the graph sequence. Suitable such parameters can be chosen in several equivalent ways. The most natural one is sampling:

Definition 1. For a fixed k, choose k random nodes from G_n uniformly and independently. The subgraph they span gives you a random labeled graph on the node set $[k] = \{1, 2, \ldots, k\}$. Define (G_n) to be convergent if the distribution of this random labeled graph is convergent for any fixed k.

It turns out that an equivalent, but more convenient formalism for this can be given based on counting homomorphisms between graphs.

Definition 2. A homomorphism between finite, simple, undirected graphs F and G is a map $V(F) \to V(G)$ such that whenever i and j are adjacent nodes in F, then their images are adjacent nodes in G. The number of homomorphisms from F to G is written $\hom(F,G)$. The homomorphism density is the proportion of homomorphisms among all maps:

$$t(F,G) = \frac{\hom(F,G)}{\mathrm{v}(G)^{\mathrm{v}(F)}},$$

where the v stands for the number of nodes.

Proposition 3. *The graph sequence (G_n) is convergent if and only if $t(F,G_n)$ converges for any fixed F.*

1.2 Graphons

Given a convergent graph sequence, can we define a "graph-like", but infinite, "measure theoretical" object the sequence converges to? It was shown by Lovász and Szegedy that this is possible. The limit object is called a graphon (this word is a contraction of 'graph-function').

Definition 4. A graphon is a symmetric, measurable function $W : [0,1]^2 \to [0,1]$.

We can think of it as a graph-like object: the node set is $[0,1]$, and adjancency of two nodes x, y is 'fuzzy': it can be any number $0 \le W(x,y) \le 1$ rather than just 0 or 1. Any graph G on the node set $V(G) = [n]$ gives rise to a 0-1 valued graphon: let $W(x,y) = 1$ if $\lceil nx \rceil$ and $\lceil ny \rceil$ are adjacent in G and $W(x,y) = 0$ otherwise. This leads to a generalization of homomorphism densities to graphons:

Definition 5.

$$t(F,W) = \int_{[0,1]^k} \prod_{ij \in E(F)} W(x_i, x_j) \mathrm{d}x_k \cdots \mathrm{d}x_1,$$

where F is a simple graph on the node set $[k]$.

The homomorphism density $t(F,G)$ does not depend on how we label the nodes of the graphs F and G. Similarly, homomorphism densities of a graphon are invariant with respect to permutations of the node set $[k]$ of F, and also under measure preserving self-maps of $[0,1]$.

Definition 6. A measurable map $\phi : [0,1]^k \to [0,1]^k$ is measure preserving if $\phi^{-1}(A)$ has the same Lebesgue measure as A for all measurable sets $A \subset [0,1]^k$. Here $\phi^{-1}(A) = \{b : \phi(b) \in A\}$ is the inverse image of A under ϕ.

Proposition 7. *If $\phi : [0,1] \to [0,1]$ is measure preserving, then*

(a) so is the map $\phi^{\otimes k} : [0,1]^k \to [0,1]^k$ defined by

$$\phi^{\otimes k}(x_1, \ldots, x_k) = (\phi(x_1), \ldots, \phi(x_k)),$$

for any k;

(b) for any graph F and any graphon W, we have $t(F,W) = t(F,W^\phi)$, where $W^\phi(x,y) = W(\phi(x),\phi(y))$.

Definition 8. We say that $G_n \to W$ if $t(F,G_n) \to t(F,W)$ for all F. More generally, we can define convergence of a sequence of graphons : $W_n \to W$ if $t(F,W_n) \to t(F,W)$ for all F.

1.3 Cut distance

A key tool for understanding this convergence notion is the so-called cut norm: for an integrable function $W \in \mathfrak{L}_1([0,1]^2)$ on the unit square, we write

$$\|W\|_\square = \sup_{S,T\subseteq[0,1]} \left| \int_S \int_T W(x,y)\mathrm{d}y\mathrm{d}x \right|,$$

where the sets S and T are measurable. It turns out that this is a norm. Moreover, we have

Lemma 9.

$$\left| \int_0^1 \int_0^1 f(x)W(x,y)g(y)\mathrm{d}y\mathrm{d}x \right| \leq \|W\|_\square$$

for any measurable functions $f, g : [0,1] \to [0,1]$.

Definition 10. The cut distance of two graphons U and W is

$$\delta_\square(U,W) = \inf \|U^\phi - W^\psi\|,$$

where the infimum is taken over all pairs ϕ, ψ of measure preserving maps $[0,1] \to [0,1]$.

This is a pseudometric (distinct graphons can have cut distance zero, but all other axioms of a metric hold).

Proposition 7 and Lemma 9 help us to establish a main link between the cut norm and homomorphism densities:

Counting lemma (Lovász and Szegedy, 2006). *For any two graphons W and W', and any graph F, the inequality*

$$|t(F,W) - t(F,W')| \leq \mathrm{e}(F)\delta_\square(W,W')$$

holds. Here $\mathrm{e}(F)$ is the number of edges in F.

The proof relies on the more general

Lemma 11 (Lovász and Szegedy, 2006). Assign two graphons W_e and W'_e to each edge e of the graph F on the node set $[k]$. Then the inequality

$$\left| \int_{[0,1]^k} \left(\prod_{ij\in E(F)} W_e(x_i,x_j) - \prod_{ij\in E(F)} W'_e(x_i,x_j) \right) \mathrm{d}x_k \cdots \mathrm{d}x_1 \right| \leq \sum_{e\in E(F)} \|W_e - W'_e\|_\square$$

holds.

1.4 Szemerédi partitions

Another cornerstone of understanding convergence of graphs or graphons is the regularity lemma. It was discovered by Endre Szemerédi in the 1970's. He used it to prove a conjecture of P. Erdős and P. Turán: any set of integers with positive upper density contains arithmetic progressions of any finite length. In the decades that followed, the lemma gradually turned out to play a central role in the study of large dense graphs. Informally, the lemma says that, given an arbitrarily small prescribed margin of error ϵ, the node set of any large graph can be partitioned into k classes of approximately equal size so that between all but ϵk^2 pairs of classes, the graph 'looks random' up to an error of ϵ, and the number k of classes needed to do this depends only on ϵ, not on the number of nodes of the graph. There are many ways to make this precise. We shall prove a weak version of this, for graphons rather than graphs.

Weak Regularity Lemma for Graphons (Frieze and Kannan, 1999). *For any graphon W, and any positive integer k, there exists a partition of $[0,1]$ into measurable subsets $P_1, \ldots, P_{4^k}$, and a graphon U that is constant on each $P_i \times P_j$, such that*

$$\|W - U\|_\square < \frac{1}{2\sqrt{k}}.$$

The proof relies on

Lemma 12 (Frieze and Kannan, 1999). For any square integrable function $W \in \mathfrak{L}_2([0,1]^2)$ on the unit square, and any positive integer k, there exist measurable subsets $S_1, T_1, \ldots, S_k, T_k$ of $[0,1]$ and real numbers $a_1, \ldots, a_k$ such that

$$\left\| W - \sum_{i=1}^k a_i \mathbb{1}_{S_i\times T_i} \right\|_\square < \frac{\|W\|_2}{\sqrt{k}}.$$

2 Local limits of graphs of bounded degree

2.1 Local convergence

Consider a sequence of graphs (G_n) such that the degree of each vertex in each G_n is $\leq d$, where d is a fixed number independent of n.

If $\mathrm{v}(G_n) \to \infty$, then $t(F, G_n) \to 0$ for all F, so in the sense of dense graph convergence discussed in the previous section, G_n converges to the identically zero graphon. This is not interesting. However, if we study bounded degree graphs with a different method of sampling, then we get a different convergent notion which is meaningful (in particular, not all sequences converge).

Definition 13. If we fix $r > 0$, and look at the isomorphism type of the r-neighborhood of a uniform random vertex in G_n, then we get a random rooted graph of radius r and degree bound d. If the distribution of this random rooted graph converges as $n \to \infty$ for any fixed r, then we say that the graph sequence (G_n) is locally convergent or Benjamini–Schramm convergent.

Example 14. If P_n is the path on n nodes, then the sequence P_n is locally convergent, and so is the sequence of 2-dimensional grid graphs $Q_n = P_n \square P_n$, but the sequence $P_1, Q_1, P_2, Q_2, \ldots$ is not.

Similarly to dense graph convergence, an equivalent definition can be given via homomorphism counting.

Definition 15. Let us define the homomorphism frequency

$$t^*(F, G) = \hom(F, G)/\mathrm{v}(G),$$

where the graph F is assumed to be connected.

Proposition 16. *The bounded degree graph sequence (G_n) is locally convergent if and only if $t^*(F, G_n)$ converges for any connected F.*

2.2 Random rooted graphs

What is the limit of a locally convergent sequence? Let us consider the set $\mathfrak{G}^\bullet$ of rooted, connected, possibly infinite graphs with degree bound d (up to root-preserving isomorphism). We can turn this into a metric space: the distance of two such graphs is $1/r$ if the r-neighborhood of the root is isomorphic but the $(r+1)$-neighborhood is not. This yields a compact space. Any finite graph G with degree bound d yields a Borel probability measure λ_G on $\mathfrak{G}^\bullet$, i.e., yields a random rooted graph: simply choose a uniform random node o of G and take the random rooted graph (G, o). This probability measure is of course very special: it is supported on finitely many finite graphs. It also has a less obvious, but more important special property called involution invariance.

Definition 17. Given a Borel probability measure λ on $\mathfrak{G}^\bullet$, choose a random point (G, o) with distribution λ, and then choose the node p of G to be each neighbor of the root o with probability $1/d$. We say that λ is involution invariant if the subrandom birooted graphs (G, o, p) and (G, p, o) have the same sub-probability distribution.

The reason sub-probability distributions arise here is that the degree of the root o is not necessarily d, it can be smaller, and in this case there is a positive probability of not choosing any neighbor of o to be p.

Proposition 18. *The bounded degree graph sequence (G_n) is locally convergent if and only if the sequence (λ_{G_n}) of probability measures on $\mathfrak{G}^\bullet$ is weakly convergent. If this is the case, then the weak limit is an involution invariant probability measure.*

The most famous open problem in this area is

Conjecture 19 (Aldous and Lyons). Every involution invariant probability measure on $\mathfrak{G}^\bullet$ arises as the weak limit of (λ_{G_n}) for a suitable locally convergent graph sequence (G_n).

By Proposition 18, one way of thinking of the limit of a locally convergent graph sequence is as an involution invariant random rooted graph, and this limit object is unique. There is also a less unique, but more graph-like limit object called a graphing. Time permitting, it will be discussed in the minicourse.

The minicourse will be based entirely on a tiny portion of László Lovász's monograph Large Networks and Graph Limits, as are these notes. There is no claim of originality, except for possible errors.

Introduction

In the below lecture notes and during the Summer School, I will often use functional analytic notions and tools that not everyone may be very familiar with, but I will strive to provide concrete examples that can be considered "prototypical", and inserted in place of the more abstract counterparts.

In particular, Banach spaces and their duals play a key role in formulating a general framework in which the combinatorics of homomorphism densities can be transformed to a "universal" analytic language in which the theorems and proofs can be more easily handled. The following "dictionary" may be helpful.

- Banach space $(X, \|\cdot\|_X)$: think of the compact sets $K=[0,1]$ or $K=\{0,1,\ldots,k\}$, and consider the linear space of continuous functions over these sets, with the norm ("vector length notion") $\|f\| := \max_{y\in K}|f(y)|$.
- Dual space $(X', \|\cdot\|_{X'})$ of the above: for the examples given above, this corresponds to the space of (Radon) measures on K, with the norm corresponding to the total variation.
- The weak evaluation $\langle f,\mu\rangle \in \mathbb{R}$ with $f\in X$, $\mu\in X^*$: in the above examples, this means the integral of f with respect to μ.
- Weak-* convergence in X^*: a sequence μ_n in X^* is convergent in the weak-* sense if for any $f\in X$ the numerical sequence $\langle f,\mu_n\rangle$ converges. In probability theory, this is often simply called weak convergence, or convergence in the vague topology.

1 A heuristic approach to the limit objects

Given that we want to understand what a limit object for (dense) graph sequences can be, how do we approach the question? What even constitutes a reasonable limit object? Formally, if $(G_n)_{n\in\mathbb{N}}$ is such that $(t(F,G_n))_{n\in\mathbb{N}}$ converges for every test-graph F, then the sequence

$$(c_F)_{F \text{ simple graph}} := \left(\lim_{n\to\infty} t(F,G_n)\right)_{F \text{ simple graph}}$$

fully encodes the limit. In addition, reformulating to such countably infinite sequences with values between 0 and 1 adds an important insight: the convergence we speak of is metrizable, and the closure of graphs in the metric space is automatically compact. On the other hand, characterizing the exact sequences that appear in this closure is anything but straight forward, and – from the point of view of the initial motivation, even more importantly – we seem to lose all semblance of combinatorial structure.

The definition of homomorphism densities, however, provide a formal expression that hints at a way out, leading straight to the notion of graphons that we have become familiar with in the other minicourses. As a mild spoiler, I shall here only mention one word: *averages*. The full details and motivations from the point of view of an analyst will be presented during the minicourse.

2 From counting homomorphisms to topologies on function spaces

At its core, the convergence of graph sequences is defined through homomorphisms, which encode *counting* maps from a graph to another (or to a function) according to certain rules. Reformulated, to each vertex map $V(F)\to V(G)$ we assign a *weight* that depends on our combinatorial approach. In the classical homomorphism count, this weight is 1 iff the map preserves connection by edges and 0 otherwise, whereas for induced homomorphisms, we also require non-connectedness to be preserved.
Different assignments of weights lead to different densities, and, à priori, to possibly different topologies on the space of $[0,1]$-valued functions on $[0,1]^2$. For the two cases mentioned, an easy inclusion-exclusion formula can be used to relate homomorphism and induced homomorphism numbers, showing that they lead to an equivalent convergence notion. How can we in general decide whether two different weightings lead to the same topology?

It turns out that graph decorations are a first step on the road to answering this question. Recall Lemma 11 from the lecture *Homomorphism numbers and graph convergence* which is a key step in proving the Counting lemma. There we allowed each edge in F to be assigned its own graphon (this is roughly what we mean by *decorating* the edges), thus generalizing the notion of density. More precisely, let us look at how homomorphism numbers are obtained both in the classical and in the induced setting. Let $\varphi : V(F) \to V(G)$ be any map. We have

$$\begin{aligned}\hom_\varphi(F,G) &= \prod_{ij\in E(F)} \mathbb{1}(\varphi(i)\varphi(j)\in E(G)) \\ &= \left(\prod_{ij\in E(F)} \mathbb{1}(\varphi(i)\varphi(j)\in E(G))\right)\left(\prod_{ij\notin E(F)} 1\right)\end{aligned}$$

whereas

$$\hom_{\varphi,ind}(F,G) = \left(\prod_{ij\in E(F)} \mathbb{1}(\varphi(i)\varphi(j)\in E(G))\right)\left(\prod_{ij\notin E(F)} \mathbb{1}(\varphi(i)\varphi(j)\notin E(G))\right).$$

In both cases, for each pair of vertices $i \neq j$ in F, we look at the image $\varphi(i)\varphi(j)$, and based on these two, we assign a value that is then made part of the product. Essentially, we have the following general formalism.

$$\hom_{\varphi,\mathcal{C}}(F,G) = \prod_{i<j} \mathcal{C}(ij, \varphi(i)\varphi(j)),$$

where $\mathcal{C}$ is a real-valued function.

We are now ready to introduce decorated graphs and the corresponding homomorphisms/densities.

Definition 1. If $\mathcal{X}$ is any set, an $\mathcal{X}$-*decorated graph* is a graph where every edge ij is decorated by an element $X_{ij} \in \mathcal{X}$. An $\mathcal{X}$-decorated graph will be denoted by (G,g), where G is a simple graph (possibly with loops), and $g :\ E(G) \to \mathcal{X}$.

In our setting, we shall be working with graphs decorated with elements of the Banach spaces X and X^*. At first sight, this may seem odd. Why do we decorate some graphs with elements of a Banach space, and other graphs with elements of the dual? Recall that the values a graphon $W : [0,1]^2 \to [0,1]$ takes can be interpreted as probabilities, and the limit object therefore suggests that edges along a sequence of simple graphs G_n behave like probability measures (on $\{0,1\}$). In addition, analysis becomes much nicer if one can extend the theory from the (convex) space of graphons to the whole linear space of bounded functions in a linear way.

Exercise 2. Extend the function $\mathcal{C}$ introduced above to graphons instead of graphs G for both the hom and $\hom_{ind}$ cases. Show that in each case the extended function $\mathcal{C}$ is linear.

Thus the real valued functions $\mathcal{C}(\cdot,\cdot)$ are linear in the second component, and its it a simple step to do a linear extension in the first variable as well. As we are doing limit theory, continuity of these functions is also a natural requirement, and so we have arrived at Banach spaces and duals.

Definition 3. For every X-decorated simple graph $\mathbb{F} = (F, f)$ and X^*-decorated complete graph $\mathbb{G} = (G,g)$, we define

$$\hom(\mathbb{F},\mathbb{G}) =: \sum_{\varphi:V(F)\to V(G)} \prod_{e\in E(F)} \langle f(e), g(\varphi(e))\rangle$$

The homomorphism density $t(F,G)$ is defined by

$$t(F,G) = \mathbb{E}\Big(\prod_{e\in E(F)} \langle f(e), g(\varphi(e))\rangle\Big), \tag{2.1}$$

where the expectation is taken over uniform random maps $\varphi :\ V(F) \to V(G)$.

Exercise 4. Give a reason why we would want to decorate F with elements of X and G with elements of X^*, and not the other way around. (The definitions given obviously would still work.)

Exercise 5. Reformulate homomorphism and induced homomorphism densities in the above language. (Hint: $X = X^* = \mathbb{R}^2$.)

Definition 6. A symmetric weak-* measurable function $W :\ [0,1]^2 \to X^*$ is called a X^*-*graphon* if the function $(x,y) \mapsto \|W(x,y)\|_{X^*}$ lies in

$$\bigcap_{1\le p<\infty} L_p^{\mathrm{sym}}([0,1]^2).$$

Note that this function is measurable, since X is separable, and for a countable dense subset $\mathscr{F} \subset X$ we have

$$\|W(x,y)\|_{X^*} = \sup_{f \in \mathscr{F}\setminus\{0\}} \frac{|\langle f, W(x,y)\rangle|}{\|f\|_X}.$$

Let the space of X^*-graphons be denoted by $\mathcal{W}_{X^*}$. We set

$$\|W\|_p := \left\| \|W(.,.)\|_X^* \right\|_p.$$

(i.e., we take the X^*-norm of $W(x,y)$ for every $x,y \in [0,1]$, and then take the L^p-norm of the resulting function).

The question of equivalence of different counting methods is settled by the following lemma.

Lemma 7. *Let $W_1, W_2 \ldots \in \mathcal{W}_X^*$ be X^*-graphons and $\mathscr{F} \subseteq X$ be a generating subset. Assume that the sequence $(\|W_n\|_p)$ is bounded by some $0 \le c_p$ for each $1 \le p < \infty$. Then the following are equivalent:*

(i) *For every X-decorated graph $\mathbb{F}$, the sequence $t(\mathbb{F}, W_n)$ is convergent;*

(ii) *For every $\mathscr{F}$-decorated graph $\mathbb{F}$, the sequence $t(\mathbb{F}, W_n)$ is convergent.*

Also, the existence and representation theorem for limits of simple graphs extends in the following form.

Theorem 8 (K.-K., Lovász and Szegedy, 2014). *Let $\mathscr{F}$ be a countable generating subset of X and $W_1, W_2, \ldots$ be X^*-graphons with finite range satisfying the following conditions:*

(i) *for every $1 \le p < \infty$ there is a $c_p > 0$ such that $\|W_n\|_p \le c_p$ for all n;*

(ii) *the sequence $(t(F, W_n) : n \in \mathbb{N}^*)$ is convergent for every $\mathscr{F}$-decorated graph F.*

Then there exists a X^*-graphon W such that $t(F, W_n) \to t(F, W)$ for every X-decorated graph F.

To illustrate how this very abstract setting can be useful, we shall turn our attention to multigraph sequences.

Example 9 (Multigraphs and simple test-graphs). Let us consider multigraphs with unbounded edge-multiplicity, and simple graphs as test-graphs. It turns out that in this case, multigraphs can be thought of as $\mathbb{N}$-decorated simple graphs, and the fact that the edgeweights are nonnegative integers plays no role; so we can take $X^* = X = \mathbb{R}$, and consider the basis $\mathscr{F} = \{1\}$ in X. If F is a simple ($\mathscr{F}$-decorated) graph and G is an edge-weighted (complete) graph, then $\hom(F, G)$ is the homomorphism number into G as a multigraph.

Example 10 (Multigraphs and multi-test-graphs). Consider multigraphs with unbounded edge-multiplicity, and multi-graphs as test-graphs. We have already seen that homomorphisms of a multigraph into another can be defined in different ways. In the bounded case, these notions turned out to be essentially the same, but in the unbounded case, the correspondence is more subtle.

We sketch the idea how to fit convergence according to node-end-edge homomorphism densities into our framework. Let $X = \mathbb{R}[Y]$ be the space of polynomials in one variable, and let X^* be the space of real sequences with finite support. For $f \in X$ and $a = (a_1, a_2, \ldots) \in X^*$, let us define

$$\langle f, a\rangle = \sum_{i=0}^{\infty} a_i f(i).$$

We encode a multigraph F by decorating each edge $e \in E(F)$ with multiplicity m by the polynomial Y^m, to get an edge-decorated simple graph $\widehat{F}$. We encode a "target" multigraph G by labeling each edge $e \in E(F)$ with multiplicity m by sequence e_m with a single 1 in the m-th position, to get an edge-decorated complete graph $\widetilde{G}$. Then $\hom(\widehat{F}, \widetilde{G}) = \hom(F, G)$.

The problem with this construction is that X and X^* are not Banach spaces, and our theory needs the Banach space structure, but this can be amended

Exercise 11. Try to find a way to introduce appropriate Banach spaces X and X^* for the above example.

3 Cut norm versus L^1, and other norms on kernels

Finally, we shall take a closer look at the cut norm. Recall that

$$\|W\|_\square = \sup_{S,T\subset[0,1]} \left| \int_{S\times T} W \right|.$$

The cut norm also has an alternative form, namely

Lemma 12.

$$\|W\|_\square = \max_{f,g:[0,1]\to[0,1]} \left| \int_{[0,1]^2} W(x,y)\cdot(f(x)g(y))dxdy \right|.$$

Now, the integral can also be considered as the weak evaluation $\langle W, f\otimes g\rangle$, with $W\in L^1$ and $f,g\in L^\infty$. If we were to allow *all* two-variable functions with values in $[0,1]$, we would obtain the L^1 norm, but this restriction to positive (or, at least, constant sign) rank-one functions drastically changes the properties of the norm in question.

The big difference between the cut norm and the L^1 norm is illustrated by the following result.

Theorem 13 (Lovász and Szegedy, 2007). *The metric space $(\widetilde{W_0}, \delta_\square)$ of equivalence classes of graphons under the cut distance is compact.*

This is not true for the L^1-distance $\delta_1(U,V) := \inf\|U^\psi - V^\psi\|_1$, and the lack of compactness (that the space of limit objects automatically possesses in the graph density approach) makes δ_1 generally unsuited to for graph limit theory.

What alternative norms may be reasonable in the graphon setting is a not very well explored question, but some results are known.

Theorem 14. *Let $W:[0,1]^2\to\mathbb{R}$ be bounded. Then*

$$\|W\|_{S_{2n}} := t(C_{2n}, W)$$

is a norm, where C_{2n} is the cycle of length $2n$. This corresponds to the Schatten-von Neumann norm.

If time permits, some further cases and approaches will be presented.

This minicourse is partly based on the paper *Multigraph limits, unbounded kernels, and Banach space decorated graphs* (K.-K., Lovász, and Szegedy, 2014)

1 Introduction

The flag algebra method was initiated by Razborov [26], and it changed the landscape of extremal combinatorics [29] by providing solutions and substantial progress on many long-standing open problems, see, e.g. [1,2,3,4,5,13,15,16,17,18,24,25,26,27,28]. In my talk, I will give an overview of the method and discuss some applications.

If there is time, I will also talk about finite forcibility, and some recent results, which can be thought of as "meta" extremal graph theory, i.e., what can the structure of the solution to an extremal problem look like?

2 Flag algebras

2.1 Definitions

Recall that given two graphs H and G, $\hom(H,G)$ denotes the number of homomorphisms from H to G, and this can be normalized to be between 0 and 1; we denote this as $t(H,G)$. This extends to graphons as

$$t(H,W) = \int_{[0,1]^{V(H)}} \prod_{\{i,j\}\in E(H)} W(x_i,x_j)dx_{V(H)}.$$

Suppose now that S is a subset of $V(H)$ of size s, and $\theta : S \to [s]$ is a bijection. Suppose also that $v_1, v_2, \cdots, v_s \in V(G)$. We then let $t_S(H_\theta, G(v_1,v_2,\cdots,v_s))$ be the probability that a random map from H to G that maps each $i \in S$ to v_i is a homomorphism. We extend this to graphons:

$$t_S(H_\theta, W(x_1,x_2,\cdots,x_s)) = \int_{[0,1]^{V(H)\setminus S}} \prod_{\{i,j\}\in E(H)} W(x_i,x_j)dx_{V(H)\setminus S}.$$

Given two graphs H_1 and H_2 with subsets of size s labeled θ_1, θ_2, we define their product $H_1 \cdot H_2$ as follows. Start with a disjoint union of H_1 and H_2. For each i, identify $\theta_1^{-1}(i)$ and $\theta_2^{-1}(i)$, and let θ map the new vertex to i. Take the union of edges on $\theta^{-1}([s])$, taking an edge twice if it appears in both. It can be checked that we then have

$$t_S(H_1\cdot H_2, W(x_1,x_2,\cdots,x_s)) = t_S(H_1, W(x_1,x_2,\cdots,x_s)) \cdot t_S(H_2, W(x_1,x_2,\cdots,x_s)).$$

Fix s. We define the algebra $\mathbb{A}_s$. Let $\mathcal{F}_s$ consist of all multigraphs with a subset of size s labeled with a function θ. The elements are finite linear combinations of graphs in $\mathcal{F}_s$:

$$\sum_H c_H H, c_H \in \mathbb{R}.$$

We extend the function t linearly. We also extend the product linearly, and thus obtain an algebra $\mathbb{A}_s$.

Given $T \subset [s]$ with $|T| = t$, we can define an "averaging map", denoted by $[[.]]_{s\to T}$, $\mathbb{A}_s \to \mathbb{A}_t$, by forgetting the labels in $[s]\setminus T$. Note that we have

$$t([[H]]_{s\to T}, W(x_{\sigma'})) = \int_{[0,1]^{[s]\setminus T}} t(H, W(x_\sigma)).$$

If T is the empty set, we will just write $[[.]]$.

Suppose now that

$$\sum_{H\in\mathcal{F}_s} c_H H \in \mathbb{A}_s.$$

We then have that for almost every $x_1, x_2, \cdots, x_s$,

$$0 \le \left(t\left(\sum_{H\in\mathcal{F}_s} c_H H, W(x_1,x_2,\cdots,x_s)\right)\right)^2 = t\left(\left(\sum_{H\in\mathcal{F}_s} c_H H\right)^2, W(x_1,\cdots,x_s)\right).$$

If we then have

$$\left[\left(\sum_{H\in\mathcal{F}_s} c_H H\right)^2\right] = \sum_{H\in\mathcal{F}_0} a_H H,$$

then integrating over all choices of $x_1, \cdots, x_s$, we obtain that for any graphon W, we have

$$0 \le \sum_{H \in \mathcal{F}_0} a_H t(H, W).$$

Obtaining such equations can give proofs of extremal graph theory problems.

2.2 Mantel's theorem

Here we give a proof of the asymptotic version of Mantel's theorem. Although it only proves it asymptotically, it serves to illustrate the concepts in flag algebras. One can then use a "blowup" argument to prove it exactly.

$$0 \le \left(t(\,, W(x)) - 1/2\right)^2 = t(\,, W(x)) - t(\,, W(x)) + 1/4.$$

Therefore,

$$0 \le t(\,, W) - t(\,, W) + 1/4.$$

Note also that

$$t(\,, W) = t(\,, W)$$

Finally, we claim that in any graph,

$$t(\,, W) - 2t(\,, W) + t(\,, W) \ge 0.$$

Indeed, the above expression is the induced density of the graph on three vertices with a single edge, with labeled vertices, i.e.

$$\int_{[0,1]^3} W(x,y)(1 - W(x,z))(1 - W(y,z)).$$

In our case, we also have that

$$t(\,, W) = 0$$

Combining these, we obtain

$$t(\,, W) \le t(\,, W) + 1/4 \le t(\,, W)/2 + 1/4 = t(\,, W)/2 + 1/4.$$

Rearranging gives

$$t(\,, W) \le 1/2.$$

We can also see above that we have equality if and only if

- every vertex has degree $1/2$, and
- The induced density of is 0.

It is clearly true for graphs and not difficult to show for graphons that the above implies that we have a complete bipartite graph or graphon.

2.3 Quasirandom graphs

In this section, we give a proof of a classical result on quasirandom graphs [6, 30, 31]. Suppose we have that $t(\,, W) = p$. What is the minimum of $t(\,, W)$? We have

$$\left(t(\,, W(x)) - p\right)^2 = t(\,, W(x)) - 2pt(\,, W(x)) + p^2.$$

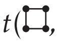
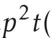
This implies that

$$t(\,, W) \ge 2pt(\,, W) - p^2 = p^2.$$

We also have that

$$\left(t(\,$$$$\,, W(x,y)) - p^2\right)^2 = t(\,$$$$\,, W(x,y)) - 2p^2 t(\,$$$$\,, W(x,y)) + p^4.$$

We therefore obtain

$$t(\,$$$$\,, W) \geq 2p^2 t(\,$$$$\,, W) - p^4 \geq p^4.$$

Note that we have equality if and only if for almost every x,

$$\int_{[0,1]} W(x,y)dy = p$$

and for almost every x and y,

$$\int_{[0,1]} W(x,z)W(y,z)dz = p^2$$

In this case, for almost every x and y, we have

$$\int_{[0,1]} (W(x,z)-p)(W(z,y)-p)\,dz = 0.$$

Let $U = W - p$. Let T be any subset of $[0,1]$ with positive measure. We then have

$$0 = \int_{T\times T}\int_{[0,1]} U(x,z)U(y,z)dzdxdy = \int_{[0,1]}\left(\int_T U(x,z)dx\right)^2 dz.$$

We have obtained that for any T,

$$\int_T U(x,z)dx = 0$$

for almost every $z \in [0,1]$. This implies that $U = 0$ almost everywhere, therefore, $W = p$ almost everywhere.

2.4 Systematic approach

Let us try to approach this in a systematic way. This is simpler to implement with *induced densities*, as was done by Razborov [26]. Furthermore, if we permute the vertices of a graph, the number of induced copies is the same, so the program is simpler to do with *unlabeled* copies. Given a graph H, let $d(H,G)$ be equal to the probability that a randomly chosen subset of $V(G)$ of size $|V(H)|$ induces a graph isomorphic to H. We extend this to W as

$$d(H,W) = \frac{|V(H)|!}{\mathrm{Aut}(H)}\int_{[0,1]^{V(H)}}\prod_{(u,v)\in E(H)} W(x_u,x_v)\prod_{(u,v)\in E(\overline{H})}(1-W(x_u,x_v))dx_{V(H)}.$$

We also define, if $\theta : S \to [s]$ is a bijection as before,

$$d(H_\theta, W(x_1,x_2,\cdots,x_s)) = \frac{|V(H)|!}{\mathrm{Aut}(H)}\int_{[0,1]^{V(H)\setminus S}}\prod_{(u,v)\in E(H)} W(x_u,x_v)\prod_{(u,v)\in E(\overline{H})}(1-W(x_u,x_v))dx_{V(H)\setminus S}.$$

It is not difficult to see that with the above definition, G_n converges to W if and only if for every graph F, $d(F,G_n) \to d(F,W)$. Working with induced densities corresponds to a change of basis in $\mathbb{A}_s$. For example, , with induced densities, corresponds to  –  with homomorphism densities. We now have the property that for a fixed k, the sum of the densities of all possible isomorphism classes of graphs is 1. We next define types and flags.

Definition 1. A *type* σ is a (labeled) graph on $[s]$.

Given a type, let $\mathcal{F}_s^\sigma \subseteq \mathcal{F}_s$ consist of those partially labeled graphs $(H,\theta) \in \mathcal{F}_s$ which are simple, and θ induces an isomorphism.

Definition 2. Given a type σ, a σ-*flag* is a graph H with a subset S of vertices that induce a copy of σ with a function $\theta : S \to [s]$.

Working with induced densities makes the product more complicated, however, we can still express it as

$$\begin{aligned} &d(H_1(\sigma), W(x_1,x_2,\cdots,x_s))d(H_2(\sigma), W(x_1,x_2,\cdots,x_s)) \\ &\qquad = d(\sigma, W(x_1,x_2,\cdots,x_s))\sum_{H\in\mathcal{F}_s^\sigma} c(H_1,H_2,H)d(H(\sigma), W(x_1,x_2,\cdots,x_s)). \end{aligned}$$

The advantage is that we then have the condition that any induced subgraph has nonnegative density, and the sum of densities of all graphs of a fixed size is 1. We then define

$$H_1 \cdot H_2 = \sum_{H \in \mathcal{F}_s^\sigma} c(H_1, H_2, H) H$$

Let $\mathcal{F}_m^\sigma$ be the set of isomorphism classes of graphs. Let $l = 2m - |\sigma|$. Suppose that $Q = (q_{i,j})_{i,j \in \mathcal{F}_m^\sigma}$ is a symmetric positive semidefinite matrix. We then obtain that for every $x_1, x_2, \cdots, x_s$,

$$\sum_{i,j \in \mathcal{F}_l^\sigma} q_{ij} d(H_i \cdot H_j, W(x_1 \cdots x_s)) \geq 0.$$

We do divide by $d(\sigma, W(x_1, x_2, \cdots, x_s))$ here, but it is easy to check that if this is 0, then the expression above is also 0. Applying the forgetting operator, we obtain an inequality of the form

$$\sum_{H \in \mathcal{F}_l^0} c_H(Q) d(H, W) \geq 0.$$

Here each $c_H(Q)$ is a linear function of the entries of Q. Note that it may be negative. We also have that each $d(H, W) \geq 0$ and

$$\sum_{H \in \mathcal{F}_l^0} d(H, W) = 1.$$

If we then want to maximize an expression of the form $\sum_{H \in \mathcal{F}_l^0} a_H d(H, W)$, we obtain an upper bound of

$$\sum_{H \in \mathcal{F}_l^0} a_H d(H, W) \leq \sum_{H \in \mathcal{F}_l^0} (a_H + c_H) d(H, W) \leq \max_{H \in \mathcal{F}_l^0} (a_H + c_H).$$

Note that if we are maximizing say the number of edges, it can be expressed as $\sum_{H \in \mathcal{F}_l^0} a_H d(H, W)$ for any l, for some values of a_H. If we have constraints, we can also add those constraints. For example, if the density of triangles is zero, then we can omit all graphs from $\mathcal{F}_m^\sigma$ and $\mathcal{F}_l$ that have triangles.

Let us revisit Mantel's theorem in the above framework. Let σ consist of a single vertex, and $m = 2$. We then have two graphs, and . We have

$$d(, W(x)) d(, W(x)) = d(, W(x)).$$

$$d(, W(x)) d(, W(x)) = d(, W(x))/2 + d(, W(x))/2.$$

$$d(, W(x)) d(, W(x)) = d(, W(x)) + d(, W(x)).$$

If we take

$$Q = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

we obtain

$$0 \leq d(, W(x))/2 - d(, W(x))/2 - d(, W(x))/2 + d(, W(x))/2 + d(, W(x))/2.$$

We also have

$$\int_{[0,1]} d(, W(x)) dx = d(, W)/3.$$

$$\int_{[0,1]} d(, W(x)) dx = 2d(, W)/3.$$

$$\int_{[0,1]} d(, W(x)) dx = 2d(, W)/3.$$

$$\int_{[0,1]} d(, W(x)) dx = d(, W)/3.$$

$$\int_{[0,1]} d(, W(x)) dx = d(, W).$$

This gives

$$0 \leq -d(, W)/6 - d(, W)/6 + d(, W)/2.$$

We also have that

$$d(, W) = 2d( )/3 + d( )/3.$$

Adding the two we obtain that

$$d(\;, W) \leq d(\;, W)/2 + d(\;, W)/6 + d(\;, W)/2 \leq 1.$$

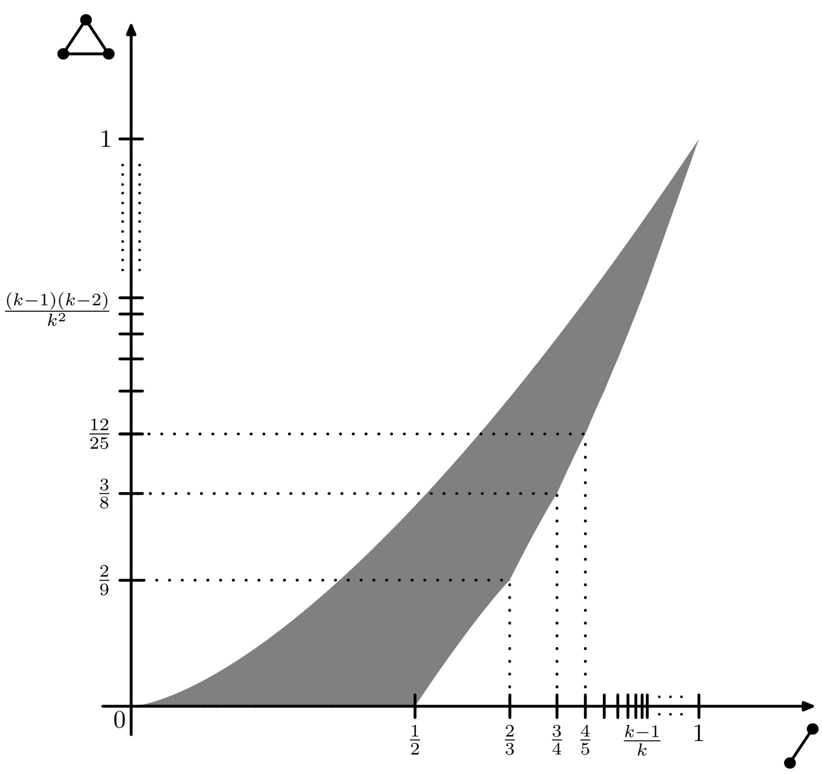
This also makes it easier to see when we have equality. In the above inequality, the second holds with equality if and only if $d(\;, W) = 0$. The first holds if and only if for almost every x, we have

$$d(\;, W(x)) = d(\;, W(x))$$

for almost every x. Combining these, one can show that W must be weakly isomorphic to the graphon that is the limit of complete bipartite graphs with parts of equal size.

3 Triangle edge count

Flag algebras were used by Razborov to prove bounds on triangle vs edge counts. Suppose $t(\;, W) = p$. What is $\min t(\;, W)$? If $p = 1 - 1/k$ for $k \in \mathbb{N}$, then it is not difficult to prove that it is maximized by taking a complete k-partite graph, this was shown by Goodman [12]. What about for other values? This was settled recently by Razborov [28]. It is obtained by taking a complete multipartite graph with all parts except one the same size, and the last part smaller.



4 Other applications

If there is time, I will also discuss other applications of the flag algebra method.

5 Finite forcibility

A graphon is *finitely forcible* if there exist graphs $H_1, \ldots, H_\ell$ with the following property: if $(G'_n)_{n\in\mathbb{N}}$ is another convergent sequence of graphs such that

$$\lim_{n\to\infty} d(H_i, G_n) = \lim_{n\to\infty} d(H_i, G'_n)$$

for every $i = 1, \ldots, \ell$, then

$$\lim_{n\to\infty} d(H, G_n) = \lim_{n\to\infty} d(H, G'_n)$$

for every graph H. Turán's theorem, for example, implies that the complete k-partite graphon with parts of equal size is finitely forcible. We have also seen that for any $p \in [0,1]$ the constant p graphon (quasirandom graphon) is finitely forcible (by densities of 4-vertex subgraphs). Lovász and Sós [22] generalized this result to graph limits corresponding to stochastic block models (which are represented by step graphons). Additional examples of finitely forcible sequences can be found, e.g., in [23], constructive examples of non-finitely forcible sequences, e.g., in [9].

Finitely forcible graphons form a meager set (in the Baire category sense) in the set of all graphons, as shown in [23]. Therefore, there should be something "nice" that can be said about them. This was formalized by Lovász and Szegedy, who conjectured the following [23, Conjectures 9 and 10]:

Conjecture 3. The space of typical vertices of every finitely forcible graphon is compact.

Conjecture 4. The space of typical vertices of every finitely forcible graphon has finite dimension.

It is not difficult to show that a graphon is finitely forcible if and only if it is the unique solution to a minimization or maximization problem, such as the ones considered above. The converse is not true, for example, in Razborov's result, if $1-1/k < p < 1-1/(k+1)$, then there are multiple extremal examples. However, we can add a constraint so that the minimum does not increase, and the solution is unique. In that case, the constraint we would add is $d(\ ,W) = 0$.

One of the most commonly cited problems concerning graph limits is the following conjecture of Lovász, which is often referred to as saying that "every extremal problem has a finitely forcible optimum", see [21, p. 308]. The conjecture appears in various forms, also sometimes as a question, in the literature; see e.g. [19, Conjecture 3], [20, Conjecture 9.12], [21, Conjecture 16.45], and [23, Conjecture 7].

Conjecture 5. Let $H_1,\dots,H_\ell$ be graphs and $d_1,\dots,d_\ell$ reals. If there exists a convergent sequence of graphs with the limit density of H_i equal to d_i, $i = 1,\dots,\ell$, then there exists a finitely forcible such sequence.

Unfortunately, none of the above conjectures are true. A series of papers [7, 8, 10, 11] showed that finitely forcible graphons can in fact be very complicated. In addition, recently, Conjecture 5 has been disproven by the Grzesik, Král', and the author [14]. I will briefly discuss some of these results.

[1] R. Baber: *Turán densities of hypercubes*, available as arXiv:1201.3587.

[2] R. Baber and J. Talbot: *A solution to the $2/3$ conjecture*, SIAM J. Discrete Math. **28** (2014), 756–766.

[3] R. Baber and J. Talbot: *Hypergraphs do jump*, Combin. Probab. Comput. **20** (2011), 161–171.

[4] J. Balogh, P. Hu, B. Lidický and H. Liu: *Upper bounds on the size of 4- and 6-cycle-free subgraphs of the hypercube*, European J. Combin. **35** (2014), 75–85.

[5] J. Balogh, P. Hu, B. Lidický and F. Pfender: *Maximum density of induced 5-cycle is achieved by an iterated blow-up of 5-cycle*, European J. Combin. **52** (2016), 47–58.

[6] F.R.K. Chung, R.L. Graham and R.M. Wilson: *Quasi-random graphs*, Combinatorica **9** (1989), 345–362.

[7] J. W. Cooper, T. Kaiser, D. Král′ and J. A. Noel: *Weak regularity and finitely forcible graph limits*, Trans. Amer. Math. Soc **370** (2018), 3833–3864.

[8] J. W. Cooper, D. Král′ and T. Martins: *Finitely forcible graph limits are universal*, available as arXiv:1701.03846.

[9] R. Glebov, A. Grzesik, T. Klimošová and D. Král′: *Finitely forcible graphons and permutons* J. Combin. Theory Ser. B **110** (2015), 112–135.

[10] R. Glebov, T. Klimošová and D. Král′: *Infinite dimensional finitely forcible graphon*, available as arXiv:1404.2743.

[11] R. Glebov, D. Král′ and J. Volec: *Compactness and finite forcibility of graphons*, available as arXiv:1309.6695.

[12] A. W. Goodman: *On sets of acquaintances and strangers at any party*, Amer. Math. Monthly **66** (1959), 778–783.

[13] A. Grzesik: *On the maximum number of five-cycles in a triangle-free graph*, J. Combin. Theory Ser. B **102** (2012), 1061–1066.

[14] A. Grzesik and D. Král′ and L. M. Lovász *Elusive extremal graphs*, in preparation.

[15] H. Hatami, J. Hladký, D. Král′, S. Norine and A. Razborov: *Non-three-colorable common graphs exist*, Combin. Probab. Comput. **21** (2012), 734–742.

[16] H. Hatami, J. Hladký, D. Král′, S. Norine and A. Razborov: *On the number of pentagons in triangle-free graphs*, J. Combin. Theory Ser. A **120** (2013), 722–732.

[17] D. Král′, C.-H. Liu, J.-S. Sereni, P. Whalen and Z. Yilma: *A new bound for the 2/3 conjecture*, Combin. Probab. Comput. **22** (2013), 384–393.

[18] D. Král′, L. Mach and J.-S. Sereni: *A new lower bound based on Gromov's method of selecting heavily covered points*, Discrete Comput. Geom. **48** (2012), 487–498.

[19] L. Lovász: *Graph homomorphisms: Open problems*, manuscript, 2008, available as `http://www.cs.elte.hu/~lovasz/problems.pdf`.

[20] L. Lovász: *Very large graphs*, Curr. Dev. Math. **2008** (2009), 67–128.

[21] L. Lovász: Large networks and graph limits, AMS, Providence, RI, 2012.

[22] L. Lovász and V.T. Sós: *Generalized quasirandom graphs*, J. Combin. Theory Ser. B **98** (2008), 146–163.

[23] L. Lovász and B. Szegedy: *Finitely forcible graphons*, J. Combin. Theory Ser. B **101** (2011), 269–301.

[24] O. Pikhurko and A. Razborov: *Asymptotic structure of graphs with the minimum number of triangles*, Combin. Probab. Comput. **26** (2017), 138–160.

[25] O. Pikhurko and E.R. Vaughan: *Minimum number of k-cliques in graphs with bounded independence number*, Combin. Probab. Comput. **22** (2013), 910–934.

[26] A. Razborov: *Flag algebras*, J. Symbolic Logic **72** (2007), 1239–1282.

[27] A. Razborov: *On 3-hypergraphs with forbidden 4-vertex configurations*, SIAM J. Discrete Math. **24** (2010), 946–963.

[28] A. Razborov: *On the minimal density of triangles in graphs*, Combin. Probab. Comput. **17** (2008), 603–618.

[29] A. Razborov: *Flag algebras: an interim report*, in R. Graham, J. Nešetřil, S. Butler (eds.): The mathematics of Paul Erdős II, Springer, 2013, pp. 207–232.

[30] A. Thomason: *Pseudo-random graphs*, Ann. Discrete Math. **33** (1987), 307–331.

[31] A. Thomason: *Random graphs, strongly regular graphs and pseudorandom graphs*, in: C. Whitehead (ed.), Surveys in Combinatorics 1987, London Math. Soc. Lecture Note Ser. **123** (1987), 173–195.