BUDAPEST

E·L·T·E

SUMMER SCHOOL IN MATHEMATICS

On the crossroads of topology, geometry and algebra

Eötvös Loránd University, Budapest

June 24–28, 2019

Titles of talks:

Balázs Csikós: Curvature and topology

László Fehér: Enumerative geometry: classical and new problems

Gábor Moussong: From Poincaré to Thurston and Perelman: one hundred years of a conjecture

András Némethi: Projective algebraic plane curves

András Stipsicz: Invariants of knots: polynomials and homologies

András Szűcs: Three Fields medalists of topology: Smale (immersions), Thom (cobordisms), Milnor (exotic spheres)

Contents

Preface

This volume contains the more or less faithful lecture notes of five minicourses (consisting of three 60 minute lectures) of the summer school held at Eötvös Loránd University (ELTE) in Budapest between June 24 and 28, 2019. The summer school was hosted by the Institute of Mathematics at ELTE and it was the seventh such event organized by the university. The topic of the school was well reflected by its title: *On the crossroads of topology, geometry and algebra*. The aim of the lectures was to introduce the students – some of them already experts and some of them "newcomers" – into these exciting intertwined areas of modern mathematics. Besides the minicourses a special lecture was devoted to the mathematical ideas which led to the solution of the famous conjecture made by Poincaré more than one hundred years ago. (This individual lecture is represented in this volume only by a short abstract.)

The summer school had 36 registered participants from 20 different universities, representing 12 countries. By counting the home countries of participants, this latter number climbs to 18. The speakers of the summer school are professors at Eötvös Loránd University (Balázs Csikós, László Fehér, Gábor Moussong, András Némethi, András Szűcs) and at the Rényi Mathematical Institute (András Stipsicz) in Budapest. Besides the lectures, three special practice sessions were organized with the help of PhD students at ELTE and the Central European University (Tamás Ágoston, Viktória Földvári, Ákos Matszangosz and András Sándor) and a professor at ELTE (Tamás Terpai). Besides helping with the problem solving sessions most of the lecture notes were also taken by them. A large part of the TEXnical editing of the lecture notes was done by Tamás Ágoston while the format of the cover page is based on the design of Dénes Balázs.

The organizers wish to express their gratitude to all lecturers and contributors of this volume but also to the audience without whose active participation the summer school would have been far less successful.

The content of this volume together with the volumes of the previous summer schools can be downloaded from

`http://bolyai.cs.elte.hu/summerschool/?page=download`

Budapest, September 6, 2019

The organizers

Curvature and topology

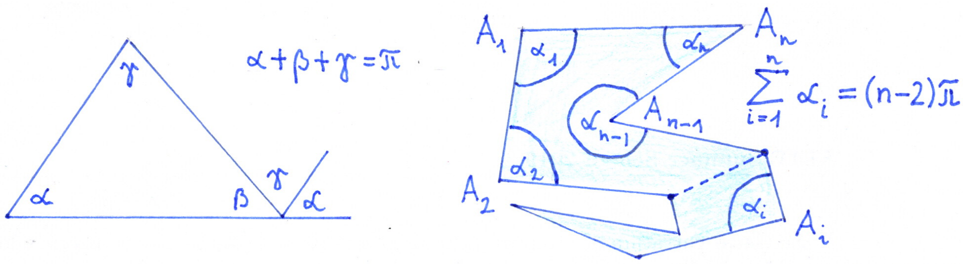
By Balázs Csikós

(Notes by Balázs Csikós)

1.1 The "Umlaufsatz"

Proposition 32 in Book I of Euclid's *Elements* claims that the sum of the angles of a triangle in the Euclidean plane is equal to π. This proposition, which is equivalent to Euclid's fifth postulate on parallel lines, can be considered as the simplest variant of the theorems we want to focus on in these notes. The angle of a triangle is a local measure of the curvedness of the boundary at a vertex. Deforming the shape of the triangle, the angles can change their values simultaneously, but the sum of the angles is an invariant quantity not sensitive to these deformations.

The proposition can be extended to simple polygons.

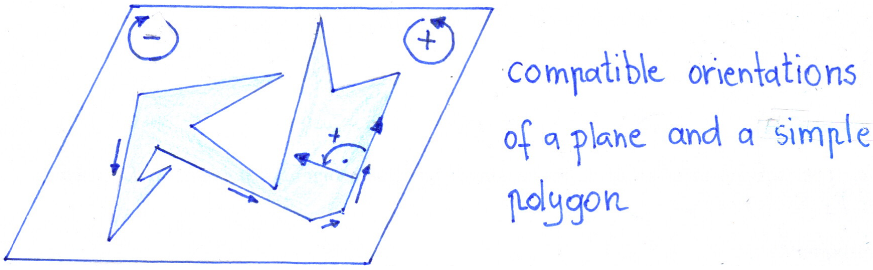


Proposition 1. *The sum of the inner angles of a simple n-gon in the Euclidean plane is $(n-2)\pi$.*

Proof. Induction by n. It can be shown that there is a diagonal of the polygon that is lying in the interior, and cuts the n-gon into a k-gon and an l-gon, where $n = k + l - 2$. Then the sum of the angles is $(k-2)\pi + (l-2)\pi = (n-2)\pi$. □

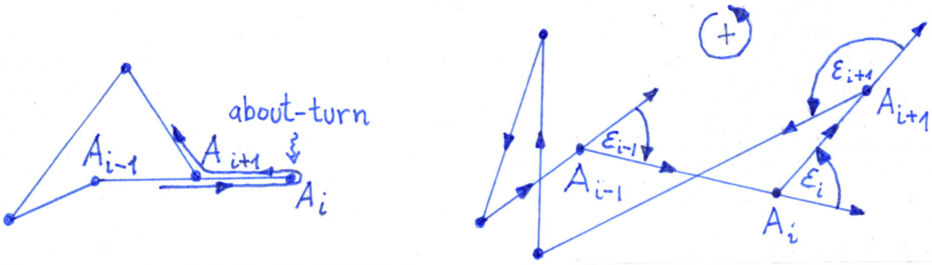
The precise definition of inner angles of a simple polygon rests upon Jordan's theorem, which says that a simple closed curve in the Euclidean plane cuts the plane into two parts: a bounded one, called the interior, and an unbounded one, called the exterior. If a closed polygon is not simple, then we cannot speak about its interior and inner angles. However, fixing an orientation of the plane and an orientation of the polygon, we can define the exterior angles of the polygon properly, assuming that the polygon has no about-turns.

Intuitively, orientation means the following. Given a point in a plane, the plane can be rotated about the point in two opposite directions. Looking at the plane from a given side, we can rotate either clockwise or anti-clockwise. An orientation of the plane is a choice of one of these directions, which is called the positive direction of a rotation. Similarly, we can go around a closed (possibly self-intersecting) polygon in two opposite direction. An orientation of the polygon is selecting one of these directions, which will be called the positive direction of going around the polygon. If the closed polygon is *simple*, then the orientation of the



plane induces a compatible orientation of the polygon by the following rule. We say that the orientation of the polygon is compatible with the orientation of the plane if moving in the positive direction along a side, the rotation of our velocity vector by $+90°$ points towards the interior of the polygon (locally).

Definition 2. Take an oriented polygon with no about-turns in the oriented Euclidean plane. Moving around the polygon with unit speed in the positive direction, our velocity vector is constant while we move along a side, but at each vertex A_i it turns by a signed angle $\epsilon_i \in (-\pi, \pi)$. The angle ϵ_i is called the *exterior angle* at A_i.



It is clear that the sum $\sum_{i=1}^{n} \epsilon_i$ of all the exterior angles of a closed polygon with n vertices measures the total amount of turns made by the velocity vector as we move around the closed polygon. At the end of the motion the velocity returns back to its original position, so the total amount of turns is an integer multiple of 2π, that is

$$\sum_{i=1}^{n} \epsilon_i = 2\pi k$$

for some $k \in \mathbb{Z}$.

Definition 3. The number k in the above equation is the turning number (of the tangent) of the oriented polygon.

Exercise 4. Construct a closed polygon with turning number k for any integer k.

Exercise 5. How can we determine the turning number of an entangled polygon with many self-intersections?

Proposition 1 can be rewritten in terms of the turning number.

Corollary 6. *The turning number of a simple closed polygon is ± 1. It is $+1$ if and only if the orientation of the plane and the polygon are compatible.*

How can we generalize these propositions for polygons with curved sides? For such polygons, we have to consider also the rotation of the tangent along the sides. The definitions below tell us how to measure this rotation.

Definition 7. A regular (parameterized) curve in $\mathbb{R}^n$ is a smooth map $\gamma\colon I \to \mathbb{R}^n$ from an interval I into $\mathbb{R}^n$, for which $\gamma'(t) \neq \mathbf{0}$ for all $t \in I$.

For a regular curve $\gamma\colon I \to \Sigma$ lying in an oriented plane Σ, we shall use the notation

- $v = \|\gamma'\|$ for the *speed* of γ,
- $\mathbf{T} = \frac{\gamma'}{v}$ for *the unit tangent vector field* of γ, and
- $\mathbf{N}$ for the *unit normal vector field* of γ defined as the pointwise rotation of $\mathbf{T}$ within Σ by $+90°$.

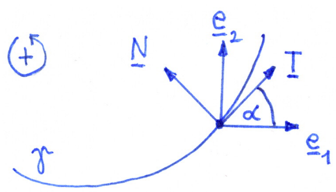
Differentiating $\langle \mathbf{T}, \mathbf{T}\rangle \equiv 1$ we obtain $2\langle \mathbf{T}, \mathbf{T}'\rangle \equiv 0$. Thus, there is a smooth function $\kappa\colon I \to \mathbb{R}$ such that

$$\frac{1}{v}\mathbf{T}' = \kappa\mathbf{N}.$$

- The function κ is called the *curvature* of γ.

Exercise 8. Show that

$$\frac{1}{v}\mathbf{N}' = -\kappa\mathbf{T}.$$



Choose two orthogonal unit vectors $\mathbf{e}_1, \mathbf{e}_2$ parallel to Σ such that $\mathbf{e}_2$ is the $+90°$ rotation of $\mathbf{e}_1$. Since $\mathbf{T}$ is a smooth unit vector field, there is a smooth direction angle function $\alpha\colon I \to \mathbb{R}$ for $\mathbf{T}$ with respect to the reference frame $\mathbf{e}_1, \mathbf{e}_2$. Then

$$\mathbf{T} = (\cos\circ\alpha)\mathbf{e}_1 + (\sin\circ\alpha)\mathbf{e}_2.$$

Differentiating, we obtain

$$v\kappa\mathbf{N} = \mathbf{T}' = \alpha'\Big((-\sin\circ\alpha)\mathbf{e}_1 + (\cos\circ\alpha)\mathbf{e}_2\Big) = \alpha'\mathbf{N},$$

thus, if $I = [a, b]$, then the total amount of the rotation of $\mathbf{T}$ along γ is

$$\alpha(b) - \alpha(a) = \int_a^b \kappa(t)v(t)dt.$$

The integral on the right is called the *total curvature* of the curve γ.

Consider an oriented closed polygon with curved sides lying in an oriented plane Σ. Parameterize the sides regularly by the maps $\gamma_i\colon [a_i, b_i] \to \Sigma$ moving in the positive direction. Assume that the unit tangents of the sides meeting at a vertex are not opposite to one another. Then we can define the exterior angles $\epsilon_1, \dots, \epsilon_n$ at the vertices as usual, and the total amount of the rotation of the unit tangent as we move around the curve is

$$\sum_{i=1}^n \epsilon_i + \sum_{i=1}^n \int_{a_i}^{b_i} \kappa_{\gamma_i}(t)v_{\gamma_i}(t)dt.$$

This number must be an integer multiple of 2π, say $2\pi k$. Just as for ordinary polygons, k is called the *turning number* of the curvilinear polygon.

The curved version of Corollary 6 is the famous

Theorem 9 (Umlaufsatz). *The turning number of a simple closed polygon with curved sides is* ± 1*, that is*

$$\sum_{i=1}^{n} \epsilon_i + \sum_{i=1}^{n} \int_{a_i}^{b_i} \kappa_{\gamma_i}(t) v_{\gamma_i}(t) dt = \pm 2\pi.$$

Proof. We can approximate polygons with curved sides with ordinary polygons with the same turning number. If a curvilinear polygon is simple, then we can approximate by simple polygons. Details are left to the reader. □

Exercise 10. Let $\Phi\colon D_1 \to D_2$ be an orientation preserving diffeomorphism between two open subsets of the plane, both diffeomorphic to an open disk. Show that the turning number of an oriented curvilinear polygon in D_1 coincides with the turning number of its image under Φ. Is the statement true if D_1 and D_2 are not necessarily diffeomorphic to a disk?

1.2 Total absolute curvature of closed space curves

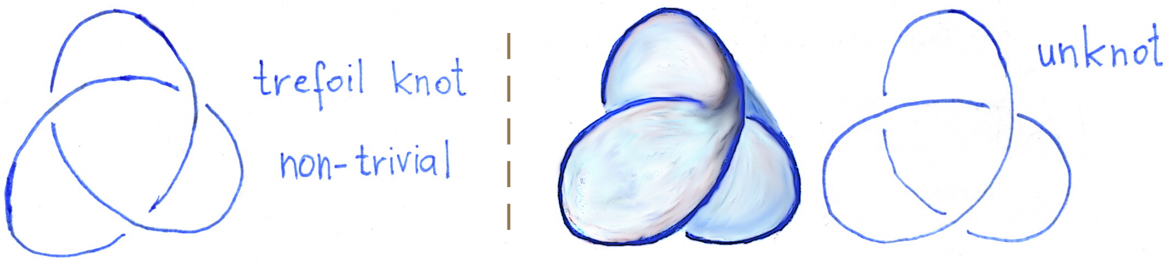
What can we say about space curves? In contrast to the planar case, though we can measure the angle between two non-zero vectors also in the space, there is no natural way to assign an orientation depending sign to these angles. Due to the lack of sign, exterior angles of a polygon in space can be defined only as non-negative numbers in the interval $[0, \pi)$, and a spatial version of the curvature function will take only non-negative values. For simplicity, we deal only with the case of smooth closed curves in this section.

Definition 11. Let $\gamma\colon [a, b] \to \mathbb{R}^3$ be a regular space curve. Then the *absolute curvature function* k of γ is defined by the equality $k = \frac{1}{v}\|\mathbf{T}'\|$, where $v = \|\gamma'\|$, $\mathbf{T} = \frac{\gamma'}{v}$. The *total absolute curvature* of γ is the integral $\int_a^b k(t)v(t)dt$.

The total absolute curvature of a closed smooth regular space curve can change its value continuously. We can always increase its value and make it as large as we want by making an arc of it curlier. Thus, we cannot expect that the total absolute curvatures of smooth closed regular space curves take quantized values, but there is a lower bound for them.

Theorem 12 (Fenchel). *The total absolute curvature of a closed smooth regular space curve is at least* 2π*. Equality holds exactly for convex plane curves (gone around once).*

A closed space curve without self-intersection is a knot. A knot is trivial if it is the boundary of an embedded topological disk. Simple closed plane curves, in particular convex plane curves are trivial knots (also called unknots). The complexity of a knot gives a stronger lower bound for the total absolute curvature.

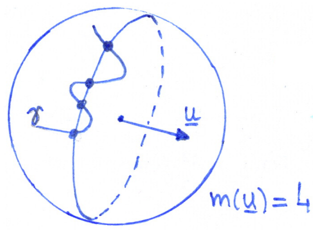


Theorem 13 (Fáry–Milnor). *If a γ is a regular smooth parameterization of a non-trivial knot, then its total absolute curvature is greater than 4π.*

The weak inequality $\geq 4\pi$ is due to Fáry. Shortly after the appearance of Fáry's paper Milnor published a paper proving $> 4\pi$. Both Fenchel's and Fáry's theorems can be proved by an application of the spherical Crofton formula.

Theorem 14 (Spherical Crofton Formula). *Let $\gamma\colon [a,b] \to S^2$ be a spherical curve, and define the function $m\colon S^2 \to \mathbb{N} \cup \{\infty\}$ by $m(\mathbf{u}) = \#\{t \in [a,b] : \gamma(t) \perp \mathbf{u}\}$. Then the length of γ is*

$$l_\gamma = \frac{1}{4}\int_{S^2} m(\mathbf{u})d\mathbf{u}.$$



Exercise 15. Prove Crofton's formula for the arcs of a great circle, and for spherical polygons.

Proof of Fenchel's and Fáry's theorems (sketch). The total absolute curvature of a curve is the length of the spherical curve drawn by the unit tangent vectors $\mathbf{T}$, called the *tangent indicatrix*. If γ is closed, then $\mathbf{T}$ is closed. Almost all great circles intersect a closed smooth curve in an even number of points. Thus, if the total absolute curvature of a closed curve were less than 2π, then applying Crofton's formula, we could find an open hemisphere centered at $\mathbf{u} \in S^2$ which contains $\operatorname{im}\mathbf{T}$. However, that would mean that $\langle \gamma, \mathbf{u}\rangle' = v\langle \mathbf{T}, \mathbf{u}\rangle > 0$, so $\langle \gamma, \mathbf{u}\rangle$ would be strictly increasing, but then γ could not be closed, a contradiction.

If the total absolute curvature of a closed smooth regular curve is less than 4π, then again by Crofton's formula, there exists a great circle with spherical center $\mathbf{u}$ which intersects the tangent indicatrix in two points. Thinking of $\mathbf{u}$ as the vertical direction, $\operatorname{im}\gamma$ can be split into two arcs, along which γ moves monotonously up and down respectively. Then the segments connecting points of equal height of $\operatorname{im}\gamma$ sweep out an embedded disk with boundary $\operatorname{im}\gamma$. Thus, $\operatorname{im}\gamma$ is a trivial knot. $\square$

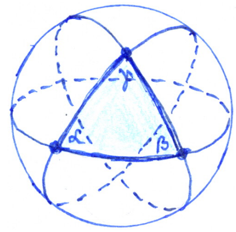
Exercise 16. Prove the second part of Fenchel's theorem on the case of equality.

1.3 Gauss–Bonnet theorem on the sphere

Our goal in this section is to extend the Umlaufsatz to simple closed spherical curves. Spherical geometry is a close relative of Euclidean geometry with many similarities, but also with many crucial differences. In spherical geometry, the role of straight lines is taken by the great circles. As any two great circles intersect in exactly 2 points, there are no parallel lines on the sphere, and we can create a spherical triangle with a given side c and arbitrarily given inner angles $\alpha, \beta \in (0, \pi)$ lying on it. In particular, Euclid's theorem on the sum of the angles of a triangle is false on the sphere. However, the "error term" has a nice geometrical meaning.

Theorem 17 (Girard). *The sum of the angles of a spherical triangle $\triangle$ is bigger than π. The excess is proportional to the area of the triangle, namely, if the angles are α, β, γ, and the radius of the sphere is R, then*

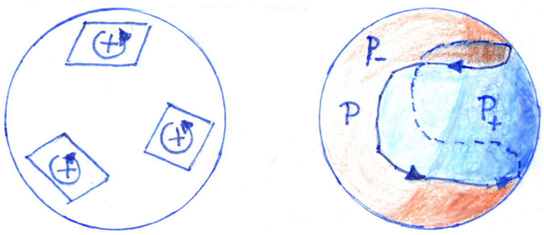
$$\alpha + \beta + \gamma - \pi = \frac{1}{R^2} area(\triangle).$$



Observe that Euclid's theorem can be obtained as the limit of Girard's formula as R tends to infinity.

As a consequence of Jordan's theorem, a simple closed curve on the sphere cuts the sphere into two parts, both parts are homeomorphic to an open disk. In the case of spherical triangles, we can define the interior of the triangle as the component having the smaller area, but polygons with more sides, there is no way to make a natural distinction between the two parts, unless we make some artificial choices.

An orientation of a surface is a continuous choice of the orientations of its tangent planes. The sphere has a standard orientation, for which the positive rotation of the tangent planes are the anti-clockwise rotations looking at the sphere from outside (which coincide with the clockwise rotations if we look at the tangent plane from inside the sphere). If P is a simple closed polygon on the sphere, and we choose an orientation of it, then using this orientation, we can define the interior P_+ by the rule that if we move along P in the positive direction, the $+90°$ rotation of our velocity vector should point toward P_+ (locally). Then the other component of the complement P_- of P is called the exterior of P. It is clear that flipping the orientation of P flips the role of P_+ and P_-.



Once the orientation of P is fixed, we can speak about the inner and exterior angles of P just as in the Euclidean case. Splitting the interior of the polygonal domain into spherical triangles by suitably chosen diagonals, we can extend Girard's theorem to oriented spherical polygons.

Corollary 18. *The sum of the inner angles $\alpha_1, \dots, \alpha_n$ of an oriented simple spherical n-gon P is bigger than $(n-2)\pi$. The excess is proportional to the area of the interior of the polygon, that is,*

$$\left(\sum_{i=1}^{n} \alpha_i\right) - (n-2)\pi = \frac{1}{R^2} area(P_+).$$

In terms of (signed) external angles $\epsilon_i = \pi - \alpha_i$,

$$\sum_{i=1}^{n} \epsilon_i = 2\pi - \frac{1}{R^2} area(P_+).$$

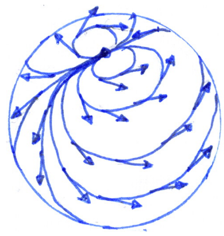
To extend this equation to spherical polygons with curved sides, we need a generalization of the notion of curvature for spherical curves, or more generally, for curves lying on an oriented surface.

The curvature function of a unit speed curve in $\mathbb{R}^2$ is the angular speed of the unit tangent vector relative to a fixed orthonormal basis.

It is a crucial difference between the Euclidean plane and the sphere that while in the Euclidean plane, using parallel translation, we can extend an orthonormal basis to a global orthonormal frame, that is, to two orthogonal unit vector fields, there is no global orthonormal frame on the sphere. This fact is the consequence of the

Theorem 19 (Hedgehog Theorem). *A continuous tangential vector field on the sphere S^2 must vanish at least at one point.*

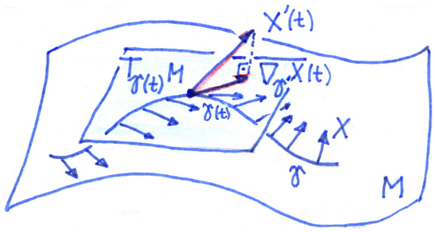
The *hedgehog theorem* (called also as the *hairy ball theorem*) got its name from the popular formulation that if you take a hirsute hedgehog which has a spine growing out from each point of its skin, then it is not possible to comb all the spines of the hedgehog into a tangent direction of the surface of its body in a continuous way. No matter how we try, cowlicks will appear.



There are two methods to get around this problem, we shall use both. One is that we can always choose an orthonormal frame *locally*, on any open set diffeomorphic to a disk. The other is that given a parameterized curve on a surface, there is a canonical way to transport a tangent vector "parallelly" along the curve from one curve point to another curve point. Thus, an orthonormal basis in the tangent plane at a curve point can be extended to a unique "parallel" orthonormal frame *along the curve*.

Definition 20. Let $\gamma\colon I \to M \subset \mathbb{R}^3$ be a smooth parameterized curve on a surface M, $X\colon I \to \mathbb{R}^3$ be a tangential vector field along γ. (This means that $X(t) \in T_{\gamma(t)}M$ for all $t \in I$, where T_pM denotes the tangent plane of M at $p \in M$.)

The *(Levi-Civita) covariant derivative* of X is the tangential vector field $\nabla_{\gamma'}X$ along γ for which $\nabla_{\gamma'}X(t)$ is the orthogonal projection of $X'(t)$ onto $T_{\gamma(t)}M$.



Exercise 21. Prove the identity

$$\langle X, Y\rangle' = \langle \nabla_{\gamma'}X, Y\rangle + \langle X, \nabla_{\gamma'}Y\rangle,$$

where X and Y are arbitrary smooth tangential vector fields along γ.

Exercise 22. Let $\mathbf{r}\colon \Omega \to M$ be a smooth map defined on an open set $\Omega \subset \mathbb{R}^2$, $\partial_1\mathbf{r}$ and $\partial_2\mathbf{r}$ be its partial derivatives. For $(u_0, v_0) \in \Omega$, let $\nabla_1\partial_2\mathbf{r}(u_0, v_0)$ be the value of the covariant derivative of the vector field $u \mapsto \partial_2\mathbf{r}(u, v_0)$ along the curve $u \mapsto \mathbf{r}(u, v_0)$ evaluated at u_0, and define $\nabla_2\partial_1\mathbf{r}(u_0, v_0)$ similarly. Prove that

$$\nabla_1\partial_2\mathbf{r}(u_0, v_0) = \nabla_2\partial_1\mathbf{r}(u_0, v_0).$$

Definition 23. A tangential vector field X along γ is *parallel* if $\nabla_{\gamma'}X = 0$. Any tangent vector $X_0 \in T_{\gamma(t_0)}M$ defines a unique parallel vector field X along γ for which $X(t_0) = X_0$. Then the vector $X(t)$ is called the *parallel transport of X_0 to $\gamma(t)$ along γ*.

Exercise 24. Prove that parallel transport along a curve $\gamma\colon [a,b] \to M$ gives a linear transformation $\Pi_\gamma\colon T_{\gamma(a)}M \to T_{\gamma(b)}$ which preserves the length of vectors, and the angle between vectors.

Definition 25. A smooth parameterized curve on a surface is a *geodesic curve* if its velocity vector field γ' is parallel.

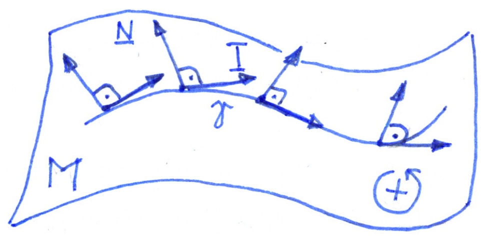
Equivalently, γ is a geodesic curve, if the acceleration vector γ'' is always orthogonal to the surface. By Newton's laws, if no force acts on a point, then the point moves along a straight line with constant speed, or stays at a given point. Thus, if we want to keep a moving point on a surface, then we have to apply some force $\mathbf{F}$ on it unless the surface contains straight lines. The force applied to a point is proportional to the acceleration by Newton's $\mathbf{F} = m\gamma''$ equation. Thus, a point moves along a geodesic curve if and only if the force which keeps the point on the surface is orthogonal to the surface. In particular, the force does not do any work on the point, hence the kinetic energy $\frac{1}{2}m\|\gamma'\|^2$ of the point is constant.

Exercise 26. Prove that $\|\gamma'\|$ is constant for any geodesic curve.

Exercise 27. Show that a regular spherical curve is a geodesic if and only if it parameterizes a great circle with constant speed.

Similarly to the planar case, for a regular curve $\gamma\colon I \to M$ lying on an oriented surface M, we set

- $v = \|\gamma'\|$, the speed of γ;
- $\mathbf{T} = \frac{\gamma'}{v}$, the unit tangent vector field of γ;
- $\mathbf{N}(t) =$ the rotation of $\mathbf{T}(t)$ within $T_{\gamma(t)}M$ by $+90^\circ$ using the orientation of M, the unit normal vector field of γ.



- Covariantly differentiating $\langle \mathbf{T}, \mathbf{T}\rangle \equiv 1$ we obtain $2\langle \mathbf{T}, \nabla_{\gamma'}\mathbf{T}\rangle \equiv 0$. Thus, there is a smooth function $\kappa_g\colon I \to \mathbb{R}$ such that
$$\frac{1}{v}\nabla_{\gamma'}\mathbf{T} = \kappa_g \mathbf{N}.$$
The function κ_g is called the *geodesic curvature* of γ.

Exercise 28. Show that
$$\frac{1}{v}\nabla_{\gamma'}\mathbf{N} = -\kappa_g \mathbf{T}.$$

Exercise 29. Show that a smooth parameterized curve on a surface is a geodesic curve if and only if its geodesic curvature is 0.

Exercise 30. Let $\mathbf{P}_1, \mathbf{P}_2$ be a parallel orthonormal frame along $\gamma\colon [a,b] \to M$ such that $\mathbf{P}_2$ is the $+90°$ rotation of $\mathbf{P}_1$. Then there is a smooth direction angle function $\alpha\colon [a,b] \to \mathbb{R}$ such that $\mathbf{T} = (\cos\circ\alpha)\mathbf{P}_1 + (\sin\circ\alpha)\mathbf{P}_2$. Show that $\alpha' = \kappa_g v$, hence the change of α along γ is $\int_a^b \kappa_g(t)v(t)dt$.

Definition 31. The total geodesic curvature of a regular curve $\gamma\colon [a,b] \to M$ on an oriented surface M is the integral $\int_a^b \kappa_g(t)v(t)dt$.

Exercise 32. Let $\gamma\colon I \to S^2_R$ be a smooth unit speed curve on the sphere of radius R, $t_0 \in \operatorname{int} I$. Show that for small $|s|$, we have

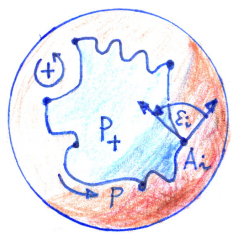
$$\gamma(t_0+s) = \gamma(t_0) + s\mathbf{T}(t_0) + \frac{s^2}{2}\left(\kappa_g(t_0)\mathbf{N}(t_0) \pm \sqrt{k^2(t_0) - \kappa_g^2(t_0)}\frac{\gamma(t_0)}{R}\right) + O(s^3).$$

Using this, prove that for the exterior angle $\epsilon(s)$ of the spherical broken line $\gamma(t_0-s)$, $\gamma(t_0)$, $\gamma(t_0+s)$ at $\gamma(t_0)$, we have

$$\epsilon(s) = s\kappa_g(t_0) + O(s^2).$$

A simple spherical n-gon with curvilinear sides can be approximated by simple inscribed spherical polygons, the vertices of which cut the sides into N equal parts, where N is sufficiently large. Applying Corollary 18 to these polygons and taking the limit as N goes to infinity, using also Exercise 32, we obtain the following theorem.

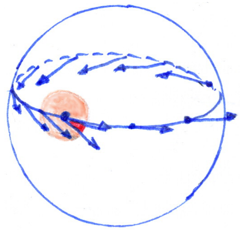
Theorem 33 (Gauss–Bonnet theorem on the sphere). *Let $P \subset S^2_R$ be a simple spherical n-gon with curvilinear sides. Fix an orientation of the polygon. Let the exterior angles of P be ϵ_i and parameterize the sides of P by the regular curves $\gamma_i\colon [a_i,b_i] \to S^2$ moving in the positive direction. Denote by P_+ the domain surrounded by P toward which the unit normal vector fields of the sides γ_i point. Then*



$$\sum_{i=1}^n \epsilon_i + \sum_{i=1}^n \int_{a_i}^{b_i} \kappa_g^{\gamma_i}(t)v^{\gamma_i}(t)dt = 2\pi - \frac{1}{R^2}area(P_+). \tag{1.1}$$

The left hand side of this formula defined an integer multiple of 2π for any closed curvilinear polygon in the Euclidean plane, but it can change continuously in the spherical case if the area enclosed by the polygon changes. Why is this difference? In the Euclidean case, the sum of the exterior angles and the total geodesic curvatures of the sides measures the turning of the unit tangent vector field relative to a globally defined orthonormal frame, while on the sphere, we use a parallel orthonormal frame along γ as the reference frame. The parallel transport of an orthonormal basis around a closed loop does not coincide with the original basis in general. Thus, as we go around the curve, the unit tangent arrives back to its original position, but the reference frame does not. That is why the turning of the unit tangent of a closed polygon relative to a parallel frame is not necessarily an integer multiple of 2π.

Definition 34. Let $p \in M$ be a point on M, $\gamma\colon [a,b] \to M$ be a piecewise smooth loop based at $\gamma(a) = \gamma(b) = p$. The parallel transport Π_γ along γ is an element of the orthogonal group $O(T_pM)$, that is either a rotation of a reflection in a line. Parallel transports along all possible piecewise smooth loops form a subgroup of $O(T_pM)$ called the *holonomy group* of M at p.



Exercise 35. Show that the parallel transport of a vector around an oriented simple curved sided polygon P on the sphere is a rotation of the original vector by the angle $\frac{1}{R^2}area(P_+)$.

1.4 Local Gauss–Bonnet theorem for surfaces

What can we say about closed curves on an arbitrary surface $M \subset \mathbb{R}^3$? Assume first that M is oriented and diffeomorphic to an open disk. Then we can choose a nowhere zero tangential unit vector field E_1 on M. Let E_2 be the pointwise rotation of E_1 by $+90°$ in the tangent plane. For a regular parameterized curve $\gamma\colon [a,b] \to M$, we can compute the rotation of the unit tangent *relative to the frame* E_1, E_2 as follows. There is a smooth function $\omega\colon [a,b] \to \mathbb{R}$ such that

$$\nabla_{\gamma'} E_1^\gamma = \omega E_2^\gamma, \qquad \nabla_{\gamma'} E_2^\gamma = -\omega E_1^\gamma,$$

where $E_i^\gamma = E_i \circ \gamma$, and we can choose a smooth direction angle function $\alpha\colon [a,b] \to \mathbb{R}$ such that

$$\mathbf{T} = (\cos\circ\alpha)E_1^\gamma + (\sin\circ\alpha)E_2^\gamma.$$

Differentiating covariantly, we obtain

$$v\kappa_g \mathbf{N} = \mathbf{T}' = (\alpha' + \omega)\mathbf{N},$$

which implies

$$\alpha' = v\kappa_g - \omega.$$

Lemma 36. *The turning number of a simple closed positively oriented polygon with curved sides in M relative to the frame E_1, E_2 is* 1.

Proof. This Lemma is true because of our assumption that M is a diffeomorphic to a disk. By this assumption, the curve can be shrunk to a point $p \in M$ continuously. During the shrinking, the turning number must change continuously, on the other hand it is an integer number, so it remains constant. Zooming in onto smaller and smaller neighborhoods of p the surface looks more and more like a flat plane, and the contracted curve will be closer and closer to a positively oriented simple curvilinear polygon in that flat plane. Thus, the limit of the unchanging turning number of these curves is 1 by the Umlaufsatz. □

As a consequence,

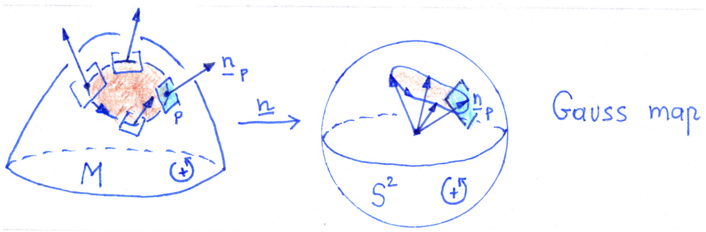
$$\sum_{i=1}^n \epsilon_i + \sum_{i=1}^n \int_{a_i}^{b_i} \kappa_g^{\gamma_i}(t) v^{\gamma_i}(t)dt - \sum_{i=1}^n \int_{a_i}^{b_i} \omega^{\gamma_i}(t)dt = 2\pi.$$

This formula is applicable also in the situation described in the Gauss–Bonnet theorem on the sphere. If $P_+ \subset M \subset S^2$, then comparing the above formula with equation 1.1, we obtain

$$-\sum_{i=1}^n \int_{a_i}^{b_i} \omega^{\gamma_i}(t)dt = \frac{1}{R^2} area(P_+).$$

A useful tool to compare the geometry of an arbitrary surface to the geometry of a sphere is the *Gauss map.*

Definition 37. Let $M \subset \mathbb{R}^3$ be an oriented surface, $\mathbf{n}$ its unit normal vector field for which looking at the tangent plane T_pM facing $\mathbf{n}_p$, the positive rotation is counter-clockwise. Then the *Gauss map* of M is the map $\mathbf{n}\colon M \to S^2$, $p \mapsto \mathbf{n}_p$.



The Gauss map has a simple but important property: the tangent plane T_pM of M at p is parallel to the tangent plane $T_{\mathbf{n}_p}S^2$ of the sphere at the image point. Thus, the two tangent planes can be identified by a translation which preserves the given orientations of these planes. This implies a simple relation between the covariant derivations of M and S^2.

Proposition 38. *Let M be an oriented surface, $\mathbf{n}\colon M \to S^2$ its Gauss map. Then for any smooth parameterized curve $\gamma\colon I \to M$, we can consider its Gauss image $\gamma^* = \mathbf{n} \circ \gamma\colon [a,b] \to S^2$. Then a map $X\colon [a,b] \to \mathbb{R}^3$ is a tangential vector field of M along γ if and only if it is a tangential vector field of S^2 along γ^*, and in that case, we have*

$$\nabla_{\gamma'} X = \nabla_{\gamma^{*\prime}} X.$$

The local behaviour of the Gauss map is described by its derivative.

Definition 39. The *Weingarten map* of the surface at $p \in M$ is the map

$$L_p = -T_p\mathbf{n}\colon T_pM \to T_{\mathbf{n}_p}S^2 \cong T_pM, \qquad \mathbf{v} \mapsto -\partial_{\mathbf{v}}\mathbf{n}.$$

Exercise 40. Show that the Weingarten map is self-adjoint. As a consequence, the eigenvalues of L_p are real and T_pM has an orthonormal basis consisting of eigenvectors of L_p. Eigenvalues of L_p are the *principal curvatures*, (the directions of) the eigenvectors of L_p are the *principal directions* of M at p.

Definition 41. The *Gauss curvature* $K(p)$ of M at p is the determinant of L_p. It is the product of the principal curvatures.

If the Gauss curvature of a surface is not 0 at a point p, then the Gauss map is a local diffeomorphism by the inverse function theorem. In other words, we can find open neighborhoods, $D \subset M$ and $D^* \subset S^2$ of p and $\mathbf{n}_p$ respectively, such that the Gauss map is a diffeomorphism between D and D^*. Assume that D is diffeomorphic to a disk.

Take a simple positively oriented n-gon P in D, which bounds the domain P_+, and denote by P^* and P^*_+ the Gauss image of P and P_+, respectively. P^* is a simple polygon in D^*. It is positively oriented if and only if the Gauss curvature is positive on D. Choose an orthonormal reference frame E_1, E_2 on D and let E^*_1, E^*_2 be the corresponding reference frame on D^* given by $E^*_i(\mathbf{n}_p) = E_i(p)$.

If $\gamma_i\colon [a_i, b_i] \to M$ is a parameterization of a side of P, γ_i^* is the parameterization of its Gauss image, then $\omega^{\gamma_i} = \omega^{\gamma_i^*}$ by Proposition 38. For this reason,

$$\sum_{i=1}^{n} \int_{a_i}^{b_i} \omega^{\gamma_i}(t)dt = -\sum_{i=1}^{n} \int_{a_i}^{b_i} \omega^{\gamma_i^*}(t)dt = -\operatorname{sgn} K \cdot area(P_+^*).$$

By the multivariable integration by substitution, the area of the image of P_+ can be obtained by integrating over P_+ the weight function $|\det(T_p\mathbf{n})| = |K(p)|$, which tells how the Gauss map stretches or shrinks the area infinitesimally around a point. Hence

$$area(P_+^*) = \int_{P_+} |K| d\sigma. \tag{1.2}$$

Thus, we proved

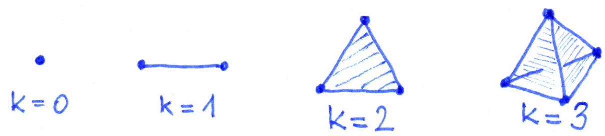
Theorem 42 (Local Gauss–Bonnet Formula). *Under the above assumptions on the surface D and the polygon P, we have*

$$-\sum_{i=1}^{n} \epsilon_i + \sum_{i=1}^{n} \int_{a_i}^{b_i} \kappa_g^{\gamma_i}(t) v^{\gamma_i}(t)dt + \int_{P_+} K d\sigma = 2\pi.$$

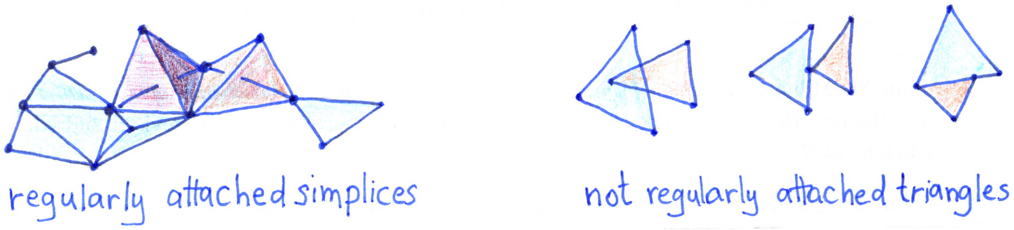
We proved the local Gauss Bonnet formula only when the Gauss map is a diffeomorphism between P_+ and P_+^*. This condition is, in fact, not necessary, it can be replaced by the weaker assumption that P_+ is diffeomorphic to a disk. We shall discuss this generalization later.

1.5 Euler Characteristic of Simplicial Complexes

A *k-dimensional simplex* is the convex hull of $k+1$ affinely independent points. For $k = 0, 1, 2, 3$, a k-dimensional simplex is a point, segment, triangle, tetrahedron, respectively.



An *l-dimensional face* of a simplex is the convex hull of $l+1$ vertices of the simplex. We say the two simplices are *regularly attached* if their intersection is a face of both simplices.



Definition 43. A *finite simplicial complex* is a finite collection $\mathcal{P}$ of simplices with the following properties:

- any two simplices in $\mathcal{P}$ are regularly attached;
- if a simplex belongs to $\mathcal{P}$, then so do all its faces.

The union $|\mathcal{P}|$ of the simplices of a simplicial complex $\mathcal{P}$ is a topological space with the topology inherited from the ambient affine space. The topological space $|\mathcal{P}|$ is called the *body* of $\mathcal{P}$.

Definition 44. Let s_k denote the number of k-dimensional simplices in the finite simplicial complex $\mathcal{P}$. Then the Euler characteristic of $\mathcal{P}$ is the alternating sum

$$\chi(\mathcal{P}) = \sum_k (-1)^k s_k.$$

Homology theory is a branch of algebraic topology. It requires some work to build its machinery, but once one has it, it provides simple proofs of some otherwise non-trivial theorems. One of the first such consequences of homology theory is the homotopy invariance of Euler characteristic.

Theorem 45. *If the bodies of the finite simplicial complexes $\mathcal{P}$ and $\mathcal{Q}$ are homotopy equivalent (e.g. if they are homeomorphic), then $\chi(\mathcal{P}) = \chi(\mathcal{Q})$.*

According to the theorem, we may define the Euler characteristic of a topological space X homeomorphic to the body $|\mathcal{P}|$ of a finite simplicial complex $\mathcal{P}$ by the equation

$$\chi(X) := \chi(\mathcal{P}).$$

A useful tool in the computation of the Euler characteristic is the following inclusion-exclusion formula.

Proposition 46. *If $\mathcal{K}_1$ and $\mathcal{K}_2$ are simplicial subcomplexes of a finite simplicial complex $\mathcal{P}$, then*

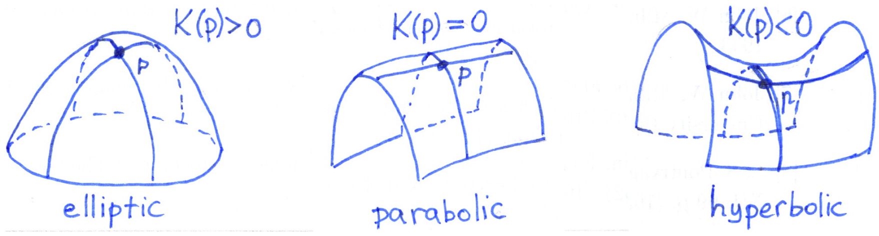
$$\chi(\mathcal{K}_1 \cup \mathcal{K}_2) = \chi(\mathcal{K}_1) + \chi(\mathcal{K}_2) - \chi(\mathcal{K}_1 \cap \mathcal{K}_2).$$

Exercise 47. Show that

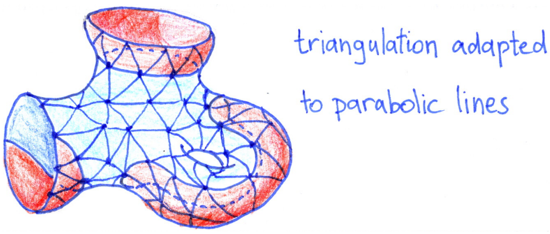
- $\chi(k\text{-dimensional simplex}) = 1$;
- $\chi(S^k) = \chi(\text{boundary of a } (k+1)\text{-dimensional simplex}) = 1 + (-1)^k$;
- $\chi(|\mathcal{P}| \times |\mathcal{Q}|) = \chi(|\mathcal{P}|) \cdot \chi(|\mathcal{Q}|)$ for any finite simplicial complex $\mathcal{P}$ and $\mathcal{Q}$;
- $\chi(\text{Möbius band}) = 0$;
- $\chi(\text{handle}) = -1$, where a handle is a 2-torus $T^2 = S^1 \times S^1$ from which an open disk is removed;
- $\chi(\text{a sphere from which } g \text{ disjoint open disks are removed }) = 2 - g$;
- $\chi(\text{sphere with } g \text{ handles}) = 2 - 2g$.

1.6 Gauss–Bonnet theorem for surfaces with boundary

Consider a compact surface $M \subset \mathbb{R}^3$ with smooth boundary ∂M. Depending on the sign of the Gauss curvature, a point $p \in M$ is called *elliptic* if $K(p) > 0$, *parabolic* if $K(p) = 0$ and *hyperbolic* if $K(p) < 0$. The surface looks like a bump around an elliptic point, and looks like a saddle around a hyperbolic point.



Elliptic and hyperbolic points form two disjoint open subsets of M separated by the set of parabolic points. Any surface can be approximated by surfaces for which the set of parabolic points is a disjoint union of some simple closed curves and some simple arcs with endpoints in ∂M. For this reason, we shall assume that M has this property. Let us take a triangulation of M, for which the boundary of M and the lines of parabolic points are 1-dimensional subcomplexes. Then K is either positive or negative on the interior of a triangle of the triangulation. Refining the triangulation if necessary, we may also assume that the Gauss map is a diffeomorphism between the interior of any triangle of the triangulation and its Gauss image. Then the local Gauss–Bonnet theorem is applicable to each of these triangles.



Let $\triangle_1, \dots, \triangle_f$ be the triangles of the triangulation, $\alpha_i, \beta_i, \gamma_i$ be the inner angles of $\triangle_i$. Then the local Gauss–Bonnet formula for $\triangle_i$ says

$$(\alpha_i + \beta_i + \gamma_i - \pi) = \text{“sum of the total geodesic curvatures of the sides of } \triangle_i\text{”} + \int_{\triangle_i} K(p)\, dp.$$

We remark that M is not assumed to be oriented. A small neighborhood of $\triangle_i$ can be oriented randomly for the sake of the proof of the formula, but eventually the terms in the formula do not depend on the choice of the orientation.

Now add the local formulae for all the triangles. If the number of vertices of the triangulation is $v = v_b + v_i$, where v_b is the number of vertices on the boundary ∂M, v_i is the number of vertices in the interior $M \setminus \partial M$, then since the inner angles of all triangles fill a complete angle at each vertex in $M \setminus \partial M$ and a straight angle at each vertex in ∂M, we have

$$\sum_{j=1}^{f} (\alpha_j + \beta_j + \gamma_j - \pi) = \pi(2v_i + v_b - f).$$

Exercise 48. Show that if $\triangle_i$ and $\triangle_j$ share a common side, then the total geodesic curvature of the common side appears in the local Gauss–Bonnet formula for $\triangle_i$ and $\triangle_j$ with opposite signs.

By the exercise, summing the total geodesic curvatures of the sides of all the triangles, only the total geodesic curvatures of those sides will not be cancelled, which are lying on ∂M, and they add up to the total geodesic curvature of the boundary. Thus we get the formula

$$\pi(v_0 + 2v_+ - f) = \text{“total geodesic curvature of } \partial M\text{”} + \int_M K(p)\, dp.$$

Let's play a little with the combinatorics of the triangulation. Denote by $e = e_b + e_i$ the number of edges, where e_b is the number of edges in ∂M. Since every triangle has 3 edges, $3f$ counts all the edges with some multiplicity: it counts edges on the boundary once all the other edges twice. Hence $3f = 2e_i + e_b$. Since the boundary of ∂M is the disjoint union of some circles, $\chi(\partial M) = v_b - e_b = 0$, that is $v_b = e_b$. According to these,

$$v_b + 2v_i - f = (2v - v_b) - 3f + 2f = 2v - (e_b + 3f) + 2f = 2(v - e + f) = 2\chi(M).$$

The last two equations give the following theorem.

Theorem (Global Gauss–Bonnet Theorem). *For a compact surface with boundary, the following equation holds:*

$$2\pi\chi(M) = \text{“total geodesic curvature of } \partial M\text{''} + \int_M K(p)\,dp.$$

Corollary 50. *If M is a compact surface with no boundary, then*

$$\chi(M) = \frac{1}{2\pi}\int_M K(p)\,dp.$$

Let us give a consequence of the last equation. A point $\mathbf{u} \in S^2$ is said to be a regular value of the Gauss map $\mathbf{n}\colon M \to S^2$, if the preimages of $\mathbf{u}$ under $\mathbf{n}$ are elliptic, or hyperbolic. It is a consequence of Sard's lemma, that almost all points of S^2 are regular values of $\mathbf{n}$. A regular value $\mathbf{u}$ can have only a finite number of preimages $p_1, \dots, p_m$. The number m can depend on $\mathbf{u}$, but if we count each preimage p_i with the sign of $K(p_i)$, then the total number of signed preimages, that is the sum $\sum_{i=1}^m \operatorname{sgn} K(p_i)$ gives a number not depending on the choice of the regular value $\mathbf{u}$. This is a special case of the first theorems proved in differential topology and gives rise to the definition of the degree of a map.

Definition 51. The degree of the Gauss map $\mathbf{n}$ of a compact surface M with no boundary is

$$\deg \mathbf{n} = \sum_{p\in\mathbf{n}^{-1}(\mathbf{u})} \operatorname{sgn} K(p),$$

where $\mathbf{u}$ is a regular value of $\mathbf{n}$.

If $\mathbf{u}$ is a regular value with preimages $p_1, \dots, p_m$, then we can find open neighborhoods $U_1, \dots, U_m \subset M$ of $p_1, \dots, p_m$ respectively, such that $\mathbf{n}$ maps each U_i onto the same open neighborhood $U^* \subset S^2$ of $\mathbf{u}$, and $\mathbf{n}^{-1}(U^*) = \bigcup_{i=1}^m U_i$. Then applying (1.2), we get

$$\int_{\mathbf{n}^{-1}(U^*)} K(p)dp = \sum_{i=1}^m \int_{U_i} K(p)dp = \sum_{i=1}^m \operatorname{sgn} K(p_i)\cdot area(U^*) = \deg \mathbf{n}\cdot area(U^*).$$

With some analysis, we get from this formula the following

Theorem 52. *The total Gauss curvature of a closed surface $M \subset \mathbf{R}^3$ is the product of the degree of the Gauss map and the total area of the unit sphere, that is*

$$\int_M K(p)dp = 4\pi \deg \mathbf{n}.$$

A comparison of the equation and the Gauss–Bonnet formula yields

Corollary 53.

$$\deg \mathbf{n} = \frac{1}{2}\chi(M).$$

1.7 Recommended reading

In the above notes, looking for more and more general versions of Euclid's theorem on the sum of the angles of a triangle, we arrived at the Gauss–Bonnet theorem for surfaces in $\mathbb{R}^3$. Though the latter is really much more general than Euclid's theorem, it is not at all the end of the story. Trying to keep the proofs on an elementary level, the proofs were sometimes sketchy, skipping the technical details. The references below can be a good source deepening the knowledge of the interested reader in this direction.

A detailed proof of the Umlaufsatz can be found in [2]. For a proof of the spherical Crofton formula, Fenchel's theorem, and the Fáry–Milnor theorem see [1], [6] and [5].

Basic facts on simplicial complexes and the Euler characteristic are proved in [8]. A good introduction to the techniques of differential topology is [4]

It was understood already at the time of Gauss that one can make a distinction between the *intrinsic* and *extrinsic* geometry of a surface. A notion or quantity belongs to the intrinsic geometry of a surface, if it can be defined or measured by intelligent creatures who live inside the surface, and able to measure the lengths of curves on the surface, but can get no information from the ambient space. When a surface is bent, the lengths of curves on the surface do not change, so the surface inhabitants do not observe any change in the intrinsic geometry, however, certain quantities, e.g., the principal curvatures of the surface can change during the bending. These quantities belong to the extrinsic geometry of the surface. It can happen, that the definition of a notion uses objects in the ambient space, but it can be defined also by the creatures living on the surface using intrinsic terms. For example the notions of Levi-Civita derivation, parallel transport, geodesic curve, geodesic curvature, and holonomy group belong to the intrinsic geometry of the surface. Theorema Egregium, that is the "Remarkable Theorem" of Gauss claims that although the principal curvatures are extrinsic, their product, the Gauss curvature is an intrinsic quantity (see [1]). Riemann announced in his habilitation thesis the program to work out a higher dimensional generalization of the intrinsic geometry of surfaces. This led to a new branch of geometry what we call Riemannian geometry today. As all the quantities that appear in the Gauss–Bonnet formula are intrinsic quantities, the formula can be generalized and proved for arbitrary compact 2-dimensional Riemannian manifolds with boundary. An intrinsic proof of the theorem can be found in [3].

The book [9] contains a generalization of the Gauss–Bonnet formula for compact hypersurfaces of $\mathbb{R}^n$ with no boundary. In this generalization, the Gauss curvature is replaced by the Gauss-Kronecker curvature defined as the product of the $n-1$ principal curvatures of the hypersurface.

The search for generalizations of the Gauss–Bonnet theorem raises the natural question how we can extract information on the topology of a manifold, or vector bundle from the curvature of a connection on it. The answer to this question led to the theory of characteristic classes, which is a powerful tool of modern differential topology. See Appendix C of [7].

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Problem session

(András Sándor)

Day 1

1. Give a practical way of computing the turning number of an entangled regular plane curve. In other words, given a drawing of such a curve, how can you tell quickly its turning number?

2. What is the minimal total absolute curvature of a regular plane curve? Prove properly that your upper bound holds.

3. The Crofton formula shows us a way to compute the length of a spherical curve by integrating the intersection numbers of the curve with the great circles on the sphere. Precisely

$$l_\gamma = \frac{1}{4}\int_{S^2} m(\mathbf{u})d\mathbf{u}$$

with the notations from the lecture. Let's prove this!

a) Prove the formula for a segment of a great circle and for spherical polygons.

b) Show that you can approximate a spherical curve with spherical polygons sufficiently.

4. Prove that the area of a spherical triangle on the unit sphere is

$$\alpha + \beta + \gamma - \pi$$

expressed in terms of the interior angles of the triangle.
(Hint: First, express the area of a spherical 'biangle'.)

Day 2

1. What are the geodesic curves on

a) a sphere?

b) a cone?

Consider both embedded in the space. Apply the definition given at the lecture: a curve is geodesic if its derivative is parallel along the curve, or in other words if the speed vector changes only perpendicularly to the surface.

2. Consider any circle (not necessarily a great one) on the unit sphere.

a) What is the surface area of the inside of the circle?

b) Compute its geodesic curvature.

Convince yourself that the relation appearing in the lecture is true: the total geodesic curvature of the circle equals 2π minus the area.

3. Consider the torus embedded in the space. Describe its Gauss maps. Is it surjective? How many preimages do the points of the sphere have?

Enumerative geometry: classical and new problems

By László Fehér

(Notes by Ákos Matszangosz)

2.1 Motivation

Example 1 (Apollonius circles). How many circles are tangent to 3 given circles in the plane? (About 2200 years old)

Example 2 (4 lines in space). How many lines intersect four given lines L_i in 3-space?

Example 3 (Conics). In how many points do two conics intersect?

Homework 4 (Triangle). How many triangles exist with vertices on given lines and sides through given points? (Geogebra help for the triangle problem)

Such problems were a starting point for the development of modern algebraic geometry. They are also related to algebraic topology, representation theory, algebraic combinatorics... However, developing the precise theory requires a lot of time. In this mini-course we will sacrifice some of the precision, and rather rely on intuition and black boxes. [1]

2.2 Dimension counting

In the example of four lines why do we take 4 lines, and not 3 or 5? The reason is that we want such questions to have a finite answer. Let us verify this for the four lines problem. The space parameterizing lines in 3-space is four dimensional: look at the intersection with two parallel planes, it gives four independent coordinates, see Figure 2.1.

The subspace of lines intersecting a given line L_i is given by one condition. One can expect that each of the conditions reduces the dimension by one, so the number of lines intersecting four lines is finite.

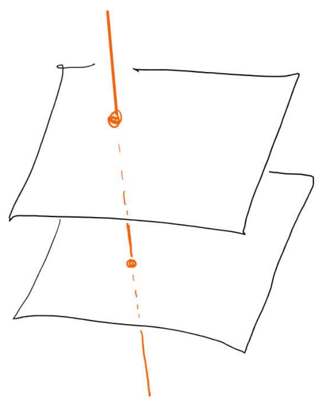


Figure 2.1: The space of lines is 4 dimensional

[1] *"Behind every successful mathematical idea there is a strong proof"* Groucho Marx.

Problem 5. Find n, m such that the answer to each of the following problems is finite:

- How many conics pass through n given points?
- How many conics are tangent to m conics?

In the following we will always formulate questions in the following manner:

> How many
>
> - points/lines/conics
> - intersect/are tangent to/pass through
> - n *generic* points/lines/conics
>
> in *projective* m*-space*?

Both highlighted conditions eliminate cases which are in some sense degenerate. Now we will discuss projective space in some more detail.

2.3 Projective space

2.3.1 Definition

As a set,

$$\mathbb{P}^n := \{L \leq \mathbb{F}^{n+1} \text{ lines through the origin}\}$$

where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. What is the topology of this space? For $\mathbb{F} = \mathbb{R}$ and $n = 2$, consider Figure 2.2.

Take a plane parallel to the x, y-plane. Each line which is not parallel to this plane intersects it in a single point. In this way, we identified an affine plane sitting inside $\mathbb{R}P^2$. However, $\mathbb{R}P^2$ contains other points, namely the lines parallel to the x, y plane – these are the 'points at infinity'. In this way, one can think of $\mathbb{R}P^2$ as obtained by gluing 'points at infinity' to an affine plane. The affine plane is also called an *affine chart*.

These descriptions using points at infinity also adapt to $\mathbb{P}^n$. The reader is encouraged to show that

$$\mathbb{R}P^1 \cong S^1, \qquad \mathbb{C}P^1 \cong S^2, \qquad \mathbb{R}P^2 \cong S^2 \big/ (a \sim -a).$$

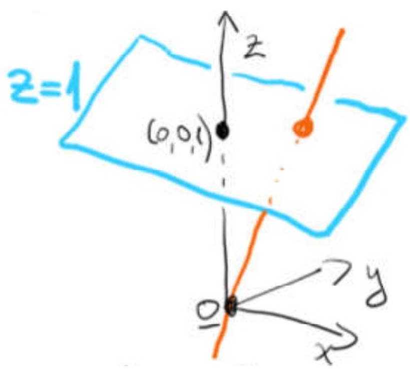


Figure 2.2: Topology of $\mathbb{R}P^2$

Another equivalent definition of $\mathbb{C}P^n$ is obtained by taking

$$\mathbb{C}P^n = \mathbb{C}^{n+1}\backslash\{0\}/\operatorname{GL}(1).$$

Homogeneous coordinates parametrize points of $\mathbb{C}P^n$ (not uniquely!):

$$x = [x_0 : x_1 : \ldots : x_n]$$

where two such $n+1$-tuple of coordinates x and y corresponds to the same point of $\mathbb{C}P^n$ iff $x = \lambda \cdot y$ for some λ. (We assume that not all x_i are zero.)

2.3.2 Subspaces

The same definitions make sense for a vector space V; its *projectivization* $\mathbb{P}(V)$ is a $\dim(V)-1$ dimensional projective space. A $k+1$-dimensional linear subspace $W \leq V$, gives rise to a *k-dimensional projective subspace* $\mathbb{P}(W) \subseteq \mathbb{P}(V)$. For example, in the case $n = 2$ described above, the lines lying in the plane $z = 0$ form a *projective line* (projective 1-dimensional subspace). Projective subspaces behave similarly to linear subspaces:

Lemma 6. *The intersection of projective subspaces is a projective subspace. Furthermore, if* $\mathbb{P}^a, \mathbb{P}^b \subseteq \mathbb{P}^n$, $\mathbb{P}^a \cap \mathbb{P}^b = \mathbb{P}^c$, *then* $c \geq a + b - n$.

If $\mathbb{P}^a$ and $\mathbb{P}^b$ are in general position, then $c = a + b - n$. This can be the definition of general position in this situation. The expression *transversal* can also be used. If $a + b < n$ then transversality implies that the intersection is empty.

Corollary 7. *Any two lines in* $\mathbb{P}^2$ *intersect.*

2.3.3 Symmetries of projective spaces

The linear invertible transformations $\operatorname{GL}(n+1)$ of $\mathbb{C}^{n+1}$ send lines to lines; therefore they also transform $\mathbb{P}^n$. In modern language we say that $\operatorname{GL}(n+1)$ *acts on* $\mathbb{P}^n$. Notice that the scalar transformations don't do anything:

$$\lambda \cdot [x_0 : x_1 : \ldots : x_n] \sim [x_0 : x_1 : \ldots : x_n],$$

so in fact, one has an action of the group

$$\operatorname{PGL}(n+1) := \operatorname{GL}(n+1)/\{\lambda \operatorname{Id}_{\mathbb{C}^{n+1}}\}.$$

Example 8. The $\operatorname{GL}(2)$-action on $\mathbb{P}^1$ can be written in coordinates as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} [u : v] = [au + bv : cu + dv]$$

On the affine chart $x_0 = 1$ the transformation is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} [1 : v] = [a + bv : c + dv] = \left[1 : \frac{c + dv}{a + bv}\right]$$

Homework 9.

- Show that 'usually' such a transformation has two fixed points.
- Let $p, q, r \in \mathbb{P}^1$ be 3 distinct points. Show that there exists $g \in \operatorname{GL}(2)$ such that $g(p) = [0 : 1]$, $g(q) = [1 : 0]$ and $g(r) = [1 : 1]$.

2.4 Shapes and equations

Let $f(x_0, x_1, \ldots, x_n)$ be a *homogeneous polynomial* (i.e. $f(\lambda x) = \lambda^d f(x)$ for all λ for some d $\Leftrightarrow$ all monomials have degree d). Although such polynomials are not functions on $\mathbb{P}^n$, they have a well-defined zero-set:

$$(f = 0) \subseteq \mathbb{P}^n.$$

One can also consider the zero-set of several homogeneous polynomials $f_1, \ldots, f_r$. Such zero-sets are called *varieties.*

Example 10.

- The line defined by $z = 0$ is a variety in $\mathbb{P}^2$.
- The points satisfying the equations $x_0x_3 = x_1x_2$, $x_1^2 = x_0x_2$ and $x_2^2 = x_1x_3$ is a (famous) curve in $\mathbb{P}^3$ (the *rational normal curve*).

2.5 Classical proof of 4 lines in 3-space

Lemma 11. *Given 3 lines L_1, L_2, L_3 in $\mathbb{P}^3$, there exists a quadratic surface containing them.*

Proof.

i) The space of quadric surfaces is a $\mathbb{P}^9$: the coefficients $[a_{ij}]$ of

$$q = \sum_{i \leq j} a_{ij} x_i x_j, \tag{2.1}$$

or more formally, let V be the vector space of degree 2 homogeneous polynomials and identify the space of quadric surfaces with $\mathbb{P}(V)$.

ii) The condition 'containing a line' defines a $\mathbb{P}^6$ in $\mathbb{P}^9$. Indeed, take the line $L = (x_0 = 0, x_1 = 0)$, let q be a quadric equation as in (2.1) and let $Q := (q = 0)$ be its zero-set. Then

$$Q \supseteq L \qquad \Longleftrightarrow \qquad \mathrm{subs}(q, x_0 \mapsto 0, x_1 \mapsto 0) \equiv 0 \qquad \Longleftrightarrow \qquad a_{22} = a_{23} = a_{33} = 0$$

iii) $\cap_{i=1}^3 \mathbb{P}^6 \neq \emptyset$ by Lemma 6. (And in fact for generic lines L_i, there is exactly one intersection point: there is exactly one quadric surface containing 3 generic lines.)

$\square$

Homework 12. Show the converse: Any smooth quadric surface contains 3 disjoint lines.

Lemma 13. *Given 2 lines $L_1 \neq L_2$ and a point $p \notin L_1, L_2$ there exists a unique line L incident to them.*

Homework 14. Prove Lemma 13.

Corollary 15. *A quadric surface Q containing 3 lines in generic position is* ruled*: a disjoint union of lines.*

Figure 2.3: Two rulings on a hyperboloid

Proof. Let $p \in Q$. By Lemma 13, there is a unique line L incident to L_1, L_2 and p.

Claim: If L' is a line, and $|L' \cap Q| \geq 3$ then $L' \subseteq Q$.

Indeed, a degree 2 polynomial has two roots. Applying the claim to L, we get that there is a unique line through any point of Q which intersects both L_1 and L_2. □

Lemma 16. *There is another ruling of the quadratic surface and any two lines from opposite rulings intersect.*

The easy proof is left to the reader. See also Figure 2.3. Where is this sculpture?

Digression: classification of smooth quadric surfaces. See Figure 2.4 for the classification of quadric surfaces over $\mathbb{R}$. Over $\mathbb{C}$ it is harder to draw. However, topology can help us: by the previously seen argument, the parameter space of generic configurations is connected, as the space of degenerate configurations has real codimension 2. Alternatively, one can use algebra, and show that all non-degenerate quadratic forms are equivalent over $\mathbb{C}$.

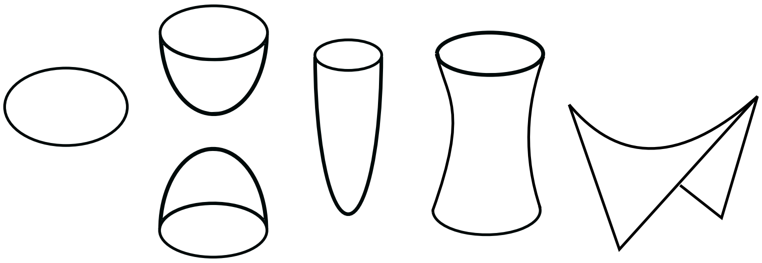


Figure 2.4: Classification of real quadrics

Theorem 17. *Let L_1, L_2, L_3, L_4 be four given generic lines in $\mathbb{P}^3$ over $\mathbb{F}$. Then for $\mathbb{F} = \mathbb{C}$, there are exactly 2 lines in $\mathbb{P}^3$ intersecting each of these. For $\mathbb{F} = \mathbb{R}$, we get either 0 or 2 such lines, depending on the configuration.*

Proof. Let L_1, L_2, L_3, L_4 be four given generic lines. By Lemma 11 there exists a quadric Q containing the three lines L_1, L_2, L_3.

Let the field be $\mathbb{C}$ now. Then the intersection of the quadric with the fourth line L_4 is two points, denote them $\{A, B\} := Q \cap L_4$. The ruled surface is a disjoint union of lines, so A and B each determine a line in the same ruling as L_1, L_2, L_3. In the *other* ruling, there is also a line passing through each point, which intersects L_1, L_2, L_3 and also L_4 (in A, B). So the answer is 2.

In the real case, the answer depends on the configuration, see Figure 2.5. □

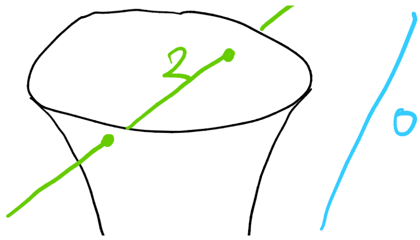


Figure 2.5: Four lines: real case

This solution has the advantage that it is elementary, however it is difficult to generalize to other problems. Topology can be used to deal with the other problems, which is what we will discuss next.

2.6 Degree and Bézout's theorem(s)

Definition 18. Let $H = (f = 0)$ be a *hypersurface* in $\mathbb{P}^n$ (i.e. a variety defined by one equation). Then $\deg H := \deg f$ where we assume that f is minimal, i.e. not a power of an other polynomial.

Theorem 19 (Bézout 1). *Let $H = (f = 0)$ be a* hypersurface *in $\mathbb{P}^n$ (i.e. a variety defined by one equation) and let $L \subseteq \mathbb{P}^n$ be a line. Then for a generic line $\#(H \cap L) = \deg f$.*

Proof. The restriction $f|_L$ is a polynomial on the line of degree $\deg f$, so by the fundamental theorem of algebra it has $\deg f$ roots. (Think about $\mathbb{C}$ versus $\mathbb{R}$!.) □

Remark 20. From the proof we see that here generic means that $f|_L$ cannot have multiple roots.

Theorem 21 (Bézout 2). *Let C_1, C_2 be complex plane curves in generic position. Then*

$$\#(C_1 \cap C_2) = \deg C_1 \cdot \deg C_2.$$

Notice that this theorem solves Example 3, the intersecting conics.

Proof. (sketch) Let $c_i := \deg C_i$ $(i = 1, 2)$. Let A_i be a set of c_i many lines. There are $c_1 \cdot c_2$ many points of intersection in $A_1 \cap A_2$. A_i is a variety given by a polynomial f_i of degree c_i. By slightly perturbing the equations f_i, one gets two curves, and the points of intersection are also slightly perturbed (see Figure 2.6). You can also play with Geogebra here.

We claim that this implies the theorem for generic curves. The key idea is that the parameter space of degree d curves is a $\mathbb{C}P^N$ (show that $N = \binom{d+1}{2} - 1$), and the subset of curves in generic position with a fixed curve in $\mathbb{C}P^N$ is connected, since the set of non-generic ones is a subvariety (defined by polynomial equations) and any subvariety has complex

Figure 2.6: Bézout 2 proof

codimension at least 1, therefore real codimension at least 2. *This* is the essential point where we use that we are in the complex situation and which is not true in the real situation. □

We arrive at a central definition:

Definition 22 (Degree). Let $X \subseteq \mathbb{P}^n$ be a variety of $\dim X = d$. Let $\mathbb{P}^{n-d} \subseteq \mathbb{P}^n$ be in generic position. The *degree of* X is

$$\deg X := \#(X \cap \mathbb{P}^{n-d})$$

Notice that this is indeed a generalization of Definition 18.

Theorem 23 (Bézout in higher dimensions). *Let $X_1, X_2 \subseteq \mathbb{P}^n$ be complex subvarieties in generic position. Then*

$$\deg(X_1 \cap X_2) = \deg(X_1) \cdot \deg(X_2).$$

This can be proved by induction. This is not enough for our purposes; the parameter space of our objects is not always a projective space. For example,

$$\{\text{conic curves}\} \cong \mathbb{P}^5,$$

$$\{\text{quadric surfaces}\} \cong \mathbb{P}^9,$$

but

$$\{\text{lines in } \mathbb{P}^3\} \not\cong \mathbb{P}^4,$$

so we will need to generalize Bézout's theorem to more general spaces.

2.7 Cohomology

Let M be a topological space. Then one can associate to it a graded ring called the *cohomology ring of* M

$$H^*(M) = \bigoplus_i H^i(M).$$

The elements of $H^*(M)$ are called *cohomology classes*. For us, M is a smooth variety. (We will not define smoothness.) Using cohomology rings, one has extra geometry:

Given a subvariety $X \subseteq M$, one can associate to it a cohomology class $[X] \in H^{2k}(M)$, where $k = \operatorname{codim}_{\mathbb{C}}(X \subseteq M)$.

Theorem 24 (Cohomology Bézout). *If $X_1 \pitchfork X_2$ (read: intersect transversally) then*

$$[X_1 \cap X_2] = [X_1] \cdot [X_2]. \tag{2.2}$$

2.7.1 CW-complexes and their cohomology

Theorem 25. *Suppose M is a disjoint union of 'complex cells': $M = \coprod B_i$ where $B_i \cong \mathbb{C}^{n_i}$ are $2n_i$-cells. Then $H^{2k}(M)$ is freely generated by the classes $[\overline{B_i}]$, where $k = \dim_{\mathbb{C}} M - n_i$.*

Example 26. $\mathbb{C}P^n = \mathbb{C}P^{n-1} \coprod \mathbb{C}^n$ where $\mathbb{C}^n$ are the ordinary points and $\mathbb{C}P^{n-1}$ are the points at infinity. Iterating,

$$\mathbb{C}P^n = \mathbb{C}^n \coprod \mathbb{C}^{n-1} \coprod \dots \coprod \mathbb{C}^0$$

Claim 27.

$$H^*(\mathbb{C}P^n) \cong \mathbb{Z}[g]/g^{n+1}$$

Proof. (idea) Let $g := [\mathbb{C}P^{n-1}] \in H^2(\mathbb{C}P^n)$ and use Cohomology Bézout. $\square$

Remark 28. If $X \subseteq \mathbb{C}P^n$, $\operatorname{codim}_{\mathbb{C}} X = d$, then $[X] = \deg(X) \cdot g^d$, so the higher dimensional Bézout is a special case of the Cohomology Bézout.

2.7.2 Grassmannians and their cell decomposition

Definition 29 (Grassmannian).

$$\operatorname{Gr}_k(\mathbb{C}^n) := \{k\text{-dimensional subspaces in } \mathbb{C}^n\}$$

For example, $\mathbb{P}^n = \operatorname{Gr}_1(\mathbb{C}^{n+1})$. Now we have a name for the space of projective lines in $\mathbb{P}^3$: $\operatorname{Gr}_2(\mathbb{C}^4)$.

Fix the *reference flag in $\mathbb{C}^n$*

$$F_\bullet = (F_1 = \langle e_1 \rangle \le F_2 = \langle e_1, e_2 \rangle \le \ldots \le F_n = \mathbb{C}^n)$$

(an ordered set of subspaces). We will decompose the Grassmannian into subsets, according to their relative position with respect to $F_\bullet$. Given $V \in \operatorname{Gr}_k(\mathbb{C}^n)$, introduce the *dimension vector*:

$$\operatorname{di}_V(i) := \dim_{\mathbb{C}}(V \cap F_i).$$

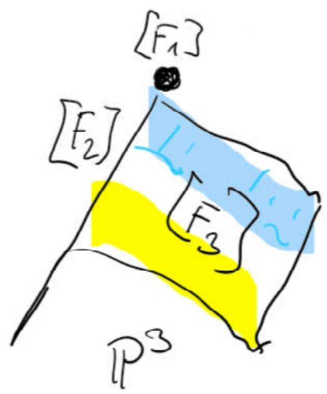


Figure 2.7: A 'flag' in $\mathbb{P}^3$

Proposition 30.

i) Let $f : \{1, \dots, n\} \to \{1, \dots, k\}$ be a monotonic function. Then

$$\{V \in \mathrm{Gr}_k(\mathbb{C}^n) : \mathrm{di}_V = f\} \tag{2.3}$$

is a cell, i.e. it is isomorphic to $\mathbb{C}^d$, for some d, or empty.

ii) Every such cell, which is nonempty, contains exactly one coordinate k-space.

Proof. Essentially Gauss elimination. □

Homework 31.

1. Check Proposition 30 for $\mathbb{P}^n$.

2. What are the possible dimension vectors?

By the Proposition, to index the cells, one has to index the coordinate subspaces. For example, in $\mathrm{Gr}_2(\mathbb{C}^4)$, there are $\binom{4}{2} = 6$ coordinate 2-planes in $\mathbb{C}^4$: $V_{ij} = \langle e_i, e_j \rangle$. More generally one has V_I for $I \in \binom{n}{k}$. The dimension function is given for some I in Table 2.1.

i	1	2	3	4
$\mathrm{di}_{V_{24}}(i)$	0	1	1	2
$\mathrm{di}_{V_{13}}(i)$	1	1	2	2

Table 2.1: Dimension function

The cells given by (2.3) are called *Schubert cells*:

$$\Omega_I := \{V \in \mathrm{Gr}_k(\mathbb{C}^n) : \mathrm{di}_V = \mathrm{di}_{V_I}\}$$

Remark 32. In an alternative formulation, the action of $\mathrm{GL}(n)$ on $\mathbb{C}^n$ induces a $\mathrm{GL}(n)$-action on the Grassmannian. The Schubert cells are the orbits of the subgroup of upper triangular matrices $B^+ \leq \mathrm{GL}(n)$.

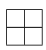
The closures σ_I of the Schubert cells Ω_I are called *Schubert varieties* (show that they are unions of Schubert cells).

2.7.3 Schubert varieties in $\mathrm{Gr}_2(\mathbb{C}^4)$

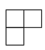
We explicitly determine the Schubert varieties for $\mathrm{Gr}_2(\mathbb{C}^4)$. We will think of it as projective lines in $\mathbb{P}^3$. Recall that $[F_1]$ is a point, $[F_2]$ a line, $[F_3]$ a plane in $[F_4] = \mathbb{P}^3$. Now we describe the Schubert varieties in $\mathrm{Gr}_2(\mathbb{C}^4)$ (for the moment ignore the boxes on the right-hand side).

The smallest Schubert variety is a point:

$$\sigma_{12} = \{V_{12}\} = \{[F_2]\}$$



$$\sigma_{13} = \{L \leq \mathbb{P}^3 : [F_1] \subseteq L \subseteq [F_3]\}$$



This is isomorphic to $\mathbb{P}^1$, namely, $\mathbb{P}(F_3/F_1)$.

$$\sigma_{14} = \{L \leq \mathbb{P}^3 : [F_1] \subseteq L\} \qquad \square\square$$

i.e. projective lines passing through a point, which can be identified with a projective plane $\mathbb{P}^2$, namely $\mathbb{P}(\mathbb{C}^4/F_1)$.

$$\sigma_{23} = \{L \leq \mathbb{P}^3 : L \subseteq [F_3]\} \qquad \begin{smallmatrix}\square\\\square\end{smallmatrix}$$

i.e. projective lines contained in a plane, which can be identified with a projective plane $\mathbb{P}^2$, namely the dual of $\mathbb{P}(F_3)$.

$$\sigma_{23} = \{L \leq \mathbb{P}^3 : L \cap [F_2] \neq \emptyset\} \qquad \square$$

This is less straightforward to identify; it is the cone of the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$ (so this is the first example of a non-smooth Schubert variety). Finally,

$$\sigma_{34} = \mathrm{Gr}_2(\mathbb{C}^4) \qquad \bullet$$

Comparing the dimensions of the cells to the cell-decomposition of $\mathbb{P}^4$, we see that $\mathrm{Gr}_2(\mathbb{C}^4)$ has an extra cell in dimension 2.

2.7.4 Schubert varieties in $\mathrm{Gr}_k(\mathbb{C}^n)$

As we have mentioned above, we can index the cells in an alternative way. Let

$$I = (i_1 < \ldots < i_k) \in \binom{n}{k}.$$

Set

$$\lambda_i = n - k + j - i_j,$$

for $i = 1, \ldots, k$. The sequence of λ_i's forms a *partition*: a non-strictly decreasing sequence of nonnegative integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k).$$

The partitions obtained from $I \in \binom{n}{k}$ also satisfy $\lambda_1 \leq n - k$. To such a partition, one can associate a Young diagram by drawing λ_i boxes in the ith row; see the Schubert cells of $\mathrm{Gr}_2(\mathbb{C}^4)$ for the two kinds of indexing. The condition $\lambda_1 \leq n - k$ implies that Schubert cells are indexed by partitions that fit in a $k \times (n-k)$ box. The complex codimension of the Schubert variety σ_λ is the area of the Young diagram, which we will denote

$$|\lambda| = \sum \lambda_i.$$

Recall, that given a subvariety $\sigma_\lambda \subseteq \mathrm{Gr}_k(\mathbb{C}^n)$, one can associate to it a cohomology class

$$[\sigma_\lambda] \in H^{2|\lambda|}(\mathrm{Gr}_k(\mathbb{C}^n)).$$

As we have seen in Theorem 2.2, the product of such classes (under some genericity conditions), corresponds to the class of their intersection. To give some motivation for the upcoming calculations, let us state that in $H^*(\mathrm{Gr}_2(\mathbb{C}^4))$

$$[\sigma_{\square}]^4 = u \cdot [\sigma_{\boxplus}],$$

and u is the answer to the four-line problem! Indeed, as we saw $\sigma_{\square}$ is the set of projective lines intersecting $[F_2]$, so the cohomology Bézout implies the statement above. Keeping this in mind, we will examine the multiplication table of $H^*(\mathrm{Gr}_2(\mathbb{C}^4))$.

2.7.5 Cohomology of $\mathrm{Gr}_2(\mathbb{C}^4)$

We will determine the multiplication table using the geometric description of Schubert varieties in Section 2.7.3. First, by Theorem 2.2.

$$[\sigma_{\ydiagram{1,1}}]^2 = [\sigma_{\ydiagram{1,1}}(F_\bullet) \cap \sigma_{\ydiagram{1,1}}(F'_\bullet)] = [\{\text{lines in two planes}\}] = [\sigma_{\ydiagram{2,2}}] = [pt]. \tag{2.4}$$

Similarly,

$$[\sigma_{\ydiagram{2}}]^2 = [\sigma_{\ydiagram{2,2}}] = [pt]. \tag{2.5}$$

$$[\sigma_{\ydiagram{2}}] \cdot [\sigma_{\ydiagram{1,1}}] = [\sigma_{\ydiagram{1,1}}] \cdot [\sigma_{\ydiagram{2}}] = [\sigma_{\ydiagram{1,1}}(F_\bullet) \cap \sigma_{\ydiagram{2}}(F'_\bullet)] = 0, \tag{2.6}$$

since for generic flags $F_\bullet$, $F'_\bullet$, $F_1 \not\leq F'_3$ (and H^{2*} is commutative).

Proposition 33.

$$[\sigma_{\square}]^2 = [\sigma_{\ydiagram{2}}] + [\sigma_{\ydiagram{1,1}}]. \tag{2.7}$$

Proof. Write

$$[\sigma_{\square}]^2 = a[\sigma_{\ydiagram{2}}] + b[\sigma_{\ydiagram{1,1}}].$$

Notice, that Lemma 13 (Figure 2.8) together with (2.5) and (2.6) can be used to determine a:

$$a[pt] = [\sigma_{\square}]^2[\sigma_{\ydiagram{2}}] = [\sigma_{\square}(F_\bullet) \cap \sigma_{\square}(F'_\bullet) \cap \sigma_{\ydiagram{2}}(F''_\bullet)] = [pt]$$

Similarly, one can show $b = 1$. □

Theorem 34. *Given four generic lines in $\mathbb{P}^3$, there are 2 lines passing through each one.*

Proof. Using equations (2.4)–(2.7):

$$[\sigma_{\square}]^4 = ([\sigma_{\ydiagram{2}}] + [\sigma_{\ydiagram{1,1}}])^2 = 2[pt].$$

□

Homework 35. Determine the rest of the multiplication table.

It is not difficult to see that the complex dimension of $\mathrm{Gr}_k(\mathbb{C}^{n-k})$ is $k(n-k)$. For example, $\dim_{\mathbb{C}} \mathrm{Gr}_2(\mathbb{C}^5) = 6$.

Problem 36. Find the number n of planes in $\mathbb{P}^4$ intersecting 6 given lines! Answer: $[\sigma_{\square}]^6 = n[pt]$.

We will return to this problem a little later.

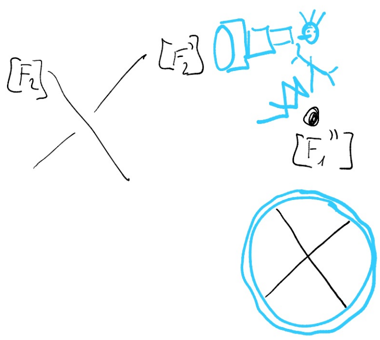


Figure 2.8: Proof of Lemma 13

2.7.6 Schubert calculus

The general multiplication table of $[\sigma_\lambda] \in H^*(\mathrm{Gr}_k(\mathbb{C}^n))$ is also called *Schubert calculus*. Without proof, we now state the general rules of the multiplication table.

Theorem 37 (Giambelli formula).

$$[\sigma_\lambda] = \det([\sigma_{\lambda_i+j-i}]) = \begin{vmatrix} [\sigma_{\lambda_1}] & [\sigma_{\lambda_1+1}] & \dots & [\sigma_{\lambda_1+k-1}] \\ [\sigma_{\lambda_2-1}] & [\sigma_{\lambda_2}] & \dots & [\sigma_{\lambda_2+k-2}] \\ \vdots & \vdots & \ddots & \vdots \\ [\sigma_{\lambda_k-k+1}] & \dots & \dots & [\sigma_{\lambda_k}] \end{vmatrix} =: \Delta_\lambda(\sigma) \tag{2.8}$$

For example $[\sigma_{\ydiagram{1,1}}] = [\sigma_{\ydiagram{1}}]^2 - [\sigma_{\ydiagram{2}}]$.

Theorem 38 (Pieri formula). *Given a partition $\lambda \subseteq k \times (n-k)$ and $\mu \in \mathbb{Z}$,*

$$[\sigma_\lambda] \cdot [\sigma_\mu] = \sum_{\lambda' \in L} [\sigma_{\lambda'}]$$

where L consists of all Young diagrams λ' obtained by adding μ boxes to λ by the following rules:

- *no two squares in the same column,*
- *$\lambda' \subseteq k \times (n-k)$,*
- *has to be a valid Young diagram.*

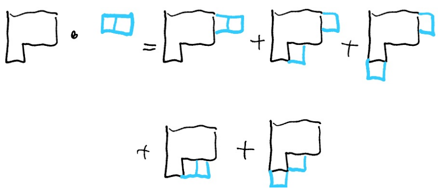


Figure 2.9: Pieri rule

Using Pieri formula, one can solve Problem 36:

$$\ydiagram{1}^6 = \left(\ydiagram{2} + \ydiagram{1,1}\right)^3 = \left(\ydiagram{3,1} + \ydiagram{2,2} + 2 \cdot \ydiagram{3,1} + \ydiagram{2,2}\right)\left(\ydiagram{2} + \ydiagram{1,1}\right) = (1+2+1+1)[pt] = 5[pt].$$

Alternatively,

$$\ydiagram{1}^3 = \left(\ydiagram{2} + \ydiagram{1,1}\right) \cdot \ydiagram{1} = \ydiagram{3} + \ydiagram{2,1} + \ydiagram{2,1} + \ydiagram{1,1,1},$$

where $\ydiagram{1,1,1} = 0$, since $\not\subseteq \ydiagram{3,3}$. Using that H^3 is self-dual, we get that the answer is $2^2 + 1 = 5$.

Remark 39. It is not difficult to show that if σ_λ and σ_μ are half dimensional, then

$$[\sigma_\lambda] \cdot [\sigma_\mu] = \delta_{\lambda,\mu}[pt].$$

This is what I meant on self-duality.

Homework 40. Find out and solve an enumerative problem using Schubert calculus.

Homework 41.** Generalization of four lines: Given 4 projective $2n-1$-dimensional subspaces V_i in $\mathbb{P}^{4n-1}$. How many $2n-1$-dimensional subspaces intersect all V_i in an $n-1$-dimensional subspace? (Try $n=2$ first.)

More generally, one can multiply any two Schubert classes, and express them in a basis:

$$[\sigma_\lambda]\cdot[\sigma_\mu]=\sum c^\nu_{\lambda\mu}[\sigma_\nu].$$

The $c^\nu_{\lambda\mu}$ are called *Littlewood-Richardson coefficients*. They can be calculated using the Giambelli and Pieri formulas. A faster direct method is enumerating *Young tableaux*.

2.8 Symmetric polynomials

Recall that the symmetric group S_n acts on the polynomial ring $\mathbb{Z}[z_1,\dots,z_n]$ by permuting the variables. The subring of invariant polynomials is called the ring of symmetric polynomials. It is generated as a ring by the elementary symmetric polynomials $E_i=E_i(z_1,\dots,z_n)$:

$$\mathbb{Z}[z_1,\dots,z_n]^{S_n}=\mathbb{Z}[E_1,\dots,E_n]$$

Recall their definition:

$$\prod_{i=1}^n(1+tz_i)=\sum_{i=0}^n E_it^i.$$

For example,

$$E_1=z_1+z_2+\dots z_n,\qquad E_2=z_1z_2+z_1z_3+\dots+z_{n-1}z_n.$$

Another convenient additive basis in the ring of symmetric polynomials is given by *Schur polynomials*. Schur polynomials

$$s_\lambda(z)\in\mathbb{Z}[z_1,\dots,z_n]^{S_n}$$

are indexed by partitions λ and are defined by

$$s_\lambda(z):=\Delta_{\lambda^T}(E)$$

where λ^T denotes the mirror of λ (see Figure 2.10) and recall (2.8) for the definition of the polynomial Δ_λ (the variables were named $[\sigma_i]$, and now E_i).

Theorem 42.

$$s_\lambda s_\mu=\sum c^\nu_{\lambda\mu}s_\nu$$

where $c^\nu_{\lambda\mu}$ are the Littlewood-Richardson coefficients introduced above, and the sum is for all partitions of $|\lambda|+|\mu|$.

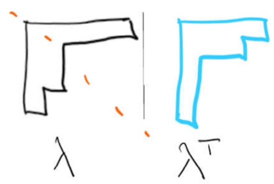


Figure 2.10: $\lambda\Rightarrow\lambda^T$

2.9 Representation theory

It is a fact, that irreducible complex representations V_λ of $\mathrm{SL}(n)$ can be parametrized by partitions λ of length n. Given such a representation

$$\rho_\lambda : \mathrm{SL}(n) \to \mathrm{GL}(V_\lambda),$$

its character can be written as

$$\chi_\lambda = \frac{\det |z_j^{\lambda_j+n-i}|}{\det |z_j^{n-i}|}$$

Theorem 43 (Jacobi formula).

$$\chi_\lambda(z) = s_\lambda(z)$$

This has the representation theoretic consequence that

$$\rho_\lambda \otimes \rho_\mu = \bigoplus c^\nu_{\lambda\mu} \rho_\nu.$$

Let us examine the three flavours of what we have seen so far through an example.

Example 44.

i) **Schubert calculus:**

$$\square^2 = \square\square + \begin{smallmatrix}\square\\\square\end{smallmatrix}.$$

ii) **Symmetric polynomials:**

$$s_1^2 = s_{1,1} + s_2,$$

since $E_i = s_{1^i}$, so $s_1 = E_1 = (z_1 + \ldots + z_n)$, $s_{1,1} = E_2 = \sum_{i<j} z_i z_j$ and $s_2 = \sum_{i\leq j} z_i z_j$.

iii) **Representation theory:**

$$V \otimes V = \mathrm{Sym}^2 V \oplus \Lambda^2 V.$$

2.10 Real enumerative problems

The 4 spaces problem of Homework 41 can be solved also by using linear algebra. This can be used to study the question over the reals. E.g. for $n = 2$ the complex answer is 6, the real is 2 or 6. For more details, see [2]. Notice that in the real case the answer to an enumerative problem is not a number but a set of numbers. The reason for this phenomenon was discussed in the proof of Theorem 21.

An other classical problem is the Steiner problem: how many conics is tangent to 5 conics? It has a fascinating story see e.g [1]. Steiner's original argument was the following: the set of conics tangent to a given conic is defined by a single degree 6 equation (try to prove this!).

Remark 45. Theorem 19 implies that this degree being 6 is equivalent to the fact that there are six members of a generic *pencil of conics* tangent to a given conic, see Figure 2.11.

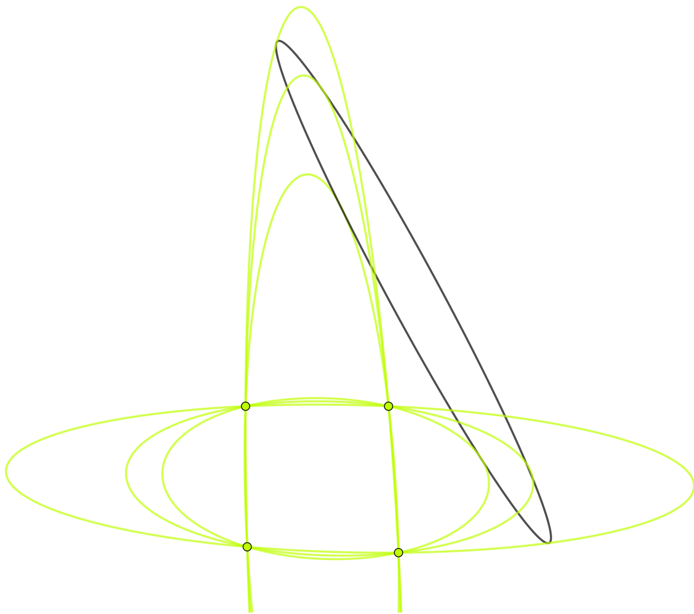


Figure 2.11: 6 ellipses of a pencil tangent to an ellipse

Applying Bézout's theorem Steiner claimed the answer should be 6^5. The answer is wrong, since these 5 degree 6 hypersurfaces will not be in generic position (double lines are conics, and they are tangent to all conics!). The correct answer is that in general there are 3264 smooth conics tangent to 5 smooth conics.

The real case is not completely understood. The maximum 3264 can be obtained by [3]. For the small solution numbers we know that the answer can be 0: five concentric circles. It can be 32: e.g. five small circles in the vertices of a pentagram. On Figure 2.12 you can see six ellipses symmetric to the vertical symmetry axis, you can rotate them 5 ways, and you have a circle tangent to the small circles from the inside and one from the outside: $6 \cdot 5 + 2 = 32$. Notice that $32 = 2^5$: given any subset of the 5 small circles the is a solution containing exactly them (Notice the similarity to the Apollonius circle problem!). We are working on this problem with Tamás Ágoston. We recently found a configuration with 16 solutions.

References

[1] Bashelor, A., Ksir, A., Traves, W.: Enumerative Algebraic Geometry of Conics, *The American Mathematical Monthly* 115 (2008), 701–728.

[2] Fehér L. M., Matszangosz Á.: Real solutions of a problem in enumerative geometry, *arXiv:1401.4638v2* (2014), 1–20.

[3] Ronga, F., Tognoli, A., Vust, T.: The number of conics tangent to 5 given conics: the real case, *Revista Matemática de la Universidad Complutense de Madrid* 10 (1997), 391–421.

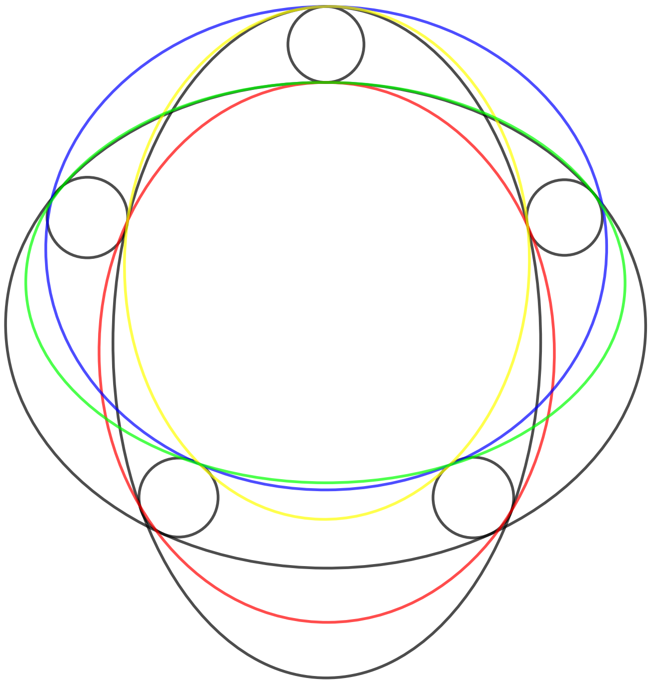


Figure 2.12: 6 of the 32 conics

Problem session

(Ákos Matszangosz)

Day 1

1. How many triangles exist with vertices on given lines and sides through given points? Geogebra help: `https://www.geogebra.org/m/u379yjtw`

2. Find n, m such that the answer to each of the following problems is finite:

a) How many conics pass through n given points?

b) How many conics are tangent to m conics?

3. a) Show that 'usually' the $\mathrm{GL}(2)$-action on $\mathbb{P}^1$ has two fixed points.

b) Let $p, q, r \in \mathbb{P}^1$ be 3 distinct points. Show that there exists $g \in \mathrm{GL}(2)$ such that $g(p) = [0 : 1]$, $g(q) = [1 : 0]$ and $g(r) = [1 : 1]$.

4. Prove that given 2 lines $L_1 \neq L_2$ and a point $p \notin L_1, L_2$ in $\mathbb{P}^3$ there exists a unique line L incident to them ($p \in L, L_i \cap L \neq \emptyset$).

Day 2

1. Identify the cells in $\mathbb{P}^n$ given by the general cell decomposition of Grassmannians (i.e. using the dimension vector $\mathrm{di}_V(i) = \dim(V \cap F_i)$ with respect to a reference flag $F_\bullet$).

2. Verify and complete the geometric description of the cells in $\mathrm{Gr}_2(\mathbb{C}^4)$ seen in class (again, using the dimension vector).

From Poincaré to Thurston and Perelman: one hundred years of a conjecture

By Gábor Moussong

Abstract:

Modern-day topology grew out of the mathematical works of Henri Poincaré. His famous conjecture, put forward in 1904, was about characterizing the three-dimensional sphere in terms of its homotopy type. The Poincaré conjecture withstood all attempts of proof for nearly 100 years, and functioned as the primary motivating force behind many new developments in twentieth century topology and geometry.

The lecture will explain the conjecture and its significance in the topology of manifolds. A few historically famous unsuccessful early attempts to prove the conjecture will be mentioned. In the 1980's Thurston's theory of geometrization generalized the Poincaré conjecture in a geometric context, and opened up new directions for proving it. Without going into technicalities, the lecture will sketch Hamilton's program and Perelman's results which in the early 2000's led to the proof of both Thurston's and Poincaré's conjectures.

Projective algebraic plane curves

By András Némethi

(Notes by Tamás Ágoston)

4.1 Intersection theory of algebraic curves

In algebraic geometry we study algebraic varieties. Over the course of these lectures we will be working with 1-dimensional varieties: algebraic curves. We will denote the base field with k, which will usually be $\mathbb{C}$ (in general the same question could be studied over any field, for example $\mathbb{R}$, $\mathbb{F}_p$ etc.). Let us first recall the basic terminology.

4.1.1 Basics

Definition 1. The affine plane over k is

$$\mathbb{A}^2 = k^2 = \{(x,y) \mid x, y \in k\}.$$

An affine algebraic plane curve is a subset

$$\mathcal{C} = \left\{(x,y) \in k^2 \,\middle|\, f(x,y) = 0\right\} \subset \mathbb{A}^2$$

for some polynomial $f \in k[x,y]$. (The ring $k[x,y]$ is called the coordinate ring of $\mathbb{A}^2$.)

The degree of the curve is $\deg \mathcal{C} = \deg f$.

Example 2. The following all define plane curves:

a) $x^2 + y^2 - 4 = 0$ in $\mathbb{R}^2$ (a circle with radius 2)

b) $x + y = 10$ in $\mathbb{R}^2$ (a line)

c) $x^{10} + \pi y^{200} + e = 0$.

Our main goal in today's lecture is to understand how curves intersect each other.

Example 3. Let us look at the simplest case: the intersection of lines $\mathcal{L}_1 \neq \mathcal{L}_2$. We have 2 cases:

a) $\mathcal{L}_1 \cap \mathcal{L}_2 = \{P\}$ for some point P;

b) $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$ (they are parallel).

To get rid of the exceptional case b), we consider the projective plane:

Definition 4. The projective plane over k is

$$\mathbb{P}^2 = {k^3 \setminus \{0\}} \Big/ {x \sim \lambda x \ (\lambda \in k^*)}$$

(the space of lines in k^3 through the origin). A projective algebraic plane curve is a subset

$$\mathcal{C} = \left\{[x,y,z] \in \mathbb{P}^2 \,\middle|\, f(x,y,z) = 0\right\} \subset \mathbb{P}^2$$

for some homogeneous polynomial $f \in k[x,y,z]$. Note that this is well-defined: for f homogeneous, f is 0 at the same time on the points of a line through the origin in k^3. (The graded ring $k[x,y,z]$ is called the homogeneous coordinate ring of $\mathbb{P}^2$.)

The degree of the curve is $\deg \mathcal{C} = \deg f$.

Remark 5. This can also be considered as $\mathbb{P}^2 = \mathbb{A}^2 \cup \mathcal{L}_\infty$ where $\mathcal{L}_\infty$ is the *line at infinity*, via the identification

$$\mathbb{A}^2 \ni (x, y) \mapsto [x, y, 1] \in \mathbb{P}^2.$$

(points of type $[x, y, 0] \in \mathbb{P}^2$ form $\mathcal{L}_\infty$).

Then the homogeneous equation $f(x, y, z) = 0$ on $\mathbb{P}^2$ defines the affine curve $f(x, y, 1) = 0$ on $\mathbb{A}^2$. Conversely, an affine curve $g(x, y) = 0$ of degree d can be considered as a projective curve of the same degree by filling up the monomial terms with enough z's that it becomes homogeneous of degree d.

Example 6. The homogeneous equations for the curves in Example 2 are:

$$x^2 + y^2 - 4z^2 = 0, \quad x + y - 10z = 0, \quad x^{10}z^{190} + \pi y^{200} + ez^{200} = 0.$$

Note that when we consider projective plane curves of degree 1 (projective lines) $\mathcal{L}_1 \neq \mathcal{L}_2$, we always have $\mathcal{L}_1 \cap \mathcal{L}_2 = \{P\}$ (parallel lines in $\mathbb{A}^2$ intersect on $\mathcal{L}_\infty$).

Both $k[x, y]$ and $k[x, y, z]$ are unique factorization domains, so any polynomial f can be written in the form $\prod_i f_i^{\alpha_i}$ with f_i irreducible. These f_i define the *components* of the curve $\mathcal{C} = \{f = 0\} = \bigcup_i \{f_i = 0\}$, and the α_i can be interpreted as some kind of multiplicities of these components (though we will mostly assume f to be square-free). The curve $\mathcal{C}$ is said to be *irreducible* if f is.

Example 7.

$x^2 + y^2 - 1$ irreducible $\quad$ xy has 2 components

To better understand intersections, we wish to look at small neighborhoods of points on a curve instead of the global picture. For example consider the following 2 curves at the origin:

Example 8.

$xy = 0$ $\quad$ $x^3 + y^3 + xy = 0$

Ideally, we would find that the two are "the same" near $(0, 0)$, since there are two "smooth" branches with the same tangents in both cases – or at the very least, we should see that $x^3 + y^3 + xy = 0$ has two local components near the origin.

It turns out that the way to do this is to consider the defining function f as an element of the local coordinate ring $k[\![x, y]\!]$ instead of $k[x, y]$. Indeed it is a UFD as well, so the definition of components carries over: the *curve germs*, defined by elements of $k[\![x, y]\!]$ split into components too.

Remark 9. The ring $k[\![x,y]\!]$ is a *local ring*: the non-invertible elements form the unique maximal (proper) ideal $\mathfrak{m}$. The invertible elements in this case are the $f \in k[\![x,y]\!]$ with nonzero constant term, and the rest (where the corresponding curve goes through the origin) clearly form an ideal. This corresponds to the fact that analytic functions f not vanishing at $(0,0)$ are locally invertible there.

The above described connection between the affine, projective and local geometry, and the algebraic properties of the respective coordinate rings are summarized below:

Geometry	**Algebra**
$\mathbb{A}^2$ $\mathcal{C}$ affine curve $\downarrow$ $\mathcal{C} = \bigcup_i \mathcal{C}_i$ irred. components	$k[x,y]$ coordinate ring $f \in k[x,y]$ $\downarrow$ UFD $f = \prod_i f_i^{\alpha_i}$
$\mathbb{P}^2$ $\mathcal{C}$ projective curve $\downarrow$ $\mathcal{C} = \bigcup_i \mathcal{C}_i$ irred. components	$k[x,y,z]$ homog. coordinate ring $f \in k[x,y,z]$ homogeneous $\downarrow$ UFD $f = \prod_i f_i^{\alpha_i}$
$(\mathbb{A}^2, 0)$ $\mathcal{C}$ curve germ $\downarrow$ $\mathcal{C} = \bigcup_i \mathcal{C}_i$ irred. components	$k[\![x,y]\!]$ local coordinate ring $f \in k[\![x,y]\!]$ $\downarrow$ UFD $f = \prod_i f_i^{\alpha_i}$

As we noted in Remark 5, the affine and projective curves roughly correspond to each other, and they also have matching components – with the exception that if the homogeneous equation $f = 0$ of a projective curve contains z as a factor (maybe multiple times), then it gets lost upon transition to the affine plane; in other words, if $\mathcal{L}_\infty$ is a component of $\{f = 0\}$ then it disappears when we view the affine equation, since $\mathbb{A}^2 \cap \mathcal{L}_\infty = \emptyset$.

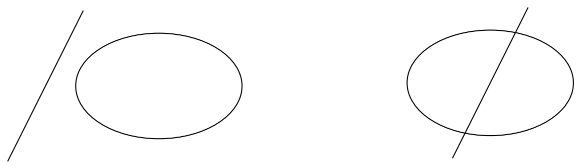
The situation changes though in the case of the local picture: an irreducible affine (or projective) curve may have multiple local components:

Exercise 10. Verify that among the curves in Example 8, xy has the same irreducible components in $k[x,y]$ and $k[\![x,y]\!]$, but while x^3+y^3+xy is irreducible in $k[x,y]$, it decomposes in $k[\![x,y]\!]$. Furthermore, the components are associated elements to the two components of xy.

4.1.2 Intersections and singularities

Let us now resume working towards our previously stated aim of understanding the intersection of curves. We already reviewed the simplest case in Example 3, so let us move on to a slightly more interesting example:

Example 11. Let $\mathcal{L}$ be a line and $\mathcal{C}$ a conic ($\deg \mathcal{L} = 1$, $\deg \mathcal{C} = 2$). They may intersect each other in 0 or 2 points as shown below:



This only happens for $k = \mathbb{R}$ though – for $k = \mathbb{C}$, in the projective plane we *almost* always have 2 intersection points (due to $\mathbb{C}$ being algebraically closed). In particular we always have at least 1. The exceptional case being when $\mathcal{L}$ is tangent to $\mathcal{C}$ – but then we feel that this point should have multiplicity 2 (among other things, slightly moving either $\mathcal{L}$ or $\mathcal{C}$ will result in 2 intersection points near the original point of tangency). So the number of points in $\mathcal{L} \cap \mathcal{C}$, counted with multiplicity is always 2.

Because of what we see here, from now on we assume $k = \mathbb{C}$, and we are working in the projective plane $\mathbb{P}^2 = \mathbb{CP}^2$ (or locally at a point). (It would in fact be enough to assume that k is algebraically closed, and $\operatorname{char} k = 0$.) And then the following will be true in general:

Theorem 12 (Bézout). *Let $\mathcal{C}, \mathcal{D} \subset \mathbb{P}^2$ projective curves,* $\deg \mathcal{C} = c$*,* $\deg \mathcal{D} = d$*. If they have no common irreducible component then*

$$\sum_{P \in \mathcal{C} \cap \mathcal{D}} i_P(\mathcal{C}, \mathcal{D}) = cd$$

where $i_P(\mathcal{C}, \mathcal{D})$ is the intersection multiplicity defined below.

Definition 13. Let $f, g \in k[\![x, y]\!]$. The intersection multiplicity at the point P of the curve germs defined by f and g is $i_P(f, g)$. For $P = 0$,

$$i_0(f, g) = \dim_k \frac{k[\![x, y]\!]}{(f, g)}$$

where (f, g) is the ideal generated by f and g, and for general $P = (a, b)$, let $i_P(f, g) = i_0\Big(f(x + a, y + b), g(x + a, y + b)\Big)$.

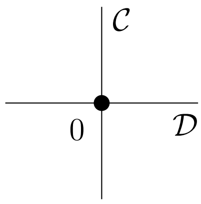
Remark 14. If either curve does not contain 0, then f or g is invertible, and $(f, g) = k[\![x, y]\!]$, hence $i_0(f, g) = 0$. Otherwise $(f, g) \subset \mathfrak{m}$, so $i_0(f, g) \geq \dim \dfrac{k[\![x, y]\!]}{\mathfrak{m}} = 1$.

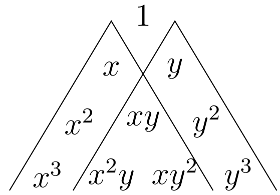
Let us look at some examples of how this definition works:

Example 15. Let $\mathcal{C} = \{x = 0\}$, $\mathcal{D} = \{y = 0\}$. Then (x, y) consists of those $f \in k[\![x, y]\!]$ where all monomial terms are divisible by either x or y – in other words where the constant term is 0, hence $(x, y) = \mathfrak{m}$. So

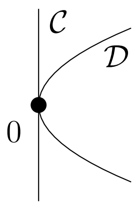
$$i_0(x, y) = 1.$$

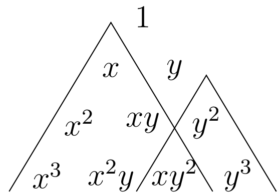
The allowed monomial terms in (x, y) can be visualized as those below x and y in the following diagram:





Example 16. Let $\mathcal{C} = \{x = 0\}$, $\mathcal{D} = \{x - y^2 = 0\}$. Then $(x, x - y^2) = (x, y^2)$ consists of those $f \in k\llbracket x, y \rrbracket$ where all monomial terms are divisible by either x or y^2. The quotient ring $\dfrac{k\llbracket x, y \rrbracket}{(x, y^2)}$ is generated as a k-vector space by the monomials not under either of them in the below diagram: 1 and y.





Therefore

$$i_0(x, x - y^2) = 2.$$

Exercise 17. Calculate $i_0(x^2 + y^3, x^3 + y^2) = 4$.

Exercise 18. Show that $i_P(f_1 f_2, g) = i_P(f_1, g) + i_P(f_2, g)$.

Another local property we will be looking at more deeply later on is whether for a point P on a single curve $\{f = 0\}$, it is smooth (as all points of a circle for instance, like in Example 7), or singular (like the origin on curves xy or $x^3 + y^3 + xy$, as in Example 8). We should first define though what exactly we mean by that.

Definition 19. Let $f \in k\llbracket x, y \rrbracket$ be a curve germ, and write it as the sum of homogeneous parts: $f = f_0 + f_1 + f_2 + \cdots$ where $f_d \in k[x, y]$ is homogeneous of degree d. This is actually the Taylor expansion of f:

$$f_d = \sum_{k=0}^{d} \frac{1}{k! \cdot (d-k)!} \frac{\partial^d f}{\partial x^k \partial y^{d-k}}(0) \cdot x^k y^{d-k}.$$

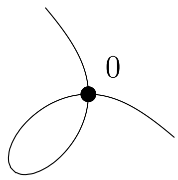
Assume $f_0 = 0$ (i.e. that $0 \in \mathcal{C} = \{f = 0\}$). Then 0 is called a smooth point of $\mathcal{C}$ if $f_1 \neq 0$, i.e. if $\operatorname{grad} f(0) \neq 0$. It is called a singular point otherwise.

The definition can be naturally extended to define when a point $P = (a, b)$ on a curve $\{f = 0\}$ is smooth or singular (consider the germ of $f(x + a, y + b)$ at 0, or rather simply look at $\operatorname{grad} f(P)$). A curve $\mathcal{C}$ is said to be smooth if all of its points are smooth, singular otherwise.

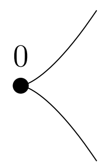
Remark 20. Note that if P is a smooth point of a curve $\mathcal{C} = \{f = 0\}$, then $\operatorname{grad} f(P) \neq 0$, so by the implicit function theorem $\mathcal{C}$ at P as embedded in $\mathbb{A}^2$ is locally diffeomorphic to a line in the plane – hence the name smooth point.

Essentially, smooth points are most often considered "boring" in a sense: the structure of the curve is simple at these points. The global structure of a smooth curve can still be of interest though.

Example 21. The following curves are both singular at 0:



$x^3 + y^3 + xy = 0$



$x^3 - y^2 = 0$

Another property inspired by this, differentiating certain singular points is the following:

Definition 22. For $0 \neq f \in k[\![x, y]\!]$, let the multiplicity of f at 0 be

$$\operatorname{mult}(f) = \min \{d \mid f_d \neq 0\}.$$

The multiplicity of a point P on an affine/projective curve $\{f = 0\}$ can again be naturally defined as before. (Then P is on $\{f = 0\}$ if $\operatorname{mult}_P(f) \geq 1$, and P is a singular point of this curve if $\operatorname{mult}_P(f) \geq 2$.)

Remark 23. Clearly $\operatorname{mult}_P(fg) = \operatorname{mult}_P(f) + \operatorname{mult}_P(g)$.

Corollary 24. *If the projective curve $\mathcal{C}$ is reducible then it is singular.*

Proof. Suppose $\mathcal{C}$ decomposes as $\mathcal{C}_1 \cup \mathcal{C}_2$. Then by Bézout's theorem there is a $P \in \mathcal{C}_1 \cap \mathcal{C}_2$. This satisfies

$$\operatorname{mult}_P(\mathcal{C}) = \operatorname{mult}_P(\mathcal{C}_1) + \operatorname{mult}_P(\mathcal{C}_2) \geq 1 + 1 = 2,$$

so $\mathcal{C}$ is singular at P. $\square$

Multiplicity is also connected to intersection multiplicity:

Exercise 25. Let $f, g \in k[\![x, y]\!]$. Then $i_0(f, g) \geq \operatorname{mult}(f) \cdot \operatorname{mult}(g)$.

Example 26. Consequently, given for example $f = x + y^2$ and $g = x^2 + y^3 + x^{10}y^{1000}$, we can conclude that $i_0(f, g) \geq 1 \cdot 2 = 2$.

It is also worth observing that for $\deg \mathcal{C} = 2$, the only way $\mathcal{C}$ can be singular is as in Corollary 24 (not so for $\deg C \geq 3$, as evidenced by Example 21):

Corollary 27. *All degree 2 irreducible projective curves are smooth.*

Proof. Suppose $\mathcal{C}$ is a projective curve, $\deg \mathcal{C} = 2$, and $P \in \mathcal{C}$ singular. Then let $Q \in \mathcal{C}$, $Q \neq P$, and take the line $\mathcal{L}$ through P and Q. We thus have

$$i_P(\mathcal{L} \cap \mathcal{C}) + i_Q(\mathcal{L} \cap \mathcal{C}) \geq \operatorname{mult}_P(\mathcal{L}) \cdot \operatorname{mult}_P(\mathcal{C}) + 1 \geq 1 \cdot 2 + 1 = 3,$$

so by Bézout's theorem we have a common component. Hence $\mathcal{C}$ has $\mathcal{L}$ as a component, making it reducible. $\square$

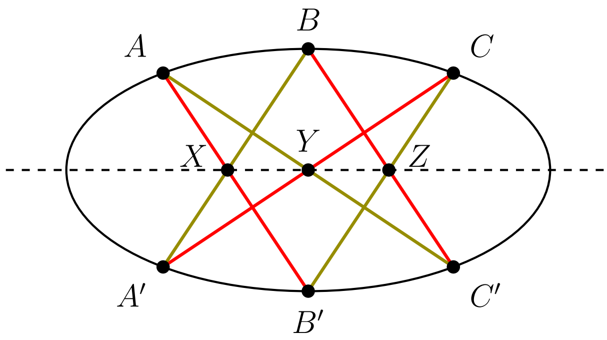
To finish today's lecture, we will prove a statement that will prove to be of crucial importance later on:

Theorem 28 (Splitting Lemma). *Let $\mathcal{C}, \mathcal{D} \subset \mathbb{P}^2$ projective curves, $\deg \mathcal{C} = \deg \mathcal{D} = d$, and suppose they do not have common components. Then by Bézout, $\mathcal{C} \cap \mathcal{D} = P_1 + \cdots + P_{d^2}$ (meaning the set of these points, listed with proper multiplicities).*

Assume that for some irreducible curve $\mathcal{E}$ with $\deg \mathcal{E} = e$, we have $P_1, \cdots, P_{de} \in \mathcal{E}$. Then there is a curve $\mathcal{F}$ with $\deg \mathcal{F} = d - e$ such that $P_{de+1}, \ldots, P_{d^2} \in \mathcal{F}$.

Remark 29. This is automatically satisfied if $\mathcal{D}$ decomposes in the form $\mathcal{E} \cup \mathcal{F}$ for some curve $\mathcal{F}$ of degree $d - e$.

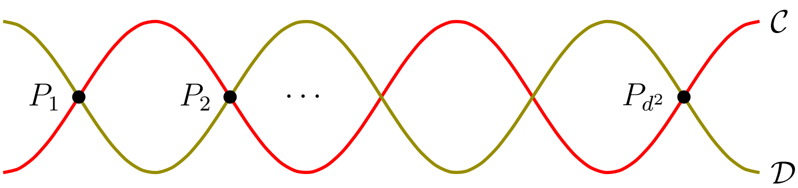
Corollary 30 (Pascal's theorem). *Let $\mathcal{Q}$ be a smooth degree 2 curve (a conic), and $A, B, C, A', B', C' \in \mathcal{Q}$ distinct points. Define X, Y, Z as intersection points of lines according to the diagram. Then X, Y, Z are collinear.*



Proof. Let the union of the 3 red lines be the cubic curve $\mathcal{C}$, that of the 3 green lines be $\mathcal{D}$, and $\mathcal{E} = \mathcal{Q}$. Then applying the splitting lemma, the 9 points are $A, B, C, A', B', C', X, Y, Z$, and $\mathcal{E}$ contains A, B, C, A', B', C', so the degree 1 curve must contain X, Y and Z – exactly what we wanted to prove. □

Remark 31. Pappus' theorem follows via a similar argument: instead of the irreducible conic $\mathcal{Q}$ we need to consider a reducible one, the union of 2 lines.

Proof (Theorem 28).



Let $\mathcal{C} = \{f = 0\}$ and $\mathcal{D} = \{g = 0\}$ (we have $\deg f = \deg g = d$). Let us now consider linear combinations of these, forming a so-called pencil of curves of degree (at most) d:

$$\mathbb{P}^1 \ni [\alpha, \beta] \mapsto \{\alpha f + \beta g = 0\} = \mathcal{C}_{[\alpha,\beta]}.$$

This is well-defined, since $\{\alpha f + \beta g = 0\} = \{\lambda\alpha f + \lambda\beta g = 0\}$ for $\lambda \in \mathbb{C}^*$.

It is obvious that these curves all contain all the P_i's, since f and g both vanish at these points. Furthermore for any $Q \notin \{P_1, \ldots, P_{d^2}\}$ there is a unique $[\alpha, \beta] \in \mathbb{P}^1$ such that $Q \in C_{[\alpha,\beta]}$: we merely need to solve the equation $\alpha f(Q) + \beta g(Q) = 0$, which we can uniquely do since $f(Q)$ and $g(Q)$ are not both 0.

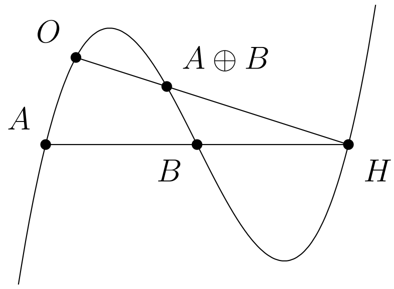
Let us now fix a point $Q \in \mathcal{E} \setminus \{P_1, \ldots, P_{de}\}$, and take a curve $\mathcal{H} = \mathcal{C}_{[\alpha,\beta]} \ni Q$. We then have $\mathcal{H} \cap \mathcal{E} \supseteq \{P_1, \ldots, P_{de}, Q\}$ (as a multiset), and by Bézout, $\mathcal{H}$ and $\mathcal{E}$ must have a common component. But $\mathcal{E}$ is irreducible, so $\mathcal{E}$ *is* a component of $\mathcal{H}$, hence $\mathcal{H} = \mathcal{E} \cup \mathcal{F}$ for some curve $\mathcal{F}$ of degree $d - e$, and this will satisfy our requirements. □

4.2 Group structure on elliptic curves

In this lecture, we will be working with smooth cubic curves: so-called elliptic curves. We will construct an abelian group operation on points of such a curve, making it a Lie group.

Definition 32. Let $\mathcal{C} \subset \mathbb{P}^2$ be a smooth (in particular irreducible) projective curve, $\deg \mathcal{C} = 3$, and fix a point $O \in \mathcal{C}$. For any points $X, Y \in \mathcal{C}$, let $\mathcal{L}_{XY}$ denote the line through them – or if $X = Y$ then the tangent $T_X\mathcal{C}$. Also, we set $\mathcal{L}_{XY} \cap \mathcal{C} = \{X, Y, T(X,Y)\}$ as a multiset (in other words, $T(X,Y)$ is the third intersection point of $\mathcal{L}_{XY}$ with $\mathcal{C}$).

Take any two points $A, B \in \mathcal{C}$. Then let $H = T(A,B)$, and $A \oplus_O B = T(H,O)$ (or simply $A \oplus B$ for ease of notation).



Proposition 33. *This operation has the following properties:*

1) $A \oplus B = B \oplus A$,

2) $O \oplus A = A$,

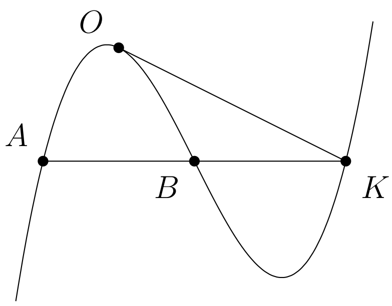
3) $\forall A \exists B : \ A \oplus B = O$,

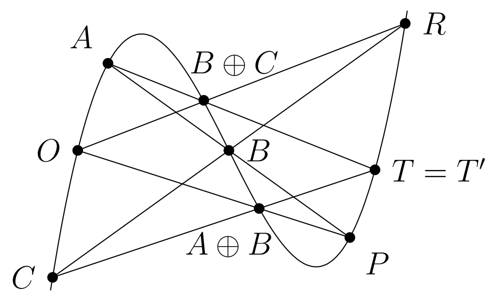
4) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

Proof. Part 1) is of course trivial since $\mathcal{L}_{AB} = \mathcal{L}_{BA}$, so $T(A,B) = T(B,A)$.

Part 2) is also easy, since A, O and $H = T(A,O)$ are collinear by the notation used in the definition above.

For part 3), consider $K = T(O,O)$. Then let $A \in \mathcal{C}$, and $B = T(A,K)$. It is straightforward to check that $A \oplus B = O$, since we get $T(A,B) = K$, and then $T(K,O) = O$.





Lastly, let $A, B, C \in \mathcal{C}$ and verify 4). Use the notation $P = T(A, B)$, $R = T(B, C)$. Then $A \oplus B = T(P, O)$ and $B \oplus C = T(R, O)$. Let $T = T(A \oplus B, C) = T$ and $T' = T(A, B \oplus C)$. Since $(A \oplus B) \oplus C = T(T, O)$, $A \oplus (B \oplus C) = T(T', O)$, and conversely $T = T\big(O, (A \oplus B) \oplus C\big)$, $T' = T\big(O, A \oplus (B \oplus C)\big)$, we simply need $T = T'$.

Once again, we can apply the Splitting Lemma. The two curves in this case are $\mathcal{C}$ and $\mathcal{L}_{A,B} \cup \mathcal{L}_{O,B\oplus C} \cup \mathcal{L}_{A\oplus B,C}$. The intersection points are $A, B, C, O, P, R, T, A \oplus B, B \oplus C$. We then get that $A, B \oplus C, T$ must be collinear, therefore $T = T'$ indeed. $\square$

Thus $(\mathcal{C}, \oplus_O, O)$ forms an abelian group, where $-A = T(A, K)$ with $K = T(O, O)$. Clearly the operations are smooth, so

Proposition 34. *$(\mathcal{C}, \oplus_O, O)$ is a Lie group.*

Corollary 35. *$(\mathcal{C}, \oplus_O, O) \simeq S^1 \times S^1$ (where S^1 is the multiplicative group of the complex unit circle).*

Proof. Since $\mathcal{C} = \{f = 0\}$ is smooth, 0 is a regular value of f, so $\mathcal{C}$ topologically is 2-manifold. Being defined by a complex equation automatically implies orientability, and a Lie group has Euler characteristic 0, so $\mathcal{C}$ is a topological torus. All Lie group structures on the torus are isomorphic, so we are done. $\square$

This implies that for $O, O' \in \mathcal{C}$ we have $(\mathcal{C}, \oplus_O, O) \simeq (\mathcal{C}, \oplus_{O'}, O')$. It is not obvious though what the isomorphism actually is.

Exercise 36. Construct an isomorphism $\Phi : (\mathcal{C}, \oplus_O, O) \to (\mathcal{C}, \oplus_{O'}, O')$.

From now on fix a point $O \in \mathcal{C}$ and a corresponding group operation $\oplus = \oplus_O$. Let us also fix the notation $K = T(O, O)$ as before.

The group structure allows us to rephrase certain properties of points on $\mathcal{C}$, and prove a number of interesting statements seemingly unrelated to it.

Proposition 37. *Let $A, B, C \in \mathcal{C}$. Then*

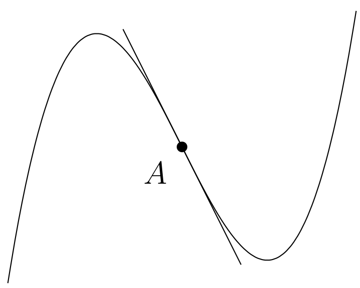
$$C = T(A, B) \text{ (i.e. } A, B, C \text{ are collinear)} \quad \Longleftrightarrow \quad A \oplus B \oplus C = K.$$

Proof. If $T(A, B) = C$, so $A \oplus B = T(O, C)$. Then

$$(A \oplus B) \oplus C = T\big(T(A \oplus B, C), O\big) = T\Big(T\big(T(O, C), C\big), O\Big) = T(O, O) = K.$$

Conversely, let $C' = T(A, B)$. Then by the previously proved direction, $K = A \oplus B \oplus C'$. But we also have $K = A \oplus B \oplus C$, so $C = K \ominus A \ominus B = C'$. Thus $C = T(A, B)$. $\square$

Definition 38. A smooth point A on a curve $\mathcal{C}$ is called an inflection point if the tangent $T_A\mathcal{C}$ intersects $\mathcal{C}$ with multiplicity at least 3. (In this case, $\mathcal{C}$ is an irreducible cubic, so the multiplicity has to be exactly 3 by Bézout's theorem.)



Remark 39. By Proposition 37, $A \in \mathcal{C}$ is an inflection point if and only if $A \oplus A \oplus A = K$.

Proposition 40. *$\mathcal{C}$ has exactly 9 inflection points.*

Proof. Since $(\mathcal{C}, \oplus) \simeq S^1 \times S^1$, we can identify the points on $\mathcal{C}$ with pairs (x, y) where $x, y \in S^1$ – let $A = (a_1, a_2)$ and $K = (k_1, k_2)$. Then by the previous statement, we simply need to solve the equation

$$(3a_1, 3a_2) = (k_1, k_2)$$

for $a_1, a_2 \in S^1$. Clearly, both a_1 and a_2 can have 3 distinct values independently of each other, so altogether we get 9 solutions. □

Proposition 41. *Let $A \in \mathcal{C}$. If A is not an inflection point then there are exactly 5 lines $\mathcal{L} \ni A$ that are tangent to $\mathcal{C}$. If it is, then there are 4.*

Proof. The line $\mathcal{L}$ can be tangent either at A or at another point. In other words, if $\mathcal{L} \cap \mathcal{C} = \{A, X, Y\}$ as a multiset then we need 2 of the points A, X and Y to be equal.

If $A = X$ or $A = Y$ then we get the tangent line $T_A\mathcal{C}$ (and the third point is $K \ominus A \ominus A$ by Proposition 37).

If on the other hand $X = Y$ then similarly to the previous proof, we need to solve the equation

$$A \oplus X \oplus X = K \qquad \Longleftrightarrow \qquad (2x_1, 2x_2) = (k_1 - a_1, k_2 - a_2)$$

where A, X and K correspond to $(a_1, a_2), (x_1, x_2)$ and (k_1, k_2) respectively in $S^1 \times S^1$. We always get 4 solutions here.

But if A is an inflection point then $X = A$ is one of these solutions, so we already counted that. Otherwise, all the solutions result in new lines. This completes the proof. □

Exercise 42. Prove that if $I_1, I_2 \in \mathcal{C}$ are inflection points then $T(I_1, I_2)$ is one too.

We will now state Splitting Lemma in a slightly different form (provable in the same way):

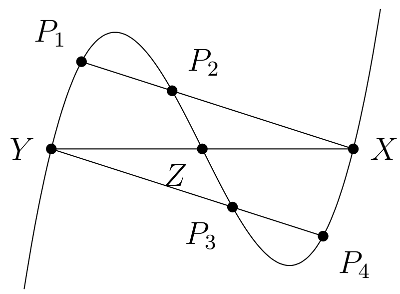
Theorem 43. *Let $\mathcal{C}, \mathcal{D} \subset \mathbb{P}^2$ projective curves, $\deg \mathcal{C} = c$, $\deg \mathcal{D} = d$, and $C \cap D = \{P_1, \ldots, P_{cd}\}$ as a multiset. Assume that $P_1, \ldots, P_d \in \mathcal{L}$ for some line $\mathcal{L}$. Then there is a curve $\mathcal{F}$ of degree $c - 1$ such that $P_{d+1}, \ldots, P_{cd} \in \mathcal{F}$.*

As a consequence of that, we can prove the following statement on elliptic curves:

Theorem 44. *Let $\mathcal{C}$ be a smooth cubic as before, and $\mathcal{D}$ a complex projective curve of degree d, for which $\mathcal{C} \cap \mathcal{D} = \{P_1, \ldots, P_{3d}\}$. Then*

$$P_1 \oplus P_2 \oplus \cdots \oplus P_{3d} = \underbrace{K \oplus \cdots \oplus K}_{d}.$$

Proof. We prove by induction on d. For $d = 1$ we already know the statement by Proposition 37, so assume $d \geq 2$, and that we know the statement for all smaller d's.



Let $X = T(P_1, P_2)$, $Y = T(P_3, P_4)$, $Z = T(X, Y)$, and consider $\mathcal{D}' = \mathcal{D} \cup \mathcal{L}_{XY}$, a curve of degree $d + 1$. Then apply the previous theorem for $\mathcal{C}, \mathcal{D}'$ and $\mathcal{L} = \mathcal{L}_{P_1P_2}$. This yields the d-curve $\mathcal{E}$ containing $P_3, P_4, \ldots, P_{3d}, Y, Z$.

Apply the splitting lemma again to $\mathcal{C}, \mathcal{E}$ and $\mathcal{L}_{P_3P_4}$ to get the $(d-1)$-curve $\mathcal{F}$ containing $P_5, \ldots, P_{3d}, Z$.

Then by induction

$$
\begin{aligned}
P_5 \oplus \cdots \oplus P_{3d} \oplus Z &= K^{\oplus(d-1)}, \\
P_1 \oplus P_2 \oplus X &= K, \\
P_3 \oplus P_4 \oplus Y &= K.
\end{aligned}
$$

Putting them all together yields

$$P_1 \oplus \cdots \oplus P_{3d} \oplus X \oplus Y \oplus Z = K^{\oplus(d+1)}.$$

Subtracting $X \oplus Y \oplus Z = K$, we get the statement, finishing the inductive step. □

Corollary 45. *Let $\mathcal{C}, \mathcal{D}$ be smooth cubics, and $\mathcal{C} \cap \mathcal{D} = \{P_1, \ldots, P_9\}$. Assume $\mathcal{E}$ is also a cubic containing $P_1, \ldots, P_8$. Then we also must have $P_9 \in \mathcal{E}$.*

Proof. Either $\mathcal{E}$ contains – in this case implying being equal to – the irreducible cubic $\mathcal{C}$, making the statement obvious, or the 9th intersection point P_9' (existing by Bézout) satisfies

$$P_1 \oplus \cdots \oplus P_8 \oplus P_9' = K^{\oplus 3} = P_1 \oplus \cdots \oplus P_8 \oplus P_9,$$

implying $P_9' = P_9$ and completing the proof. □

The statement Theorem 44 also answers the following natural question when $\mathcal{C}$ is a smooth cubic.

Question 46. *Given a curve $\mathcal{C}$ of degree c and an integer $d > 0$, can we get any set of points $\{P_1, \ldots, P_{cd}\}$ as $\mathcal{C} \cap \mathcal{D}$ for some curve $\mathcal{D}$ of degree d? (Essentially, is the statement of Bézout's theorem all we can say on the intersection?)*

As we see, for $\mathcal{C}$ being a smooth cubic, the answer is negative for any d. There are two main ways we could investigate this further: we could consider other curves $\mathcal{C}$, or we could ask if the condition of Theorem 44 is sufficient. For example:

Question 47. *Given a line $\mathcal{L} \subset \mathbb{P}^2$ and $P_1, \ldots, P_d \in \mathcal{L}$, is there curve $\mathcal{D}$ of degree d such that $\mathcal{L} \cap \mathcal{D} = \{P_1, \ldots, P_d\}$?*

Question 48. *Given a smooth conic $\mathcal{Q} \subset \mathbb{P}^2$ and $P_1, \ldots, P_{2d} \in \mathcal{Q}$, is there curve $\mathcal{D}$ of degree d such that $\mathcal{Q} \cap \mathcal{D} = \{P_1, \ldots, P_{2d}\}$?*

The answer to the first one is trivially positive. We can assume $\mathcal{L}$ to be the x-axis, and then we take $\mathcal{D}$ to be the graph of a polynomial with roots at the x-coordinates of the P_i's. It is an easy exercise to show that the second question has a positive answer too. It will not be so though for $c \geq 3$.

As for the sufficiency of the sum property for a smooth cubic $\mathcal{C}$, it is trivially so for $d = 1$ by Proposition 37. For $d = 2$ and 3 we also get that $P_1 \oplus \cdots \oplus P_{3d} = K^{\oplus d}$ is enough to ensure the existence of a curve $\mathcal{D}$. For $d = 4$, let us entertain the original question for one more moment:

Question 49. *Let $\mathcal{C}$ be a smooth cubic, and $P_1, \dots, P_{12} \in \mathcal{C}$. Is there a curve $\mathcal{D}$ of degree 4 such that $\mathcal{C} \cap \mathcal{D} = \{P_1, \dots, P_{12}\}$?*

Answer. Clearly, as we observed before, the answer is **NO**, since we must have $P_1 \oplus \cdots \oplus P_{12} = K^{\oplus 4}$ (and equality does not always hold, the sum can be any point on $\mathcal{C}$).

On the other hand, the answer is **YES**. The moduli space of degree 4 curves is the 14-dimensional projective space, as these curves are defined by equations of the form

$$a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 + a_8x^2y + a_9xy^2 + a_{10}y^3 + \\ + a_{11}x^4 + a_{12}x^3y + a_{13}x^2y^2 + a_{14}xy^3 + a_{15}y^4 = 0.$$

The coefficient vectors $\mathbf{a} = (a_1, \dots, a_{15})$ are nonzero, and the curve is invariant up to multiplication of this vector with nonzero scalars, so we get $\mathbb{P}^{14} = \mathbb{A}^{15} \big/ x \sim \lambda x \ (\lambda \neq 0)$.

The condition that a point $P_i = (x_i, y_i)$ is on such a curve $\mathcal{D}$ means that substituting (x_i, y_i), the equation is satisfied. This is a homogeneous linear equation on $\mathbf{a}$, so it defines a projective hyperplane $\mathcal{H}_i \subset \mathbb{P}^{14}$. Then we need

$$\mathcal{D} \in \bigcap_{i=1}^{12} \mathcal{H}_i.$$

The intersection is not empty (in fact it is infinite), so there is such a $\mathcal{D}$. $\square$

Exercise 50. Resolve the apparent contradiction in the above argument.

4.3 Topology of complex plane curves

In this last lecture we will try to understand how the topology of complex projective curves, in particular singular curves, behaves, and make an effort to classify the singularities.

Let $\mathcal{C} = \{f = 0\} \subset \mathbb{CP}^2$ be a projective algebraic curve of degree d, $f = \sum_{i+j+k=d} a_{i,j,k} x^i y^j z^k$.

If $\mathcal{C}$ is smooth then it is a closed manifold with (real) dimension 2 embedded in the closed 4-manifold $\mathbb{CP}^2$. Being defined by complex algebraic equations automatically implies orientability, so the topology of $\mathcal{C}$ is determined by the genus $g \geq 0$ (the number of "holes" in the surface).

Theorem 51 (Genus formula). *If the $\mathcal{C}$ is a smooth complex projective curve of degree $d > 0$ then it has genus*

$$g = \frac{(d-1)(d-2)}{2}.$$

Example 52. For $d = 3$ this implies that a cubic curve is topologically a torus (which we have proved before in Corollary 35).

This theorem solves the question for smooth curves. We also want to understand however how this topology changes when

$$\operatorname{Sing} \mathcal{C} = \left\{ [a,b,c] \,\middle|\, f(a,b,c) = \frac{\partial f}{\partial x}(a,b,c) = \frac{\partial f}{\partial y}(a,b,c) = \frac{\partial f}{\partial z}(a,b,c) = 0 \right\} \neq \emptyset.$$

Remark 53. Observe that for a homogeneous polynomial f of degree $d > 0$,

$$d \cdot f(x,y,z) = x \cdot \frac{\partial f}{\partial x}(x,y,z) + y \cdot \frac{\partial f}{\partial y}(x,y,z) + z \cdot \frac{\partial f}{\partial z}(x,y,z),$$

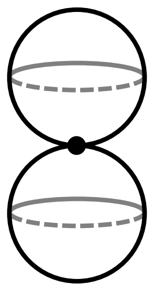
so

$$[a,b,c] \in \operatorname{Sing} \mathcal{C} \quad \Longleftrightarrow \quad \frac{\partial f}{\partial x}(a,b,c) = \frac{\partial f}{\partial y}(a,b,c) = \frac{\partial f}{\partial z}(a,b,c) = 0.$$

Example 54. Consider $f = xy$ and $\mathcal{C} = \{f = 0\}$. Here $\mathcal{C}$ has two components: $\mathcal{L}_1 = \{x = 0\}$ and $\mathcal{L}_2 = \{y = 0\}$. These are intersecting (complex projective) lines.

Locally at 0, $\mathcal{C}$ then looks like two 2-dimensional disks intersecting each other at one point in the interior (since complex lines have real dimension 2).

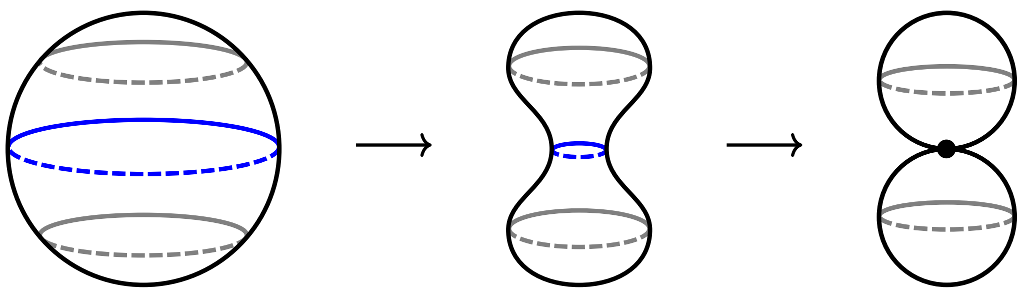
Globally, $\mathcal{L}_i \simeq \mathbb{CP}^1 \approx S^2$. So we get $\mathcal{C} \approx S^2 \vee S^2$.



(Here $X \vee Y$, the *wedge sum* of topological spaces X and Y means gluing together a single point on each. We would generally need to choose these basepoints, but for homogeneous spaces like S^2, we get the same space regardless.)

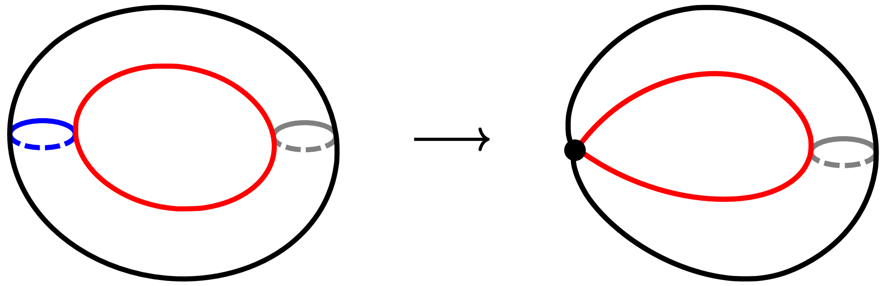
Example 55. Consider $f_t = xy + tz^2$ and $\mathcal{C}_t = \{f_t = 0\}$. It is easy to see that for $t \neq 0$, $\mathcal{C}_t$ is smooth. So by Theorem 51, $\mathcal{C}_t \approx S^2$ for $t \neq 0$, and $\mathcal{C}_0 \approx S^2 \vee S^2$ by the previous example. We can then ask how the sphere S^2 degenerates into $S^2 \vee S^2$ as $t \to 0$.

It turns out that the process can be visualized as illustrated below: the equator of the sphere (blue) gets gradually thinner, until it collapses into a single point, which will be the glued together point in the wedge sum.



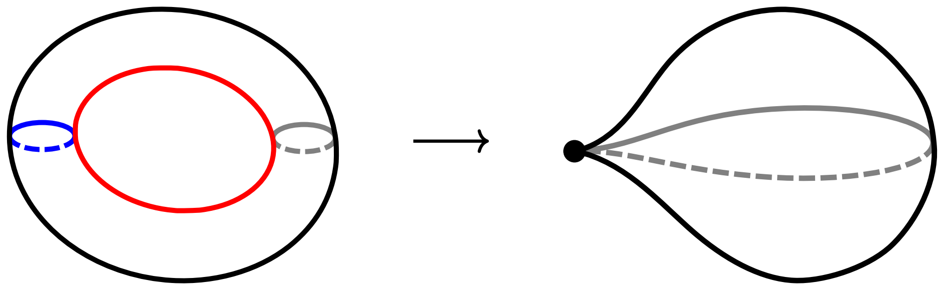
Exercise 56. Verify that (as stated above) $f_t = xy + tz^2$ defines a smooth curve for $t \neq 0$.

Example 57. Now let $f_t = x^3 + y^3 + xyz + tz^3$ and $\mathcal{C}_t = \{f_t = 0\}$. For small $t \neq 0$ we have $\mathcal{C}_t$ smooth, but $\mathcal{C}_0$ has a unique singular point at $[0,0,1]$. Then as $t \to 0$, $\mathcal{C}_t \approx S^1 \times S^1$ by Theorem 51, and we would like to know what the topology of $\mathcal{C}_0$ is.



The first homology group of the torus is $H_1(S^1 \times S^1; \mathbb{Z}) \simeq \mathbb{Z}^2$: the two generators (red and blue) are seen above. What happens in this case is that *one* of the generating cycles (the blue one) collapses into a point: we get $H_1(\mathcal{C}_0; \mathbb{Z}) \simeq \mathbb{Z}$.

Example 58. Let $f_t = x^2z + y^3 + tz^3$ and $\mathcal{C}_t = \{f_t = 0\}$. This curve is smooth for all $t \neq 0$, and $\mathcal{C}_0$ again has the unique singular point $[0, 0, 1]$. This time however both generators of $H_1(S^1 \times S^1; \mathbb{Z})$ collapse into a point – $\mathcal{C}_0$ will be homeomorphic to S^2.



We see that the topology of a singular curve is not uniquely determined by the degree, as was the case with smooth curves. If we converge towards said curve with smooth ones, the topology can degenerate in different ways. The exact way this happens is related to the so-called Milnor number, defined below:

Definition 59. Let $\mathcal{C} = \{f = 0\}$ be an affine plane curve and $P \in \mathcal{C}$. The *Milnor number* of $\mathcal{C}$ at P is

$$\mu(f, P) = i_P\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

In particular for $P = 0$ we have

$$\mu(f, 0) = i_0\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \dim_k \frac{k\,[\![x, y]\!]}{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)}.$$

Remark 60. If P is a smooth point of $\mathcal{C}$ then $\operatorname{grad} f \neq (0, 0)$, hence $\mu(f, P) = 0$. Otherwise $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ is contained in the maximal ideal $\mathfrak{m}$ of $k\,[\![x, y]\!]$ so $\mu(f, P) \geq 1$.

Theorem 61. *Let $\mathcal{C} = \{f = 0\}$ be an affine plane curve and $P \in \operatorname{Sing} \mathcal{C}$. Let $\mathcal{C}_t = \{f = t\}$ for all $t \in \mathbb{C}$. Then for all sufficiently small $t \neq 0$ we have that $\mathcal{C}_t$ is smooth. Additionally, looking at a small neighborhood of P we find that for all $\varepsilon > 0$ small enough and all (depending on ε) sufficiently small $t \neq 0$ we have*

$$H_1(\mathcal{C}_t \cap B(P, \varepsilon); \mathbb{Z}) \simeq \mathbb{Z}^{\mu(f,P)}.$$

Therefore the number of vanishing cycles, i.e. the rank of this homology group (all of whose generators collapse into a point as $t \to 0$) is $\mu(f, P)$.

Example 62. Let us check the previous examples:

$$\mu(xy,0) = \dim \frac{\mathbb{C}[\![x,y]\!]}{(x,y)} = 1$$
$$\mu(x^3+y^3+xy,0) = \dim \frac{\mathbb{C}[\![x,y]\!]}{(3x^2+y,3y^2+x)} = 1$$
$$\mu(x^3+y^2,0) = \dim \frac{\mathbb{C}[\![x,y]\!]}{(3x^2,2y)} = 2$$

More generally, it is an easy exercise to check that for $a, b > 0$ and $f = x^a + y^b$ we have

$$\mu(f,0) = \dim \frac{\mathbb{C}[\![x,y]\!]}{(x^{a-1},y^{b-1})} = (a-1)(b-1).$$

In our goal to characterize all degree d plane curves with singularities, this results in a useful theorem. Namely, it turns out that if f is irreducible of degree d then all $P \in \operatorname{Sing}\mathcal{C}$ will kill $\mu(f,P)$ distinct generators of $H_1(\{f=t\}\,;\mathbb{Z})$ (for a small $t \neq 0$). Here the genus of $\{f=t\}$ is $g = \dfrac{(d-1)(d-2)}{2}$, so $H_1(\{f=t\}\,;\mathbb{Z}) \simeq \mathbb{Z}^{2g} = \mathbb{Z}^{(d-1)(d-2)}$. As a consequence, we have that

Theorem 63. *For an irreducible plane curve $\{f=0\}$ of degree d,*

$$\sum_{P \in \operatorname{Sing}\mathcal{C}} \mu(f,P) \leq (d-1)(d-2).$$

This will allow us to easily eliminate certain configurations of singularities. For example, we instantly get

Corollary 64. *Let $\{f=0\}$ be an irreducible plane curve of degree d. Then* $\operatorname{Sing}\mathcal{C}$ *is finite, in particular*

$$|\operatorname{Sing}\mathcal{C}| \leq (d-1)(d-2).$$

Remark 65. Finiteness of $\operatorname{Sing}\mathcal{C}$ holds for $\mathcal{C}$ reducible as well, since the number of intersection points of the components can be bounded by Bézout's theorem.

At the very least, as far as the Milnor numbers are concerned there are only finitely many ways the singularities of a degree d curve can look like.

If we want to move further, we need to understand what we mean by describing a local singularity type. The Milnor number is relevant information, but does not (necessarily) describe the local topology fully. The tool generally used to grasp this topology is the *link* of the singularity which we will define shortly. But let us first recall what a knot is:

Definition 66. A *knot* is an embedding $K : S^1 \hookrightarrow S^3$. Two knots K_1 and K_2 are considered *isotopic* if one can move one continuously to the other while preserving the embedding property throughout the process. In precise terms, we need a map $\varphi : S^1 \times [0,1] \to S^3$ such that

$$\varphi(p,0) = K_1(p), \quad \varphi(p,1) = K_2(p) \quad \text{and} \quad S^1 \ni p \mapsto \varphi(p,t) \in S^3 \text{ is a knot for all } t \in [0,1].$$

A *link* is an embedding $L : \bigsqcup_{i=1}^{k} S^1 \hookrightarrow S^3$, with link isotopy defined the same way as for knots.

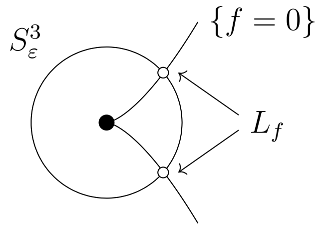
Let us now consider a plane curve singularity at 0 corresponding to $f \in \mathbb{C}[\![x, y]\!]$. In some small neighborhood of 0, $\mathcal{C}$ will be smooth everywhere except at 0, so $\mathcal{C} \setminus \{0\}$ is a (smooth) 2-manifold locally. By intersecting it with a small sphere centered at 0, we would expect to get a 1-dimensional manifold. Indeed:

Theorem 67. *Let $f \in \mathbb{C}[\![x, y]\!]$ irreducible, $0 \in \{f = 0\} = \mathcal{C}$. For $\varepsilon > 0$ let $L_f^\varepsilon = S_\varepsilon^3 \cap \mathcal{C}$. For sufficiently small ε, this is a smooth compact 1-manifold, and the diffeomorphism type of $(S_\varepsilon^3, L_f^\varepsilon)$ is invariant in ε. Furthermore – again for sufficiently small ε –,*

$$\Big(B(0,\varepsilon), B(0,\varepsilon)\cap\mathcal{C}\Big) \overset{homeo}{\approx} C(S_\varepsilon^3, L_f^\varepsilon) \ \text{ and } \ \Big(B(0,\varepsilon)\setminus\{0\}, B(0,\varepsilon)\cap\mathcal{C}\setminus\{0\}\Big) \overset{diff}{\approx} (S_\varepsilon^3, L_f^\varepsilon)\times(0,1)$$

where $C(X, Y)$ denotes the cone over the pair $Y \subset X$.

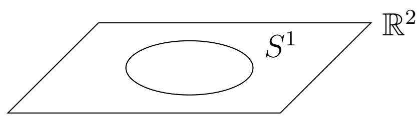
Definition 68. Using the above notations, we define the *link* of f at 0 as $L_f = L_f^\varepsilon$, together with its embedding into S^3 for sufficiently small $\varepsilon > 0$.



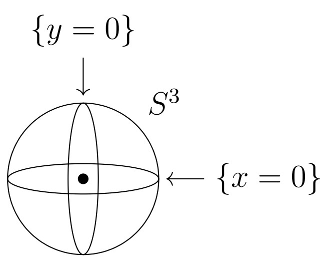
Remark 69. This theorem and definition works for higher dimensions as well – for an n-dimensional complex algebraic variety its link at an isolated singularity will be a $(2n - 1)$-dimensional manifold.

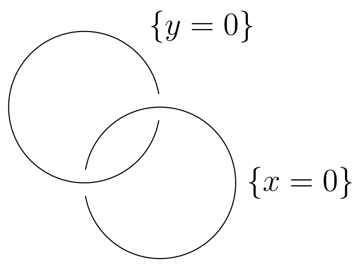
The link of a plane curve singularity is thus a compact 1-manifold, i.e. the disjoint union of circles embedded into S^3 – that is a link as per Definition 66. The consequence of Theorem 67 is that studying the link is for all intents and purposes equivalent to studying the local singularity type as far as topology is concerned.

Example 70. Let $f = x$. Then $\mathcal{C}$ is a 2-plane in the 4-space, and $L_f \hookrightarrow S^3$ is the unknot (the standard embedding $S^1 \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3$).

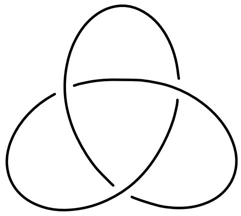


Example 71. Let $f = xy$. As we have ascertained in Example 54, locally $\mathcal{C}$ looks like two disks intersecting each other at their centers. Therefore $L_f = S^1 \sqcup S^1$. The two circles are embedded in a way that their linking number is 1.

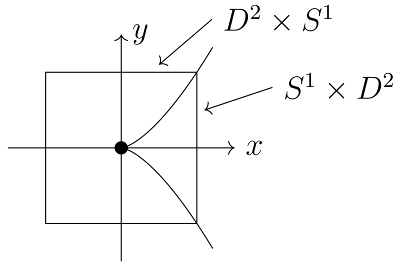




Example 72. Consider $f = x^3 - y^2$. Its link is $L_f \approx S^1$, the embedding $S^1 \to S^3$ being the torus knot $T(2,3)$, i.e. the trefoil.



This can be derived as follows. Consider – instead of the actual geometric ball – the body $B_\varepsilon = \{(x,y) \,|\, |x| < \varepsilon^2, |y| < \varepsilon^3\}$ and its boundary R_ε for small $\varepsilon > 0$. Note that $B_\varepsilon \approx D^4$ and $R_\varepsilon \approx S^3$. In fact, the disjoint union of the surfaces $R_{\varepsilon'}$ for $\varepsilon' < \varepsilon$ is B_ε, in the same way concentric spheres form a ball.



Here we have

$$R_\varepsilon = \left\{(x,y) \,\middle|\, |x| = \varepsilon^2, |y| \le \varepsilon^3\right\} \underset{S^1_{\varepsilon^2} \times S^1_{\varepsilon^3}}{\cup} \left\{(x,y) \,\middle|\, |x| \le \varepsilon^2, |y| = \varepsilon^3\right\} = (S^1 \times D^2) \underset{S^1 \times S^1}{\cup} (D^2 \times S^1),$$

i.e. the surface $R_\varepsilon \approx S^3$ can be written as two solid tori glued together on their boundary: a torus T_ε. It is clear that $\{f = 0\} \cap R_\varepsilon \subset T_\varepsilon$. Furthermore

$$\{f = 0\} \cap R_\varepsilon = \left\{(\varepsilon^2 e^{2\alpha}, \varepsilon^3 e^{3\alpha}) \,\middle|\, \alpha \in [0, 2\pi)\right\} \subset T_\varepsilon,$$

which is indeed exactly the trefoil knot embedded in a torus in the 3-sphere.

It is an easy exercise to show that we can construct a diffeomorphism $R_\varepsilon \to S^3_\varepsilon$ that also maps points of the curve $\{f = 0\}$ on one surface to those on the other, which implies that even when considering the actual geometric sphere S^3_ε, we really get the knot $T(2,3)$ for $L_f \hookrightarrow S^3$. (An isomorphism between the two trivial fibrations $B_\varepsilon \setminus \{0\} \to (0, \varepsilon)$ and $D^4_\varepsilon \setminus 0 \to (0, \varepsilon)$, which further maps points of the curve in one set to those in the other can also be constructed – the before mentioned diffeomorphism is obtained by restricting to one of the fibers R_ε.)

Remark 73. Generally, for any $f \in \mathbb{C}[\![x,y]\!]$ (locally) irreducible we have that L_f is a knot. In fact, it is a so-called *iterated torus knot*: we can get the embedding by taking a torus knot $T(a_1, b_1)$ then drawing a sufficiently thin "pipe", i.e. an embedded torus along it, and drawing a torus knot with (a_2, b_2) on that etc. In finitely many steps we can get the link of any irreducible algebraic curve singularity.

Remark 74. If f is reducible then each irreducible component will correspond to a connected component of the link. The linking numbers will be equal to the respective intersection multiplicities (with the sign dependent on the orientation of the circles).

We could then ask

Question 75. *Given a finite list of iterated torus knots, when does there exist a degree d curve $\mathcal{C}$ with singularities isotopic to the elements of this list respectively?*

A necessary condition arises from the following:

Theorem 76. *Let f define a locally irreducible singularity at 0. Then*

$$\deg \Delta(L_f \hookrightarrow S^3) = \mu(f, 0)$$

where Δ denotes the Alexander polynomial of a knot (see the lecture of András Stipsicz). From now on we will also refer to this simply as the Alexander polynomial of the singularity.

Corollary 77. *Let $\mathcal{C} = \{f = 0\}$ be an irreducible plane curve of degree d, $\operatorname{Sing} \mathcal{C} = \{P_1, \dots, P_k\}$ and $\Delta_1, \dots, \Delta_k$ the Alexander polynomials at each P_i. Then by Theorem 63,*

$$\sum_{i=1}^{k} \deg \Delta_i \leq (d-1)(d-2).$$

Question 78. *If the list of knots in Question 75 satisfies the above condition, does there always exist a corresponding degree d curve $\mathcal{C}$? What if we only fix the Alexander polynomials, but not the knots themselves?*

The Alexander polynomial happens to be rather difficult to work with in this situation, hence we will introduce another, related invariant:

Theorem 79. *Let $\Delta(t)$ be the Alexander polynomial of a locally irreducible plane curve $\{f = 0\}$ at P, multiplied by a power of t to have no negative exponents and a nonzero constant term. Then*

$$\frac{\Delta(t)}{1-t} = \sum_{s \in \mathcal{S}} t^s \in \mathbb{Z}\llbracket t \rrbracket$$

for some $\mathcal{S} \subseteq \mathbb{Z}_{\geq 0}$, which will form a semigroup under addition.

Definition 80. We call the above defined $\mathcal{S}$ the *semigroup* of $\{f = 0\}$ at P.

Example 81. Let us calculate the semigroup of a singularity with link $T(2, 3)$, i.e. the trefoil (for example that of $f = x^3 - y^2$ at 0). The normalized Alexander polynomial is (as shown in the lecture of András Stipsicz)

$$\Delta(t) = 1 - t + t^2.$$

So we get

$$\frac{\Delta(t)}{1-t} = \frac{1}{1-t} - t = 1 + t^2 + t^3 + t^4 + \cdots = \sum_{s \in \mathcal{S}} t^s,$$

thus $\mathcal{S} = \langle 2, 3 \rangle \subset \mathbb{Z}_{\geq 0}$.

If we considered the torus knot $T(a, b)$ instead (with $(a, b) = 1$), we would get

$$\Delta(t) = \frac{(t^{ab} - 1)(t - 1)}{(t^a - 1)(t^b - 1)}$$

and

$$\mathcal{S} = \langle a, b \rangle .$$

It is true in general, and can also easily be verified for the above examples that

Theorem 82. *Let $\mathcal{S}$ and Δ be the semigroup and Alexander polynomial of a locally irreducible plane curve singularity $\{f = 0\}$ at P. Then*

$$2 \cdot |\mathbb{Z}_{\geq 0} \setminus \mathcal{S}| = \mu(f, P) = \deg \Delta.$$

Armed with this tool we will attempt to classify all rational, unicuspidal curves $\mathcal{C} = \{f = 0\}$ of degree d. Here *rational* means that equality holds in Theorem 63, i.e.

$$\sum_{P \in \operatorname{Sing} \mathcal{C}} \mu(f, P) = (d-1)(d-2),$$

so $\mathcal{C} \overset{homeo}{\approx} S^2$ since all 1-cycles vanish. And *unicuspidal* means that the curve has a single singularity, and that is locally irreducible. Hence, we have a unique singular point P, and

$$\mu(f, P) = (d-1)(d-2).$$

Taking into account Theorem 82, we can then ask the following:

Question 83. *Given an integer $d > 0$ and a semigroup $\mathcal{S} \subset \mathbb{Z}_{\geq 0}$ satisfying*

$$2 \cdot |\mathbb{Z}_{\geq 0} \setminus \mathcal{S}| = (d-1)(d-2), \tag{$*$}$$

when is there a rational, unicuspidal curve $\mathcal{C}$ with degree d and semigroup $\mathcal{S}$? What is the case if we actually fix a knot whose semigroup has the property $(*)$, and want the curve to have that knot as the link?

When a knot can or can not be achieved for a given degree d can be quite complicated.

Example 84. Let $d = 5$, and consider a torus knot $T(a, b)$. We want to find out for what values of a and b does there exist a rational unicuspidal curve of degree 5 with this link. By Theorem 76 and Example 81 we have

$$(d-1)(d-2) = \mu = \deg \Delta(t) = \deg \frac{(t^{ab}-1)(t-1)}{(t^a-1)(t^b-1)} = ab + 1 - a - b = (a-1)(b-1).$$

(Confirm also the last part of Example 62. The curve $x^a + y^b$ will actually have $T(a, b)$ as a link, by basically the same reasoning as that seen in Example 72 for $x^3 - y^2$.)

So $(a-1)(b-1) = 12$, and without loss of generality we can assume $a < b$. Thus we can have $(a, b) = (2, 13)$, $(3, 7)$ or $(4, 5)$. By the above remark we get that $x^4 + y^5$ realizes $T(4, 5)$. It is also possible to have $T(2, 13)$ as the link. It turns out however that this is not the case for $T(3, 7)$: there is not rational, unicuspidal curve of degree 5 with this link.

Observe that by Example 81 we have $\mathcal{S} = \langle a, b \rangle$, which satisfies $(*)$ for all these pairs, so that condition is not enough to guarantee the existence of a curve.

We have seen that the set of good pairs (a, b) for a given degree d can be highly nontrivial to describe. For instance in the case $d = 5$ above, there is no clear reason to distinguish $(3, 7)$ from the other possible pairs – not even a simple bound, since it is in the middle. Nonetheless, a complete characterization does exist, as follows:

Theorem 85. *Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence:*

$$F_0 = F_1 = 1, \ \forall n \geq 0 : \ F_{n+2} = F_{n+1} + F_n.$$

Given integers $d > 0$ and $a, b > 1$ with $(a, b) = 1$, the torus knot $T(a, b)$ is realized as the link of a rational unicuspidal curve of degree d in exactly the following cases:

(1) $(a, b) = (d, d-1)$*;*

(2) $(a, b) = (\frac{d}{2}, 2d-1)$ *with d even;*

(3) $(a, b) = (F_{n-1}^2, F_{n+1}^2)$*, $d = F_n^2 + 1$ with $n \geq 4$ even;*

(4) $(a, b) = (F_{n-2}, F_{n+2})$*, $d = F_n$ with $n \geq 5$ odd;*

(5) $(a, b) = (F_4, F_8 + 1) = (3, 22)$ *with $d = F_6 = 8$;*

(6) $(a, b) = (2F_4, 2F_8 + 1) = (6, 43)$ *with $d = 2F_6 = 16$.*

Surprisingly enough, there is also a general description for any knot, underlying this theorem.

Definition 86. Let $\mathcal{S} \subseteq \mathbb{Z}_{\geq 0}$ be a semigroup under addition, and fix an integer $d > 0$. We say that $\mathcal{S}$ satisfies the *semigroup distribution property* with respect to d if all of the following are true:

$$\begin{aligned} |\mathcal{S} \cap (-\infty, 0]| &= 1 \\ |\mathcal{S} \cap (0, d]| &= 2 \\ |\mathcal{S} \cap (d, 2d]| &= 3 \\ &\vdots \\ \left|\mathcal{S} \cap (d^2 - 2d, d^2 - d]\right| &= d \end{aligned}$$

Note that if these equalities are satisfied then $(d^2 - 2d, \infty) \subset \mathcal{S}$ clearly follows since $\mathcal{S}$ is closed under addition and contains an element $1 \leq k \leq d$.

Theorem 87. *Let $K : S^1 \hookrightarrow S^3$ be an algebraic knot, $\mathcal{S}$ the corresponding semigroup, and $d > 0$ an integer. If there exists a rational unicuspidal curve of degree d with link K then $\mathcal{S}$ satisfies the semigroup distribution property with respect to d.*

Remark 88. Observe that this property trivially implies the equality $(*)$ in Question 83, since altogether there are

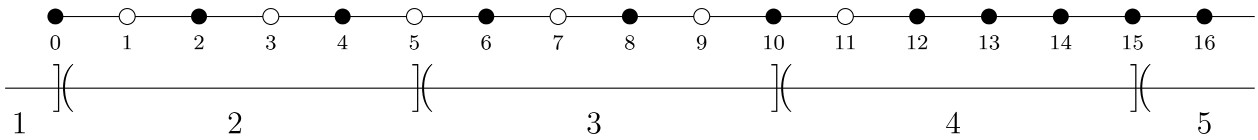
$$(d-2) + (d-3) + \cdots + 1 = \frac{(d-1)(d-2)}{2}$$

nonnegative numbers left out.

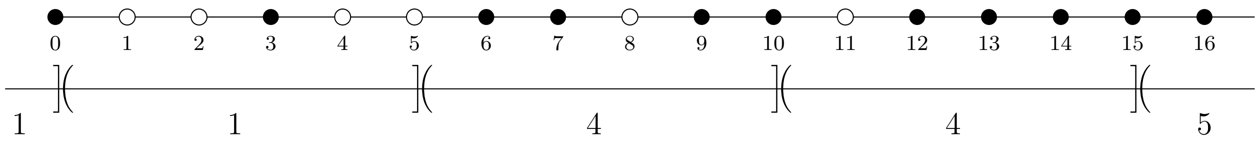
This is a very strong restriction on the semigroup, and checking whether or not it is satisfied can be highly nontrivial, as exemplified by Theorem 85 – which is essentially the case of $\mathcal{S}$ having 2 generators.

Example 89. We can verify that in Example 84, out of the 3 considered semigroups $\langle 2,13\rangle$, $\langle 3,7\rangle$ and $\langle 4,5\rangle$, the first and last ones satisfy the semigroup distribution property with respect to $d=5$, but $\langle 3,7\rangle$ does not:

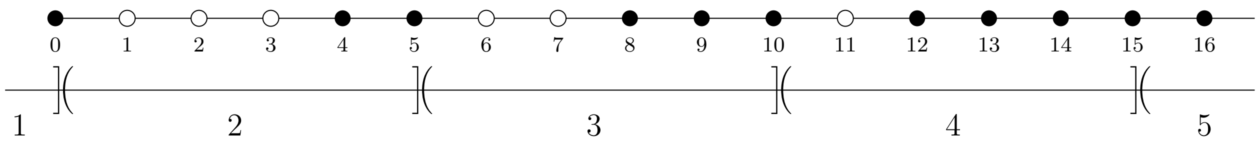
$\mathcal{S}=\langle 2,13\rangle$:



$\mathcal{S}=\langle 3,7\rangle$:



$\mathcal{S}=\langle 4,5\rangle$:



Problem session

(Tamás Ágoston)

Day 1

1. a) Let $f=x^2+y^3$ and $g=x^2-y^3$. What is $i_0(f,g)$?

b) Let $f,g\in k[[x,y]]$ be relatively prime. Show that $k[[x,y]]/(f,g)$ is a finite dimensional vector space over k.

2. a) Given $P_1,P_2\in\mathbb{P}^2$, $P_1\neq P_2$, find the number of lines $\mathcal{L}$ with $P_i\in\mathcal{L}$.

b) Given $P_1,\dots,P_5\in\mathbb{P}^2$, find the number of curves $\mathcal{C}$ with $\deg\mathcal{C}=2$ such that $P_i\in\mathcal{C}$ for all i.

c) Given $P_1,\dots,P_9\in\mathbb{P}^2$, find the number of curves $\mathcal{C}$ with $\deg\mathcal{C}=3$ such that $P_i\in\mathcal{C}$ for all i.

3. Let $\mathcal{C}$ be a smooth, projective curve in $\mathbb{CP}^2$. Prove:

a) If $\deg\mathcal{C}=1$ then $\mathcal{C}\simeq\mathbb{CP}^1$ ($\overset{top}{\sim} S^2$).

b) If $\deg\mathcal{C}=2$ then $\mathcal{C}\simeq\mathbb{CP}^1$.

4. a) Let $\mathcal{C}$ be smooth, $\deg\mathcal{C}=2$ and fix $P\notin\mathcal{C}$. Find the number of tangent lines $\mathcal{L}$ of $\mathcal{C}$ such that $P\in\mathcal{L}$.

b) Let $\mathcal{C}$ be smooth, $\deg \mathcal{C} = 2$ and fix $P \in \mathcal{C}$. Find the number of tangent lines $\mathcal{L}$ of $\mathcal{C}$ such that $P \in \mathcal{L}$.

c) Solve the same questions for $\deg \mathcal{C} = 3$.

5. If $\deg \mathcal{C} = c$, $\deg \mathcal{D} = d$, $\mathcal{C}$ smooth, then $\mathcal{C} \cap \mathcal{D} = \{P_1, \ldots, P_{cd}\}$. Is it true that for any fixed choice of points $P_1, \ldots, P_{cd}$ on $\mathcal{C}$ there exists a $\mathcal{D}$ of degree d such that $\mathcal{C} \cap \mathcal{D} = \{P_1, \ldots, P_{cd}\}$.

Case 1: $c = 1$, d points on a line, does there exist an equation of degree d with given d roots?

Case 2: $c = 2$?

Case 3: $c = 3$?

Day 2

1. Let $\mathcal{C} \subset \mathbb{P}^2$ be a smooth degree 3 curve, $O, O' \in \mathcal{C}$ and $\oplus, \oplus'$ the corresponding group operations. Give an isomorphism

$$\Phi : (\mathcal{C}, O, \oplus) \longrightarrow (\mathcal{C}, O', \oplus').$$

2. Show that a compact, positive dimensional Lie group $\mathcal{G}$ (i.e. a smooth manifold with a group structure where multiplication and taking inverse are smooth) has Euler characteristic 0.

3. Let $\mathcal{Q} \subset \mathbb{P}^2$ be a smooth conic, and fix the points $0, 1, \infty \in \mathcal{Q}$. Define the following operations:

$$P = \mathcal{L}_{\infty,\infty} \cap \mathcal{L}_{X,Y} = P, \quad X \oplus Y = \mathcal{L}_{0,P} \cap \mathcal{Q} \qquad (X, Y \in \mathcal{Q} \setminus \{\infty\}$$
$$P = \mathcal{L}_{0,\infty} \cap \mathcal{L}_{X,Y}, \quad X \odot Y = \mathcal{L}_{1,P} \cap \mathcal{Q} \qquad (X, Y \in \mathcal{Q} \setminus \{0, \infty\})$$

Show that these define groups on $\mathcal{Q} \setminus \{\infty\}$ and $\mathcal{Q} \setminus \{0, \infty\}$ respectively. What are these groups?

4. Let $\mathcal{C} \subset \mathbb{P}^2$ be smooth, $\deg \mathcal{C} = 3$, $I_1 \neq I_2$ inflection points and $\mathcal{L}_{I_1 I_2} \cap \mathcal{C} = \{I_1, I_2, X\}$ (with multiplicity). Show that X is also an inflection point.

5. What is the problem with the contradicting proofs at the end of the lecture?

Day 3

1. Confirm that $f_t = xy + tz^2$ and $g_t = x^3 + y^3 + xyz + tz^3$ both define smooth projective curves for all $t \neq 0$ and sufficiently small $t \neq 0$ respectively.

2. Find the singular points and their respective Milnor numbers of the projective plane curve $f = (x^2 + y^2)^3 + (x^3 + z^3)^2$.

3. Let $a \geq 2$ be an integer, and $d = a^2 + 1$. Verify that

$$\mathcal{S} = \left\langle a^2 - a, a^2, a^3 + 2a + 1 \right\rangle$$

defines a semigroup satisfying the semigroup distribution property with d as the degree.

Invariants of knots: polynomials and homologies

By András Stipsicz

(Notes by Viktória Földvári)

5.1 Knot invariants

5.1.1 Introduction

Definition 1. A *knot* K is a $\mathcal{C}^\infty$ function $K : S^1 \to S^3 (= \mathbb{R}^3 \cup \{\infty\})$, where $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$. K has three $\mathcal{C}^\infty$ coordinate functions $(x(\vartheta), y(\vartheta), z(\vartheta))$ giving embeddings, that is, K is an embedding of the circle into the 3-sphere/ S^1 into S^3.

Definition 2. Two knots K_0 and K_1 are *isotopic* if there is a smooth map $K : S^1 \times [0, 1] \to \mathbb{R}^3 \times [0, 1]$, such that

- $K_t = K |_{S^1 \times \{t\}}$ is a knot,
- for $t = 0$ and 1, K_0 and K_1 are the given knots.

This means that K_0 can be moved into K_1 without cutting.

The main problem of knot theory is to distinguish knots from each other. There are several ways to study them: the most topological one is to view knots as subspaces of $\mathbb{R}^3$, while a rather combinatorial idea is to consider their projection to a plane. For a generic choice of this plane, we can assume that the projection of the knot has at most double points. Moreover, it is an immersion with finitely many double points. At the double points, we illustrate the strand passing under as an interrupted curve segment, see Figure 5.1. The resulting diagram D is called the *knot diagram* of K, and the neighborhood of a double point a *crossing*. It is obvious that the knot diagram determines the knot up to isotopy.

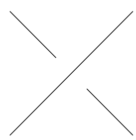


Figure 5.1: Crossing in a diagram

The trivial knot is usually called the unknot and is denoted by U. Figure 5.2 shows two diagrams of U.

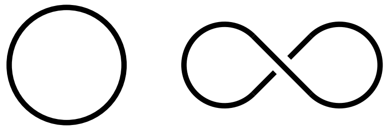


Figure 5.2: Two diagrams of the unknot

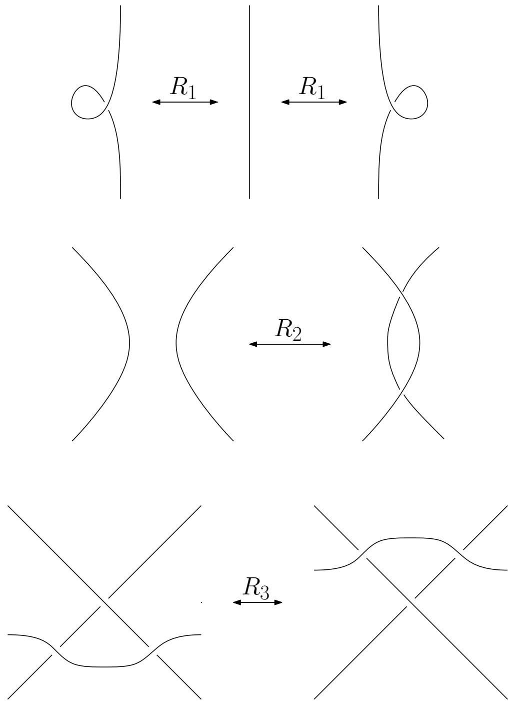


Figure 5.3: Reidemeister moves

Definition 3. An *l-component link* is the disjoint union of l knots.

There are three important local modifications of a knot diagram, called the Reidemeister moves, shown on Figure 5.3:

- R_1: Twisting or untwisting a strand,
- R_2: Moving a loop over another strand or removing it from that,
- R_3: Sliding a string over or under a crossing

The other parts of the diagram stay unchanged. It is easy to see that the Reidemeister moves preserve the knot up to isotopy, but also more is true:

Theorem 4 (Reidemeister). *Two knot diagrams D_1 and D_2 represent equivalent knots if and only if they can be transformed to each other by a finite sequence of Reidemeister moves and planar isotopies.*

To classify knots, is very useful to introduce quantities that stay unchanged under isotopy. These are called *knot invariants.* In this series of lectures we observe how knot theory and knot invariants has evolved in the past years. Our main goal is to measure the complexity of knots. To this, we focus on the following questions:

1. How to identify the unknot $\mathcal{U}$? Can we give an invariant that detects $\mathcal{U}$? (We are going to see such an example.)

2. If a knot is not the trivial one, how far is it from being the unknot? To determine this, we introduce three numbers:

   (a) The unknotting number $u(K)$
   $u(K)$ is the minimal number of crossing changes needed to transform K to U. It can be shown as an exercise, that $u(K)$ always exists.

(b) The crossing number $cr(K)$
$cr(K)$ is the minimal number of crossings K has in any diagram. The crossing number is always a non-negative integer, and equals 0 if and only if K is the unknot.

Remark 5. There is no non-trivial knot with 1 or 2 crossings.

(c) The genus (Seifert genus) $g(K)$
According to Seifert's theorem, every knot bounds a surface in $\mathbb{R}^3$. What is more, there is always a diagram for K for which this surface is orientable. We call an embedded, oriented, connected surface Σ, such that $\delta\Sigma = K$ a *Seifert surface* of K.
Let $g(\Sigma)$ denote the genus of the surface Σ. The *Seifert genus* of a knot is $g(K) = \min\{g(\Sigma)|\Sigma$ is a Seifert surface of $K\}$.

These provide upper bounds on the complexity of knots. However, to obtain more precise results, for example to know the minimal values of the above numbers, we need lower estimates too. Now we show some ways to get lower bounds.

5.1.2 Three-colorings

Definition 6. A diagram D is *three-colorable* if we can associate a color out of {red, white, green} to every arc of D such that

- we use at least two colors,
- at each crossing either only one or all the three colors are used to color the meeting arcs.

Example 7. The following diagrams of the unknot and the figure-8 knot are not three-colorable, while the ones for the left-handed and the right-handed trefoil are.

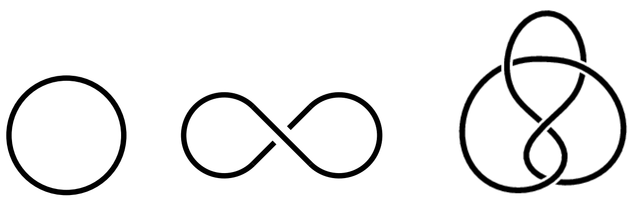


Figure 5.4: Not three-colorable diagrams of the unknot and the figure-8 knot

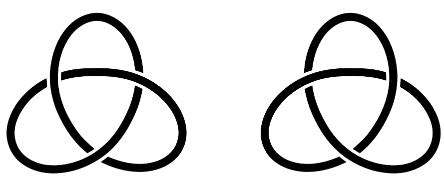


Figure 5.5: Three-colorable diagrams of the left-handed and the right-handed trefoil knot

It is obvious that this property depends on the chosen diagram of the knot. But how do we know in case of a not three-colorable diagram if there exists another choice of the projection that is three-colorable?

Theorem 8. *If D_1 and D_2 are two diagrams corresponding to the same knot, then D_1 is three-colorable if and only if D_2 is three-colorable.*

Proof. Isotopies obviously do not change this property of a diagram. Therefore it is enough to check that nor do Reidemeister moves, see Figure 5.6. □

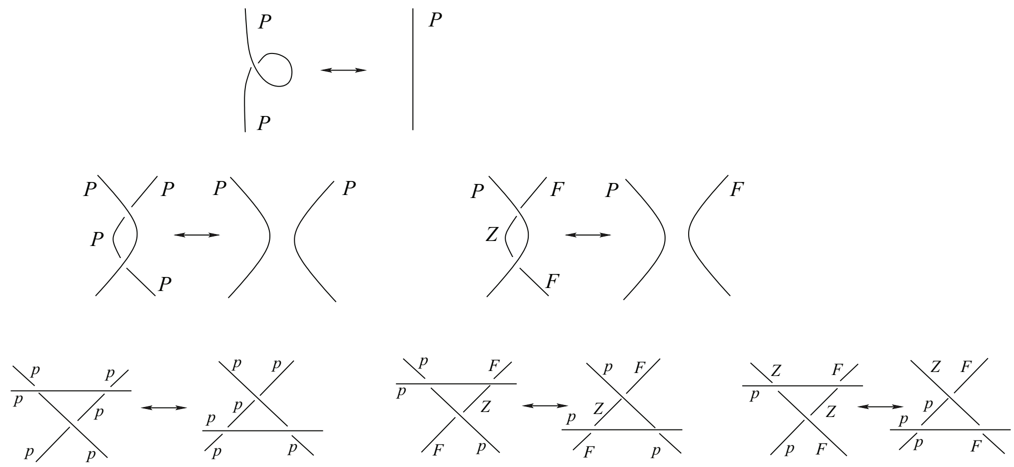


Figure 5.6: Invariance of three-colorability under Reidemeister moves

As a consequence, we see that there are at least two different knots: the unknot is not three-colorable, while the trefoil knot is.

We can improve this idea by using more colors, as follows:

Definition 9. Let q be an odd prime. A diagram D is *q-colorable* if the arcs of D can be colored by $\{0, ..., q-1\}$ such that

- we use at least two colors,
- at each crossing colored with a, b and c, $a + c \equiv 2b \pmod{q}$.

Example 10. The trefoil knot is not 5-colorable, but the figure-8 knot is.

Example 11. For p and q odd primes the diagram with p twists in Figure 5.7 admits a q-coloring if and only if $p = q$.

We start coloring the arcs with colors a and b according to the figure. Then, at the first crossing we have to use color $2b-a$ to satisfy $a+c \equiv 2b \pmod{q}$. At the next crossing we can only choose color $3b - 2a$, etc. After reaching the last crossing, we get that the first colored arcs have to be of the colors $pb - (p-1)a$ and $(p+1)b - p$. These give us two equations:

$$pb - (p-1)a \equiv a \pmod{q}$$

$$(p+1)b - p \equiv b \pmod{q}.$$

We get $p(b-a) \equiv 0 \pmod{q}$, that is, $q|p(b-a)$. This means that either $q|b-a$, meaning that all the arcs were of the same color, or $q|p$, meaning that $p = q$.

We can generalize the idea of q-colorings by allowing to use only one color. Let $\mathcal{C}(q, K)$ denote the set of all generalized q-colorings.

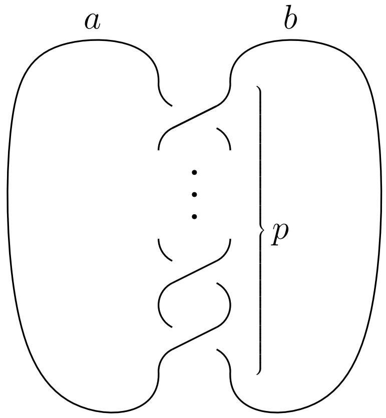


Figure 5.7: This diagram admits a q-coloring if and only if $p = q$.

Theorem 12.

- $\mathcal{C}(q, K)$ *forms a vector space over the finite field* $\mathbb{F}_q = \{0, ..., q-1\}$.
- $\mathcal{C}(q, K)$ *is independent of the choice of* D*, so it is a knot invariant.*
- *There exists a* q*-coloring if and only if* $\dim \mathcal{C}(q, K) > 1$.
- $\dim \mathcal{C}(q, K_1 \# K_2) = \dim \mathcal{C}(q, K_1) + \dim \mathcal{C}(q, K_2) - 1$
- *For the trefoil knot* T *and the connected sum operation* $\#$*,*

$$nT = \underbrace{T\#T\#\cdots\#T}_{n}$$

has $\dim \mathcal{C}(3, nT) = n + 1$ *- different for every value of* n*. Therefore, the knots obtained this way are all different.*

Theorem 13. *For the unknotting number* $u(K) \geq \dim \mathcal{C}(3, K) - 1$.

The idea of the proof is that a crossing change can change $\mathcal{C}(3, K)$ by at most 1 dimension. As a corollary, we get that nT has $u(nT) \geq n$.

Now we introduce a more systematic knot invariant:

5.1.3 The Alexander polynomial

The Alexander polynomial is one of the first knot invariants. The definition presented here relies on the introduction of Kauffman states and a state sum formula.

An explicit expression for the Alexander polynomial can be given in terms of a diagram D for the oriented knot K, equipped with the following additional choice. Distinguish an edge in D by marking it with a point p. The diagram, together with this choice of edge, is called a *marked diagram* (D, p).

Let $Cr(D)$ denote the set of crossings in the diagram and let $Dom(D)$ denote the set of domains in the plane (i.e. the connected components of the complement of D) which do not contain the marking p on their boundary.

Proposition 14.

- *For a knot K the cardinality $|Cr(D)|$ is equal to $|Dom(D)|$.*
- *For a disconnected diagram of a split link L we have $|Cr(D)| \neq |Dom(D)|$.*

Definition 15. A *Kauffman state* is a map that associates to each crossing in $Cr(D)$ one of the four quadrants around that crossing, so that the induced map $\kappa\colon Cr(D) \to Dom(D)$ is a bijection. The set of Kauffman states in a decorated marked diagram (D, p) is denoted $Kauf(D)$.

When illustrating Kauffman states, we mark the quadrant associated to the crossing in the diagram, as shown in Figure 5.8.

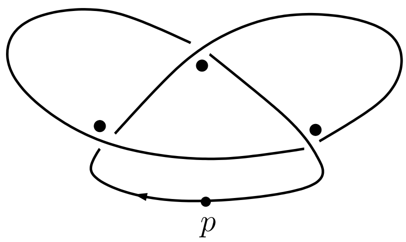


Figure 5.8: **A Kauffman state of a marked diagram of the left-handed trefoil knot.** The Kauffman state is indicated by a dot placed in the chosen quadrant at each crossing. The arrow indicates an orientation on the knot.

Remark 16. If K admits a diagram with a single Kauffman state then K is the unknot.

Remark 17. The diagrams of the trefoil knots shown in Figure5.5 have 3 Kauffman states.

Two quantities can be associated to a Kauffman state κ of D:

$$A(\kappa) = \sum_{c_i \in Cr(D)} A(\kappa(c_i)); \qquad M(\kappa) = \sum_{c_i \in Cr(D)} M(\kappa(c_i)),$$

where the local coefficients $A(\kappa(c_i)) \in \{0, \pm 1/2\}$ and $M(\kappa(c_i)) \in \{0, \pm 1\}$ for $\kappa \in Kauf(D)$ at a crossing $c_i \in Cr(D)$ are shown in Figure 5.9.

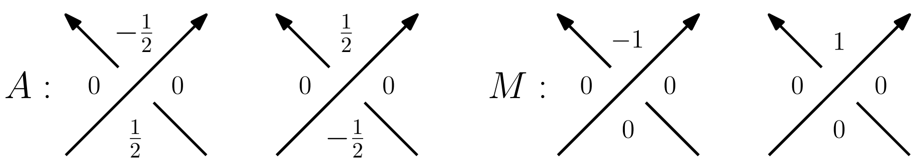


Figure 5.9: Local coefficients A and M at a crossing c_i.

The Alexander polynomial for K can be expressed in terms of Kauffman states, as follows:

Proposition 18. *Let K be an oriented knot, and consider a marked diagram (D, p) for K. Then, the Alexander polynomial is computed by the expression*

$$\Delta_K(t) = \sum_{\kappa \in Kauf(D)} (-1)^{M(\kappa)} \cdot t^{A(\kappa)} \in \mathbb{Z}[t, t^{-1}]. \tag{5.1}$$

Example 19.

$$\Delta_{\text{Unknot}} = 1,$$
$$\Delta_{\text{RH trefoil}} = t - 1 + t^{-1},$$
$$\Delta_{\text{Figure}-8} = -t + 3 - t^{-1}.$$

Theorem 20. *The Alexander polynomial is a knot invariant (and is independent of the choice of a marked knot diagram).*

We will prove this using an alternate perspective on the Alexander polynomial. Note that the Alexander polynomial has a natural generalization for oriented links.

Consider the polynomial $A(D, p)$ associated to a marked knot diagram (D, p), defined by the expression from Equation (5.1), verify that the sum is invariant under Reidemeister moves and the placement of the marked point p, and verify that the resulting oriented knot invariant satisfies the following equation, called the skein relation:

Proposition 21. *Let L_+, L_- and L_0 be oriented links that differ in a single crossing according to Figure 5.10. Then, the Alexander polynomials of these three links are related by the* skein relation

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta_{L_0}(t). \tag{5.2}$$

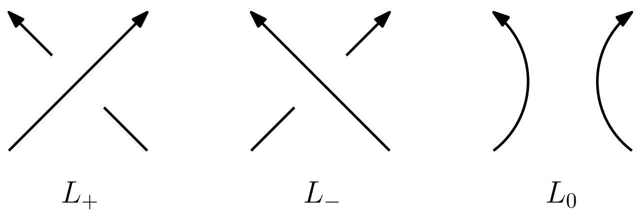


Figure 5.10: Local picture of the links in the skein relation

Proof. The diagram L_+ (and similarly of L_-) admits two types of Kauffman states, depending on the behaviour of the state at the crossing in the skein relation.
If the state is in the West or East quadrant, then the two states appear with the same contribution in $A(L_+, p)$ and $A(L_-, p)$, hence in the formula they cancel.
If the state is in the North or South quadrant, then the state gives rise to a unique state in L_0 as well, and indeed all Kauffman states of L_0 arise in this way.
Computing the local contributions, the formula follows. □

The skein relation is a very useful tool to compute the Alexander polynomial.

Now we turn to the proof of the invariance:

Proposition 22. *Let (D, p) be a marked diagram for an oriented knot K. The function $A(D, p)$ is independent of the choice of diagram for K, giving an oriented knot invariant Δ_K that satisfies the skein relation and the normalization $\Delta_{\text{Unknot}}(t) = 1$.*

Proof. First we show that the polynomial is invariant under those Reidemeister moves which do not contain the marked point p.

For the first Reidemeister move R_1 creating a new crossing, it is easy to see that the Kauffman states before and after the move are in bijection. The newly created crossing is the only crossing in the newly created domain, and the local contributions here are trivial

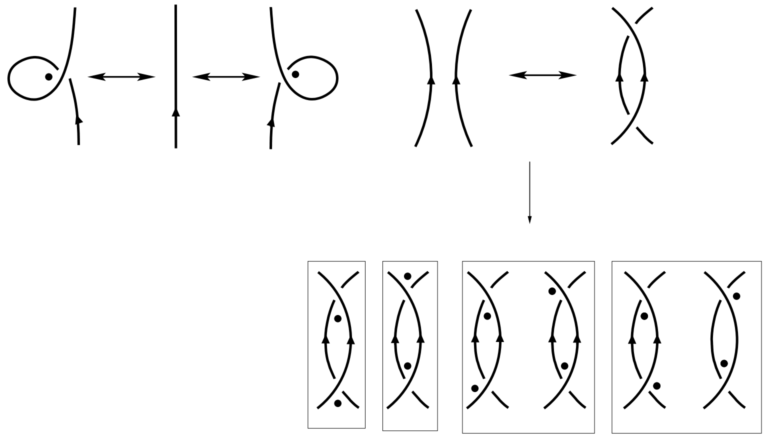


Figure 5.11: Kauffman states and the first and second Reidemeister moves. The contribution change under the first Reidemeister move is clearly zero. In the second Reidemeister move the groups in the boxes provide vanishing contribution.

($M = 0$ and $A = 0$); the remaining crossings correspond, along with their local contributions, to the Kauffman states before R_1.

For invariance under the second Reidemeister move R_2, there are two cases, corresponding to the relative orientations of the two strands. The two verifications are similar, and we discuss only one. The projection having the extra two crossings has two types of Kauffman states, which we call *symmetric* and *asymmetric* ones, depending on the local structure at the two new crossings. The symmetric Kauffman states are in one-to-one correspondence with the Kauffman states before R_2 (so that all the local contributions coincide), while the asymmetric ones come in pairs as instructed by Figure 5.11.

Their contributions to $A(D, p)$ cancel within these pairs: their A-values coincide and their M values differ by one.

In studying the invariance under the third Reidemeister move R_3 there are a number of cases for the possible orientations of the arcs. We will discuss only one such choice (and leave the analysis of the rest of the cases to the reader).

First notice that in the disk around the move there are seven domains and three crossings (and recall that the marking p is not in the disk). Therefore four of the domains get their markings (which can be the p, or a marking at a crossing) from outside the disk. We distinguish three cases, depending on whether these four domains are consecutive, three are next to each other and one is separated, or are grouped in two groups having two adjacent domains each. (See the domains containing empty circles in the projections of Figure 5.12.) In the first case (with four consecutive marked regions) the markings in the disk are unique, and their contributions coincide before and after the move.
In the second case (three consecutive domains and one separated) there is either one Kauffman state or there are three; the contributions again coincide.
Finally in the third case (two pairs of consecutive domains) we have two Kauffman states before and after the move, with coinciding contributions. (See Figure 5.12 for some orientation of the link, other orientations differ only in the actual values of the contributions.)

Next we show that the polynomial is invariant under moving the marking p under (or over) a crossing. To this end, we note that we can consider the knot diagram (i.e. the projection

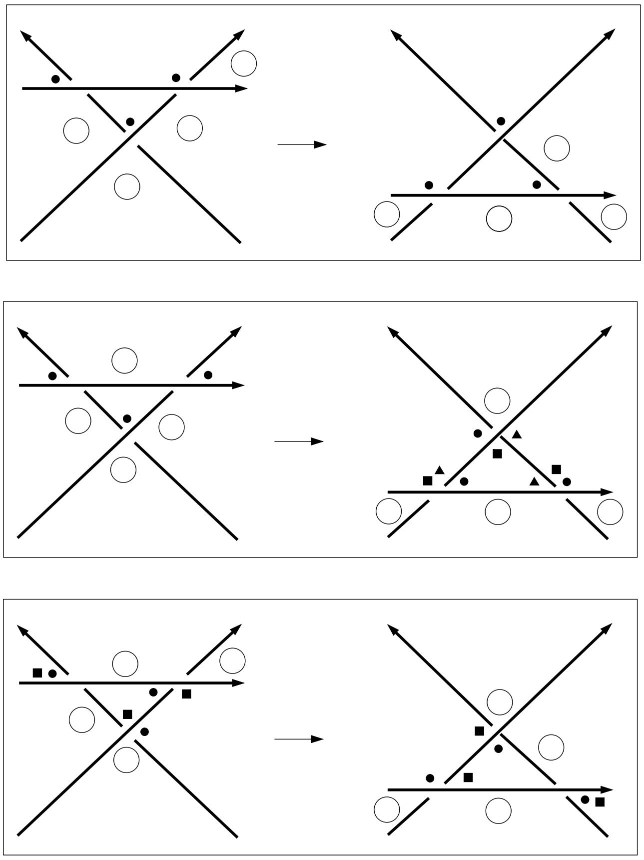


Figure 5.12: Kauffman states and the third Reidemeister move.

to a plane) as a map to the sphere S^2 in S^3 rather to plane $\mathbb{R}^2$ in $\mathbb{R}^3$, still picturing the part of the projection in the 'finite' part $\mathbb{R}^2 \subset S^2$. Although moving a strand across the point at infinity seems like an additional move, it is not hard to see that it can be given as a composition of Reidemeister moves, see Figure 5.13(a).

Choose a projection to S^2 with the property that c is mapped into the point at infinity. Perturbing the strand without p into the two possible directions, and applying our previous observation, the claim easily follows, see Figure 5.13(b).

For a knot K the above principle of moving p across a crossing shows that the value of the invariant is independent of the chosen arc distinguished by p. For an oriented link one need additional work to move the marking from one component to the other - using the skein

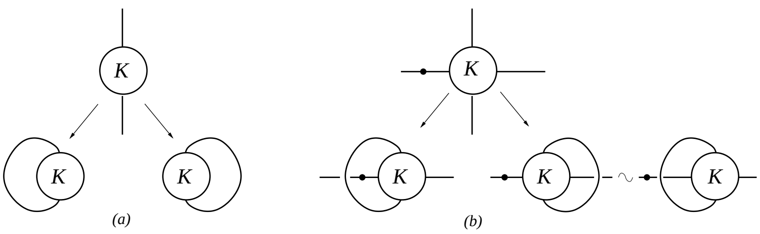


Figure 5.13: **Moving a strand across the point at infinity.** In (a) we show the two possibilities when perturbing the arc passing through the point at infinity. These diagrams can be transformed into each other by Reidemeister moves. In (b) we show how by moving the crossing through the point at infinity, we can move the marking through a crossing.

relation the independence follows by induction on the number of component:

For knots this has been already established.
An n-component link can be easy put into a skein triple with the two other knots having $(n-1)$-components, for which the invariance follows by induction, and since their polynomials determine the value for the n-component link, the moving of the distinguished point follows.

Finally, we need to show that the polynomial is unchanged under *any* Reidemeister moves. Indeed, if the marking is in the disk in which the Reidemeister move is to be performed, then first move p away, perform the Reidemeister move and then move p back. With this last assertion the proof of independence is complete.

The claim about the polynomial of the unknot U follows easily:
take the diagram D of U which has no crossings. The single Kauffman state of this decorated projection (the bijection between the two empty sets $Cr(D)$ and $Dom(D)$) has A- and M-values equal to zero, giving $A(D,p) = (-1)^0 \cdot t^0 = 1$. □

Next we summarize some further facts about the Alexander polynomial.

Proposition 23.

- $\Delta_K(t) = \Delta_K(t^{-1})$, *so* $\Delta_K(t) = a_0 + \sum_{i=0}^{d} a_i(t^i + t^{-i})$ *where* $a_d \neq 0$ *and* $d =$*deg*Δ.
- *deg*$(\Delta) \leq g(K)$, *i.e. the Alexander polynomial gives the desired lower bound.*
- *If K is alternating, meaning that in its diagram undercrossings and overcrossings alternate one after the other, then degΔ=$g(K)$.*
- *If K admits a q-coloring for a prime q, then* $q||\Delta_K(-1)| =$*det*(K).

5.2 The Jones polynomial

The Alexander polynomial used to be the most powerful knot invariant until around 1985, when Jones introduced a new one: his polynomial was able to distinguish the right-handed trefoil knot from the left-handed one.

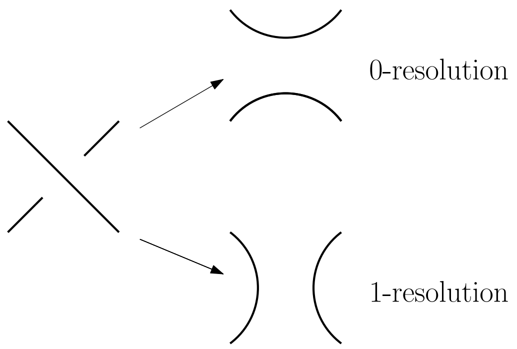


Figure 5.14: The two ways of resolving a crossing

The first idea of the definition is to turn the diagram of the knot K into a diagram of (maybe more) unknots by *resolving* its crossing. A resolution means that we eliminate the crossing by connecting the participating arcs in a different way. We have two choices to do this, shown in Figure 5.14. These two types are called *0-resolution* and *1-resolution.* If

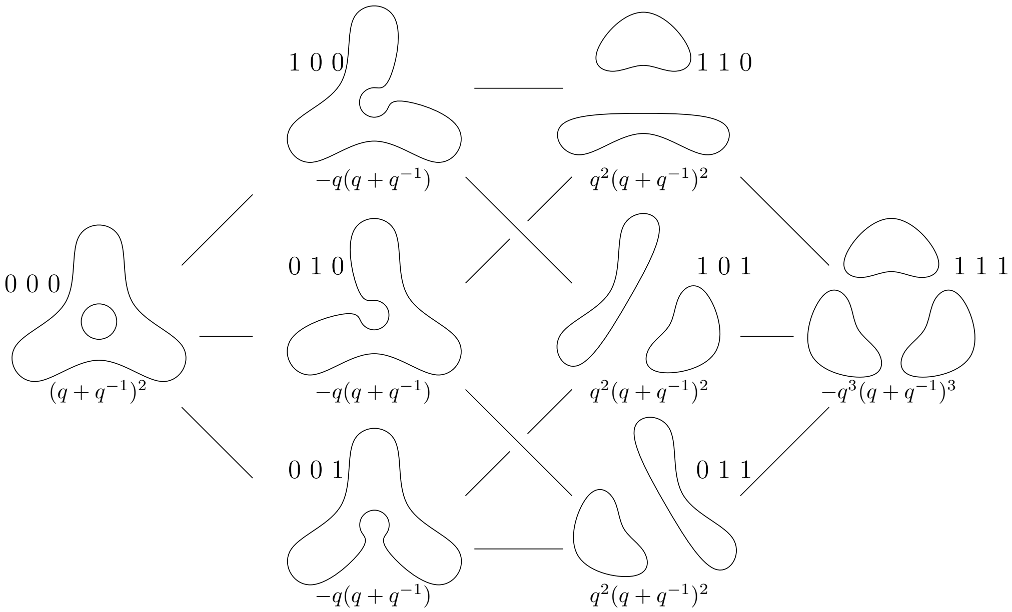


Figure 5.15: The cube of resolutions for the right-handed trefoil knot

the diagram of K had n crossings, then there are 2^n resolutions. Take all of them, and think of every possible stage of reaching the full resolution of the knot diagram (i.e. the one containing only unknot components), as vertices. Organizing them according to the number of 1-resolutions we made to get that stage, these can be arranged to form a cube of dimension n, see Figure 5.15.

Now consider the cube of resolutions, and at each vertex α, let $r(\alpha)$ denote the number of 1-resolutions done, and $k(\alpha)$ the number of circle components in resolution α. We associate a polynomial of a formal variable q to every resolution to define the *bracket* of a knot diagram D:

$$\langle D \rangle = \sum_{\text{resolutions}} (-q)^r(\alpha) \cdot (q+q^{-1})^k(\alpha)$$

This is a Laurent polynomial in $\mathbb{Z}[q, q^{-1}]$ with the following properties:

- For the empty diagram, $\langle \emptyset \rangle = 1$,
- for the diagram $\bigcirc$ of the unknot, $\left\langle \bigcirc \right\rangle = q + q^{-1}$,
- for a two-component diagram with an unknot, $\left\langle D \cup \bigcirc \right\rangle = (q+q^{-1}) \cdot \langle D \rangle$,
- $\left\langle \times \right\rangle = \left\langle \asymp \right\rangle - q \cdot \left\langle)(\right\rangle$

Despite of its nice properties, the bracket of a diagram is not a knot invariant, which is easy to see by checking its behaviour under Reidemeister moves:

$$\left\langle \overset{\circ}{\times} \right\rangle = \left\langle \overset{\circ}{\asymp} \right\rangle - q \cdot \left\langle \Omega \right\rangle = (q+q^{-1}) \cdot \langle \frown \rangle - q \cdot \langle \frown \rangle = q^{-1} \cdot \langle \frown \rangle$$

$$\left\langle \overset{\circ}{\times} \right\rangle = \left\langle \Omega \right\rangle - q \cdot \left\langle \overset{\circ}{\asymp} \right\rangle = \langle \frown \rangle - q(q+q^{-1}) \cdot \langle \frown \rangle = -q^2 \cdot \langle \frown \rangle$$

$$\left\langle \overset{\times}{\times} \right\rangle = \left\langle \overset{\asymp}{\times} \right\rangle - q \cdot \left\langle \overset{)(}{\times} \right\rangle = -q^2 \cdot \left\langle)(\right\rangle - q \left(\left\langle \asymp \right\rangle - q \cdot \left\langle \asymp \right\rangle \right) = -q \cdot \left\langle)(\right\rangle$$

$$\langle \ \rangle = \langle \ \rangle - q \cdot \langle \ \rangle = \langle \ \rangle - q \left(-q(-q^{-1}) \cdot \langle \ \rangle \right) = \langle \ \rangle$$

Definition 24. Suppose the knot K has D as a diagram. Orient D, and let n_+ denote the number of crossings of type (positive crossing) and n_- the number of crossings of type (negative crossing).

$$\tilde{V}_K(q) = (-1)^{n_-} \cdot q^{n_+ - 2n_-} \cdot \langle D \rangle$$

is the *unnormalized Jones polynomial* of K.

The unnormalized Jones polynomial is a Laurent polynomial in $\mathbb{Z}[q, q^{-1}]$.

Theorem 25. *$\tilde{V}_K(q)$ is a knot invariant.*

Proof. We verify that the polynomial does not change under Reidemeister moves. Checking the invariance under R_3 is trivial. Under R_2, both n_+ and n_- increases by 1... Under R_1, either n_+ or n_- increases by 1, and the proof goes the same way as for R_2. □

Definition 26. $\tilde{V}_K(q)$ is divisible by $q + q^{-1}$, since there is always at least one circle component in every stage. Therefore, we can consider

$$V_K(q) = \frac{1}{q + q^{-1}} \cdot \tilde{V}_K(q)$$

, the *normalized Jones polynomial* of K.

Theorem 27. *$V_K(q)$ is a knot invariant.*

Example 28.

$$\begin{aligned} V_{\text{Unknot}}(q) &= 1, \\ V_{\text{RH trefoil}}(q) &= q^2 + q^6 - q^8, \\ V_{\text{LH trefoil}}(q) &= q^{-2} + q^{-6} - q^{-8}, \\ V_{\text{Figure}-8}(q) &= q^{-4} - q^{-2} + 1 - q^2 + q^4. \end{aligned} \tag{5.3}$$

Remark 29. Now we see that the Jones polynomial is different from the Alexander polynomial: while the latter one never distinguishes a knot from its mirror image, the Jones polynomials of the right-handed and the left-handed trefoil are not equal.

Theorem 30 (Skein relation). *For oriented links L_-, L_+ and L_0 which differ only in a crossing according to Figure 5.10, the* skein relation *holds:*

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot \Delta_{L_0}(t).$$

Let us use the following notations:
$M(V_K(q))$ =the maximal degree of q in $V_K(q)$,
$m(V_K(q))$ =the minimal degree of q in $V_K(q)$,
$B(V_K(q)) = M(V_K(q)) - m(V_K(q))$, the span of the polynomial.

Theorem 31. *For an n-crossing diagram of K, $n \geq \frac{1}{2}B(V_K(q))$. If the diagram is alternating and has no untwistable crossing , then $n = \frac{1}{2}B(V_K(q))$.*

So the Jones polynomial bounds, and sometimes determines the minimal crossing number. However, it is still open, whether the Jones polynomial detects the unknot:

Conjecture 32. *If $V_K(q) = 1$, then K is the unknot.*

Remark 33. The Alexander polynomial does not detect the unknot. While $\Delta_U(t) = 1$, we can construct other, non-trivial knots that have Alexander polynomial 1:

Consider the pretzel knot $P(a, b, c)$ of Figure 5.16. For $a = 2a_1 + 1$, $b = 2b_1 + 1$ and $c = 2c_1+1$ this knot P is non-trivial with Alexander polynomial $\Delta_P(t) = Bt+(1-2B)+Bt^{-1}$, where $B = b_1b_2 + b_1b_3 + b_2b_3 + b_1 + b_2 + b_3 + 1$.

For $b_1 = -2$, $b_2 = 2$ and $b_3 = 3$, $B = 0$ therefore $\Delta_P(t) = 1$, when P is still non-trivial.

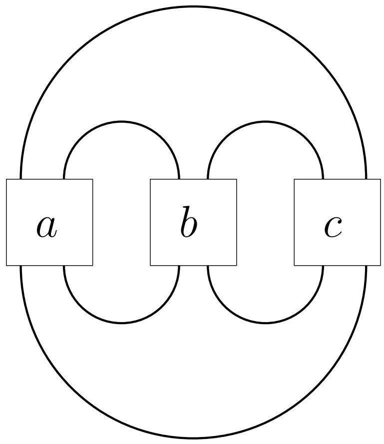


Figure 5.16: The pretzel knot $P(a, b, c)$. The letters in boxes indicate the number of half twists.

Next, we mention some connections of the Jones polynomial to the invariants introduced in Lecture 1:

Remark 34. Connection to three-colorings:
$\dim\mathcal{C}(3, K) = 3|V_K^2(e^{\frac{2\pi i}{6}})|$.

Remark 35. The Alexander polynomial and the Jones polynomial are also related to each other: There is a 2-variable Laurent polynomial $P(\alpha, z) \in \mathbb{Z}[\alpha, \alpha^{-1}, z, z^{-1}]$, called the HOMFLY-PT polynomial, satisfying $\alpha P_{L_+} - \alpha^{-1}P_{L_-} = zP_{L_0}$ for links L_-, L_+ and L_0 which differ only in a crossing according to Figure 5.10.

The HOMFLY-PT polynomial at $\alpha = 1$ and $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ gives the Alexander polynomial, while at $\alpha = q^{-2}$ and $z = q - q^{-1}$ it gives the Jones polynomial. That is,

$$\Delta_K(t) = P(\alpha = 1, z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}),$$

$$V_K(q) = P(\alpha = q^{-2}, z = q - q^{-1}).$$

5.3 Graded vector spaces

Let $\mathbb{F}_2$ denote the field of two elements.

Definition 36. A *$\mathbb{Z}$-graded vector space* is a decomposition of a vector space V as $V = \bigoplus_{k\in\mathbb{Z}} V_k$ with $V_k \leq V$. The elements of V_k are called homogeneous elements of degree k.

Graded vector spaces have *graded dimension*: gr-dim$V(q) = \sum\limits_k \dim V_k \cdot q^k \in \mathbb{Z}[q, q^{-1}]$, and $\dim V =$gr-dim$V(1)$.

Remark 37. We can generalize this definition to get *bigraded vector spaces*: $V = \bigoplus\limits_{(a,b)\in\mathbb{Z}\times\mathbb{Z}} V_{a,b}$. Here the graded dimension is a polynomial of two variables.

Example 38. Consider $V = \mathbb{F}_2^2 = (\mathbb{F}_2)_{-1} \oplus (\mathbb{F}_2)_1$. This bigraded vector field has four elements: $\{0,\ v_+,\ v_-,\ v_+ + v_-\}$. Element 0 has any degree, $v_+ = (\mathbb{F}_2)_1$ is of degree 1, $v_- = (\mathbb{F}_2)_{-1}$ is of degree -1, and $v_+ + v_-$ has no degree because it is non-homogeneous. Here, gr-dim$V = q + q^{-1}$.

We can consider the *shifted grading* of a graded vector space V: let $V\{l\}$ be the graded vector space V with its grading shifted by l, that is, $\mathrm{gr}(V\{l\})_k = V_{k-l}$.
For this, gr-dim$V\{l\} = q^l\cdot$gr-dim$V$ holds.

Definition 39. Let V be a graded vector space and $\partial : V \to V$ endomorphism with $\partial(V_k) \subset V_{k-1}$. (V, ∂) is a *chain complex* if $\partial \circ \partial = \partial^2 = 0$.
Then we call the elements of Ker ∂ *cycles*, the elements of Im ∂ *boundaries*.

Definition 40. The *homology* of a chain complex (V, ∂) is a graded vector space

$$H(V, \partial) = {}^{\textstyle \mathrm{Ker}\,\partial}\Big/_{\textstyle \mathrm{Im}\,\partial}\ .$$

Remark 41. If V is bigraded, then $\partial(V_{a,b}) \subset V_{a-1,b}$ and $H(V, \partial)$ is also bigraded.

Definition 42. $f : (V_1, \partial_1) \to (V_2, \partial_2)$ is a *chain map* if $f \circ \partial_1 = \partial_2 \circ f$.
$f, g : (V_1, \partial_1) \to (V_2, \partial_2)$ are *chain homotopic* maps if there exists $h : V_1 \to V_2$ such that $f - g = \partial_2 \circ h + h \circ \partial_1$.

Proposition 43.

- *Every chain map $f : (V_1, \partial_1) \to (V_2, \partial_2)$ induces a map $f_* : H(V_1, \partial_1) \to H(V_2, \partial_2)$ on the homologies.*
- *If f and g are chain homotopic, then $f_* = g_*$.*

Proof.

- If $[a] \in H(V_1)$, then $[f(a)] = [f(a + \partial b)] = [f(a) + \partial f(b)]$, therefore f_* is well-defined.
- For a cycle a $\in$Ker∂_1, consider the difference $f(a) - g(a) = \partial_2 \circ h(a) + h \circ \partial_1(a) = \partial_2 \circ h(a)$ is a boundary element, thus $[f(a)] = [g(a)]$ meaning that $f_* = g_*$.

□

Our goal is to associate to every knot a graded vector space so that the homology is a knot invariant.

To this, take a knot diagram D and its cube of resolutions. First, we associate a graded vector space to each resolution α:

Definition 44. For $V = \langle v_1, ..., v_m\rangle$ and $W = \langle w_1, ...w_n\rangle$ graded vector spaces, let $V \otimes W = \langle v_1 \oplus w_1, v_1 \oplus w_2, ..., v_1 \oplus w_n, v_2 \oplus w_1, ...\rangle$.

Let $V = (\mathbb{F}_2)_{-1} \oplus (\mathbb{F}_2)_1 = \langle v_-, v_+ \rangle$.

Example 45. The vector field

$$V \otimes V = V^{\otimes 2} = \langle v_+^1 \otimes v_+^2, v_+^1 \otimes v_-^2, v_-^1 \otimes v_+^2, v_-^1 \otimes v_-^2 \rangle = (\mathbb{F}_2)_2 \oplus (\mathbb{F}_2)_0^2 \oplus (\mathbb{F}_2)_{-2}$$

has 16 elements. (Here, the upper indices help seeing which generator belongs to which vector field.)

Let k denote the number of circles and r the number of 1-resolutions in vertex α. To α we associate $V^{\otimes k}\{r\}$, a graded vector space with graded dimension $q^r \cdot (q + q^{-1})^k$.

For all resolutions of r 1-resolutions, consider

$$[\![D]\!]^r = \sum_{|s|=r} V_s^{\otimes k}\{r\}.$$

Now

$$Ch(D) = \bigoplus_0^n [\![D]\!]^r$$

is a bigraded vector space.

We would like to have a chain complex. To this, we need a map $\partial : Ch(D) \longrightarrow Ch(D)$. We will define

$$\partial^r : [\![D]\!]^r \longrightarrow [\![D]\!]^{r+1} :$$

For a resolution s of r 1-resolutions, consider all the resolutions s' of $r+1$ 1-resolutions obtained from s by changing the resolution type of one crossing from 0 to 1, that is $\asymp \longmapsto)($.

The idea is that for a term $V_s^{\otimes k}\{r\}$ of $[\![D]\!]^r$, ∂^r assigns the sum of graded vector spaces $V_{s'}$ corresponding to the resolutions s'. We should do this for every term s and sum them to get $\mathrm{Im}(\partial^r) = \sum_s \sum_{s'} V_{s'}$. However, finding $V_{s'}$ is not obvious, since changing the type of a resolution may also change the number of components. This can happen in the following two ways:

1. *Merge* decreases the number of circles by 1, so $k(s') = k(s) - 1$.
2. *Split* increases the number of circles by 1, so $k(s') = k(s) + 1$. See Figure 5.17.

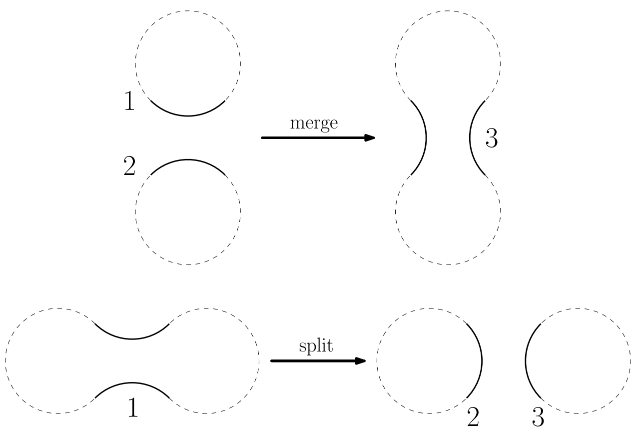


Figure 5.17: Merge and split

Note that both are only local changes, the other parts of the diagram stay the same. To obtain $V_{s'}$, we need to introduce operations:

1. To merge, we need to define a multiplication $m : V_1 \otimes V_2 \to V_3$.

$$v_+^1 \otimes v_+^2 \mapsto v_+^3,$$

$$v_+^1 \otimes v_-^2 \mapsto v_-^3,$$

$$v_-^1 \otimes v_+^2 \mapsto v_-^3,$$

$$v_-^1 \otimes v_-^2 \mapsto 0.$$

   This gives a ring structure on V, the truncated polynomial ring.

$$V = \mathbb{F}_2[x]\Big/(x^2) = \{\alpha + \beta x| \ \alpha, \beta \in \mathbb{F}_2\}.$$

2. To split, we define a co-multiplication $\Delta : V_1 \to V_2 \otimes V_3$, making V a Hopf algebra.

$$v_+^1 \mapsto v_+^2 \otimes v_-^3 + v_-^2 \times v_+^3,$$

$$v_-^1 \mapsto v_-^2 \otimes v_-^3.$$

Putting all together, V has a grading, a ring structure and a Hopf algebra structure, therefore V is a Frobenius algebra with 4 elements. Now we also have $\partial : Ch(D) \longrightarrow Ch(D)$.

Theorem 46. *$(Ch(D), \partial)$ is a bigraded chain complex, $\partial \circ \partial = 0$. On one grading ∂ is constant, on the other grading ∂ increases: $\partial(Ch(D)_{a,b}) \subset Ch(D)_{a,b+1}$.*

Theorem 47. *$Kh(D) = H_*(Ch(D), \partial) = \bigoplus\limits_{a,b} Kh^{a,b}(D)$ is a bigraded vector space, which is an invariant of the knot K.*

Theorem 48.

- *Let $P_K(t,q) = \sum\limits_{a,b \in \mathbb{Z}\times\mathbb{Z}} \dim Kh^{a,b}(D) \cdot t^a \cdot q^b \in \mathbb{Z}[t, t^{-1}, q, q^{-1}]$. This is a knot invariant.*
- *$P_K(-1,q) = \tilde{V}_K(q)$, so Kh categorifies the Jones polynomial.*

The following theorem shows that Khovanov homology detects the unknot:

Theorem 49 (Kronheimer–Mrovka).
$P_K(t,q) = t^0 q^{-1} + t^0 q^1 = q + q^{-1}$ if and only if K is the unknot.

The theorem of Kronheimer and Mrovka means that if for a knot K, $Kh(K) = Kh(U) = (\mathbb{F}_2)_1 \oplus (\mathbb{F}_2)_{-1}$, then K is isotopic to the unknot. Recall that the Alexander polynomial does not detect the unknot and that for the Jones polynomial this question is still open.

Problem session

(ANDRÁS STIPSICZ, VIKTÓRIA FÖLDVÁRI)

DAY 1

1. Show that $\pi_K = \{\text{3-colorings}\}$, as a vector space over $\mathbb{F}_3$ is a knot invariant.

2. Verify that the Alexander polynomial $\Delta_K(t)$ is invariant under Reidemeister moves R_1 and R_2.

3. Prove the skein rule. That is, verify for links $\overrightarrow{L_-}$, $\overrightarrow{L_+}$ and $\overrightarrow{L_0}$ which differ only in a crossing that

$$\Delta_{\overrightarrow{L_+}}(t) - \Delta_{\overrightarrow{L_-}}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot \Delta_{\overrightarrow{L_0}}(t).$$

DAY 2

1. Compute the Jones polynomial of the following:

a) negative Hopf link,

b) positive Hopf link

c) right-handed trefoil knot,

d) left-handed trefoil knot.

2. Show that $V_K(q) = V_{m(K)}(q^{-1})$.

Three Fields medalists of topology: Smale (immersions), Thom (cobordisms), Milnor (exotic spheres)

By András Szűcs

(Notes by Tamás Terpai)

6.1 Thom

We start by recalling the rotation/degree, which most of you know already. All four main topics of today's talk are natural extensions of the rotation number.

Belt trick: take a belt in a flat, vertical position, and add a certain number of full twists. Can one return the belt to the original state while only moving the top and the bottom of the belt by translations, never rotating them? How does this imply the hedgehog theorem (there is no nowhere vanishing tangent vector field on the sphere S^2)? How is it related to cobordisms? Stay tuned to find out!

6.1.1 Rotation number

Take a continuous nowhere vanishing vector field v along a circle (lying in the plane). Its rotation (or winding) number is defined by taking any sufficiently fine subdivision x_0, x_1, ..., $x_N = x_0$ of the circle and forming the sum

$$\sum_{j=1}^{N} \angle(v(x_{j-1}), v(x_j))$$

of the (signed!) angles of the successive $v(x_j)$. This sum is an integer multiple of 2π, and we say that v has rotation number k if the sum is $2k\pi$.

If we assume that the vector field v has constant length 1, then v can also be considered as a map from S^1 to S^1. **Fact:** the homotopy class of $v \in [S^1, S^1] \cong \mathbb{Z}$ is k.

Can one define a similar „rotation number" for maps $f : M^n \to N^n$ between higher dimensional manifolds? The given definition is not suited to such a generalization, but an equivalent definition is. Notice that the rotation number can be determined in a local way: observe a point p in the target S^1 and count the times v takes p as a value. Every time v goes through p in the positive (anti-clockwise) direction, add 1 to the counter, and every time v goes through p in the negative direction, subtract 1 from the counter. It is not hard to see that the result (assuming that v always visits p in one of those two ways) coincides with the rotation number of v.

This alternative definition generalizes immediately to the concept of *degree*: assume that M^n and N^n are smooth and oriented, and $f : M \to N$ is a smooth map. Take $p \in N$ a regular value of f (that is, at all f-preimages q of p the rank of the Jacobian is n, the maximal possible), assign signs to its f-preimages according to whether the Jacobian at that point has positive or negative determinant and add those ± 1's together; this is the degree $\deg f$ of the map f.

Does the statement about the rotation number uniquely determining the homotopy class of a map hold for the degree? It doesn't, however, the degree is still a well-defined map $[M, N] \to \mathbb{Z}$, and when $N = S^n$, we have

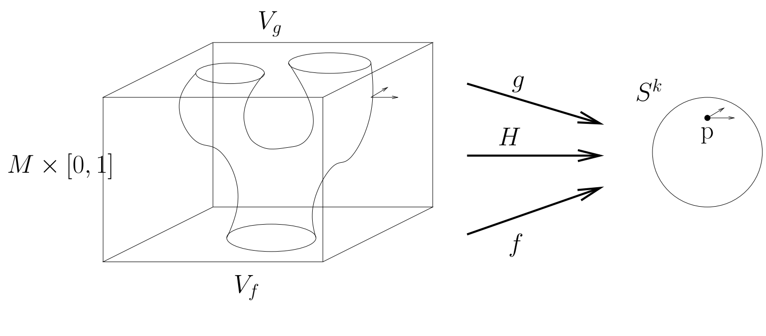
Theorem 1 (Hopf). *The degree* $\deg : [M^n, S^n] \to \mathbb{Z}$ *is a bijection.*

6.1.2 Pontryagin construction

Our next problem is: what can be done if M and N are not of the same dimension? Given a (smooth) manifold M^{n+k}, how many homotopy classes of maps $M^{n+k} \to S^k$ are there and given a (smooth) map $M^{n+k} \to S^k$, how can we determine which homotopy class does it represent? In the case when $n < 0$ there is clearly only a single homotopy class: we can pick a smooth representative $f : M \to S^k$, by Sard's lemma it is not surjective and leaves out a point $p \in S^k$, so f factors through a map from M to $S^k \setminus \{p\} \cong D^k$, which is null-homotopic.

In the case $n > 0$ we start out in a similar fashion: pick a smooth representative $f : M \to S^k$ and a regular value $p \in S^k$ (recall that it is defined by the Jacobian having maximal rank at all the preimages), and consider its preimage $V = f^{-1}(p)$. We thus get an analogue of the preimage points of the equidimensional case, what may be the analogue of the signs of those points? It's a *framing*: we pick a basis $b_1, \ldots, b_k$ in the tangent space T_pS^k and lift it to V as follows. Locally at each point $x \in V$ the tangent space T_xM splits as $T_xV \oplus \mathbb{R}^k$ – for example, if there is a metric on M, we can pick the $\mathbb{R}^k$ factor to be the tangent vectors orthogonal to TV – and the differential of f maps $\mathbb{R}^k$ isomorphically onto T_pS^k. We can therefore take the preimages of the vectors $b_1, \ldots, b_k$ under this restriction of the differential and obtain k tangent vectors $\tilde{b}_1(x), \ldots, \tilde{b}_k(x)$ that are linearly independent even in the quotient T_xM/T_xV (in the metric case, they are orthogonal to V).

Have we defined an invariant of the homotopy class of f; is it even well-defined? Not quite: if we pick another map $g : M \to S^k$ homotopic to f, then the entire construction can be repeated with a (smooth) homotopy $H : M \times [0,1] \to S^k$ that joins f and g and a value $p \in S^k$ that is regular for f, g and H as well. The preimage of p is a manifold with boundary $f^{-1}(p) \sqcup g^{-1}(p)$ and we also obtain k linearly independent normal vectors on it that extend the framings on the boundary. The result is a *framed cobordism* between the preimages $V_f = f^{-1}(p)$ and $V_g = g^{-1}(p)$.



We call two framed n-dimensional manifolds in M *framed cobordant* is there exists a framed cobordism between them.

It is easy to see that framed cobordism is an equivalence relation. We denote the set of equivalence classes of framed codimension k manifolds embedded into M by $Emb^{fr}(n, M)$; for $M = S^k$ we abbreviate it to $Emb^{fr}(n, k)$.

Theorem 2 (Pontryagin). *For any closed manifold M^{n+k} there is a bijection $Emb^{fr}(n, M) \to [M, S^k]$.*

In particular the Pontryagin construction establishes a bridge between algebraic topology and differential topology by identifying $Emb^{fr}(n, k)$ with $\pi_{n+k}(S^k)$. The homotopy groups of spheres are notoriously hard to compute in general, and the translation of the problem to calculation of cobordism groups allows, for example, the handling of cases $k = 0$, when $\pi_n(S_n) \cong \mathbb{Z}$ – recall the degree; $k = 1$, when $\pi_{n+1}(S^n)$ is $\mathbb{Z}$ for $n = 2$ and $\mathbb{Z}_2$ for $n > 2$; and $k = 2$, when $\pi_{n+2}(S^n) \cong \mathbb{Z}_2$ if $n \geq 2$ (homework: what happens when $n = 1$?). Rohlin used this construction to compute $\pi_{n+3}(S^n) \cong \mathbb{Z}_{24}$ when $n \geq 5$.

This approach belongs to the Russian school of manifold topology, which put heavy emphasis on geometry. In contrast, the French school prioritizes algebra, and in the next section we mention René Thom's work on the topic.

6.1.3 Thom construction

Can one use the Pontryagin construction in the other direction, to calculate cobordism groups from homotopy groups? It turns out that doing this can give the cobordism groups of n-manifolds:

Definition 3. The n-manifolds M_0 and M_1 are *cobordant*, if there is a *cobordism* W^{n+1} between them: a closed manifold with boundary $M_0 \sqcup M_1$ in the unoriented setting and $(-M_0) \sqcup M_1$ in the oriented setting (where $-M$ denotes M with the opposite orientation). The set of equivalence classes is denoted by $\mathfrak{N}_n$ (unoriented manifolds) or Ω_n (oriented manifolds).

Note that in contrast to the previously considered cobordism of embedded (and framed) manifolds, this cobordism is of abstract manifolds, with no maps given. The notion was introduced by Rohlin, who determined that $\mathfrak{N}_3 = \Omega_3 = 0$.

Thom used a generalization of the Pontryagin construction to compute these groups almost completely:

Theorem 4 (Thom). *The direct sum $\mathfrak{N}_* = \oplus_{n \geq 0} \mathfrak{N}_n$ admits a graded ring structure (the sum induced by disjoint union and the product induced by the product of manifolds), and as a graded ring,*

$$\mathfrak{N}_* \cong \mathbb{Z}_2[x_2, x_4, x_5, \ldots],$$

a polynomial ring in variables x_j of degree j, one for every j that is not of the form $2^m - 1$.

This result is highly nontrivial, but the generalization of the Thom construction reduces it to algebraic topology. Thom also determined the analogous graded ring $\Omega_* \otimes \mathbb{Q}$ (oriented cobordisms without the torsion part), which turns out to be $\mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \ldots, \mathbb{C}P^{2j}, \ldots]$. The torsion part – which is pure 2-torsion, $2 \cdot Tors\Omega_* = 0$ – is harder to compute, it was done later by C. T. C. Wall.

Thom's construction worked as follows. One considers the cobordism classes $Emb(n, k)$ of n-manifolds embedded into $\mathbb{R}^{n+k}$ up to cobordism (embedded into $\mathbb{R}^{n+k} \times [0, 1]$); no framing is required. Still, there is a space $T\gamma^k$ such that $Emb(n, k) \cong \pi_{n+k}(T\gamma^k)$. How does this help with the goal of classifying abstract manifolds? If $k > n+1$, then $Emb(n,k) \cong \mathfrak{N}_n$, because by Whitney's theorem any n-manifold can be embedded into $\mathbb{R}^{n+k}$ and any two such embeddings are isotopic (hence also cobordant).

Definition 5. A rank k *vector bundle* ξ is a continuous map $p : E \to B$ such that every point preimage $p^{-1}(b)$ is homeomorphic to $\mathbb{R}^k$ and the linear structure of these fibers varies continuously as a function of the point b (technically, there are local trivializations $p^{-1}(U) \cong Utimes\mathbb{R}^k$ such that on the overlapping parts the different trivializations differ in a linear map on each fiber). If additionally one chooses a metric ρ on each fiber that depends continuously on the basepoint (an averaging argument shows that such a metric always exists), then $D(\xi)$ shall denote the *disc bundle* $\{v \in E : \rho(v) \leq 1\}$ and $S(\xi)$ shall denote the *sphere bundle* $\{v \in E : \rho(v) = 1\}$; these are well-defined (independent of the choice of ρ) up to homeomorphism, and if B is a manifolds, then so is $D(\xi)$ and $S(\xi)$ is the boundary $\partial D(\xi)$.

The *Thom space* $T\xi$ of the vector bundle ξ is the quotient space $T\xi = D(\xi)/S(\xi)$.

To finish the description of $T\gamma^k$ we need to explain the vector bundle γ^k. It is the *universal* rank k vector bundle $E_k \to B_k$ in the sense that for any rank k vector bundle $E \to B$ there are homotopically unique maps $\kappa : B \to B_k$ and $\tilde{\kappa} : E \to E_k$ such that $\tilde{\kappa}$ takes fibres to fibres by linear isomorphisms and the diagram

$$\begin{array}{ccc} E & \xrightarrow{\tilde{\kappa}} & E_k \\ \big\downarrow & & \big\downarrow \\ B & \xrightarrow{\kappa} & B_k \end{array}$$

commutes. We will not explain here why does the isomorphism $Emb(n,k) \cong \pi_{n+k}(T\gamma^k)$ holds.

6.1.4 What about the hedgehog theorem?

Given a closed manifold M^n one can associate to it numbers in $\mathbb{Z}_2$, the so-called *Stiefel-Whitney characteristic numbers* of M. For every partition $n = i_1 + i_2 + \cdots + i_r$ ($i_1 \geq i_2 \geq \cdots \geq i_r \geq 1$) of n, which we denote by I for brevity, one can define the characteristic number $w_I[M] \in \mathbb{Z}_2$ that describes how complicated in a sense the tangent bundle TM is. In particular, when $I = (n)$ is the one-element partition, the corresponding characteristic number is the Euler characteristic $\chi(M)$ taken modulo 2.

Theorem 6 (Thom). *M is null-cobordant if and only if all the characteristic numbers $w_I[M]$ vanish.*

Corollary 7. *If M is null-cobordant, then $\chi(M)$ is even.*

We reproduce here a geometric proof, due to Sandro Buoncristiano, of a weaker statement, namely, that if TM is trivial: $TM = M \times \mathbb{R}^n$ (also formulated as "M is parallelizable", or equivalently M admits n linearly independent vector fields), then M is null-cobordant.

Let $\Delta \subset M \times M$ be the diagonal $\Delta = \{(x,x) : x \in M\}$. Pick a small neighbourhood $U \subset M \times M$ of Δ; it is easy to see that if U is picked as an ε-neighbourhood for some metric, then for a sufficiently small ε one can identify U with the disc bundle of TM: the points of U correspond to considering short tangent vectors $v \in T_pM$ and taking the tangent vectors $(v, -v) \in T(M \times M)$. In particular, $U \cong M \times D^n$.

Also, one has the natural involution $T : M \times M \to M \times M$ that acts as $T(x, y) = (y, x)$. Outside Δ it has no fixed points, and one can shrink U if necessary to make it T-invariant, so that the restriction of T to $M \times M \setminus U$ is a fixed point-free involution. It is not hard to prove that this implies the existence of an equivariant map $\Phi : M \times M \setminus U \to S^q$ (that is, $\Phi \circ T = -\Phi$) for some large q. Now, on the boundary $\partial U \cong M \times S^{n-1}$ the involution T is homotopic (through fixed-point free involutions) to the involution $T' : M \times S^{n-1} \to M \times S^{n-1}$, $T'(p, v) = (p, -v)$; one can check that Φ can be chosen in such a way that ∂U is mapped to $S^{n-1} \subset S^q$.

We can now quotient by the involutions and obtain a map φ from $\hat{M} = (M \times M \setminus U)/T$ to $\mathbb{R}P^q$. The boundary $\partial\hat{M} = \partial U/T = M \times \mathbb{R}P^{n-1}$ of $\hat{M}$ is mapped to $\mathbb{R}P^{n-1}$, sending every point to its projection in $\mathbb{R}P^{n-1}$ (up to homotopy, that is). But then we can pick the submanifold $\mathbb{R}P^{q-n+1}$ complementary to $\mathbb{R}P^{n-1}$ in the sense that they intersect in a single point; we may assume that φ is transverse to it and hence $W = \varphi^{-1}(\mathbb{R}P^{q-n+1})$ is a manifold with boundary $\varphi^{-1}(\mathbb{R}P^{q-n+1} \cap \mathbb{R}P^{n-1}) = M$, finishing the proof. □

Notice that this statement does not hold in reverse: while S^2 is null-cobordant (it bounds the 3-dimensional ball), the hedgehog theorem shows that S^2 is not parallelizable.

6.1.5 The belt trick revisited

The answer to the question about the belt trick is: one can return to the original belt state exactly if n, the number of full turns, is even. After showing that 2 full turns can be cancelled, one only needs to prove that one turn cannot be undone. Equip the belt with triples of vectors along the midline, the first one pointing up, the second pointing to the left, the third one pointing inside from the surface of the belt. These are orthonormal bases of $\mathbb{R}^3$ forming a path in $SO(3)$ (initially the constant one), and any movement of the belt carries along these bases to yield a homotopy of said path. For a once-twisted belt, the starting loop is the image of a generator of $\pi_1(SO(2))$ in $\pi_1(SO(3))$, and in the identification $SO(3) \cong \mathbb{R}P^3$ that sends a rotation of angle α around the unit vector v to the point $\alpha \cdot v$ in the quotient $\pi D^3/(v \sim -v$ on $v \in S^2$ this loop is clearly nontrivial in $\pi_1(\mathbb{R}^3)$.

This argument in fact can prove the hedgehog theorem. Indeed, the unit tangent vectors of S^2 form an $SO(3)$: the tangent vector v at the point $u \in S^2$ corresponds to the orthonormal basis $(u, v, u \times v)$. If there were a nowhere vanishing tangent vector field $v(u)$ on S^2, it would mean that there would be a projection $P : SO(3) \to SO(2)$ that sends the basis (p, q, r) to the oriented angle between q and $v(p)$. Composing this projection with the natural embedding $i : SO(2) \to SO(3)$ into the tangent vectors at a single point in S^2 we clearly get the identity: $P \circ i = id_{SO(2)}$. This, however, would mean that the map $i_* : \pi_1(SO(2)) \to \pi_1(SO(3))$ induced by i is at the very least injective – but the belt trick shows that the double of the generator of $\pi_1(SO(2))$ is annulled by i_*. This contradiction show that our initial assumption was false, proving the hedgehog theorem.

6.2 Smale

In 1957, Stephen Smale, then a student of Raoul Bott, told his advisor that he can prove that a sphere S^2 in $\mathbb{R}^3$ can be turned out by a regular homotopy. Bott was very skeptical, especially so because the degree of the Gauss map for the standard sphere and the turned-out sphere seemed to have different sign – after all, turning the sphere inside out is a composition of 3 reflections, right? Actually, that isn't right; homework: if an oriented even-dimensional manifold M^{2n} is immersed into $\mathbb{R}^{2n+1}$, then the degree of its Gauss map is $\frac{\chi(M)}{2}$.

Smale gave a pure existence proof, explicit regular homotopies that turn the sphere inside out were constructed only later (in particular, 1979 video by Bernard Morin; check out also "outside in" on Youtube: `https://www.youtube.com/watch?v=wO61D9x6lNY`).

We mention here a simpler case, that of circles immersed into the plane. Then we have the following:

Theorem 8 (Whitney–Graustein). *Given two immersions $f, g : S^1 \looparrowright \mathbb{R}^2$, they are regularly homotopic if and only if the rotation numbers of df and dg (as $S^1 \to S^1$ maps) are equal.*

We denote the rotation number of df by $\gamma(f)$.

Proof. The "only if" part of the statement is trivial, since the (S^1-direction) derivative of a regular homotopy between f and g is a homotopy between df and dg, therefore their rotation numbers must be equal. To prove the "if" direction, first note that without the regularity requirement the task is easy, one can pick for example the homotopy that is linear at each point: $H_t(p) = (1-t)f(p) + tg(p)$. Unfortunately, this homotopy may be singular at certain times. To fix this, draw inspiration from the classical telephone apparatus: the cord connecting the body of the phone to the receiver is coiled and this way moving it around (as is often necessary during the operation of the device) does not result (all that often, that is) in breaking the cord by introducing sharp corners in it. Indeed, if we add a circular motion (in $\mathbb{R}^3$) of sufficient speed to our curve, then its velocity will dominate its sum with the original velocity, in particular, it will nowhere turn to 0. The procedure can be performed along our chosen homotopy as well, turning it into a regular homotopy. But we need to stay in $\mathbb{R}^2$, and doing the same cannot possibly work: adding a rapid circular motion to f would cause $\gamma(f)$ to skyrocket. So instead of a circular motion, let's use a figure-8 motion: its Gauss map has degree 0 (hence $\gamma(f)$ is preserved), and in fact the velocity vector stays out of a conic area, so adding *any* multiple of such a motion (with the "wings" of the figure 8 in a normal direction with regard to the original curve) keeps the curve immersed and rescaling the excursions is a regular homotopy. The desired regular homotopy will hence be as follows: add the figure-8 movement to f, scale the addition to be become sufficiently large; move the base curve using the linear homotopy to g; rotate the figure-8 directions to match the differential of g (this is where the condition $\gamma(f) = \gamma(g)$ is used); and scale the excursions back to 0. $\square$

We remark that this method can also be generalized to turning the sphere inside out.

Having seen these two examples, the question arises: what other spheres S^n can be turned inside out in $\mathbb{R}^{n+1}$? This question is answered almost entirely by the following fact:

Proposition 9. *If S^n can be turned inside out in $\mathbb{R}^{n+1}$, then S^{n+1} is parallelizable.*

Recall that S^{n+1} is parallelizable exactly if n is 0, 2 or 6.

Proof. Let H be a regular homotopy that everts the sphere S^n. Then the trace of H is an immersion of the cylinder $S^n \times [0,1]$ into $\mathbb{R}^{n+1} \times [0,1]$ that preserves the last coordinate. Cap off the boundary spheres by standard half-spheres of S^{n+1} and notice that the Gauss map of the obtained immersion of S^{n+1} into $\mathbb{R}^{n+2}$ does not contain in its range one of the vertical vectors $(0,\dots,0,\pm 1)$, let's assume it's the upward-pointing vector $\uparrow = (0,\dots,0,1)$. But then we can take a framing of $S^{n+1}\setminus\uparrow$ (which is diffeomorphic to D^{n+1}) and pull it back to S^{n+1} using the Gauss map to obtain a framing of S^{n+1}; the existence of such a framing proves the parallelizability of S^{n+1}. $\square$

Going back to our original topic, Smale defined an invariant that determines the regular homotopy class of an immersion $f : S^n \looparrowright \mathbb{R}^q$. In order to define this invariant, we first introduce the *Stiefel manifold* $V_n(\mathbb{R}^k)$: it is the space of linearly independent ordered vector n-tuples of $\mathbb{R}^k$. The invariant, $\Omega(f)$, is an element of the homotopy group $\pi_n(V_n(\mathbb{R}^q))$. Its construction needs to assume that the immersion f is standard on a half-sphere. The use of this construction is that $\Omega(f)$ is a complete invariant, that is, $\Omega : Imm(S^n,\mathbb{R}^q) \to \pi_n(V_n(\mathbb{R}^q))$ is an isomorphism.

In the case of immersions $S^2 \looparrowright \mathbb{R}^3$, this concretizes to $Imm(S^2,\mathbb{R}^3) \cong \pi_2(V_2(\mathbb{R}^3))$. It is easy to see that $V_2(\mathbb{R}^3)$ is homotopy equivalent to the space $V_2'(\mathbb{R}^3))$ of orthonormal ordered vector pairs of $\mathbb{R}^3$. This latter space in turn is homeomorphic to $SO(3)$ since every such vector pair can uniquely be extended to an orthonormal basis of $\mathbb{R}^3$, and we have $\pi_2(SO(3)) = \pi_2(\mathbb{RP}^3) = \pi_2(S^3) = 0$. That is, *any* two immersions of the 2-sphere into $\mathbb{R}^3$ are regularly homotopic.

Now, a few words about the proof of the fact that Ω is a bijection. Define $X^q_{n,s}$ to be the space of immersions of D^n into $\mathbb{R}^q$ with s (linearly independent) normal vector fields such that the immersion is standard in a neighbourhood of a point on the boundary ∂D^n. Similarly, let $Y^q_{n,s}$ be the space of immersions of a sphere S^n into $\mathbb{R}^q$ with s normal vector fields, standard near a point. Notice that there is natural map $\rho : X^q_{n,s} \to Y^q_{n-1,s+1}$ that sends a map of D^n into its restriction to the boundary ∂D^n and adds the inward-pointing normal vector of S^{n-1} within D^n to the s inherited normal vectors. This map is a *Serre fibration*; a map $p : E \to B$ is a Serre fibration if whenever there is a homotopy $\Phi : K\times I \to B$ and a lift $\tilde\varphi : K\times\{0\} \to E$ (in the sense that $p\circ\tilde\varphi(x) = \Phi(x,0)$ for all $x\in K$), one can lift the entire homotopy to a map $\tilde\Phi : K\times I \to E$ that extends $\tilde\varphi$ and covers Φ: $p\circ\tilde\Phi = \Phi$. Another formulation of this property is that in the commutative diagram below, there is a dashed arrow that keeps the diagram commutative:

$$\begin{tikzcd} K\times\{0\} \arrow[r, "\tilde\varphi"] \arrow[d] & E \arrow[d, "p"] \\ K\times[0,1] \arrow[r, "\Phi"] \arrow[ur, dashed, "\tilde\Phi"] & B \end{tikzcd}$$

Every Serre fibration has a fibre $F = p^{-1}(b)$ (for any $b\in B$), well-defined up to homotopy equivalence; the fibration itself is sometimes denoted by $E \xrightarrow{F} B$, omitting the actual projection map. The homotopy lifting property also yields a long exact sequence

$$\cdots \to \pi_j(F) \to \pi_j(E) \to \pi_j(B) \to \pi_{j-1}(F) \to \dots$$

In our case, it is easy to check that the fibre of ρ is $Y^q_{n,s}$, and also $X^q_{n,s}$ is contractible (one can just pull the entire disc along itself within the standard neighbourhood on the boundary). The homotopy long exact sequence hence has trivial groups in every third place and therefore splits into isomorphisms $\pi_{j-1}(Y^q_{n,s}) \cong \pi_j(Y^q_{n-1,s+1})$. What we want to calculate is $\pi_0(Y^q_{n,0})$; applying the isomorphism obtain above n times, we see that this group is isomorphic to

$\pi_n(Y_{0,n}^q)$. But $Y_{0,n}^q$ is the space of point pairs in $\mathbb{R}^q$, equipped with n normal vectors each, with one of the points locked in a standard position. This means that $Y_{0,n}^q \cong \mathbb{R}^q \times V_n(\mathbb{R}^q)$, and the desired group is isomorphic to $\pi_n(V_n(\mathbb{R}^q))$, as claimed.

There remains an obvious question: how to read this invariant off a given immersion? For $q = 2n$, one may count the double points of the immersion in question. For n even, the double points of the immersion carry a natural sign, so their number is an integer; for n odd, one can count the parity of the number of double points and get a number modulo 2. This number of double points is the same as Smale's invariant except for $n = 1$, when it can be refined to an integer as expected from $Imm(S^1, \mathbb{R}^2) = \pi_1(V_1(\mathbb{R}^2)) = \pi_1(SO(2)) = \mathbb{Z}$. In general, however, there is no way to get Smale's invariant from the multiple point locus: there exist *embeddings* of S^3 into $\mathbb{R}^5$ that are not regularly homotopic. Distinguishing those can be done by calculating the signature of a Seifert surface, but we will not give the details here.

6.3 Milnor

A textbook for this day's lecture is Milnor, Stasheff: Characteristic classes.

The goal today is to construct exotic spheres. This topic was started by Milnor in 1956, when he constructed exotic spheres in dimension 7.

Definition 10. Σ^n is an exotic sphere of dimension n if it is homeomorphic to S^n, but not diffeomorphic to S^n.

6.3.1 Helper bundle

The construction relies on the existence of an auxiliary object:

Lemma 11. *There exists a rank 4 vector bundle ξ over S^4 that has Euler class $e(\xi) = a$ and first Pontryagin class $p_1(\xi) = 6a$, where $a \in H^4(S^4)$ is a generator.*

To explain the notions of the Lemma, start by recalling the Poincaré-Hopf theorem, which states that for any tangent vector field v on an oriented smooth manifold M that has isolated zeros, the sum of the indices of v at those vanishing points is the Euler characteristic of M. A tangent vector field is a *section* of the tangent vector bundle $p : TM \xrightarrow{\mathbb{R}^n} M^n$, a map $s : M \to TM$ such that $\pi \circ s = id_M$. If the section s is transverse to the zero section (the tangent spaces of the zero section and the graph of s generate the tangent space of TM at every intersection point of the zero section and the graph of s), then the index can be easily checked to coincide with the sign that describes whether the innate orientation of $T(TM)$ is the same as the orientation defined by the orientations of the zero section and s. That is, in this case the sum of indices of the zeros of s can be interpreted as an intersection number of the zero section and the graph of s.

A similar consideration can be made for any oriented vector bundle $\zeta \to B$ over a smooth manifold: take a section $s : B \to \zeta$, for a generic s the preimage $s^{-1}(0)$ is a submanifold of B and its cobordism class is a topological invariant of ζ. Indeed, taking any two generic sections s_1 and s_2, they can be deformed into one another by a homotopy (for example, linearly in each fibre); taking this homotopy to be generic as well, its vanishing locus forms a cobordism within $B \times I$ between the vanishing loci of s_1 and s_2. In practice, rather than using this cobordism class, it is easier to use the cohomology class represented by $s^{-1}(0)$, and this cohomology class is called the Euler class $e(\zeta)$ of ζ.

For the definition of the Pontryagin classes, we choose generic sections $s_1, \ldots, s_{n-2j+2}$ and consider the set of points in the base where the rank of the vector space generated by these sections is at most $n-2j$. This is a manifold of dimension $4j$ and the cohomology class it represents is the j^{th} Pontryagin class $p_j(\zeta)$ (note that the definition given in Milnor-Stasheff is different and gives a class that may differ from this one by a 2-torsion cohomology class; in our setup this difference is irrelevant).

6.3.2 From the Lemma to exotic spheres

Proposition 12. *Assuming the Lemma, the sphere bundle $S\xi$ is an exotic 7-dimensional sphere.*

First, we show that $S\xi$ is not diffeomorphic to S^7. Indirectly assume that it is; then one can glue a ball D^8 to the disc bundle $D(\xi)$ along its boundary to obtain a closed smooth manifold M^8. To arrive at a contradiction, we will use a *signature formula.* For a $4k$-dimensional manifold M one may take the "middle" integral cohomology group $H^{2k}(M)$ and define an integer-valued symmetric bilinear form on it, the intersection form:

$$\langle \alpha, \beta \rangle = \langle \alpha \cup \beta, [M] \rangle.$$

(in fact, it is not necessary to use cohomology here; considering cobordism classes of $2k$-dimensional immersed submanifolds gives an abelian group, and taking the algebraic intersection number of any representatives of two gives cobordism classes gives the form that is equivalent up to torsion to the one above). By Sylvester's rigidity theorem the intersection form has a well-defined signature, the number of positive entries minus the number of negative entries when diagonalized. The signature of M, denoted by $\sigma(m)$, is by definition the signature of its intersection form. It is a fact that we will not prove here that signature is a cobordism invariant and gives a homomorphism $\sigma : \Omega_{4k} \to \mathbb{Z}$.

A signature formula expresses σ via the cobordism invariants we saw earlier, the characteristic numbers. In dimension 8 it has the following form (discovered by Hirzebruch):

$$\sigma(M^8) = \frac{7p_2[M^8] - p_1^2[M^8]}{45}.$$

Recall that $p_2(M)$ is the class dual to the locus of points in M where a generic 6-tuple of tangent vector fields spans a subspace of dimension at most 4; while $p_1(M)$ is the class dual to the locus of points in M where a generic 8-tuple of tangent vector fields spans a subspace of dimension at most 6. The class $p_1^2(M)$ is dual to the self-intersection of this latter submanifold (its algebraic intersection number with a nearby perturbation that is transverse to the original submanifold).

To see that this signature formula holds, observe that $p_1^2[M]$ and $p_2[M]$ are cobordism invariants of M (the vector fields that define the corresponding classes extend to the cobordism), and by Thom's theorem $\Omega_8 \otimes \mathbb{Q}$ is generated by the cobordism classes of $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$ (we don't use it here, but in fact $\Omega_8 \cong \mathbb{Z} \oplus \mathbb{Z}$ with no torsion appearing). Therefore the homomorphisms σ, p_2 and p_1^2 must be linearly dependent and any relation between them that holds for $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$ holds for any 8-dimensional smooth manifold. It takes a short technical proof to show that

$$\begin{array}{lll} \sigma(\mathbb{C}P^4) = 1; & p_1^2[\mathbb{C}P^4] = 25; & p_2[\mathbb{C}P^4] = 10; \\ \sigma(\mathbb{C}P^2 \times \mathbb{C}P^2) = 0; & p_1^2[\mathbb{C}P^2 \times \mathbb{C}P^2] = 18; & p_2[\mathbb{C}P^2 \times \mathbb{C}P^2] = 9. \end{array}$$

Solving the resulting system of linear equation yields the signature formula.

Returning to our specific manifold M, we need to determine its invariants σ, p_1^2 and p_2 to check whether the signature formula holds. The signature $\sigma(M) = 1$, because $H^4(M)$ is generated by the zero section of ξ, and its self-intersection is positive. The class $p_1(M)$ is represented by a 4-dimensional submanifold, so it is enough to determine it in $D(\xi)$. There, $T(D(\xi))$ splits as the sum of (pullbacks of) TS^4 – which has trivial characteristic classes – and ξ – which has $p_1(\xi) = 6a$; hence $p_1(D(\xi)) = 6a$. The characteristic number $p_1^2[M]$ is therefore 36. Arriving finally to p_2, we run into a problem: to satisfy the signature formula, the value $p_2[M]$ would need to be $\frac{45\sigma(M)+p_1^2[M]}{7} = \frac{81}{7}$ while also being an integer (a number of intersection points). This is a contradiction, proving that we cannot glue D^8 to $D(\xi)$ keeping the result smooth and hence $S(\xi)$ is not diffeomorphic to S^7.

The second part of the proof is showing that $S(\xi)$ is homeomorphic to S^7. To achieve this, we first use the Gysin exact sequence to calculate the homology of $S(\xi)$:

$$\cdots \to H^{i-4}(S^4) \xrightarrow{\cup e(\xi)} H^i(S^4) \to H^i(S(\xi)) \to H^{i-3}(S^4) \to \ldots$$

It implies that $S(\xi)$ has the same homology groups as S^7. It is also easy to see that $S(\xi)$ is simply connected. By Hurewicz's theorem, it follows that $\pi_7(S(\xi)) = \mathbb{Z}$. A generator of this group induces isomorphisms in all homology groups, therefore it is a homotopy equivalence. Finally, we apply the generalized Poincaré conjecture (proved by Smale): if Σ^n is homotopy equivalent to S^n and $n \geq 6$, then Σ^n is homeomorphic to S^n. The proof is complete.

6.3.3 Bonus

Having constructed one example of an exotic sphere, one looks to construct more. Brieskorn has found equations that define all the exotic 7-dimensional spheres; the family

$$X_k = \{(z_1, \ldots, z_5) \in S(\mathbb{C}^5) : z_1^{6k-1} + z_2^3 + z_3^2 + z_4^2 + z_5^2 = 0\}$$

of codimension 2 submanifolds of S^9 contains all of them. Actually, the exotic spheres in any given dimension n form a group, and in the case $n = 7$ this group (determined by Milnor and Kervaire) is $\mathbb{Z}_{28}$. But then why are there infinitely many Brieskorn equations? Szűcs and Ekholm showed that the set of immersions of homotopy 7-spheres into S^9 up to regular homotopy forms a group, and that group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_4$. The regular homotopy classes that contain embeddings are a subgroup isomorphic to $\mathbb{Z}$, and all the embeddable classes that are in a certain sense "positive" are given by the Brieskorn equations.

Problem session

(Tamás Terpai)

Day 1

1. a) Show that for any $n \geq 1$ the sphere S^{2n-1} admits a nowhere vanishing tangent vector field!

b) Construct 3 linearly independent tangent vector fields on S^3!

c) Show that for any $n \geq 1$ the sphere S^{4n-1} admits 3 linearly independent nowhere vanishing tangent vector fields!

2. Prove that any positive dimensional compact Lie group is null-cobordant!

3. Show that the Pontryagin construction is a special case of the Thom construction!

Day 2

1. Calculate the homotopy group $\pi_n(V_n(\mathbb{R}^{2n}))$ using the homotopy long exact sequence of a fibration!

2. How many regular homotopy classes of immersions of a cylinder $S^1 \times I$ into $\mathbb{R}^3$ are there?

3. Let $\varphi : S^{n-1} \to S^{n-1}$ be a given diffeomorphism. Define the twisted sphere $\Sigma^n = D^n \underset{\varphi}{\cup} D^n$. Show that Σ^n is diffeomorphic to S^n if and only if φ extends to a self-diffeomorphism of D^n!

4. Prove that if $f : M^{2k} \looparrowright \mathbb{R}^{2k+1}$ is an immersion of an oriented $2k$-manifold M into the $2k+1$-dimensional Euclidean space, then the degree of the Gauss map of f (as a map from M to S^{2k}) is $\frac{1}{2}\chi(M)$!