Eötvös Loránd Tudományegyetem
Természettudományi kar

Homológia és bordizmus

Homology and bordism

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This is the section of the thesis I find the most appropriate to also recommend the `XY-pic` *package – particularly its* `XY-matrix` *subpackage – and its excellent documentation*[1] *for drawing figures and commutative diagrams. This package was widely utilized in the thesis.*

[1] User guide: `http://home.ustc.edu.cn/~xwchen/Useful%20files/xyguide.pdf`
Manual: `http://web.math.ku.dk/~holm/download/xyrefer.pdf`

About the document

The topic of the thesis turned out to be *slightly* larger than expected. There were multiple possible choices to combat this: remove several chapters (or possibly the entirety of bordism), or compress as much as possible. With the title “Homology and bordism”, it did not feel right to get rid of bordism, let alone chapter §6 as it is the one which connects the two halves of the thesis.

This lead to the conclusion that most proofs are omitted or only roughly sketched in this text. Here’s what is useful to know about the proofs of the thesis:

1. A large amount of proofs are **sketched**. These are typically more detailed than I would desire, but in turn the holes in them are easy to fill by the reader.

2. Among sketched proofs there are often ones which are entirely **missing**. These – unless otherwise noted – must be easy to prove from previous claims. Sometimes all statements in a section boil down to many unproved claims. Even in this case, the small claims should be easy to prove, and then the larger ones follow in a straightforward way. A good example for this is section §1.1, the one about absolute singular homology. While (§4.2.2) – the section about direct limits – may seem to fall under type 3, all statements there are trivial and thus it can be considered of this type.

3. There are **entire sections** where no proofs and no connections between claims are included – such as §2.2, the consequences of the Eilenberg-Steenrod axioms (§2.1), or the contents of the appendices. In this case, it is best if the reader consults the appropriate references.

4. Finally, there are sections where basically **all proofs** are provided in a nearly entirely precise way. In this case, no work from the reader’s side is required. Examples of this are sections about the Eilenberg-Steenrod axioms for the bordism functor (§5.3) and about the cap product (§4.3).

Generally speaking, algebraic proofs (and parts of proofs) are greatly compressed or omitted from the thesis for the sake of clarity. Nevertheless, if help is required, full proofs can be found in at least one of the reference books. We note here that **the thesis contains no original research**.

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The chapters of the thesis come in pairs, each concerned with a certain algebraic invariant: homology, cohomology, and bordism. The first chapters ((§1), (§3), (§5)) are typically simpler ones which present the basic concepts and tools related to the given algebraic invariants, while the second ones ((§2), (§4), (§6)) explore selected, more advanced topics.

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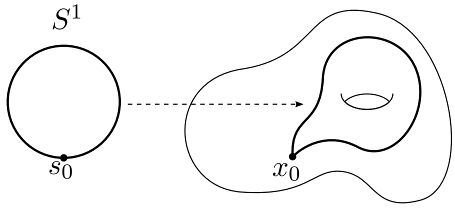
The sources of definitions and theorems are given in parentheses. Besides the external reference, additional information is provided regarding the theorem’s placement in the text. A three-letter abbreviation tells what the referenced theorem/definition is called in the original document. The original numbering of the referenced text is also provided. For example, `Thm[Hat02]:3.2` means “Theorem 3.2 of [Hat02]”.

`Prp:`	*proposition*
`Lem:`	*lemma*
`Thm:`	*theorem*
`Crl:`	*corollary*
`Def:`	*definition*

Introduction

The core concept of algebraic topology is to assign algebraic invariants to topological spaces in order to distinguish them and calculate some of their properties. There are many invariants that are applicable to arbitrary spaces, some of which are discussed below.

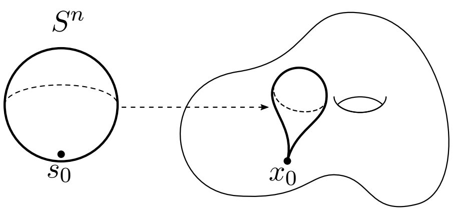
Fundamental group



The fundamental group is one of the first algebraic invariants one encounters in an introductory topology course[2]. While its concept is quite approachable and many computations involving it are feasible, it has its own problems. “Its lack of commutativity both enriches and complicates the theory” (p. 99 of [Hat02]): distinguishing noncommutative groups from each other is often complicated, so it is not the best tool to differentiate between topological spaces. Another downside is that as we obtain the group by mapping one-dimensional spheres, it mostly contains information about the low-dimensional structure of the space. This is best illustrated by the fact that the 2-skeleton determines the fundamental group of the entire CW complex.

Consequently, some algebraic invariant which uses commutative groups and higher dimensional structures is quite desirable.

Homotopy groups



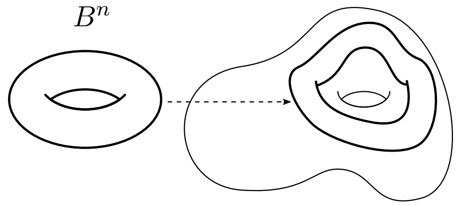
A natural generalization of the fundamental group is the n-dimensional homotopy group, where instead of one-dimensional spheres, we map n-dimensional spheres (S^n) to a space[3]. For $n > 1$, the group formed by these maps has the nice properties from earlier: it is both commutative, and is defined using high-dimensional structures. It turns out that the $n+1$-skeleton of a CW complex is necessary to determine its n-dimensional homotopy group, so these indeed detect high-dimensional features of spaces.

With new features come new problems however: computing higher homotopy groups is extremely difficult. For instance, not all homotopy groups of S^2 are known as of now. It is true furthermore that many higher homotopy groups of spheres are not trivial. The theorem of van Kampen also fails to generalize, so the toolkit of homotopy groups leaves something to be desired.

[2] At least at ELTE.

[3] In fact we map I^n's – that is, n-dimensional cubes – whose boundary maps to the basepoint. This way, an addition operation can be easily defined.

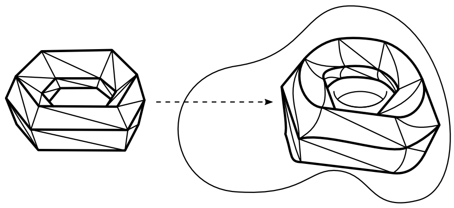
Bordism groups



In a generic setting many problems frequently become much easier to solve. In hopes of this occurring, one may leave behind the n-spheres of higher homotopy groups and instead take all maps of all n-manifolds into a given space. Settling the equivalence classes of these maps, and an addition operation on these classes takes a bit more work than for homotopy groups, but it is doable nonetheless[4]. The group formed by these equivalence classes is the n-dimensional bordism group. While van Kampen's theorem failed for higher homotopy groups, similar calculatory devices are available for bordism groups. This guarantees that once we know the groups of simple spaces, we can make calculations about the bordism groups of more complex spaces.

Only one problem remains with this approach: the fact that we use all manifolds implies that the groups we get contain information about not only the space we inspect, but also about how manifolds of a given dimension relate to each other. And these relations are far from trivial! As a result, infinitely many bordism groups of a space consisting of a single point are nontrivial. This is in many ways counter-intuitive: a simple space should have few and simple groups, not infinitely many complicated ones.

Homology groups



Finally, we arrive to the concept of homology groups. Here, instead of mapping n-manifolds, in some sense we map n-dimensional simplicial complexes into our given space. The formalism of these groups preserves the intuitive notions which led to the details of the bordism groups' definition, but also creates a more abstract and obfuscated setting. While sacrificing a great deal of intuitiveness and geometry, we leap forward in computations. Homology groups are commutative, detect high-dimensional structures but are unaffected by even higher dimensions, the groups of simple spaces are easily obtainable, and there is a large and effective toolkit for computing groups of more complex spaces.

The thesis

The goal of this thesis is to introduce the reader to the theories of (singular) homology and bordism (see chapters §1 and §5). Both of these form generalized homology theories, which means that a large part of their toolkits can be derived simultaneously (see sections §2.1, §2.2 and §2.3). Alongside this we also present the concept of (singular) cohomology in chapter §3, which is very similar to (singular) homology. A duality relation exists between homology and cohomology, which is discussed in chapter §4. Finally, we present a theorem about the existence of a classifying space for (unoriented) bordism in chapter 6, which essentially means that there is a one-to-one correspondence between bordism classes and homotopy classes of maps into this classifying space. We also note that this bijection can be exploited to express cohomology classes represented by bordism classes.

[4]Addition will be defined by disjoint union, and two maps will be equivalent – loosely speaking – if they form the restriction of a map of a manifold to its boundary.

Chapter 1

Singular homology

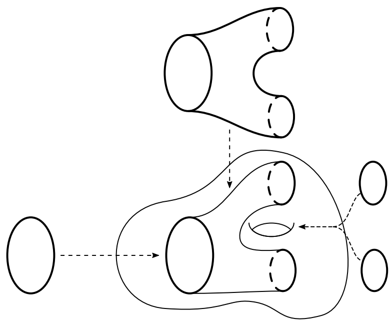
Sources. *This chapter is based entirely on chapter 2 and section 3.A of Allen Hatcher's book [Hat02]. While some parts of it have been switched around, we still follow its exposition. Many important concepts are not covered here.*

1.1 The absolute case

The construction of homology groups is quite convoluted. To gain some intuition about the process, let us foreshadow the definition of the bordism groups first. Imagine we are mapping all possible closed n-dimensional manifolds into our given space X with all their continuous maps. We make the following remarks:

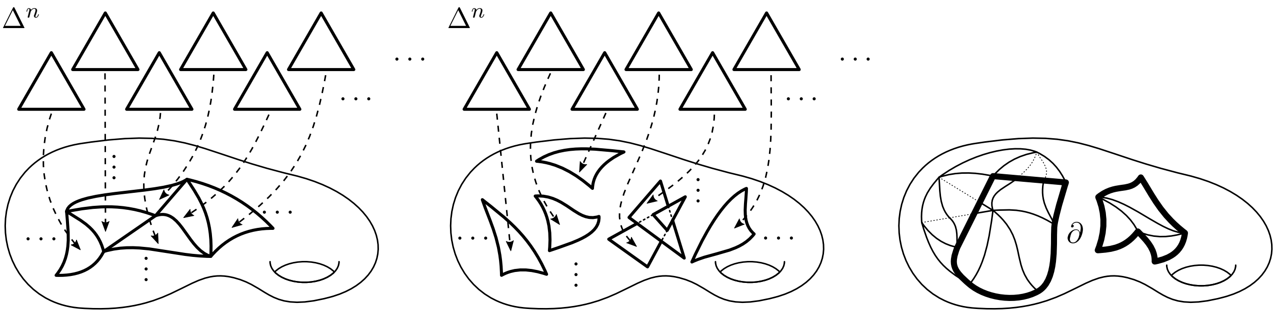
- As it is the case for the fundamental group, when measuring some qualities of a space it is practical **not to distinguish homotopic maps** – after all, they can be deformed into each other, so they are not significantly different. This also opens the door towards creating a homotopic invariant.
- However, when using many manifolds, we need a way to consider maps from different ones to be equivalent to each other, even those which come from non-homeomorphic manifolds. **So we have to quotient out not just by homotopies**, but a larger equivalence relation.
- A homotopy can be interpreted as an **extension of the two maps to a space connecting them** – to a space whose boundary they form.

With these in mind, when should we consider two maps equivalent? A natural definition would be the following: two maps are considered equivalent if there exists a higher-dimensional manifold whose boundary is the union of the two manifolds our maps come from, and a map from this higher-dimensional manifold which extends the maps defined on its boundary. This definition of course makes any two homotopic maps equivalent, but provides a way to connect maps between different manifolds too. Only a group operation is missing now, but taking the disjoint union of the two maps seems to be a natural choice for this.



But of course, we are talking about homology and not bordism – and the sketch above is closer to the concept of the latter than the former. However, it is advised to keep the ideas from above in mind when inspecting the definition of singular homology, as this structure is in some sense what we want to obtain through the process.

To avoid working with manifolds and get a more algebraic and combinatorial setting, we will not consider maps of manifolds, but "simplicial complexes". This means that a (representative of a) homology class will be a collection of maps from a standard simplex (these maps are called *singular simplices*) into our space X, which together "approximately form a closed manifold mapped into X". There are two questions about how to create a definition encapsulating this idea.

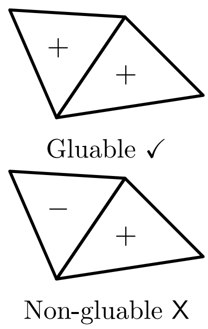


(a) ...whose faces line up. (b) ...whose faces do not line up. (c) ...and their boundary (∂).

Figure 1.1: A collection of simplices...

What should "approximately form a closed manifold" mean? Well, the simplices are already similar to $\mathbb{R}^n$ locally (at least in their interior), so the only property that separates a collection of simplices from a closed manifold is the fact that they have a boundary. We want to have a collection of maps from a standard simplex into X that *together look like a map from a manifold*, so if two maps from the simplex seem like a map from two simplices glued together, we might as well consider the collection of these two maps *not to have* a boundary at their common face. Consequently, the boundary of a larger collection of singular simplices may be defined as the faces which remain unglued after gluing together as many singular simplices as possible. And thus we receive an answer to the question at the beginning of the paragraph: a collection of simplices "approximately form a manifold" if it has no boundary, or in other words, the faces of the simplices can be partitioned into coinciding pairs.

Remark. Notice that in the paragraph above, we defined a "boundary operator" (later denoted by ∂), which assigned to a collection of maps from the standard n-simplex a collection of maps from the standard $(n-1)$-simplex.



Gluing oriented simplices

Remark. In reality, we would have liked to work with *oriented* manifolds, so we give the standard simplex an "orientation"[1]. This will not be explicitly defined as it would be superfluous to do so – instead, it will be implicit in the exact definition of singular homology. Think of an orientation of a singular simplex as its "sign": + corresponds to one orientation, while − to an other. The orientation of the simplex induces an orientation on each of its faces. To make everything consistent, we will only allow gluing together faces which have the opposite orientation – that is, their signs are opposite. The simplex with the opposite orientation/sign will also be considered the inverse of the original one.

When should two collections be considered equivalent? For manifolds, we already answered the question: if there is a higher dimensional manifold whose boundary is the union of the original manifolds, and there is a map from it which extends the original maps. In view of the last paragraph, it is easy to translate this criterion to the language of singular simplices: two collections of

[1] An orientation of the standard simplex can be given by an ordering of its vertices: two orderings represent the same orientation if an even permutation can take them into each other. However, this information will not be of use later.

singular n-simplices will be considered equivalent, if there is a collection of singular $(n+1)$-simplices whose boundary is the union of our two collections.

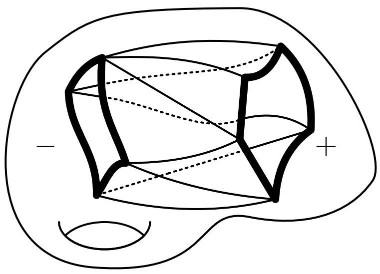


Figure 1.3: The quadrilateral and the pentagon are equivalent, as their difference is the boundary of a collection of $(n+1)$ dimensional simplices.

Remark. The question of orientations is ignored once again: if we want to account for it, we have to reverse the orientation of one of our original collections, and then ask for a higher dimensional collection which connects the two. This corresponds to the idea that “the difference of the two collections is equivalent to zero”.

Now let us summarize the above text: we define a boundary operator on the collections of singular n-simplices which has some intuitive meaning, and then take *the quotient of the kernel of the n-dimensional boundary operator by the image of the $(n+1)$-dimensional boundary operator*. This is exactly the same as the method mentioned above: the kernel is just the set of “approximately manifold” collections, while the image is just collections which form the boundaries of a higher dimensional collection, and are thus equivalent to zero.

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The discussion so far has been highly imprecise, so let us put everything on a firm algebraic footing.

Definition 1.1.1. Let $C_n = C_n(X)$ be the free abelian group with basis the collection of all continuous maps (*singular simplices*) of the form $\sigma : \Delta^n \to X$, where Δ^n is the standard n-simplex: $\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum x_i = 1, \forall i : x_i \geq 0\}$ with the subspace topology. The elements of C_n are called the **n-dimensional chains in** X.

Remark 1.1.1.1. $C_n = 0$ for $n < 0$.

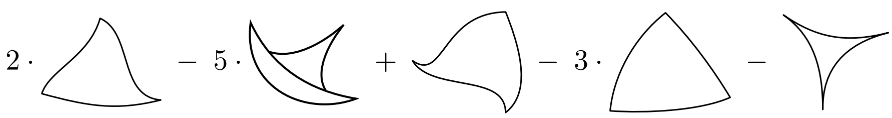


Figure 1.4: Illustration of a 2-dimensional chain.

This corresponds to what we earlier called “collections of singular simplices”: each chain is just a finite sum of simplices – with possibly negative coefficients but these can be interpreted as a reversed “orientation”.

Notation. Δ^n can be interpreted as the convex hull of its $n+1$ vertices. Name these vertices $v_0, \ldots, v_n$. This means there is a canonical ordering of the vertices for any $n \geq 0$. Take the convex hull of any $k > 0$ of these vertices: d^k. There is a canonical linear homeomorphism between d^k and Δ^k that preserves the ordering of the vertices (the ordering in d^k is inherited from the indexing in Δ^n). The order preserving nature of these homeomorphisms is connected to the concept of orientation.

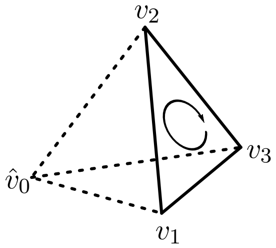


Figure 1.5: $[\hat{v}_0, v_1, v_2, v_3]$

Whenever we talk about $\sigma|_{[\text{a subset of the vertices}]}$ we mean the composition of this linear homeomorphism with the map σ, resulting in a singular simplex of the form $\Delta^k \to X$. Moreover, the notation $\{v_1, \ldots, \hat{v}_i, \ldots, v_n\}$ will mean the set, while $[v_1, \ldots, \hat{v}_i, \ldots, v_n]$ will mean the convex hull of all vertices except v_i – it is omitted from the sequence. If multiple vertices have hats, it means all of them are omitted.

Definition 1.1.2. Let $\partial = \partial_n : C_n \to C_{n-1}$ (the boundary operator) be defined on a singular n-simplex $\sigma : \Delta^n \to X$ by the formula:

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}.$$

As it is defined on the basis of the free abelian group C_n, it can be uniquely extended as a homomorphism on the entirety of C_n: the ∂ of a sum is just the sum of the ∂'s. The notation $\sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}$ is explained in above.

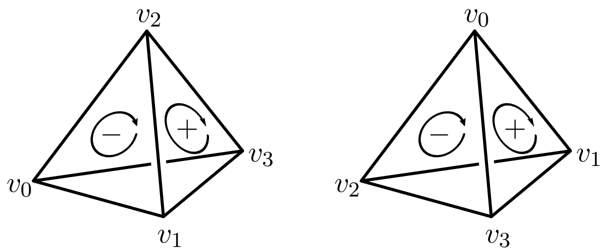


Figure 1.6: ∂_3 of a 3-simplex. The sign of each face is shown with an additional circle around it, indicating the order of the vertices: this circle is reversed for faces with a negative coefficient.

This is just the boundary operator defined earlier. Note that "gluing together simplices" simply means that their common face appears in the boundary with both a positive and negative coefficient, so it indeed cancels out from the final sum. The signs in the defining sum correspond to the (induced) "orientation" of the given face, so that it is consistent with the "orientation" of the original simplex.

Definition 1.1.3. Let $Z_n = Z_n(X)$ be $\operatorname{Ker} \partial_n \subset C_n$ (remember: $\partial_n : C_n \to C_{n-1}$). Its elements are called *cycles*.

These correspond to what we called earlier "approximately manifold collections".

Definition 1.1.4. Let $B_n = B_n(X)$ be $\operatorname{Im} \partial_{n+1} \subset C_n$ (remember: $\partial_{n+1} : C_{n+1} \to C_n$). Its elements are called *boundaries*.

These are the chains we will factor out with: two chains will be equivalent, if their difference forms the boundary of a higher dimensional chain.

Definition 1.1.5. Let $H_n(X) = Z_n(X)/B_n(X)$ be the nth **singular (absolute[2]) homology group** of the space X; the factor of the cycles with the boundaries. Its elements are called *homology classes*.

And finally, we have the homology groups we aimed to define. Of course, we still need to prove $B_n \subset Z_n$ to make $H_n(X)$ well-defined.

1.2 Elementary properties

First, we note that $B_n \subset Z_n$, so the groups $H_n(X)$ are well defined:

Claim 1.2.1 (`Lem[Hat02]:2.1`)**.** $0 = \partial\partial = \partial_{n-1} \circ \partial_n$

Corollary 1.2.1.1. *$B_n \subset Z_n$*

Next, we move on to stating that $H_n(X)$ is a functor:

Claim (`See` (1.2.4))**.** *$H_n(X)$ is a functor from the category of topological spaces and continuous maps to the category of abelian groups and their homomorphisms, for all $n \in \mathbb{Z}$.*

This means that for each continuous map between spaces there is an associated homomorphism between their homology groups, these behave well with respect to composition, and that the identity map induces the identity homomorphism:

Definition 1.2.2. Take an $f : X \to Y$ continuous map and an integer n. We will define an induced homomorphism $f_* : H_n(X) \to H_n(Y)$ in multiple steps.

1. To an arbitrary singular n-simplex $\sigma : \Delta^n \to X$ associate the singular n-simplex $f\sigma = f \circ \sigma : \Delta^n \to Y$.

$$\begin{tikzcd} \Delta^n \arrow[r, "\sigma"] \arrow[rr, bend left, dashed] & X \arrow[r, "f"] & Y \end{tikzcd}$$

2. Extend this function to a homomorphism $f_\# : C_n(X) \to C_n(Y)$. As $C_n(X)$ is free abelian with basis the singular simplices in X, this can be done uniquely: in each chain (which is a finite sum of singular simplices with certain signs), simply replace the singular simplices with their compositions with f.

3. For each $x \in H_n(X)$, take a representative $c \in Z_n(X) \subset C_n(X)$, then take the homology class of $f_\#(c) \in C_n(Y)$. **This is defined to be the element** $f_*(x)$.

Note that this last step is not sound: to take the homology class of $f_\#(c)$ we would need to have $f_\#(c) \in Z_n(Y)$, but we are yet to check this. Moreover, we still need to see that f_* is indeed well defined (independent of the choice of c) and a homomorphism.

Claim 1.2.3. *Let $f : X \to Y$ be a continuous map, and n an arbitrary integer.*

(a) $f_\#\partial = \partial f_\#$. *Note that these four letters represent four different homomorphisms, but for simplicity, they are denoted by similar symbols. In other words, the diagram below commutes:*

$$\begin{tikzcd} \dots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#"] & \dots \\ \dots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r] & \dots \end{tikzcd}$$

(b) $f_\#(Z_n(X)) \subset Z_n(Y)$

[2]For the significance of the word "absolute", see section §1.3

(c) $f_\#(B_n(X)) \subset B_n(Y)$

(d) $f_* : H_n(X) \to H_n(Y)$ *is a well defined homomorphism.*

We can now restate the functoriality of $H_n(X)$:

Claim 1.2.4. *H_n is a functor from the category of topological spaces and continuous maps to the category of abelian groupsand their homomorphisms, for all $n \in \mathbb{Z}$. That is, for arbitrary continuous maps $g : X \to Y$ and $f : Y \to Z$ we have:*

(a) $(fg)_* = f_* g_*$, *and*

(b) $\mathrm{id}_* = \mathrm{id}$, *or in other words, the identity map* $\mathrm{id} : X \to X$ *induces the identity homomorphism* $H_n(X) \to H_n(X)$.

Corollary 1.2.4.1. *Homeomorphic spaces have isomorphic homology groups.*

This functor can be thought of as the composition of the following functors:

$$\textit{Topological space} \longrightarrow \textit{Set of s. simplices} \longrightarrow \textit{Chain complex} \longrightarrow \textit{Abelian groups} \tag{E1.2.1}$$

Or with symbols:

$$X \longrightarrow (\{\sigma | \sigma : \Delta^n \to X\})_{n\in\mathbb{Z}} \longrightarrow C_*(X) = (C_n(X))_{n\in\mathbb{Z}} \longrightarrow H_*(X) = (H_n(X))_{n\in\mathbb{Z}}$$

Remark. Typically these last two sequences are instead considered as the direct sum of their elements – that is $C_*(X) = \bigoplus\limits_{n\in\mathbb{Z}} C_n(X)$ and $H_*(X) = \bigoplus\limits_{n\in\mathbb{Z}} H_n(X)$.

Of course, we still have to clarify what each category really is in this sequence. The first and last ones are evident: the category of topological spaces and continuous maps and the category of abelian groups and group homomorphisms. The second one is less interesting: it is the category of the collections of sets of singular n-simplices in a given space (for all n), and the functions on them induced by maps of the spaces. The third category is the category of chain complexes and chain maps. These (in view of the previous discussion) are defined as follows:

Definition 1.2.5. A chain complex C_* of (free) abelian groups is a sequence $(C_n)_{n\in\mathbb{Z}}$ of (free) abelian groups and maps $\partial = \partial_n : C_n \to C_{n-1}$ for each $n \in \mathbb{Z}$, such that $0 = \partial\partial = \partial_{n-1} \circ \partial_n$.

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[rr, bend left, dotted, "0"] & C_n \arrow[r, "\partial_n"] \arrow[rr, bend right, dotted, "0"'] & C_{n-1} \arrow[r, "\partial_{n-1}"] & C_{n-2} \arrow[r] & \dots
\end{tikzcd}$$

Note that $C_*(X) = (C_n(X))_{n\in\mathbb{Z}}$ is a chain complex according to the definition above and claim (1.2.1). The morphisms of these algebraic objects are:

Definition 1.2.6. A chain map $f_\# : C_* \to D_*$ between chain complexes is a sequence of homomorphisms $f_\#^{(n)} : C_n \to D_n$ such that $f_\#^{(n-1)}\partial_n = \partial_n f_\#^{(n)}$. In other words, $f_\#$ is a collection of homomorphisms such that the diagram below commutes.

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_\#^{(n+1)}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_\#^{(n)}"] & C_{n-1} \arrow[r] \arrow[d, "f_\#^{(n-1)}"] & \dots \\
\dots \arrow[r] & D_{n+1} \arrow[r, "\partial_{n+1}"] & D_n \arrow[r, "\partial_n"] & D_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

For simplicity, usually each $f_{\#}^{(n)}$ is referred to just as $f_{\#}$. With these in mind, let us go over each functor in (E1.2.1).

1. The first functor's definition and functoriality is evident: as discussed in step 1 of the definition of the induced map (1.2.2), to obtain the image of a singular simplex we just compose it with the continuous map.

2. The second functor assigns to the set of singular n-simplices the free abelian group $C_n(X)$, and to one of their morphisms induced by f the chain map $f_{\#}$. By (a) of claim 1.2.3 we have that $f_{\#}$ is indeed a chain map.

3. The third functor assigns to an arbitrary chain complex[3] its homology groups: the group at C_n will be $H_n = \operatorname{Ker}\partial_n / \operatorname{Im}\partial_{n+1}$. To a chain map it assigns a collection of homomorphisms between the appropriate homology groups as defined in step 3 of the definition of the induced map (1.2.2): from each $x \in H_n$ we take a representative $c \in C_n$, map it through $f_{\#}$, then take its equivalence class. We still have to prove that this is actually a functor.

(b)-(d) of claim 1.2.3 is true for any chain complexes and chain maps, so we know that the homomorphism of the last point is at least well defined. With this train of thought we thus have shown that we only have to see the following lemma to acquire claim 1.2.4:

Lemma 1.2.7 (`Prp[Hat02]:2.9`). *Let C_*, D_* and E_* be arbitrary chain complexes, and $g_{\#} : C_* \to D_*$ and $f_{\#} : D_* \to E_*$ be chain maps. Then we have*

(a) $(fg)_* = f_* g_*$ *and*

(b) $\mathrm{id}_* = \mathrm{id}$*, so the identity chain map induces the identity homomorphism on the homology groups.*

$\sim * \sim$

We will finish the section by first stating a direct sum theorem, then analyzing the structure of $H_0(X)$ as an illustration, calculating the homology groups of points, and finally introducing the concept of chain homotopies and a lemma regarding them:

Claim 1.2.8 (`Prp[Hat02]:2.6`). *Suppose $\{X_\alpha : \alpha \in I\}$ is the set of path-components of X, and $n \in \mathbb{Z}$ is an arbitrary integer. Then there is an isomorphism:*

$$H_n(X) \approx \bigoplus_{\alpha\in I} H_n(X_\alpha)$$

This can be easily seen, as all groups involved in the definition of $H_n(X)$ split into a direct sum.

Corollary 1.2.8.1 (`Prp[Hat02]:2.7`). *Let P be the set of path components of a topological space X. Then $H_0(X) \approx \bigoplus_{p\in P} \mathbb{Z}$.*

Remark 1.2.8.1 (`Thm[Hat02]:2A.1`). For a path-connected space X we have

$$H_1(X) \approx \pi_1(X, x_0)/\pi_1(X, x_0)',$$

that is, $H_1(X)$ is isomorphic to the abelianization of $\pi_1(X)$, the fundamental group of X. This theorem and its connections are not properly covered by the thesis, this is merely an interesting and important fact.

[3]Not necessarily constructed from a space X.

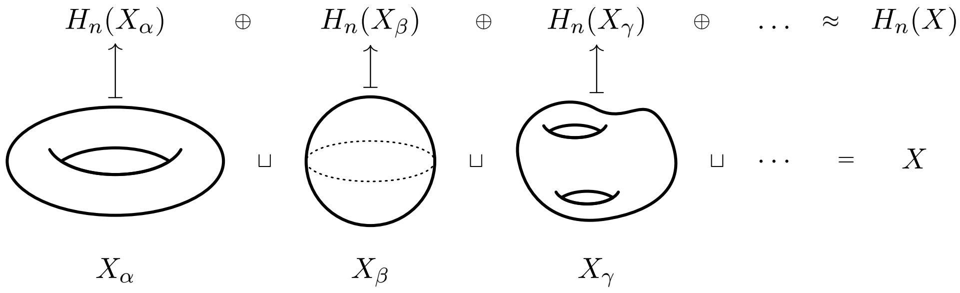


Figure 1.7: An illustration of (1.2.8).

About the homology groups of a point:

Claim 1.2.9 (`Prp[Hat02]:2.8`). *Let $*$ be a space with a single point. Then we have:*

$$H_n(*) \approx \begin{cases} 0, & n \neq 0 \\ \mathbb{Z}, & n = 0 \end{cases}$$

This can be proven by calculating explicitly the boundary operator ∂. Finally, chain homotopies:

Definition 1.2.10. Let $f_\#, g_\# : C_* \to D_*$ be chain maps of arbitrary chain complexes, and h be a collection of homomorphisms of the form $h_n : C_n \to D_{n+1}$ for each $n \in \mathbb{Z}$, such that

$$\partial h + h\partial = f_\# - g_\#.$$

(This is equivalent to $\partial_{n+1} h_n + h_{n-1}\partial_n = f_\#^{(n)} - g_\#^{(n)}$.) Then we call h a chain homotopy between $f_\#$ and $g_\#$.

Chain homotopies are best illustrated with the (non-commutative) diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, shift right, "f_\#^{(n+1)}"'] \arrow[d, shift left, "g_\#^{(n+1)}"] & C_n \arrow[r, "\partial_n"] \arrow[dl, "h_n"'] \arrow[d, shift right, "f_\#^{(n)}"'] \arrow[d, shift left, "g_\#^{(n)}"] & C_{n-1} \arrow[r] \arrow[dl, "h_{n-1}"'] \arrow[d, shift right, "f_\#^{(n-1)}"'] \arrow[d, shift left, "g_\#^{(n-1)}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial_{n+1}"] & D_n \arrow[r, "\partial_n"] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

In the section 1.4 we will see that a homotopy between the maps f and g of topological spaces induces a chain homotopy P between $f_\#$ and $g_\#$ – hence the name "chain homotopy". Applying the following lemma to this result, we get that homotopic maps induce the same homomorphism between homology groups (1.4.1):

Lemma 1.2.11 (`Prp[Hat02]:2.12`). *Suppose $f_\#, g_\# : C_* \to D_*$ are chain maps with a chain homotopy h between them. Then the homomorphisms $f_*, g_* : H_n(C_*) \to H_n(D_*)$ induced by $f_\#$ and $g_\#$ between the homology groups of the two chain complexes coincide.*

1.3 Relative, reduced, coefficients

While absolute homology groups defined earlier are easy to grasp conceptually, there are three more types of singular homology that need to be mentioned: *relative*, *reduced* and *homology with coefficients*. Moreover, we note that there are many other types of homology in general, such as *CW* homology and simplicial homology. These (in the settings where they are applicable) define the same groups as singular homology, but are more easily computable. This is what makes homology such a powerful tool. For more on them, see sections §2.1. and §2.2. of [Hat02].

Relative singular homology will be a *generalization* of absolute singular homology, which while interesting in its own right is also an essential tool in computations: without it, the toolkit of homology is not operational. Reduced homology meanwhile is a somewhat less geometrically intuitive *equivalent* of absolute homology: it provides a nearly identical language to talk about absolute homology groups which is slightly more regular in some algebraic sense. Finally, homology groups with coefficients are more or less *determined by* absolute and relative homology groups (see theorem 1.4.11), but in certain cases they conceptualize certain qualities of spaces better.

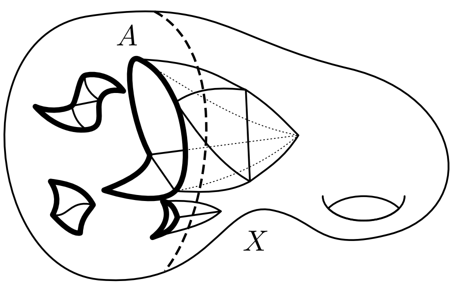
We will not detail their functoriality and homotopy invariance, as the proofs of these go analogously to the absolute case.

1.3.1 Relative homology

Suppose that there is given a space X and we only care about the structure of some subspace B. B *as a space of its own* is a simpler object than B *as a subspace of* X: the local topology around points in the "boundary" of $B \subset X$ can be quite different from the topology around those points in B alone[4]. When considering inductive calculations – computations where we obtain some algebraic property of the space using its subspaces – it is hopefully believable to the reader that B *as a subspace of* X is a more useful structure than B *alone*.

Remark. In the world of topology this idea that we only care about a certain subset is instead usually expressed by saying that there is a pair of spaces (X, A) (that is, spaces such that $A \subset X$) where we do *not* care about the structure of A: in other words, $B = X - A$.

This train of thought turns out to be quite useful. In particular, the **relative singular homology groups** measure quantities of subspaces inside larger spaces, and they are indeed applicable in inductive calculations. Furthermore, it is just generally an extremely useful tool, and after getting familiar with it one can easily convince oneself that it is a subject worth studying on its own[5].



Let us move on to the definition. There is given a pair of spaces (X, A), and we do not care about the structure of A. How could this be expressed in the language of topology, or in particular with chains, cycles and boundaries? Well, if we take a subset of $X - A$ which is a manifold, than obviously we are not interested how this manifold continues inside A. We wouldn't be bothered if this subset was not even a manifold in A, as that is a part of the space that is ignored. This idea can be translated into the language of chains: we only consider chains which are "approximately manifold in $X - A$", but may be "not approximately manifold in A" – or in other words, **relative cycles**: chains whose boundary is contained in A.

[4]For instance, if X is a sphere and B is a closed hemisphere, then of course a boundary point of B has drastically different local properties than the same point inside X.

[5]See for example the groups that make up a *CW* spectral sequence. These are described in the introductory part of section §2.3. Moreover, *CW* homology groups are defined using these.

As a boundary does not have a boundary (see claim 1.2.1), it is clear that we have to update the definition of "boundary" too, otherwise barely any chains with boundaries inside A could be considered equivalent.

If we want to follow the definition of absolute singular homology, we should consider two chains the same if they are similar in $X - A$. This "similarity" was earlier expressed by saying that their union (difference) formed a boundary. In this case, this of course means that their difference *looks like* a boundary *inside* $X - A$: inside A, it may be arbitrarily ugly, as we do not care about A. In the language of chains this can be expressed by saying that a **relative boundary** is the sum of a real boundary and some arbitrary chain inside A.

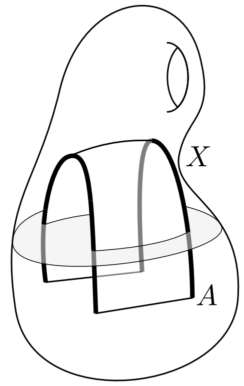


Figure 1.8: Two relative cycles connected by a relative boundary.

Remark. Both for relative cycles and relative boundaries the idea that singular simplices inside A do not matter can be algebraically expressed by saying "they are all zero": by passing to the quotient $C_n(X)/C_n(A)$.

Similar to the absolute case, the relative homology groups will be the quotient group of the relative cycles by the relative boundaries.

$\sim * \sim$

Now let us go over the definition again in a more precise manner:

Definition 1.3.1. Let (X, A) be a pair of spaces (that is, spaces such that $A \subset X$). Put $C_n(X, A) = C_n(X)/C_n(A)$. There are induced boundary operators of the form $\partial_n : C_n(X, A) \to C_{n-1}(X, A)$, and $C_*(X, A)$ with these operators forms a chain complex. The homology groups of this chain complex are denoted $H_n(X, A)$ and are called the **relative singular homology groups** of the pair of spaces. In other words:

$$H_n(X, A) = \frac{\operatorname{Ker}(\partial_n : C_n(X)/C_n(A) \to C_{n-1}(X)/C_{n-1}(A))}{\operatorname{Im}(\partial_{n+1} : C_{n+1}(X)/C_{n+1}(A) \to C_n(X)/C_n(A))}$$

Remark 1.3.1.1. This definition is compatible with the convention that we identify the pair $(X, \emptyset)$ with X: thus $H_n(X, \emptyset) = H_n(X)$.

Remark 1.3.1.2. $H_n(A, A) = 0$ for any space A and integer n. This can also be deduced from the Eilenberg-Steenrod axioms (see (§2.1), (§2.2)).

It is easy to check that a (representative of a) relative boundary and a relative cycle in $C_n(X)$ are indeed elements of $B_n(X) + C_n(A)$ and $\partial_n^{-1}(C_{n-1}(A))$ respectively. Of course, there is an implicit claim in this definition which we are yet to address:

Claim 1.3.2. *(a) $\partial_n : C_n(X, A) \to C_{n-1}(X, A)$ is well defined, and*

(b) $0 = \partial\partial$, that is $C_*(X, A)$ is a chain complex.

1.3.2 Reduced homology

The homology groups of a point are all 0, except for the 0th one, which is $\mathbb{Z}$ (see (1.2.9)). This is inconvenient for inductive arguments, as this frequently raises a special case when $n = 0$. But what if we did not know (1.2.9) (we won't know this for *singular bordism*, which operates with a very similar machinery)? Nevertheless, it would be desirable that all homology groups of a point are zero: we only want to measure properties of the space and not anomalies of the theory which produce additional groups for a point. So it is reasonable to "pass to the quotient" of the homology group by the appropriate homology group of a point, as this new group indeed does not measure superfluous properties (of the point, at least). This can be done in the following way:

Definition 1.3.3. Let X be a nonempty space. Then $\tilde{H}_n(X) = \mathrm{Ker}(H_n(X) \to H_n(*))$, where $*$ is a space consisting of a single point and the homomorphism is the one induced by the unique map $X \to *$, is the nth **reduced singular homology group** of X.

There is an alternative definition available when working with singular homology (but this does not generalize to bordism):

Claim 1.3.4. *Let $\emptyset$ denote "the (-1)-dimensional" simplex, which is defined to be the boundary ($\varepsilon = \partial_0$) of all 0-dimensional simplices. Then*

$$\ldots \longrightarrow C_n(X) \xrightarrow{\partial_n} \ldots \longrightarrow C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} C_{-1}(X) \approx \mathbb{Z} \longrightarrow C_{-2}(X) \approx 0 \longrightarrow \ldots$$

is a chain complex, whose homology group at $C_n(X), n \geq 0$ is $\tilde{H}_n(X)$.

This alternative definition gives a different interpretation of the reduced homology group too. The map ε can be given additional meaning when studied more closely (which we won't do here; if interested, see page 110 of [Hat02]).

Remark 1.3.4.1. A map $f : X \to Y$ induces a chain map between the augmented chain complexes of (1.3.4).

Remark 1.3.4.2. If $A \subset X$, then the quotient of their augmented chain complexes (1.3.4) is still the chain complex $C_*(X, A)$ defined in (1.3.1).

Finally:

Claim 1.3.5. $H_n(X, x_0) \approx \tilde{H}_n(X)$ *for arbitrary $x_0 \in X$ and $n \in \mathbb{Z}$.*

1.3.3 Homology with coefficients

While in the introduction of absolute singular homology in section §1.1 we used "oriented simplices" (where their orientation corresponded to their sign in a given chain), this orientation introduced signs in many places (such as the definition of the boundary operator (1.1.2)), complicating the theory. Moreover, it is clear that the definitions must have worked if we decided to put $\sigma = -\sigma$ for any singular simplex σ. Finally, the concept that a singular simplex can be taken many times in a chain (its coefficient can be any integer number) can feel a little artificial at first. Thus it would be probably useful to introduce an alternative homology theory where this equality holds and only 0 and 1 are allowed as coefficients, as certain calculations would become much simpler. Hence the definition of **singular homology groups with coefficients**.

What other way is there to say that singular simplices can only have coefficients in $\{0, 1\}$, and that $\sigma = -\sigma$ for any singular simplex σ? A simple answer would be to say that every simplex in a "chain"[6] should have a coefficient in $\mathbb{Z}_2$, and not in $\mathbb{Z}$. Or alternatively:

Definition 1.3.6. Let G be an abelian group, and (X, A) a pair of spaces. Put $C_n(X, A; G) = C_n(X, A) \otimes G$, that is, the direct sum of one copy of G for each singular simplex σ.

Claim 1.3.7. *There is a boundary operator $\partial_n : C_n(X, A; G) \to C_{n-1}(X, A; G)$ induced by the boundary operator $C_n(X, A) \to C_{n-1}(X, A)$, and with this operator the $C_n(X, A; G)$'s form a chain complex.*

Definition 1.3.8. Let $H_n(X, A; G)$ be the homology group at $C_n(X, A; G)$ of the chain complex $C_*(X, A; G)$. It is called the **(relative) singular homology group of** (X, A) **with coefficients in** G.

[6] Note that we redefine what chain means here!

Remark 1.3.8.1. $H_n(X, A; \mathbb{Z}) = H_n(X, A)$.

Remark 1.3.8.2. While $C_n(X, A; G) = C_n(X, A) \otimes G$, we do not have $H_n(X, A; G) \stackrel{?}{=} H_n(X, A) \otimes G$. This is in part what makes homology groups with coefficients so useful. Nevertheless, the homology groups with coefficients are determined by the homology groups with coefficients in $\mathbb{Z}$: this result is formalized by the universal coefficient theorem for homology (1.4.11), which is not proved in this thesis.

1.4 The toolkit

In this section we will introduce the most common tools of singular homology. Some of these can be acquired by manipulating chains such as:

- **Homotopy invariance** (1.4.1)**:** homotopic maps induce the same homomorphism on homologies.
- **Barycentric subdivision** (1.4.2)**:** any homology class can be represented by a chain consisting of arbitrarily small simplices (each simplex must be inside an element of an open cover $\mathcal{U}$).
- **Excision** (1.4.3)**:** for a pair of spaces (X, A) we can ignore a closed subset $Z \subset \operatorname{int} A$, as this does not change $H_n(X, A)$: $H_n(X, A) \approx H_n(X - Z, A - Z)$.

Using the zig-zag lemma (A.2.3) – a purely algebraic tool – we can get multiple different long exact sequences of homology groups (their uses are explained in the appropriate section (§1.4.2)):

- Long exact sequence (1.4.6) of homology groups associated to a **pair of spaces** (X, A) (that is, spaces such that $A \subset X$):

$$\ldots \longrightarrow H_{n+1}(X, A) \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(A) \longrightarrow \ldots$$

- Long exact sequence (1.4.7) of reduced homology groups associated to a **pair of nonempty spaces** (X, A) (that is, spaces such that $A \subset X$):

$$\ldots \longrightarrow H_{n+1}(X, A) \longrightarrow \tilde{H}_n(A) \longrightarrow \tilde{H}_n(X) \longrightarrow H_n(X, A) \longrightarrow \tilde{H}_{n-1}(A) \longrightarrow \ldots$$

- Long exact sequence (1.4.8) of homology groups associated to a **triple of spaces** (X, A, B) (that is, spaces such that $B \subset A \subset X$):

$$\ldots \longrightarrow H_{n+1}(X, A) \longrightarrow H_n(A, B) \longrightarrow H_n(X, B) \longrightarrow H_n(X, A) \longrightarrow \ldots$$

- **Mayer-Vietoris** sequence (1.4.9) of a decomposition $X = A \cup B$, where $X \subset \operatorname{int} A \cup \operatorname{int} B$:

$$\ldots \longrightarrow H_{n+1}(X) \longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \longrightarrow \ldots$$

We also note that all of these sequences are natural. Finally, we will state the universal coefficient theorem for homology (1.4.11). For an application of this toolkit, see claim 2.2.11, the calculation of the homology groups of spheres.

Remark. Note that we could spare directly proving some of these results (the last two exact sequences, all naturality statements, and the groups of spheres) from the definition of singular homology, as they can be deduced from the Eilenberg-Steenrod axioms alone. Actually, it suffices to prove only those axioms, as they characterize singular homology – at least for CW pairs (see (2.1.1)). A reminder of this fact will be written next to some (but not all) results for which this is true.

1.4.1 Manipulating chains

The goal of this section is to prove the following four theorems:

Theorem 1.4.1 (Homotopy invariance; `thm[Hat02]:2.10`, `prp[Hat02]:2.19`). *If $f, g : (X, A) \to (Y, B)$ are homotopic maps*[7]*, then $f_* = g_*$ for the induced maps $f_*, g_* : H_n(X, A) \to H_n(Y, B)$.*

Remark 1.4.1.1. Setting $A = B = \emptyset$ yields us the same result for absolute homology groups.

Corollary 1.4.1.1 (`Crl[Hat02]:2.11`). *Homotopy equivalent spaces have isomorphic homology groups. That is, singular homology is a functor of homotopy type.*

Theorem 1.4.2 (Barycentric subdivision; `prp[Hat02]:2.21`). *Suppose $\mathcal{U} = \{U_i \subset X : i \in I\}$ is a collection of spaces whose interiors together cover the space X. Put*

$$C_n^{\mathcal{U}}(X) = \{x \in C_n(X) : x \textit{ can be written in the form } x = x_{i_1} + \ldots + x_{i_k} \textit{ with } x_{i_j} \in C_n(U_{i_j})\}.$$

Then $C_*^{\mathcal{U}}(X)$ is a chain complex, and the inclusion chain map $\iota : C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X)$ is a chain homotopy equivalence, that is, there is a chain map $\rho : C_*(X) \to C_*^{\mathcal{U}}(X)$ such that $\iota\rho$ and $\rho\iota$ are chain homotopic (1.2.10) *to the identity. Moreover, we can suppose $\rho\iota = \mathrm{id}$.*

Corollary 1.4.2.1 (`Prp[Hat02]:2.21`). *Suppose we are in the situation above. Then by the chain homotopy lemma* (1.2.11) *the nth homology group of $C_*(X)$ and the nth homology group of $C_*^{\mathcal{U}}(X)$ are isomorphic: $H_n(X) \approx H_n^{\mathcal{U}}(X)$.*

Theorem 1.4.3 (Excision; `thm[Hat02]:2.20`). *We state two equivalent forms of the excision theorem.*

(a) Given subspaces $Z \subset A \subset X$ such that the closure of Z is inside $\operatorname{int} A$, the inclusion $(X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisms

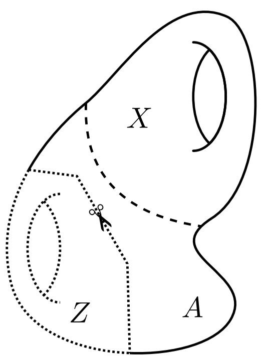
$$H_n(X - Z, A - Z) \approx H_n(X, A),$$

for all $n \in \mathbb{Z}$.

(b) For subspaces $A, B \subset X$ satisfying $\operatorname{int} A \cup \operatorname{int} B \supset X$, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms

$$H_n(B, A \cap B) \approx H_n(X, A),$$

for all $n \in \mathbb{Z}$.



Translating between the two versions is trivial: $B = X - Z$ and $Z = X - B$.

Theorem 1.4.4 (`Prp[Hat02]:2.22`). *For any good pair (X, A), the quotient map $q : (X, A) \to (X/A, A/A)$ induces isomorphisms*

$$q_* : H_n(X, A) \approx H_n(X/A, A/A) \overset{(1.3.5)}{\approx} \tilde{H}_n(X/A),$$

for all $n \in \mathbb{Z}$.

This theorem only depends on the first six Eilenberg-Steenrod axioms. Good pairs are defined as follows:

Definition 1.4.5. A pair of topological spaces (X, A) is a good pair, if A is a nonempty closed subspace in X such that there exists a $A \subset V \subset X$ open set, that deformation retracts onto A.

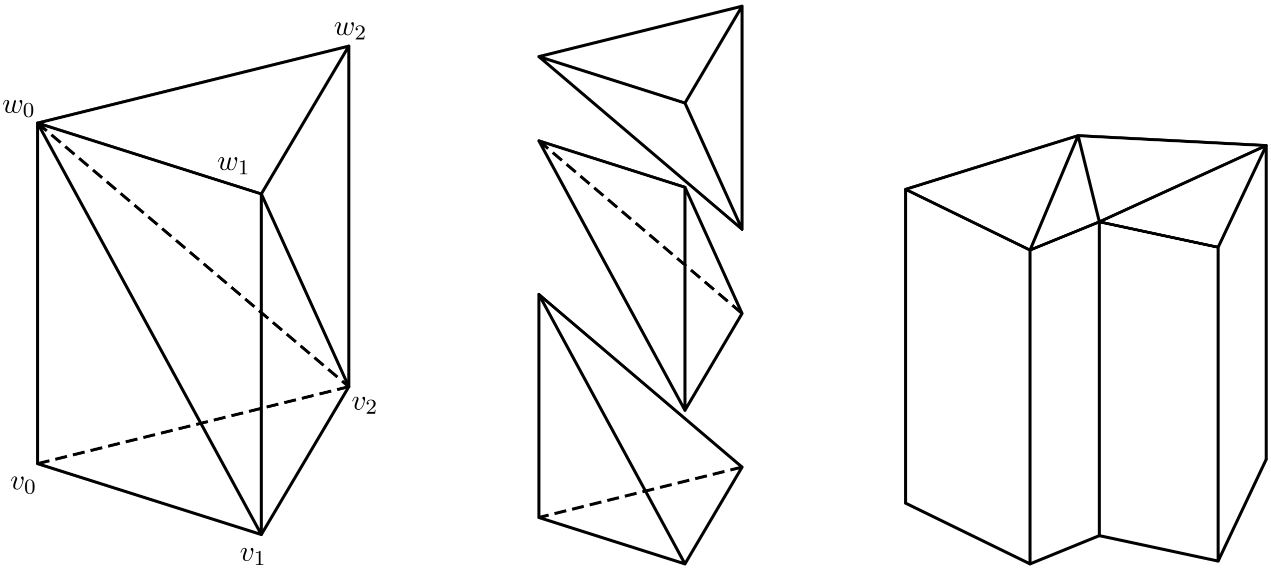
[7] So $f, g : X \to Y$, and $f(A) \subset B, g(A) \subset B$.

It is notable that CW pairs are good pairs (B.0.7).

$\sim * \sim$

First, let us prove the homotopy invariance theorem (1.4.1).

Proof of (1.4.1). We will only prove the absolute case. The proof of the relative theorem is analogous.



(a) The decomposition of $\Delta \times I$. (b) 1.9a but in detail. (c) P of a chain.

Figure 1.9: The prism operator P.

We construct a chain homotopy (1.2.10) P, that is, a collection of homomorphisms $P_n : C_n(X) \to C_{n+1}(Y)$, which shows that $f_\#$ and $g_\#$ are chain homotopic for arbitrary maps $f, g : X \to Y$ connected by a homotopy $h : X \times I \to Y$ ($I = [0, 1]$). As they are chain homotopic, they induce the same homomorphism on homology according to (1.2.11): in other words, $f_* = g_*$, and this is what we wanted to prove.

To construct P, we only have to define its value on singular simplices, as it is defined on a free abelian group. Suppose there is given a singular simplex $\sigma : \Delta^n \to X$. From this, we can create singular simplices $f\sigma, g\sigma$ in Y by composing σ with f or g, respectively. In other words, we have two maps of Δ^n into Y. However, as f and g are homotopic by h, we not only have these, but we have a map of $h(\sigma, \mathrm{id}) : \Delta^n \times I \to Y$ – the composition of (σ, id) and h. $h(\sigma, \mathrm{id})|_{\Delta^n \times \{0\}} = f\sigma$ and $h(\sigma, \mathrm{id})|_{\Delta^n \times \{1\}} = g\sigma$, so the top and bottom faces of this prism restrict to our two singular simplices in Y.

Let us decompose $\Delta \times I$ into simplices (see figure 1.9a and 1.9b), so that the top and bottom faces of the prism become faces of some simplices. This decomposes $h(\sigma, \mathrm{id})$ into multiple singular simplices: the chain formed by these will be defined as $P\sigma$. How can we describe the boundary $\partial P\sigma$? Well, it consists of:

- $-f\sigma$, the bottom face of the prism (the sign is due to orientations),
- $g\sigma$, the top face of the prism,
- $-P\partial\sigma$, the sides of the prism (it can be thought of as a prism with base the boundary of σ)

As "the prism is solid", there are no other parts of the boundary: the boundaries inside the prism cancel each other out. Summarizing these results, we recieve the formula:

$$\partial P + P\partial = g_\# - f_\#$$

As this relation holds on singular simplices, it holds for arbitrary chains. Thus P is indeed a chain homotopy (hence the definition of chain homotopy!), and $f_* = g_*$. We could of course formalize the above by defining P and calculating these results explicitly. □

The excision theorem (1.4.3) follows directly from barycentric subdivision (1.4.2), which has a significantly longer proof.

Proof of the excision theorem (1.4.3). Only (b) will be proved here, as translating between the two versions is trivial with $B = X - Z$ and $Z = X - B$.

For the cover $\mathcal{U} = \{A, B\}$ ($\operatorname{int} A \cup \operatorname{int} B \supset X$) we introduce the notation $C_n(A+B) = C_n^{\mathcal{U}}(X)$. By barycentric subdivision (1.4.2), the inclusion $\iota : C_n(A+B) \hookrightarrow C_n(X)$ is a chain homotopy equivalence, that is, it has a chain homotopy inverse ρ and a chain homotopy D which shows this inverse relation:

$$\partial D + D\partial = \operatorname{id} - \iota\rho,$$

while we have

$$\rho\iota = \operatorname{id}.$$

All maps in these formulas take chains in A to chains in A, so they induce quotient maps when passing to the quotient by $C_n(A)$. These quotient maps automatically satisfy these two formulas. Consider the quotient maps

$$C_n(B)/C_n(A \cap B) \xrightarrow{\approx} C_n(A+B)/C_n(A) \hookrightarrow C_n(X)/C_n(A),$$

where the first map is induced by inclusion, while the second is an inclusion itself. The first map is an isomorphism, as both groups are just the free abelian group with basis the singular simplices in B that are not contained in A. The second map induces an isomorphism on homology as the induced quotient maps satisfy the same formulas as the original maps, and these formulas imply the isomorphism by the chain homotopy lemma (1.2.11) (similar to how inclusion induced an isomorphism in (1.4.2.1)). □

Before we move on to barycentric subdivision, let us prove the theorem about quotient spaces (1.4.4). Note that this proof only uses the first six Eilenberg-Steenrod axioms (§2.1), so it can be stated for generalized homology theories.

Proof of (1.4.4). We have to show $q_* : H_n(X, A) \to H_n(X/A, A/A)$ is an isomorphism. Let V be a neighborhood of A in X that deformation retracts onto A. We will prove the isomorphism by showing that the normal arrows in the commutative diagram below are isomorphisms:

$$\begin{tikzcd}
H_n(X,A) \arrow[r] \arrow[d, dotted, "q_*"] & H_n(X,V) \arrow[d, dotted, "q_*"] & H_n(X-A, V-A) \arrow[l] \arrow[d, "q_*"] \\
H_n(X/A, A/A) \arrow[r] & H_n(X/A, V/A) & H_n(X/A - A/A, V/A - A/A) \arrow[l]
\end{tikzcd} \tag{E1.4.2}$$

- The horizontal two arrows on the right are isomorphisms by excision (1.4.3).
- The righmost vertical arrow is an isomorphism as q restricts to an isomorphism on the complement of A (so this arrow is induced by a homeomorphism).

- We have a deformation retraction of V onto A, and thus an induced deformation retraction of V/A onto A/A. These deformation retractions show that both (V, A) and (A, A), and $(V/A, A/A)$ and $(A/A, A/A)$ are homotopy equivalent. As $H_n(B, B) = 0$ for any space B (see (1.3.1.2)) and homotopy equivalent pairs have isomorphic homology groups, we have $H_n(V, A) \approx H_n(V/A, A/A) \approx 0$. Using the long exact sequence of the triple (1.4.8) from the next section with the triples (X, V, A) and $(X/A, V/A, A/A)$, we know that the following sequences are exact:

$$\ldots \longrightarrow H_n(V, A) \approx 0 \longrightarrow H_n(X, A) \longrightarrow H_n(X, V) \longrightarrow H_{n-1}(V, A) \approx 0 \longrightarrow \ldots$$

$$\ldots \to H_n(V/A, A/A) \approx 0 \to H_n(X/A, A/A) \to H_n(X/A, V/A) \to H_{n-1}(V/A, A/A) \approx 0 \to \ldots$$

This can be restated as the fact that the following sequences are exact for all n:

$$0 \longrightarrow H_n(X, A) \longrightarrow H_n(X, V) \longrightarrow 0$$

$$0 \longrightarrow H_n(X/A, A/A) \longrightarrow H_n(X/A, V/A) \longrightarrow 0$$

That is, the horizontal arrows on the left of (E1.4.2) are isomorphisms.

□

Finally, we shall study the barycentric subdivision theorem (1.4.2):

Proof of (1.4.2). First, we define the barycentric subdivision of the standard simplex, then the same for singular simplices. This operator (S) turns out to be a chain map, and is going to be useful as it provides a way to write a simplex as a signed sum of "smaller simplices": repeatedly applying this procedure will yield a subdivision so fine that it falls inside $C_n^{\mathcal{U}}(X)$.

Of course, we still have to show that the repeated barycentric subdivision of a chain is indeed "equivalent" in some sense to the original chain. This will be an easy consequence of the fact that applying the subdivision operator once is "equivalent" to the original chain: the operators S and id are chain homotopic.

Now we will discuss the process above in more detail. This text is designed to give a quick overview of the process, but may not provide the best environment for calculations. For a detailed proof, see the proof of porposition 2.21. on pages 119-124 of [Hat02].

We define the barycentric subdivision of standard simplices and chains in an intertwined inductive process. See figure 1.10 for a flowchart of the process.

(A) **Subdivision of Δ^n.** The barycentric subdivision of Δ^n (as well as most other concepts in this proof) will be defined using induction, with the barycentric subdivision of a 0-simplex being just the identity map into it. The result is an n-chain in Δ^n, or in other words, an element of $C_n(\Delta^n)$.

   1. Consider the identity map $\mathrm{id} : \Delta^n \to \Delta^n$ as a singular simplex, and take its boundary $\partial\,\mathrm{id}$ (a chain in $C_{n-1}(\Delta^n)$).
   2. Take the barycentric subdivision of this chain, $S\partial\,\mathrm{id}$, as defined in (B).

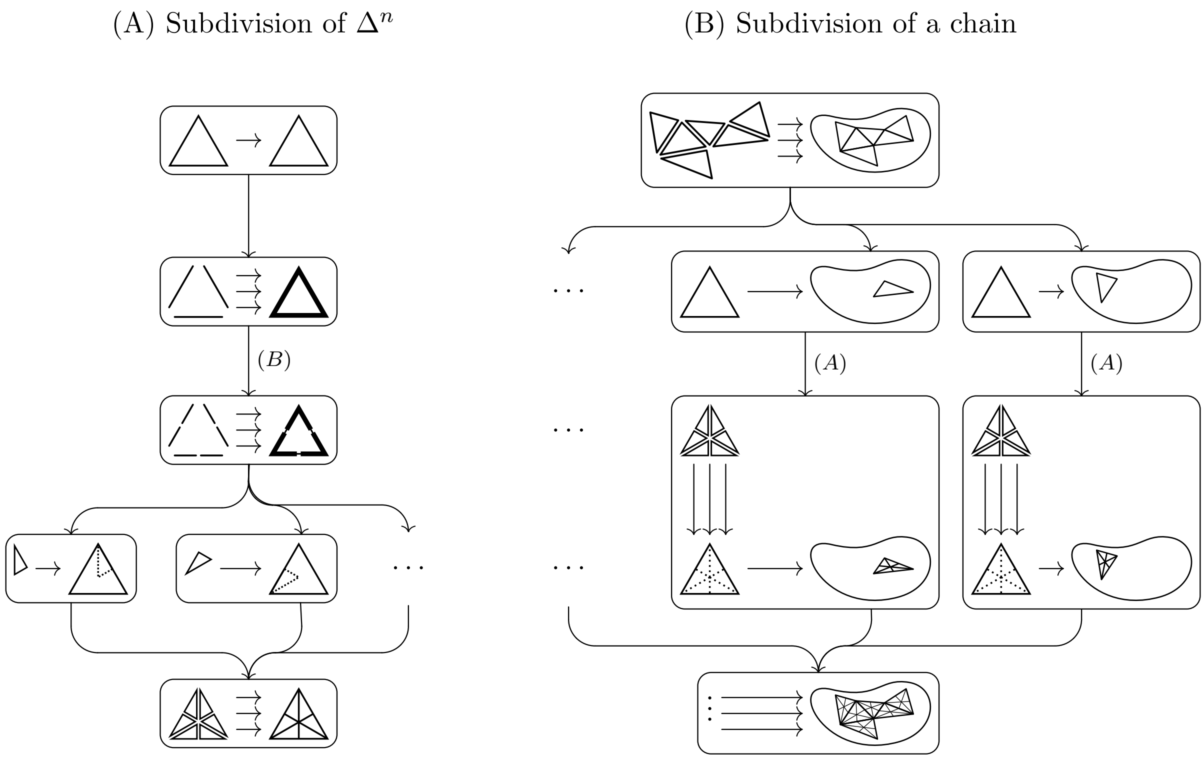


Figure 1.10: The construction of S.

3. To each singular $(n-1)$-simplex σ in this subdivision we will associate a singular n-simplex. Let b be the barycenter[8] of Δ^n. The simplex associated to σ will be the singular simplex whose image is the cone with tip b and base σ.[9]
4. The sum of these associated singular n-simplices with the same coefficients as the σ's in step 3. form the *barycentric subdivision of the standard n-simplex* Δ^n. Denote this temporarily by $S\Delta^n$.

(B) **Subdivision of a chain.** Suppose there is a singular n-simplex σ in some space X – that is, $\sigma : \Delta^n \to X$. Then σ can be viewed as a map between the spaces Δ^n and X, so it induces a homomorphism of the chain groups: $\sigma_\#$. Let $S\sigma$ – the *barycentric subdivision of a singular simplex* – be $\sigma_\# S\Delta^n$ (where $S\Delta^n$ is defined in (A)).

To calculate the barycentric subdivision of an entire chain, take the subdivisions of each simplex, and sum the resulting chains multiplied by their original signs.

The useful thing about barycentric subdivision is that the subdivided simplices are "smaller" than the original one. More precisely, the diameter of the image of each singular simplex in the subdivision of Δ^n is at most $\frac{n}{n+1}$ times the diameter of Δ^n (and this does not depend on which simplex in $\mathbb{R}^m$ we choose to represent Δ^n; this is a simple geometry result). As a consequence, using iterated subdivision we can get a chain whose singular simplices have images with arbitrarily small diameters. This means that applying subdivision sufficiently many times to a chain in X yields a chain in $C_n^{\mathcal{U}}(X)$ for any cover $\mathcal{U}$ of the type detailed in the theorem.

[8] The convex combination of all vertices with all weights being $1/(n+1)$.

[9] The precise version of this is as follows. Suppose the vertices $u_0, \dots, u_{n-1}$ of Δ^{n-1} are mapped by σ to the points $w_0, \dots, w_{n-1} \in \Delta^n$. Associate to σ the singular n-simplex which is a linear map of Δ^n and maps its vertices $v_0, \dots, v_n$ to the points $b, w_0, \dots, w_{n-1}$.

Next, we move on to define a chain homotopy T between the identity operator and the barycentric subdivision operator (on the chain complex of a space X), to show that the barycentric subdivision is indeed equivalent to the identity. This is also defined in an inductive fashion, with $T = 0$ in negative dimensions[10].

(A) T **of the standard simplex.** The T of the standard simplex will be a chain in $C_{n+1}(\Delta^n)$.

1. Take the identity singular simplex $\mathrm{id} : \Delta^n \to \Delta^n$.
2. Take the boundary of the identity singular simplex, $\partial\,\mathrm{id}$. This is an element of $C_{n-1}(\Delta^n)$.
3. Take the T of this $\partial\,\mathrm{id}$ as defined in (B): this is an element of $C_n(\Delta^n)$.
4. Let b be the barycenter of Δ^n. Then to each singular n-simplex σ in $\mathrm{id} - T\partial\,\mathrm{id}$ associate the singular simplex whose image is the cone with tip b and base σ.
5. The sum of these associated singular $(n+1)$-simplices with the same coefficients as the original σ's in step 4. form the T *of the standard n-simplex* Δ^n. Denote this temporarily by $T\Delta^n$.

(B) T **of a chain.** Suppose there is a singular n-simplex σ in some space X – that is, $\sigma : \Delta^n \to X$. Then σ can be viewed as a map between the spaces Δ^n and X, so it induces a homomorphism of the chain groups: $\sigma_\#$. Let $T\sigma$ – T *of a singular simplex* – be $\sigma_\# T\Delta^n$ (where $T\Delta^n$ is defined in (A)).

To calculate T of an entire chain, take T of each simplex, and sum the resulting chains multiplied by their original coefficients.

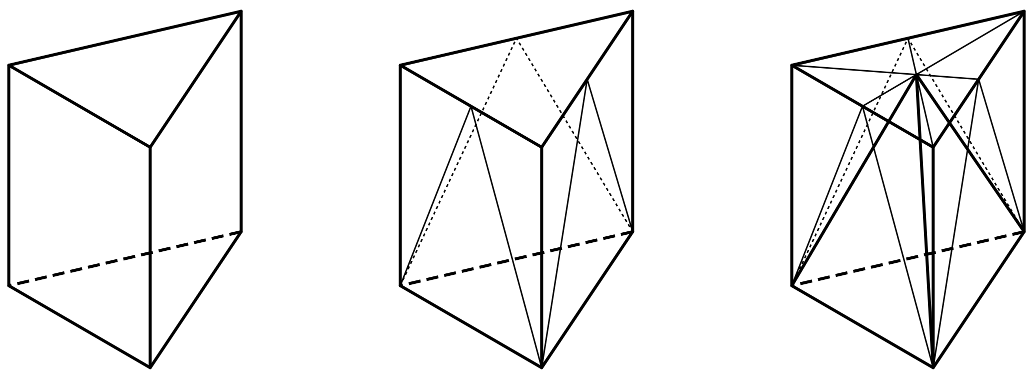


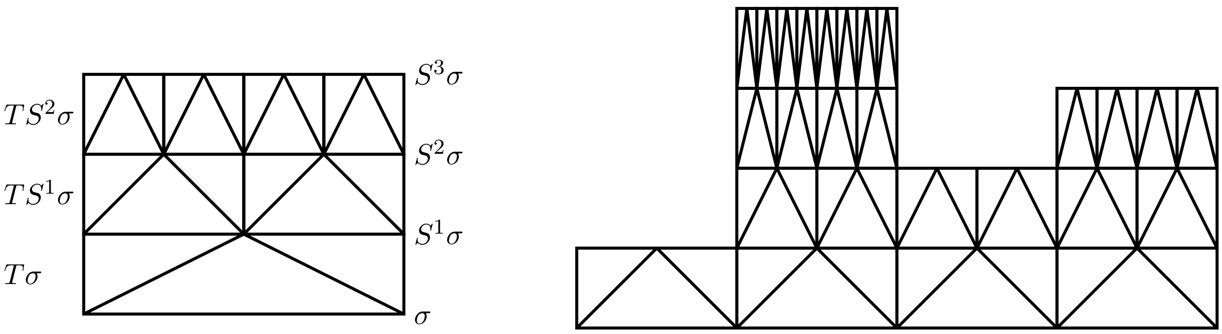
Figure 1.11: Illustration of how T connects the barycentric subdivision of a singular simplex with the original simplex. Note that the vertical projection of the third figure is what is actually "mapped" into X.

It can be checked by direct calculation that S is indeed a chain map and T is a chain homotopy between S and id, that is, we have:

$$\partial T + T\partial = \mathrm{id} - S$$

We still need two operators to finish the proof.

- The operator $D_m : C_n(X) \to C_{n+1}(X)$, which is the chain homotopy between id and S^m, the iterated barycentric subdivision operator. This is given by the formula $\sum\limits_{i=0}^{m-1} TS^i$.
- The operator $D = D^{\mathcal{U}} : C_n(X) \to C_{n+1}(X)$, which is defined on a singular simplex σ as $D_{m(\sigma)}$, where $m(\sigma)$ is the smallest natural number for which $S^m(\sigma)$ consists of simplices such that any simplex is contained in the interior of one of the sets in $\mathcal{U}$.



(a) D_m connects σ and $S^m\sigma$. (b) $D^{\mathcal{U}}$ subdivides each simplex as many times as necessary.

Figure 1.12: The operators D_m and $D^{\mathcal{U}}$.

It can be checked by direct calculation that D_m is indeed a chain homotopy. Now we would like a "chain homotopy inverse" ρ of the inclusion $\iota : C_n^{\mathcal{U}}(X) \to C_n(X)$. The chain homotopy between the two will be D, so we simply define ρ as:

$$\rho = \mathrm{id} - \partial D - D\partial$$

It can be checked by direct calculation that ρ is indeed a chain map of the form $C_n(X) \to C_n^{\mathcal{U}}(X)$.

Finally, $\rho\iota = \mathrm{id}$ as D restricts to 0 on $C_n^{\mathcal{U}}(X)$, as $m(\sigma) = 0$ here. □

1.4.2 Exact sequences

A sequence of abelian groups $(G_n)_{n\in\mathbb{Z}}$ with maps $f_n : G_n \to G_{n-1}$ forms a (long) exact sequence if $\operatorname{Ker} f_n = \operatorname{Im} f_{n+1}$ for all $n \in \mathbb{Z}$.

$$\dots \longrightarrow G_{n+1} \xrightarrow{f_{n+1}} G_n \xrightarrow{f_n} G_{n-1} \xrightarrow{f_{n-1}} G_{n-2} \longrightarrow \dots$$

Now would be a good time to go over the elementary concepts of homological algebra in section §A.1. Besides long exact sequences, we will also state the universal coefficient theorem for homology (1.4.11) later in this section. The study of spectral sequences (which can be applied to singular homology too) are postponed to the next chapter, in section §2.3.

There are many long exact sequences whose groups are (relative/reduced/absolute) homology groups. While the applicability of the tools introduced in the last section is quite self-explanatory, this is not the case for long exact sequences. For the convenience of the reader, we list two major applications here (which are exceptionally useful, as our long exact sequences will be "3-periodic" in some sense):

- Sometimes many groups in a long exact sequence turn out to be zero.

  $$\dots \longrightarrow G_{n+1} \longrightarrow 0 \longrightarrow G_{n-1} \longrightarrow G_{n-2} \longrightarrow 0 \longrightarrow \dots$$

  This can be because for example they are homology groups of contractible spaces (such as $\mathbb{R}^n$ or a point), or perhaps a relative homology group of the form $H_n(A, A)$ is included (see claim 1.3.1.2). Zeros in the sequence let us deduce the injectivity and surjectivity of certain maps. For instance (and this is a common case!) if every third group is 0, then the remaining pairs of groups are isomorphic. This lets us calculate one half of such pair from the other half.

[10] ∂ of a 0-simplex should be the empty simplex $\emptyset$ to make the induction start.

- Sometimes we already know the structure of certain groups in a sequence, but lack information about the others. Thanks to the naturality of the long exact sequences we will construct, frequently there is another long exact sequence whose terms we all understand, and there is a sequence of maps connecting the two (such that the diagram below commutes).

$$\begin{tikzcd} \ldots \arrow[r] & A_{n+2} \arrow[r] \arrow[d, dotted] & A_{n+1} \arrow[r] \arrow[d, dotted] & A_n \arrow[r] \arrow[d, dotted] & A_{n-1} \arrow[r] \arrow[d, dotted] & A_{n-2} \arrow[r] \arrow[d, dotted] & \ldots \\ \ldots \arrow[r] & B_{n+2} \arrow[r] & B_{n+1} \arrow[r] & B_n \arrow[r] & B_{n-1} \arrow[r] & B_{n-2} \arrow[r] & \ldots \end{tikzcd}$$

  The five lemma (A.2.1) states that if we take a five groups long segment of these two sequences, and all vertical homomorphisms except the one in the center are isomorphisms, then so is the one in the center. Thus if we don't know every third element of the sequence $(A_n)_{n\in\mathbb{Z}}$, but we know each B_n, then we can calculate the remaining A_n's. This is especially useful when proving the equality of different homology theories.

All of our long exact sequences will come from the zig-zag lemma (A.2.3), which states that a short exact sequence of chain complexes (their maps are the chain maps!)

$$0 \longrightarrow C_* \longrightarrow D_* \longrightarrow E_* \longrightarrow 0$$

yields a long exact sequence of their homology groups

$$\ldots \longrightarrow H_{n+1}(E_*) \xrightarrow{\partial} H_n(C_*) \longrightarrow H_n(D_*) \longrightarrow H_n(E_*) \longrightarrow \ldots,$$

where the maps are the ones induced by the chain maps, and ∂ is constructed in the proof of the zig-zag lemma (A.2.3).

Now let us mention some long exact sequences. Note that there are more long exact sequences than listed here (especially Mayer-Vietoris sequences).

Theorem 1.4.6 (LES of pair; `see page 117 of [Hat02]`). *For each pair of spaces (X, A) (that is, spaces such that $A \subset X$) there is associated a long exact sequence of homology groups:*

$$\ldots \longrightarrow H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow \ldots$$

The homomorphism ∂ can be made explicit: take a relative cycle c representing a relative homology class $x \in H_{n+1}(X, A)$. The boundary ∂c is in A. ∂x is the homology class of ∂c in $H_n(A)$.

This comes from the exactness of

$$0 \longrightarrow C_*(A) \longrightarrow C_*(X) \longrightarrow C_*(X, A) \longrightarrow 0,$$

which is basically just the definition of $C_*(X, A)$.

Theorem 1.4.7 (Reduced LES of pair; `see page 118 of [Hat02]`). *For each pair of nonempty spaces (X, A) there is associated a long exact sequence of reduced homology groups:*

$$\ldots \longrightarrow H_{n+1}(X, A) \xrightarrow{\partial} \tilde{H}_n(A) \longrightarrow \tilde{H}_n(X) \longrightarrow H_n(X, A) \longrightarrow \ldots$$

The proof of this is the same as for the previous one, but instead of $C_*(A)$ and $C_*(X)$ we use their augmented chain complexes (1.3.4), and notice that their quotient is still $C_*(X, A)$ (1.3.4.2). Alternatively, this can be deduced from the Eilenberg-Steenrod axioms (see (§2.1), (§2.2)).

Theorem 1.4.8 (LES of triple; `see page 118 of [Hat02]`). *For each triple of spaces (X, A, B) (that is, spaces such that $B \subset A \subset X$) there is associated a long exact sequence of homology groups:*

$$\dots \longrightarrow H_{n+1}(X, A) \longrightarrow H_n(A, B) \longrightarrow H_n(X, B) \longrightarrow H_n(X, A) \longrightarrow \dots ,$$

where the homomorphism $H_{n+1}(X, A) \to H_n(A, B)$ is defined by the composition:

$$H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \longrightarrow H_n(A, B) .$$

This comes from the exactness of

$$0 \longrightarrow C_*(A, B) \longrightarrow C_*(X, B) \longrightarrow C_*(X, A) \longrightarrow 0 ,$$

which is the second isomorphism theorem in the category of chain complexes and chain maps. This can be proven quite easily. Alternatively this can be deduced from the Eilenberg-Steenrod axioms: see theorem 2.2.4.

Theorem 1.4.9 (Mayer-Vietoris sequence; `see pages 149-150 of [Hat02]`). *Suppose the space X is covered by the union $\operatorname{int} A \cup \operatorname{int} B$ for some subspaces $A, B \subset X$. Then there is associated a long exact sequence of homology groups:*

$$\dots \longrightarrow H_n(A \cap B) \xrightarrow{\Phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \longrightarrow \dots ,$$

where

- $\Phi(\alpha) = (i_*(\alpha), -j_*(\alpha))$*, with inclusions* $i : A \cap B \hookrightarrow A, j : A \cap B \hookrightarrow B$.
- $\Psi(\alpha, \beta) = k_*(\alpha) + l_*(\beta)$*, with inclusions* $k : A \hookrightarrow X, l : B \hookrightarrow X$.
- ∂ *can be made explicit: take a representing chain of a given homology class. Take an equivalent chain by barycentric subdivision* (1.4.2)*, such that the result can be written as the sum of two chains: one in A and one in B. Their boundaries must coincide in $A \cap B$. The homology class of this boundary will be the the result of ∂.*

This comes from the exactness of

$$0 \longrightarrow C_*(A \cap B) \xrightarrow{\phi} C_*(A) \oplus C_*(B) \xrightarrow{\psi} C_*(A + B) \longrightarrow 0 ,$$

where $\phi(x) = (x, -x)$ and $\psi(x, y) = x + y$. According to a corollary of barycentric subdivision (1.4.2.1), the homology groups of $C_*(A + B)$ are isomorphic to the homology groups $H_*(X)$, so the theorem follows.

About the naturality of the sequences:

Theorem 1.4.10 (`See pages 127-128 of [Hat02]`). *All previously mentioned long exact sequences are natural:*

- *The LES of a pair of spaces (X, A).*
- *The reduced LES of a pair of nonempty spaces (X, A).*
- *The LES of a triple of spaces (X, A, B).*
- *The Mayer-Vietoris sequence of a decomposition $X = \operatorname{int} A \cup \operatorname{int} B$.*

This means that for maps of pairs/triples of spaces ($f : (X, A) \to (X', A'), g : (X, A, B) \to (X', A', B'), h : X = \operatorname{int} A \cup \operatorname{int} B \to X' = \operatorname{int} A' \cup \operatorname{int} B'$) there are induced maps between the corresponding groups of these sequences, and the diagram formed by the exact sequences and these maps commutes.

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & A_{n+2} & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & A_{n-1} & \longrightarrow & A_{n-2} & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & B_{n+2} & \longrightarrow & B_{n+1} & \longrightarrow & B_n & \longrightarrow & B_{n-1} & \longrightarrow & B_{n-2} & \longrightarrow & \ldots \end{array}$$

$\sim * \sim$

Finally, the universal coefficient theorem for homology:

Theorem 1.4.11 (`Thm[Hat02]:3A.3`). *If C_* is a chain complex of free abelian groups, then there are natural short exact sequences*

$$0 \longrightarrow H_n(C_*) \otimes G \longrightarrow H_n(C_*; G) \longrightarrow \operatorname{Tor}(H_{n-1}(C_*), G) \longrightarrow 0$$

for all n and G, and these sequences split, though not naturally.

We can calculate Tor using the following rules:

Claim 1.4.12 (`Prp[Hat02]:3A.5`). *Let all capital letters denote abelian groups.*

(a) $\operatorname{Tor}(A, B) \approx \operatorname{Tor}(B, A)$

(b) $\operatorname{Tor}(\bigoplus_{i \in I} A_i, B) \approx \bigoplus_{i \in I} \operatorname{Tor}(A_i, B)$

(c) $\operatorname{Tor}(A, B) = 0$ *if A or B is free (or simply torsionfree)*

(d) $\operatorname{Tor}(A, B) \approx \operatorname{Tor}(T(A), B)$, *where $T(A)$ is the torsion subgroup of A.*

(e) $\operatorname{Tor}(\mathbb{Z}_n, A) \approx \operatorname{Ker}(A \xrightarrow{n} A)$

(f) For each short exact sequence

$$0 \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0$$

there is a naturally associated exact sequence

$$0 \longrightarrow \operatorname{Tor}(A, B) \longrightarrow \operatorname{Tor}(A, C) \longrightarrow \operatorname{Tor}(A, D) \longrightarrow A \otimes B \longrightarrow A \otimes C \longrightarrow A \otimes D \longrightarrow 0\ .$$

The definition of Tor and the proofs of these claims are omitted. For details, see section 3.A of [Hat02].

Chapter 2

General homology theories

Sources. *The inclusion of this chapter in this thesis is motivated by the exposition of bordism groups followed in [CF64]. The statement of the axioms in section §2.1 is copied from [CF64], except for the 8th one, which is from [Hat02]. The consequences of these axioms in (§2.2) have been chosen primarily based on what appeared in [CF64], but the exact statements of theorems were collected from [ES52]. All references to theorems/lemmas of [ES52] are inside chapter I. of the book, so "`Thm[ES52]:5.2`" means "`Thm[ES52]:I.5.2`". The sources of section §2.3 are detailed at the start of the section.*

There is a set of axioms which are satisfied by many functors (such as the singular homology and bordism functors discussed in this thesis). Many properties and tools of these functors can be deduced purely from these axioms. This section aims to introduce them and the tools which only depend on them, including spectral sequences.

2.1 The Eilenberg-Steenrod axioms

Remark. Please note that most results stated in the next three sections can be proven much more easily for singular homology. The main reason for including these is their applicability in the context of singular bordism.

In the original work of Eilenberg and Steenrod ([ES52], pages 10–12), seven axioms are used to uniquely determine the singular homology theory of pairs of spaces:

Theorem 2.1.1 (`Thm[Hat02]:4.59`). *(a) Suppose the functors h_n satisfy the Eilenberg-Steenrod axioms (described later in this section) for finite CW pairs*[1]*. Then there are natural isomorphisms*

$$h_n(X, A) \approx H_n(X, A; H_0(*))$$

(for homologies with coefficients, see section §1.3.3) for all finite CW pairs (X, A) and $n \in \mathbb{Z}$, where $*$ is a space with a single point.

(b) If we assume the 8th axiom, then there are isomorphisms for arbitrary CW pairs.

There are many slightly different axiomatic systems for singular homology (besides this one) which are more commonly used today. In our case it is assumed that

- for each pair of spaces (X, A) and $n \in \mathbb{Z}$ there is assigned an abelian group $h_n(X, A)$,
- for each map between pairs of spaces $f : (X, A) \to (Y, B)$ and $n \in \mathbb{Z}$ there is an induced homomorphism $f_* : h_n(X, A) \to h_n(Y, B)$,

[1] Pairs (X, A) where X is a CW complex and A is a subcomplex.

- and for each pair of spaces (X, A) and $n \in \mathbb{Z}$ there is a homomorphism $\partial : h_n(X, A) \to h_{n-1}(A)$.[2]

The first two assumptions together with axioms (ES.1) and (ES.2) can be summarized as follows: there is given a functor from the category of pairs of spaces to the category of abelian groups. Axiom (ES.3) is sometimes shortened to say that ∂ is a natural transformation.

The Eilenberg-Steenrod axioms:

(ES.1) If f is the identity, then f_* is the identity as well.

(ES.2) $(gf)_* = g_* f_*$.

(ES.3) If $f|_A$ is the restriction of $f : (X, A) \to (Y, B)$ to A, then $\partial f_* = (f|_A)_* \partial$. Namely, the following diagram commutes for any $n \in \mathbb{Z}$:

$$\begin{tikzcd} h_n(X,A) \arrow[r, "f_*"] \arrow[d, "\partial"] & h_n(Y,B) \arrow[d, "\partial"] \\ h_{n-1}(A) \arrow[r, "(f|_A)_*"] & h_{n-1}(B) \end{tikzcd}$$

(ES.4) If $i : A \hookrightarrow X$ and $j : X \hookrightarrow (X, A)$ are inclusions, then the following sequence is exact:

$$\ldots \longrightarrow h_n(A) \xrightarrow{i_*} h_n(X) \xrightarrow{j_*} h_n(X,A) \xrightarrow{\partial} h_{n-1}(A) \longrightarrow \ldots$$

(ES.5) If $f, g : (X, A) \to (Y, B)$ are homotopic, then $f_* = g_*$.

(ES.6) If U is an open subset of X such that $\overline{U} \subset \operatorname{int} A$, then the inclusion map $i : (X - U, A - U) \hookrightarrow (X, A)$ induces an ismorophism (for each $n \in \mathbb{Z}$):

$$i_* : h_n(X - U, A - U) \xrightarrow{\approx} h_n(X, A)$$

(ES.7) If $*$ is a single point, then $h_n(*) = 0$ for all $n \neq 0$.

There are interesting homology theories which *do not* satisfy the last – dimensional – axiom, such as the bordism homology theory discussed later in this thesis. For such theories, the groups $h_n(*)$ (where $*$ is a single point) are called the coefficients of the theory.

These 7 axioms were originally formulated with finite CW complexes in mind. While in this setting these were sufficient to characterize singular homology, for infinite complexes they left something to be desired. This gave rise to the following axiom (due to Milnor), which is a standard part of the axiomatic system used today:

(ES.8) For any collection of spaces $\{X_\alpha\}_{\alpha \in J}$ with inclusions $i^\alpha : X_\alpha \hookrightarrow \bigsqcup_{\beta \in J} X_\beta$ into the disjoint union, the induced homomorphisms i^α_* determine an isomorphism:

$$\bigoplus_{\alpha \in J} i^\alpha_* : \bigoplus_{\alpha \in J} h_n(X_\alpha) \xrightarrow{\approx} h_n\left(\bigcup_{\alpha \in J} X_\alpha\right),$$

for all $n \in \mathbb{Z}$.

However, we will not make use of this axiom in this thesis, so it will be ignored, and the term "Eilenberg-Steenrod axioms" will refer to the first seven axioms alone.

[2] Note that pairs of the form $(Y, \emptyset)$ are abbreviated as Y.

2.2 Consequences of the axioms

Many properties of a general homology theory follow from just the first 6 axioms alone. These are:

Lemma 2.2.1 (`Lem[ES52]:8.1`). *$h_n(A, A) \approx 0$ for any space A and any integer n.*

Theorem 2.2.2 (`Thm[ES52]:4.1`). *The long exact sequence of (ES.4) is natural. That is, if there is a map $f : (X, A) \to (Y, B)$ that defines the maps $f_1 : X \to Y$ and $f|_A : A \to B$ then the following diagram is commutative:*

$$\begin{tikzcd}
\dots \arrow[r] & h_n(A) \arrow[r] \arrow[d, "(f|_A)_*"] & h_n(X) \arrow[r] \arrow[d, "(f_1)_*"] & h_n(X,A) \arrow[r] \arrow[d, "f_*"] & h_{n-1}(A) \arrow[r] \arrow[d, "(f|_A)_*"] & \dots \\
\dots \arrow[r] & h_n(B) \arrow[r] & h_n(Y) \arrow[r] & h_n(Y,B) \arrow[r] & h_{n-1}(B) \arrow[r] & \dots
\end{tikzcd}$$

Theorem 2.2.3 (`Thm[ES52]:5.2`). *If $f : (X, A) \to (Y, B)$ is a homeomorphism, then $f_* : h_n(X, A) \to h_n(Y, B)$ is an isomorphism (for all $n \in \mathbb{Z}$).*

Theorem 2.2.4 (Exact sequence of a triple; `thm[ES52]:10.2`). *If $B \subset A \subset X$ – or in other words (X, A, B) is a triple of spaces – then there is an exact sequence:*

$$\dots \longrightarrow h_n(A, B) \longrightarrow h_n(X, B) \longrightarrow h_n(X, A) \longrightarrow h_{n-1}(A, B) \longrightarrow \dots,$$

where the map $h_n(X, A) \to h_{n-1}(A, B)$ is defined as the composition

$$h_n(X, A) \xrightarrow{\partial} h_{n-1}(A) \longrightarrow h_{n-1}(A, B).$$

This is easy to deduce using a diagram chasing argument (see (A.2.4)), and by noticing that the composition $h_n(A, B) \to h_n(X, B) \to h_n(X, A)$ can instead be written as $h_n(A, B) \to h_n(A, A) \to h_n(X, A)$, where $h_n(A, A) \approx 0$ by (2.2.1).

There are also reduced homology groups:

Definition 2.2.5 (`Def[ES52]:7.3`). Let $*$ be a single point. The reduced homology group in dimension n of a space X (denoted $\tilde{h}_n(X)$) is the kernel of $\varepsilon_* : h_n(X) \to h_n(*)$ where ε collapses X to a point $*$.

Claim 2.2.6 (`Thm[ES52]:7.6`). *$h_n(X) \approx \tilde{h}_n(X) \oplus h_n(*)$ for any nonempty space X and any $n \in \mathbb{Z}$.*

Theorem 2.2.7 (`Thm[ES52]:8.4`). *For each pair (X, A) there is an exact sequence of reduced homology groups:*

$$\dots \longrightarrow \tilde{h}_n(A) \longrightarrow \tilde{h}_n(X) \longrightarrow h_n(X, A) \longrightarrow \tilde{h}_{n-1}(A) \longrightarrow \dots$$

By utilizing the excision axiom, we can also get:

Theorem 2.2.8 (Direct sum theorem; `thm[ES52]:13.2`). *Suppose that $X = X_1 \sqcup \dots \sqcup X_m$ is a disjoint union, $A_k \subset X_k$, $A = A_1 \sqcup \dots \sqcup A_m$. Then*

$$h_n(X, A) \approx \bigoplus_{k=1}^{m} h_n(X_k, A_k).$$

In other words, if $i_k : (X_k, A_k) \hookrightarrow (X, A)$, then each $u \in h_n(X, A)$ can be uniquely represented by some $u_k \in h_n(X_k, A_k)$'s in the form

$$u = \sum_{k=1}^{m} i_{k*}(u_k)$$

Theorem 2.2.9 (Mayer-Vietoris sequence; `thm[ES52]:15.3`). *Suppose the space X is covered by the union* $\operatorname{int} A \cup \operatorname{int} B$ *for some subspaces $A, B \subset X$. Then there is associated a long exact sequence of homology groups:*

$$\ldots \longrightarrow h_n(A \cap B) \xrightarrow{\Phi} h_n(A) \oplus h_n(B) \xrightarrow{\Phi} h_n(X) \xrightarrow{\partial} h_{n-1}(A \cap B) \longrightarrow \ldots,$$

where

- $\Phi(\alpha) = (i_*(\alpha), -j_*(\alpha))$, *with inclusions* $i : A \cap B \hookrightarrow A, j : A \cap B \hookrightarrow B$.
- $\Psi(\alpha, \beta) = k_*(\alpha) + l_*(\beta)$, *with inclusions* $k : A \hookrightarrow X, l : B \hookrightarrow X$.

There are additional consequences if we only consider the category of CW pairs (X, A) where A is a closed subcomplex in X.

Theorem 2.2.10 (`See page 14 of [CF64]`). *If $\varphi : (X, A) \to (Y, B)$ is a relative homeomorphism between CW-pairs, then it induces an isomorphism on the homology groups (for all $n \in \mathbb{Z}$):*

$$\varphi_* : h_n(X, A) \xrightarrow{\approx} h_n(Y, B)$$

A simple consequence is:

Corollary 2.2.10.1 (`See page 14 of [CF64]`). *For a CW pair*[3] (X, A) *there is an isomorphism (for all $n \in \mathbb{Z}$):*

$$h_n(X, A) \approx \tilde{h}_n(X/A)$$

By convention $X/\emptyset$ is the disjoint union of X with a point.

A similar result can be achieved using the concept of good pairs:

Definition (`See` (1.4.5)). A pair of topological spaces (X, A) is a good pair, if A is a nonempty closed subspace in X such that there exists an open set $A \subset V \subset X$, which deformation retracts onto A.

CW pairs (where the subspace is a subcomplex) are good pairs (B.0.7).

Theorem (`See` (1.4.4)). *For good pair (X, A) the quotient map $q : (X, A) \to (X/A, A/A)$ induces an isomorphism on the homology groups (for all $n \in \mathbb{Z}$):*

$$q_* : h_n(X, A) \xrightarrow{\approx} h_n(X/A, A/A)$$

Finally, the first four axioms also guarantee the existence of spectral sequences for CW complexes. For details, see section §2.3.

$\sim * \sim$

Now let us do some calculation using the results above.

Claim 2.2.11 (`Thm[ES52]:16.6`[4]). *1.* $\tilde{h}_n(S^k) \approx h_{n-k}(*)$

2. $h_n(S^k) \approx h_n(*) \oplus h_{n-k}(*)$

3. $h_n(I^k, S^{k-1}) \approx \tilde{h}_{n-1}(S^{k-1}) \approx h_{n-k}(*)$

By combining the relative homeomorphism property (2.2.10) and the result above (2.2.11), we get:

Claim 2.2.12. *For a CW complex X, we have*

$$h_n(\operatorname{sk}_k(X), \operatorname{sk}_{k-1}(X)) \approx \bigoplus_{i=1}^{M} h_n(I^k, S^{k-1}) \approx \bigoplus_{i=1}^{M} h_{n-k}(*),$$

if $M \in \mathbb{N}$ is the number of k-dimensional cells in X, and $\operatorname{sk}_k(X)$ is the k-skeleton.

[3] This still means that A is a closed subcomplex!

2.3 Spectral sequences

Sources. *The contents of this section are based on the spoken word introduction by my advisor Tamás Terpai, which were in turn based on page 1 of some lecture notes from ELTE [SJ]. This page is a short overview of chapter 15 of [Swi75]. However, I did not use [Swi75] for writing this thesis. The descriptions of some concepts and ideas for diagrams were borrowed from [Lyc16]. There is no citation for theorems in (§2.3.2), as that section is based entirely on the aforementioned spoken word introduction and [SJ]. For a proper source, see [Swi75], [McC01] or [Eil51].*

Generalized homology theories (which do not satisfy (ES.7), the dimension axiom) sometimes require additional tools to keep computations feasible, as the generalized homology groups even of simple spaces may be hard to compute.

One such algebraic tool is the spectral sequence of a CW complex. For now, let us ignore the "CW complex" part, and only focus on what this tool has to offer in an algebraic sense. Like a long exact sequence, a spectral sequence can be thought of as a commutative diagram with some additional connections between its objects and arrows. As it is a part of homological algebra, computations with it have a similar "feeling" to them as long exact sequences do. However, there are two important distinctions to be made between the tools mentioned. For one, both the definition of and the data contained in a spectral sequence is far-far more complex than for long exact sequences. The other – more important – contrast between the two is that while long exact sequences are used to justify **single steps** of calculations **in the context** of a larger diagram:

... as the diagram is exact here, we know that b maps to 0 ...,

spectral sequences **are these contexts**, these larger diagrams, in which **many steps** of calculation take place.

$\sim * \sim$

The motivation for these algebraic objects can be partially explained with the underlying topology. First, we will approach the topic from the viewpoint of an application of our spectral sequence: in this exposition the spectral sequence is presented as a complex tool for computing homology groups. In a remark at the end of the subsection we note that the groups which build up the sequence are also interesting in their own right: from this viewpoint the spectral sequence can be considered a collection of information creating connections between these objects.

$\sim * \sim$

As stated earlier, calculating the generalized homology groups can become troublesome for even simple spaces. As a result, we give up on determining these groups exactly and instead try to approximate them via filtrations. So instead of computing some abelian group G which arises as a generalized homology group, we determine the quotients in some filtration $0 = G_{-1} \subset G_0 \subset G_1 \subset \ldots \subset G_{n-1} \subset G_n = G$: this still provides surprisingly large amounts of information on the group G, especially with field coefficients.

When working with CW complexes and aiming for the computation of $h_n(X)$, it is natural to consider the k-skeletons of X: $\mathrm{sk}_k(X)$. While the k-skeleton's homology groups are not subgroups of $h_n(X)$, we can just take their image in it. Using this, we receive a filtration of $h_n(X)$ in a simple way. Now we only have to hope that the quotients will be calculable in some way.

It is clear that next we should be computing $h_n(\mathrm{sk}_k(X))$, or at least its image in $h_n(X)$. This will be done only in an approximate manner, but it is going to be seen that this does not pose a problem in calculating the quotients mentioned above. The main idea is that we compute $h_n(\mathrm{sk}_k(X))$ using the cycles-boundaries approach[5]: for each $r \in \mathbb{N}$ we take

[5]The general homology theory in question may lack any notion of cycles and boundaries, but this is a good mental image about the process.

- groups of *"nearly-cycles"*, which contain objects that are closer and closer to "real cycles" as r grows,

- and groups of *"restricted-boundaries"*, which contain more and more of the "real boundaries" as r grows.

Hence as r increases, the quotient of these two groups will get "closer and closer" to $h_n(\mathrm{sk}_k(X))$. The spectral sequence will be constructed from these groups, and parts of its ∞-page – which can be regarded as its *limit* – are indeed going to be the quotients of the filtration of $h_n(X)$ we were looking for.

Remark. To create a setting where we can relax "cycles" to "nearly-cycles", instead of directly approximating $h_n(\mathrm{sk}_k(X))$, we will be approximating its image in $h_n(\mathrm{sk}_k(X), \mathrm{sk}_{k-1}(X))$: in this group there are many more "cycles" than in $h_n(\mathrm{sk}_k(X))$ but we can restrict ourselves closer and closer to its image. Thanks to the algebraic trick of the bow tie lemma (A.2.2), it suffices to work in this environment.

It should also be noted that these quotients are interesting in their own right. Even when working with an infinite-dimensional CW complex, one typically only considers its skeletons in computations. It is then natural to ask what different homology groups would look like if we only cared about a "section" of the CW complex: ignoring everything below and above certain dimensions. In this sense, these groups are similar to the groups $h_n(\mathrm{sk}_k(X), \mathrm{sk}_l(X))$, and the spectral sequence is a collection of information drawing connections between them.

Remark. Up to now, we made no convincing point that any of the structures detailed above will indeed be computable. This last step of working inside $h_n(\mathrm{sk}_k(X), \mathrm{sk}_{k-1}(X))$ implies that the crudest approximation of $h_n(\mathrm{sk}_k(X))$'s image will be $h_n(\mathrm{sk}_k(X), \mathrm{sk}_{k-1}(X))$ itself. While we do not necessarily understand simple spaces, we do understand the generalized homology of spheres, as seen in the calculations of (2.2.11). This hints us that we will be able to compute the homology groups $h_n(\mathrm{sk}_k(X), \mathrm{sk}_{k-1}(X))$ for a CW complex X, and we can hope that the inner structure of a spectral sequence will enable us to proceed to better approximations of $h_n(\mathrm{sk}_k(X))$ – and finally the quotients in the filtration of $h_n(X)$.

Finally, a reminder about the general nature of mathematics:

Remark. Spectral sequences can be interpreted and constructed differently than the way discussed here. There are also many different spectral sequences (named after many well-known mathematicians) just inside algebraic topology. For more on the subject, see [McC01] or [Lyc16].

2.3.1 Algebraic definition

We will discuss *homological spectral sequences* in the category of *abelian groups* in this section. For generalizations, see [McC01] or [Lyc16]. To skip the introductory definition, go to (2.3.1), the concise definition.

$$\boxed{\text{page } 0}, \boxed{\text{page } 1}, \boxed{\text{page } 2}, \ldots, \boxed{\text{page } \infty}$$

The groups that make up a spectral sequence are organized into $\omega + 1$ *pages*. In other words, they are partitioned into sets – 1 set for each natural number, and 1 additional for ∞. This is done because no homomorphisms go between groups on different pages (but there is some additional data which connects them!). No homomorphisms go from or to the groups on page ∞ and they can be easily determined from previous pages[6], so we shall ignore them temporarily.

$$\boxed{\text{page 3}}, \boxed{\text{page 4}}, \boxed{\text{page 5}}, \ldots$$

Next, we note that the first few pages may be "missing": the pages of a spectral sequence need to be defined only for sufficiently large n. All homomorphisms on the rth page will be denoted d^r, and called the differential homomorphisms. If we are only interested in the graph of these homomorphisms, we see that it is just a union of infinitely many infinite paths. Furthermore, each of these paths is actually a **chain complex**.

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

However, the graph of homomorphisms is not the only important data about the groups of a single page. The groups of a page have a one-to-one correspondence to the lattice points of the plane: $\mathbb{Z} \times \mathbb{Z}$. The two coordinates will usually be denoted by p and q, so a group on the rth page will be denoted by E^r_{pq}. For a *homological spectral sequence/first quadrant spectral sequence* – the type of spectral sequence we are concerned with – on any page only the groups in the first quadrant (with non-negative coordinates) can be non-zero:

$$\begin{array}{ccccc}
0 & 0 & E^r_{0,2} & E^r_{1,2} & E^r_{2,2} \\
0 & 0 & E^r_{0,1} & E^r_{1,1} & E^r_{2,1} \\
0 & 0 & E^r_{0,0} & E^r_{1,0} & E^r_{2,0} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}$$

We still need to connect the coordinate system of the lattice to the graph of homomorphisms. On the rth page, exactly 1 homomorphism goes from each group, as per the graph's description above. Each of these homomorphisms go to the group r steps west and $r-1$ steps north. If we would like to differentiate between the homomorphisms of a single page, we may write d^r_{pq} for the homomorphism coming from E^r_{pq}. Combining all of this information (and replacing E^r_{pq} with $\bullet$ for clarity), we get that a spectral sequence looks somewhat like figure 2.1.

There is one last crucial property of spectral sequences. As the spectral sequence decomposes into a union of chain complexes, its homology groups can be defined. The homology group at the group E^r_{pq} is by definition the group E^{r+1}_{pq}. In other words, if we calculate the homology groups of a page we get the next page of the spectral sequence.

This concludes the introductory definition of a spectral sequence.

$\sim * \sim$

A somewhat more concise definition is as follows:

[6] Because of this property, the ∞-page can be regarded as an object derived from the spectral sequence, instead of a part of it.

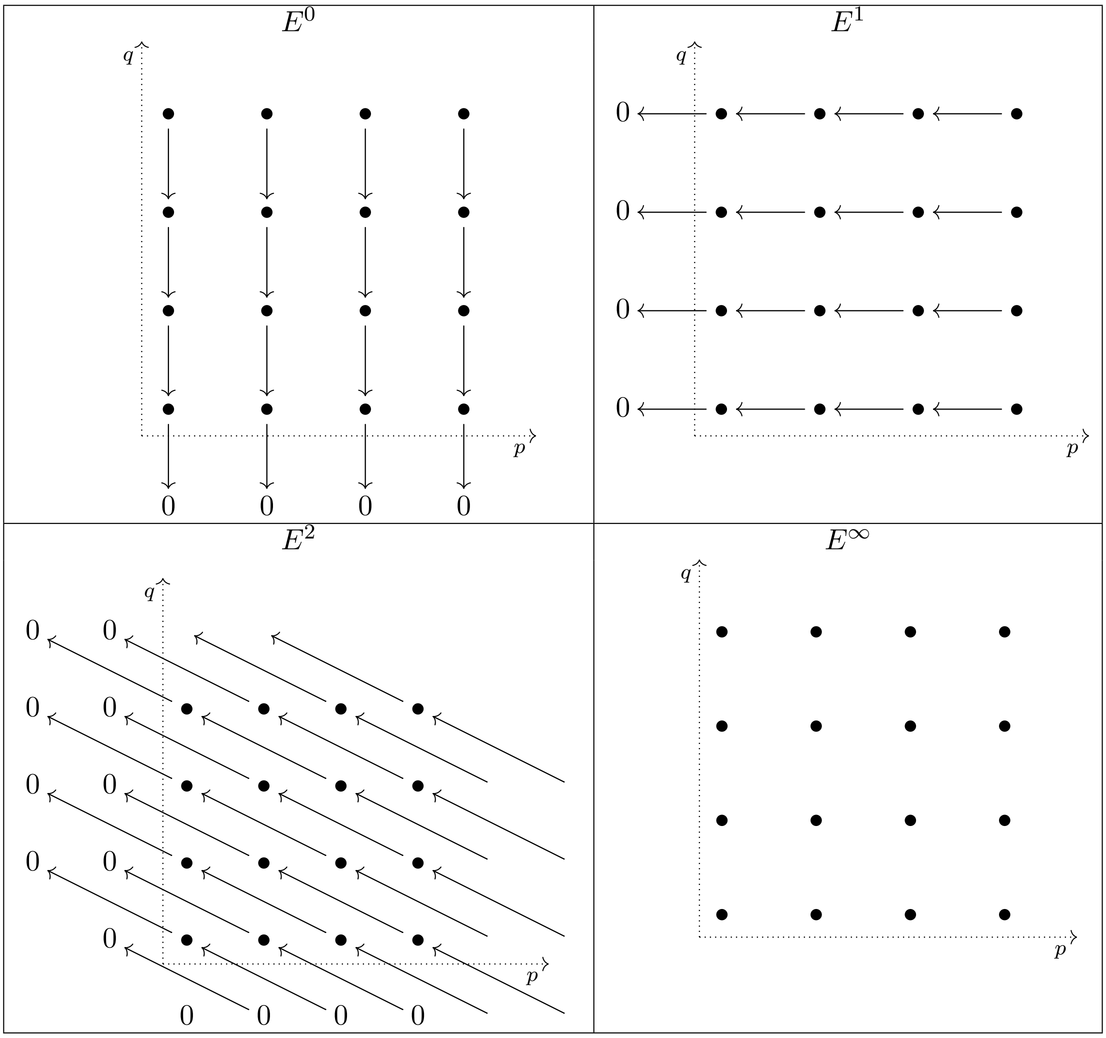


Figure 2.1: Illustration of a spectral sequence.

Definition 2.3.1 (`Def[Lyc16]:1`)**.** A (homological/first quadrant) spectral sequence consists of the following:

- an $a \in \mathbb{N}$, the index of the first defined page,
- abelian groups E^r_{pq} for each $r \geq a$ and $p, q \in \mathbb{Z}$, and
- homomorphisms $d^r_{pq} : E^r_{pq} \to E^r_{p-r,q+r-1}$ for each $r \geq a$ and $p, q \in \mathbb{Z}$.

Moreover, we suppose these have the following three properties:

1. (**Homological/first quadrant**:) $E^r_{pq} = 0$ for each $r \geq a$ and $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ where $p < 0$ or $q < 0$,
2. $d^r \circ d^r = d^r_{p-r,q+r-1} \circ d^r_{p,q} = 0$ for each $r \geq a$ and $p, q \in \mathbb{Z}$, and
3. the homology group $\operatorname{Ker} d^r_{pq} / \operatorname{Im} d^r_{p+r,q-r+1}$ at E^r_{pq} is the group E^{r+1}_{pq} for each $r \geq a$ and $p, q \in \mathbb{Z}$.

Remark. The *groups* of a page can be calculated from the previous one, but the *differentials* can not.

To derive the ∞-page, we first make an observation:

Claim 2.3.2 (`See page 2 of [Lyc16]`)**.** *For sufficiently large r in a **homological** spectral sequence we have*

$$E^r_{pq} \approx E^{r+1}_{pq} \approx E^{r+2}_{pq} \approx E^{r+3}_{pq} \approx \dots$$

for any $p, q \in \mathbb{Z}$. $r = \max(p+1, q+2)$ is sufficiently large.

Definition 2.3.3 (`See page 2 of [Lyc16]`)**.** Let E^∞_{pq} be the stabilized group $E^{\max(p+1,q+2)}_{pq}$, the group from which point on $E^r_{pq} \approx E^{r+1}_{pq} \approx \dots$

The spectral sequences we will work with have an additional property:

Definition 2.3.4 (`Def[Lyc16]:2`)**.** Let G_n be an abelian group and

$$0 = (G_n)_{-1} \subset (G_n)_0 \subset (G_n)_1 \subset \dots \subset (G_n)_{n-1} \subset (G_n)_n = G_n$$

be a filtration for each $n \in \mathbb{N}$. We say that a (homological) spectral sequence E^a_{pq} converges to G_{p+q}, denoted $E^a_{pq} \implies G_{p+q}$, iff

$$E^\infty_{pq} \approx (G_{p+q})_p / (G_{p+q})_{p-1}$$

$$E^\infty : \quad \begin{array}{r|cccccc}
n & (G_n)_0/(G_n)_{-1} \approx (G_n)_0 & & & & & \\
n-1 & & (G_n)_1/(G_n)_0 & & & & \\
n-2 & & & (G_n)_2/(G_n)_1 & & & \\
\vdots & & & & \ddots & & \\
1 & & & & & (G_n)_{n-1}/(G_n)_{n-2} & \\
q=0 & & & & & & (G_n)_n/(G_n)_{n-1} \approx G_n/(G_n)_{n-1} \\
\hline
 & p=0 & 1 & 2 & \cdots & n-1 & n
\end{array}$$

In what follows, we will use the notation $F_{pq} = (G_{p+q})_p$. With that, the previous figure[7] becomes:

$$E^\infty : \quad \begin{array}{r|cccccc}
n & F_{0,n}/F_{-1,n+1} \approx F_{0,n} & & & & & \\
n-1 & & F_{1,n-1}/F_{0,n} & & & & \\
n-2 & & & F_{2,n-2}/F_{1,n-1} & & & \\
\vdots & & & & \ddots & & \\
1 & & & & & F_{n-1,1}/F_{n-2,2} & \\
q=0 & & & & & & F_{n,0}/F_{n-1,1} \\
\hline
 & p=0 & 1 & 2 & \cdots & n-1 & n
\end{array}$$

2.3.2 Construction for complexes

We will associate a spectral sequence to a CW complex. In the construction itself the CW structure is almost irrelevant: in reality, the sequence is associated to an arbitrary filtration $\emptyset = X^{-\infty} \subset \dots \subset X^{p-1} \subset X^p \subset X^{p+1} \subset \dots \subset X^\infty = X$ of a space X. If X is a CW complex, let $X^p = \mathrm{sk}_p(X)$ be the p-skeleton of X. While this depends greatly on the CW structure, it turns out that the E^r_{pq} produced by the following process is independent of the particular cell subdivision for $r \geq 2$.

After choosing a filtration $\emptyset = X^{-\infty} \subset \dots \subset X^{p-1} \subset X^p \subset X^{p+1} \subset \dots \subset X^\infty = X$ like above, the remainder of the construction is purely algebraic. Nevertheless, we will attempt to reinforce the geometric interpretations from the beginning of the section.

Remark. The condition "$X^{-\infty}$ is empty" is not actually necessary in what follows: it could be replaced by any other set. As assuming this equality makes geometric interpretation much easier, but overall it is superfluous, we will put $\emptyset := X^{-\infty}$. This means that in this section by the symbol $\emptyset$ we do not mean the empty set, but the space $X^{-\infty}$. Take special care when dealing with absolute homology groups: in reality $h_n(X^p)$ is none other than $h_n(X^p, X^{-\infty})$. Replacing $X^{-\infty}$ with something else will come in handy when discussing relative spectral sequences.

[7] These figures were copied from [Lyc16].

The main theorem of this subsection can be stated using many different levels of assumptions. In all of them, **only the first four Eilenberg-Steenrod** axioms of section §2.1 need to be satisfied by h_*. While this is irrelevant in the proofs, always consider $X^k = \mathrm{sk}_k(X)$ for some CW complex X.

Theorem 2.3.5. *There is a* ***general*** *spectral sequence E^r_{pq} (general means that E^r_{pq} is not necessarily 0 for negative p or q) associated to a filtration $\emptyset = X^{-\infty} \subset \ldots \subset X^{p-1} \subset X^p \subset X^{p+1} \subset \ldots \subset X^\infty = X$. This is defined by:*

$$\begin{aligned} Z^r_{pq} &= \mathrm{Im}\left(j_* : h_{p+q}(X^p, X^{p-r}) \to h_{p+q}(X^p, X^{p-1})\right) \\ B^r_{pq} &= \mathrm{Im}\left(\partial : h_{p+q+1}(X^{p+r-1}, X^p) \to h_{p+q}(X^p, X^{p-1})\right) \\ E^r_{pq} &= Z^r_{pq}/B^r_{pq}. \end{aligned}$$

Specifically, we have $E^1_{pq} = h_{p+q}(X^p, X^{p-1})$.

Theorem 2.3.6. *Assume $h_n(X^k) = 0$ for all $n < 0, k \in \mathbb{Z} \cup \{\pm\infty\}$ for a filtration $\emptyset = X^{-\infty} \subset \ldots \subset X^{p-1} \subset X^p \subset X^{p+1} \subset \ldots \subset X^\infty = X$. Then*

- *the spectral sequence of* (2.3.5) *is a* ***homological*** *spectral sequence: if p or q is negative, then $E^r_{pq} = 0$,*
- *for fixed (p, q), the groups $(E^r_{pq})_{r \geq 0}$ stabilize after a sufficiently large r, defining the ∞-page.*

Theorem 2.3.7. *Assume $h_n(X^k) = 0$ for all $n < 0, k \in \mathbb{Z} \cup \{\pm\infty\}$ and $h_n(X, X^n) = 0$ for all n. Then for the homological spectral sequence of* (2.3.5) *and* (2.3.6) *we have $E^r_{pq} \implies h_{p+q}(X)$, so it converges to the groups $h_{p+q}(X)$ in the sense of* (2.3.4) *with the filtration $0 = F_{-1,n+1} \subset F_{0,n} \subset F_{1,n-1} \subset \ldots \subset F_{n,0} = h_n(X)$, where*

$$F_{pq} = \mathrm{Im}\left(i_* : h_{p+q}(X^p) \to h_{p+q}(X)\right).$$

Remark 2.3.7.1. In some generalized settings, it may make sense to replace the assumption *"$h_n(X^k) = 0$ for $n < 0, k \in \mathbb{Z} \cup \{\pm\infty\}$"* by *"for each n we have $h_n(X^k) = 0$ for sufficiently small k"*, and the assumption *"$h_n(X, X^n) = 0$ for all n"* by *"for each n we have $h_n(X, X^k) = 0$ for sufficiently large k"*. These yield similar – though not identical – results to those above. For details, see [Eil51].

Remark 2.3.7.2. We note once again that in this section $\emptyset$ means $X^{-\infty}$, an arbitrary subspace. So absolute homology groups in theorems 2.3.6 and 2.3.7 mean relative homology groups by $X^{-\infty}$: $h_n(X^k) = h_n(X^k, X^{-\infty})$, $h_n(X) = h_n(X, X^{-\infty})$, $h_{p+q}(X^p) = h_{p+q}(X^p, X^{-\infty})$ and $h_{p+q}(X) = h_{p+q}(X, X^{-\infty})$.

Of course, the theorems are true with $\emptyset$ being the empty set: this gives the absolute spectral sequences.

$\sim * \sim$

Only (2.3.7) will be proved here. The intermediate results of (2.3.5) and (2.3.6) can be extracted from the thought process below. The bow tie lemma (A.2.2) will be of great use in this section.

Now let us recall the central objects from which we will construct the spectral sequence:

Definition 2.3.8. Put

- $Z^r_{pq} = \mathrm{Im}\left(j_* : h_{p+q}(X^p, X^{p-r}) \to h_{p+q}(X^p, X^{p-1})\right)$,
- $B^r_{pq} = \mathrm{Im}\left(\partial : h_{p+q+1}(X^{p+r-1}, X^p) \to h_{p+q}(X^p, X^{p-1})\right)$,
- $F_{pq} = \mathrm{Im}\left(i_* : h_{p+q}(X^p) \to h_{p+q}(X)\right)$, and
- $E^r_{pq} = Z^r_{pq}/B^r_{pq}$.

Specifically:

- $Z^\infty_{pq} = \mathrm{Im}\left(j_* : h_{p+q}(X^p) \to h_{p+q}(X^p, X^{p-1})\right)$
- $B^\infty_{pq} = \mathrm{Im}\left(\partial : h_{p+q+1}(X, X^p) \to h_{p+q}(X^p, X^{p-1})\right)$
- $E^\infty_{pq} = Z^\infty_{pq}/B^\infty_{pq}$

Note that this E^∞_{pq} is not the same as the one defined in (2.3.3), and that ∂ comes from the exact sequence of the triple: (2.2.4).

The E^r_{pq}'s are well defined, as the following relations are present between these groups:

Claim 2.3.9. $0 = B^1_{pq} \subset B^2_{pq} \subset \ldots \subset B^\infty_{pq} \subset Z^\infty_{pq} \subset \ldots \subset Z^2_{pq} \subset Z^1_{pq} = h_{p+q}(X^p, X^{p-1})$

Verifying this is fairly straightforward using (ES.3) and the functoriality of h_*.
These groups are what have been introduced informally in the first – motivational – part of the section. So the they can be interpreted as:

- F_{pq} form the filtrations through which we aim to understand the structure of $h_n(X)$. For this, see the next claim.
- Z^r_{pq} are the "nearly-cycles" approximating $h_n(X^p)$ in $h_n(X^p, X^{p-1})$.
- B^r_{pq} are the "restricted-boundaries" of $h_n(X^p)$ in $h_n(X^p, X^{p-1})$.
- E^r_{pq} are the approximations of "$h_n(X^p)$ in $h_n(X^p, X^{p-1})$".
- E^∞_{pq} can be regarded as "the best approximation of $h_n(X^p)$ in $h_n(X^p, X^{p-1})$" using its definition above. However, we should note that (assuming the collection of E^r_{pq}'s form a spectral sequence as defined in (2.3.1)) a *different* object can also be denoted the same way: the "limit group" of $(E^r_{p,q})_{r\geq 0}$, the stabilized group of (2.3.3). These two thankfully coincide according to the next claim.

The next two claims are the ones which require the assumptions of (2.3.6) and (2.3.7).

Claim 2.3.10. *For fixed (p, q) we have*

(a) $Z^r_{pq} \approx Z^\infty_{pq}$ for sufficiently large r (by the extra assumption of (2.3.6)),

(b) $B^r_{pq} \approx B^\infty_{pq}$ for sufficiently large r (by the extra assumption of (2.3.7)), and

(c) $E^r_{pq} \approx E^\infty_{pq}$ for sufficiently large r.

The next claim states that F_{pq} really do form a filtration of $h_n(X)$. This also depends on the extra assumption of (2.3.7):

Claim 2.3.11. $0 = F_{-1,n+1} \subset F_{0,n} \subset F_{1,n-1} \subset \ldots \subset F_{n,0} = \mathrm{Im}\left(i_* : h_n(X^n) \to h_n(X)\right) = h_n(X)$

At this point, we have two things missing: we have to see that the quotients of the filtrations defined by F_{pq} are indeed the elements of the ∞-page, and prove that the groups E^r_{pq} have the structure of a spectral sequence. This latter task ensures that we can use this inner structure in later calculations. First, let us settle the question regarding the filtration – as a direct application of the bow tie lemma (A.2.2).

Claim 2.3.12. $F_{pq}/F_{p-1,q+1} \approx E^\infty_{pq}$

Proof. By applying the bow tie lemma (A.2.2) as follows:

$$\begin{tikzcd}
 & h_{p+q+1}(X, X^p) \arrow[d] \arrow[dr, "\partial"] & \\
h_{p+q}(X^{p-1}) \arrow[r] \arrow[dr, "i_1"'] & h_{p+q}(X^p) \arrow[r] \arrow[d, "i_2"] & h_{p+q}(X^p, X^{p-1}) \\
 & h_{p+q}(X) &
\end{tikzcd}$$

□

Now that this is settled, we only have to define differential homomorphisms $d^r_{pq} : E^r_{pq} \to E^r_{p-r,q+r-1}$, and show that these form chain complexes whose homology groups are the next page. These differential homomorphisms will be constructed as compositions of three homomorphisms, one of which is the following one, given by the bow tie lemma (A.2.2):

Lemma 2.3.13. $Z^r_{pq}/Z^{r+1}_{pq} \approx B^{r+1}_{p-r,q+r-1}/B^r_{p-r,q+r-1}$

Proof. By applying the bow tie lemma (A.2.2) as follows:

$$\begin{tikzcd}
 & h_{p+q}(X^p, X^{p-r-1}) \arrow[d] \arrow[dr, "j_{1*}"] & \\
h_{p+q}(X^{p-1}, X^{p-r}) \arrow[r] \arrow[dr, "\partial_1"'] & h_{p+q}(X^p, X^{p-r}) \arrow[r, "j_{2*}"'] \arrow[d, "\partial_2"] & h_{p+q}(X^p, X^{p-1}) \\
 & h_{p+q-1}(X^{p-r}, X^{p-r-1}) &
\end{tikzcd}$$

□

Finally we can define d^r:

Definition 2.3.14.

$$\begin{tikzcd}
Z^r_{pq}/B^r_{pq} \arrow[r, two heads] \arrow[d, equal] & Z^r_{pq}/Z^{r+1}_{pq} \approx B^{r+1}_{p-r,q+r-1}/B^r_{p-r,q+r-1} \arrow[r, hook] & Z^r_{p-r,q+r-1}/B^r_{p-r,q+r-1} \arrow[d, equal] \\
E^r_{pq} \arrow[rr, "d^r"] & & E^r_{p-r,q+r-1}
\end{tikzcd}$$

Claim 2.3.15. *(a)* $\operatorname{Ker} d^r = Z^{r+1}_{pq}/B^r_{pq}$

(b) $\operatorname{Im} d^r = B^{r+1}_{p-r,q+r-1}/B^r_{p-r,q+r-1}$

(c) $\operatorname{Im} d^r \subset \operatorname{Ker} d^r$

(d) $\operatorname{Ker} d^r / \operatorname{Im} d^r = Z^{r+1}_{pq}/B^{r+1}_{pq} = E^{r+1}_{pq}$

So the E^r_{pq}'s together with the d^r's form a spectral sequence, which converges to $h_n(X)$ by the definition of convergence (2.3.4).

2.3.3 Relative construction

To obtain a relative spectral sequence which converges to the groups $h_n(X, A)$, put $X^k = \mathrm{sk}_k(X) \cup A$ for all $k \in \mathbb{Z}$, and $X^{-\infty} = A, X^\infty = X$. This overrides remark 2.3.7.2 about the $\emptyset$ notation. The relative forms of the spectral sequence theorems can be formulated as follows.

Theorem 2.3.16 (Relative form of (2.3.5))**.** *There is a* ***general***[8] *spectral sequence E^r_{pq} associated to the filtration $A = X^{-\infty} \subset \ldots \subset X^p = \mathrm{sk}_p(X) \cup A \subset \ldots \subset X^\infty = X$. This is defined by:*

$$
\begin{aligned}
Z^r_{pq} &= \mathrm{Im}\left(j_* : h_{p+q}(X^p, X^{p-r}) \to h_{p+q}(X^p, X^{p-1})\right) \\
B^r_{pq} &= \mathrm{Im}\left(\partial : h_{p+q+1}(X^{p+r-1}, X^p) \to h_{p+q}(X^p, X^{p-1})\right) \\
E^r_{pq} &= Z^r_{pq}/B^r_{pq}.
\end{aligned}
$$

Specifically, we have $E^1_{pq} = h_{p+q}(X^p, X^{p-1})$.

Theorem 2.3.17 (Relative form of (2.3.6))**.** *Assume $h_n(X^k) = 0$ for all $n < 0, k \in \mathbb{Z} \cup \{\pm\infty\}$ for the filtration $A = X^{-\infty} \subset \ldots \subset X^p = \mathrm{sk}_p(X) \cup A \subset \ldots \subset X^\infty = X$. Then*

- *the spectral sequence of* (2.3.16) *is a* ***homological*** *spectral sequence: if p or q is negative, then $E^r_{pq} = 0$,*
- *for fixed (p, q), the groups $(E^r_{pq})_{r \geq 0}$ stabilize after a sufficiently large r, defining the ∞-page.*

Theorem 2.3.18 (Relative form of (2.3.7))**.** *Assume $h_n(X^k) = 0$ for all $n < 0, k \in \mathbb{Z} \cup \{\pm\infty\}$ and $h_n(X, X^n) = 0$ for all n. Then for the homological spectral sequence of* (2.3.16) *and* (2.3.17) *we have $E^r_{pq} \implies h_{p+q}(X, A)$, so it converges to the groups $h_{p+q}(X, A)$ in the sense of* (2.3.4) *with the filtration $0 = F_{-1,n+1} \subset F_{0,n} \subset F_{1,n-1} \subset \ldots \subset F_{n,0} = h_n(X, A)$, where*

$$
F_{pq} = \mathrm{Im}\left(i_* : h_{p+q}(X^p, A) \to h_{p+q}(X, A)\right).
$$

[8] *General* means that E^r_{pq} is not necessarily 0 for negative p or q.

Chapter 3

Singular cohomology

Sources. *This chapter is based entirely on [Hat02], particularly on section §3.1.*

3.1 The definition

Singular cohomology is algebraically extremely similar to singular cohomology. The main difference is that cohomology is a *contravariant* functor – that is, to a map $f : X \to Y$ it assigns a homomorphism of the cohomology groups in the other direction $f^* : H^n(Y) \to H^n(X)$. This, in many ways, makes cohomology more pleasant to work with. However, many of its great consequences – such as the existence of the cup product – will mostly be omitted from this thesis for lack of time.

While on one hand cohomology is easier to work with algebraically, on the other hand it is harder to explain it geometrically. There are two ways in which we will attempt to approach the subject: from the viewpoint of Poincaré duality and from the viewpoint of the definition.

For sufficiently nice spaces (here: topological manifolds), there exists a duality relationship between homology and cohomology. While more complicated in reality, intuitively k-dimensional homology classes in an n-manifold can be thought of as embedded submanifolds of dimension k, while cohomology is related to homomorphisms of the kth homology group to some abelian group G. One

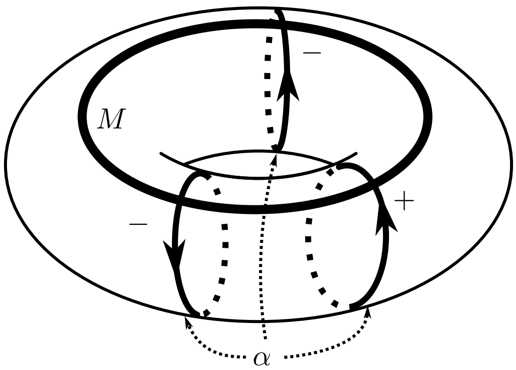


Figure 3.1: One positive and two negative intersections.

easy way to create a homomorphism of the kth homology group to let's say $\mathbb{Z}$ is to take an embedded submanifold M of *codimension* k (so dimension $n-k$), and count (with signs) the number of intersections of the representing "submanifold" of some $\alpha \in H_k(X)$ with M. Of course, if we take an ugly representative of α, the number of intersections with M may be undefined. Thankfully, we *can* take a representative for which this number is well defined, so we can construct homomorphisms this way. It turns out that for each cohomology class γ – which once again, correspond to some homomorphism $H_k(X) \to G$ – there is an "embedded $(n-k)$-submanifold M", or more precisely a homology class

$\beta \in H_{n-k}(X)$ from which we can construct the homomorphism that corresponds to γ. In some sense, this is the statement of the Poincaré duality. Poincaré duality is detailed in chapter §4. For a better and more in-depth explanation of this duality as a motivation of cohomology, see the introductory text (pages 185–189) of chapter 3 in [Hat02].

$\sim * \sim$

A different route we can take is simple *algebraic dualization*, which leads to the exact definition of cohomology. We already mentioned that

- elements of the cohomology groups correspond to homomorphisms to some abelian group G,
- cohomology is a contravariant functor, and
- there is some sort of duality relationship between homology and cohomology.

After this, it may not be surprising that the $\mathrm{Hom}(-, G)$ functor (for some fixed abelian group G; see definition A.1.9) is involved in the construction. It is indeed related to homomorphisms to some abelian group G, it is a contravariant functor (A.1.10), and it is a kind of duality functor.

In view of claim A.1.12, it is easy to guess the definition of singular cohomology.

Definition 3.1.1. Let X be an arbitrary topological space.

(a) Let $C^n(X; G)$ denote $\mathrm{Hom}(C_n(X), G)$, the group of **singular n-cochains with coefficients in G in X**.

(b) Denote the dual map $\partial^* : C^n(X; G) \to C^{n+1}(X; G)$ of $\partial : C_{n+1}(X) \to C_n(X)$ by δ, and call it the **coboundary map**.

(c) The cochain groups $C^n(X; G)$ with the coboundary maps δ form a chain complex, called the **(singular) cochain complex (with coefficient in G) of X**.

(d) The elements of $\mathrm{Ker}\,\delta$ are called **cocycles**, while the elements of $\mathrm{Im}\,\delta$ are called **coboundaries**.

(e) The homology group $\mathrm{Ker}\,\delta / \mathrm{Im}\,\delta$ of the cochain complex $C^*(X; G)$ at $C^n(X; G)$ is called the **nth cohomology of X with coefficients in G**, and is denoted by $H^n(C; G)$.

Remark 3.1.1.1. There are several remarks regarding this definition:

(a) The elements of $C^n(X; G)$ are in one-to-one correspondence with functions from the set of singular n-simplices to G.

(b) If $\psi \in C^n(X; G)$, then $\delta\psi = \psi\partial$:

$$C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\psi} G$$

(c) The cochain groups $C^n(X; G)$ indeed form a chain complex with the coboundary maps, as per claim A.1.12.

(d) A cochain $\psi \in C^n(X; G)$ is a cocycle iff $0 = \delta\psi = \psi\partial$, that is, iff it vanishes on boundaries.

Remark 3.1.1.2. Relative cohomology groups $H^n(X, A; G)$ are defined by dualizing $C_*(X, A)$ with $\mathrm{Hom}(-, G)$ and then taking its homology groups.

Remark 3.1.1.3. Reduced cohomology groups $\tilde{H}^n(X)$ are defined as the homology groups of the dual of the augmented chain complex (1.3.4).

Claim 3.1.2. *Singular cohomology is a contravariant functor.*

This can be derived similarly to the analogous claim for homology.

The universal coefficient theorem (3.2.1) – detailed in section §3.2.1 – gives a quick description of both $H^0(X;G)$ and $H^1(X;G)$ in view of previous statements (1.2.8.1) and (1.2.8.1):

Claim 3.1.3. $H^0(X;G) \overset{(3.2.1)}{\approx} \mathrm{Hom}(H_0(X),G) \overset{(1.2.8.1)}{\approx} \prod_{p\in P} G$*, if P is the set of path-components of X.*

Claim 3.1.4. $H^1(X;G) \overset{(3.2.1)}{\approx} \mathrm{Hom}(H_1(X),G) \overset{(1.2.8.1)}{\approx} \mathrm{Hom}(\pi_1(X),G)$*, as G is abelian.*

We also note that there are axioms which characterize singular cohomology, similar to the Eilenberg-Steenrod axioms (§2.1) of singular homology. These can be found for example in [ES52] (remember that just as for homology, the direct sum axiom is omitted in this book).

3.2 The toolkit

The toolkit of singular cohomology is basically the same as for singular homology, just with all arrows reversed. The following statements typically do not require a direct proof, as they can be obtained purely by dualizing the appropriate claim for homology and if necessary, invoking the naturality of the universal coefficient theorem (3.2.1) of section §3.2.1. For this reason, we will start this section by introducing that theorem and then move on to dualizing the toolkit of singular homology. However, apart from its usefulness in dualization, the universal coefficient theorem is also important from a theoretic point of view: it says that the cohomology groups of a space are determined purely by its homology groups.

3.2.1 Universal coefficient theorem for cohomology

The main purpose of this subsection is to prove the following theorem, which basically states that the homologies of a space determine its cohomologies:

Theorem 3.2.1 (`Thm[Hat02]:3.2`)**.** *If a chain complex C of free abelian groups has homology groups H_n, then the cohomology groups $H^n(C;G)$ are determined by the following split (natural) exact sequence:*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C),G) \longrightarrow H^n(C;G) \xrightarrow{h} \mathrm{Hom}(H_n(C),G) \longrightarrow 0$$

In other words:

$$H^n(C;G) \approx \mathrm{Ext}(H_{n-1}(C),G) \oplus \mathrm{Hom}(H_n(C),G)$$

However, this isomorphism is not natural.

For finitely generated H, one can compute $\mathrm{Ext}(H,G)$ using the three rules below:

- $\mathrm{Ext}(H \oplus H',G) \approx \mathrm{Ext}(H,G) \oplus \mathrm{Ext}(H',G)$
- $\mathrm{Ext}(H,G) = 0$ if H is free
- $\mathrm{Ext}(\mathbb{Z}_n,G) \approx G/nG$

So $\mathrm{Ext}(\mathbb{Z}^r \oplus \bigoplus_{i\in I} \mathbb{Z}_{n_i},G) \approx \bigoplus_{i\in I}(G/n_iG)$. A direct consequence of this is:

Corollary 3.2.1.1 (`Crl[Hat02]:3.3`)**.** *If the homology groups H_n and H_{n-1} of a chain complex C of free abelian groups are finitely generated, with torsion groups $T_n \subset H_n$ and $T_{n-1} \subset H_{n-1}$, then $H^n(C;\mathbb{Z}) \approx (H_n/T_n) \oplus T_{n-1}$*

Combining the naturality property of the short exact sequence in (3.2.1) with the five lemma gives:

Corollary 3.2.1.2 (`Crl[Hat02]:3.4`). *If a chain map between chain complexes of free abelian groups induces an isomorphism on the homology groups of the complexes, then it induces an isomorphism on the cohomology groups with any coefficient group G.*

Along the way to theorem (3.2.1) we will need the following concepts:

Definition 3.2.2. A **free resolution** of an abelian group H is an exact sequence:

$$\dots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \longrightarrow 0\ ,$$

where F_n is free for all n.

Lemma 3.2.3 (`Lem[Hat02]:3.1`). *a) Given free resolutions F and F' of abelian groups H and H', then every homomorphism $\alpha : H \to H'$ extends to a chain map $F \to F'$:*

$$\begin{tikzcd}
\dots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, "\alpha"] & 0 \\
\dots \arrow[r] & F'_2 \arrow[r, "f'_2"] & F'_1 \arrow[r, "f'_1"] & F'_0 \arrow[r, "f'_0"] & H \arrow[r] & 0
\end{tikzcd}$$

Moreover, any two chain maps extending α are chain homotopic.

b) For any two free resolutions F and F' of H, there are canonical isomorphisms $H^n(F;G) \approx H^n(F';G)$ for all n and G.

Claim 3.2.4 (`See page 195 of [Hat02]`). *Every abelian group has a free resolution of the form*

$$0 \longrightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \longrightarrow 0$$

An explicit construction can be given.

Applying b) of (3.2.3) to the specific construction in (3.2.4) we get cohomology groups $H^1(F;G)$ only dependent on H and G:

Definition 3.2.5. The cohomology groups $H^1(F;G)$ are denoted by $\mathrm{Ext}(H,G)$.

The name comes from the fact that $\mathrm{Ext}(H,G)$ can be interpreted as the isomorphism classes of G extended by H, more precisely the set of short exact sequences $0 \longrightarrow H \longrightarrow J \longrightarrow G \longrightarrow 0$ with an equivalence relation induced by isomorphisms between J's.

We also note here that the algebraic machinery above can be generalized from abelian groups to modules over an arbitrary ring.

$\sim * \sim$

The main steps in the proof of (3.2.1) are the following. First we define the homomorphisms in the diagram below, and prove it is commutative:

$$\begin{tikzcd}
H^n(C;G) \arrow[rr, "\mathrm{id}"] \arrow[dr, "h"] & & H^n(C;G) \\
& \mathrm{Hom}(H_n(C),G) \arrow[ur] &
\end{tikzcd} \tag{E3.2.1}$$

This tells us that the following sequence is exact, moreover, it splits:

$$0 \longrightarrow \operatorname{Ker} h \longrightarrow H^n(C;G) \longrightarrow \operatorname{Hom}(H_n(C),G) \longrightarrow 0 \tag{E3.2.2}$$

Clearly our goal at this point is to give an isomorphism $\operatorname{Ext}(H_{n-1}(C),G) \approx \operatorname{Ker} h$. First we will see that $\operatorname{Ker} h \approx \operatorname{Coker} i_n^*$, where $B_n \xrightarrow{i_n} Z_n$. This is done by dualizing the split short exact sequence of chain complexes $0 \longrightarrow Z_\bullet \longrightarrow C_\bullet \longrightarrow B_{\bullet-1} \longrightarrow 0$ into the split short exact sequence (A.1.13) of chain complexes $0 \longleftarrow Z_\bullet^* \longleftarrow C_\bullet^* \longleftarrow B_{\bullet-1}^* \longleftarrow 0$, and applying the zig-zag lemma to get the long exact sequence:

$$\dots \longleftarrow B_n^* \xleftarrow{i_n^*} Z_n^* \longleftarrow H^n(C;G) \longleftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \longleftarrow \dots$$

One can extract the following short exact sequences from this:

$$0 \longleftarrow \operatorname{Ker} i_n^* \longleftarrow H^n(C;G) \longleftarrow \operatorname{Coker} i_{n-1}^* \longleftarrow 0 \tag{E3.2.3}$$

$\operatorname{Ker} i_n^*$ can be naturally identified with $\operatorname{Hom}(H_n(C),G)$. Combining this with (E3.2.2) and (E3.2.3) indeed gives $\operatorname{Ker} h \approx \operatorname{Coker} i_n^*$.
Now we are looking for an isomorphism $\operatorname{Coker} i_{n-1}^* \approx \operatorname{Ext}(H_{n-1}(C),G)$. For this, we note that the short exact sequence

$$0 \longrightarrow B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \longrightarrow H_{n-1}(C) \longrightarrow 0$$

is a free resolution of $H_{n-1}(C)$, moreover, it is the same as the one given in (3.2.4). Finally, $\operatorname{Coker} i_{n-1}^*$ is precisely $H^1(F;G)$ of this free resolution, so the definition of Ext concludes the proof[1].

Now the only substantial step missing from the proof is the definition and commutativity of (E3.2.1). h can be defined as follows: an element of $H^n(C;G)$ is represented by a homomorphism $\psi : C_n \to G$ such that $\delta\psi = \psi\partial = 0$ – so in other words, ψ vanishes on B_n. Thus restricting ψ to Z_n, we get a homomorphism which induces an element $\psi_0 \in \operatorname{Hom}(H_n(C),G)$. To show this is well defined, we have to prove that if $\psi = \delta\phi$, then $\psi_0 = 0$. This however is trivial, as $\delta\phi = \phi\partial$, and ∂ already vanishes on Z_n.

Finally, the unmarked homomorphism is defined as follows. We will assign to each element $\psi_0 \in \operatorname{Hom}(H_n(C),G)$ an element $\psi \in H^n(C;G)$.

$$\begin{tikzcd}
& 0 \arrow[d] \\
& B_n \arrow[d, hook] \\
0 \arrow[r] & Z_n \arrow[r, hook] \arrow[d, two heads] & C_n \arrow[r, two heads, "\partial"] \arrow[d, "\psi"] \arrow[l, bend right, two heads, "p"'] & B_{n-1} \arrow[r] & 0 \\
& H_n(C) \arrow[r, "\psi_0"] \arrow[d] & G \\
& 0
\end{tikzcd}$$

[1] Naturality is inherited from the naturality of each step in the construction of the short exact sequence.

Short exact sequences with a free abelian group at the end split (A.1.7.1), so by the definition of a split exact sequence (A.1.3) p exists, as B_{n-1} is free (because it is a subgroup of the free abelian group C_{n-1} (A.1.8)). Composing ψ_0, $Z_n \twoheadrightarrow H_n(C)$, and p we get ψ. ψ vanishes on B_n by diagram chasing, so it represents an element of $H^n(C;G)$. It is easy to check that $\psi_0 \mapsto \psi$ is a homomorphism $\mathrm{Hom}(H_n(C);G) \to H^n(C;G)$. The composition of (E3.2.1) is indeed the identity, as h simply undoes what the other homomorphism did.

3.2.2 Manipulating chains

From here on, we will just copy the exposition of section §1.4, excluding explanatory texts. Our exposition (and the statements of the theorems) follows pages 198–204 of [Hat02].

Theorem 3.2.6 (Homotopy invariance)**.** *If $f, g : (X, A) \to (Y, B)$ are homotopic maps*[2]*, then $f^* = g^*$ for the induced maps $f^*, g^* : H^n(Y, B) \to H^n(X, A)$.*

Remark 3.2.6.1. Setting $A = B = \emptyset$ yields us the same result for absolute cohomology groups.

Theorem 3.2.6 can be proved by dualizing the formula in the proof of the corresponding theorem for homology (1.4.1):

$$\partial P + P\partial = g_\# - f_\#$$

into

$$\delta P^* + P^*\delta = g^\# - f^\#,$$

then once again applying the lemma regarding chain homotopies (1.2.11).

Corollary 3.2.6.1. *Homotopy equivalent spaces have isomorphic cohomology groups. That is, singular cohomology is a functor of* homotopy type*.*

We note here that both the barycentric subdivision theorem (1.4.2) and its corollary regarding the isomorphism of homology groups (1.4.2.1) can be dualized, as the dual of a chain homotopy equivalence is a chain homotopy equivalence.

Theorem 3.2.7 (Excision)**.** *We state two equivalent forms of the excision theorem.*

(a) Given subspaces $Z \subset A \subset X$ such that the closure of Z is inside $\operatorname{int} A$*, the inclusion $(X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisms*

$$H^n(X, A) \approx H^n(X - Z, A - Z),$$

for all $n \in \mathbb{Z}$.

(b) For subspaces $A, B \subset X$ for whom $\operatorname{int} A \cup \operatorname{int} B \supset X$*, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms*

$$H^n(X, A) \approx H^n(B, A \cap B),$$

for all $n \in \mathbb{Z}$.

This follows directly from the naturality of the universal coefficient theorem (3.2.1): just take the induced homomorphisms between the groups for the inclusion map, and apply the five lemma (A.2.1). Of course, the original proof could be adapted instead of appealing to the universal coefficient theorem (3.2.1).

[2]So $f, g : X \to Y$, and $f(A) \subset B, g(A) \subset B$, and the homotopy $H : X \times [0, 1] \to Y$ connecting f and g maps A to B at all times: $H(A \times [0, 1]) \subset B$.

3.2.3 Exact sequences

Theorem 3.2.8 (LES of pair)**.** *For each pair of spaces (X, A) (that is, spaces such that $A \subset X$) there is associated a long exact sequence of cohomology groups:*

$$\dots \longleftarrow H^{n+1}(X, A; G) \xleftarrow{\delta} H_n(A; G) \longleftarrow H_n(X; G) \longleftarrow H_n(X, A; G) \longleftarrow \dots$$

The dual of a split short exact sequence is a split short exact sequence (by claim A.1.13), so dualizing

$$0 \longrightarrow C_*(A) \longrightarrow C_*(X) \longrightarrow C_*(X, A) \longrightarrow 0 \ ,$$

gives a short exact sequence

$$0 \longleftarrow C^*(A; G) \longleftarrow C^*(X; G) \longleftarrow C^*(X, A; G) \longleftarrow 0 \ ,$$

which yields theorem 3.2.8 by the zig-zag lemma (A.2.3).

Theorem 3.2.9 (Reduced LES of pair)**.** *For each pair of nonempty spaces (X, A) there is associated a long exact sequence of reduced cohomology groups:*

$$\dots \longleftarrow H_{n+1}(X, A; G) \xleftarrow{\delta} \tilde{H}_n(A; G) \longleftarrow \tilde{H}_n(X; G) \longleftarrow H_n(X, A; G) \longleftarrow \dots$$

The reduced cohomology groups are just the homology groups of the dual of the augmented chain complex (1.3.4), so we can just apply the same process as above for this chain complex again to obtain theorem 3.2.9.

Theorem 3.2.10 (LES of triple)**.** *For each triple of spaces (X, A, B) (that is, spaces such that $B \subset A \subset X$) there is associated a long exact sequence of cohomology groups:*

$$\dots \longleftarrow H^{n+1}(X, A; G) \longleftarrow H^n(A, B; G) \longleftarrow H^n(X, B; G) \longleftarrow H^n(X, A; G) \longleftarrow \dots ,$$

where the homomorphism $H^{n+1}(A, B) \to H^n(X, A)$ is defined by the composition:

$$H^{n+1}(X, A; G) \xleftarrow{\delta} H_n(A; G) \longleftarrow H_n(A, B; G) \ .$$

As the following short exact sequence is exact, we can apply the same dualization machinery as for theorem 3.2.8:

$$0 \longrightarrow C_*(A, B) \longrightarrow C_*(X, B) \longrightarrow C_*(X, A) \longrightarrow 0 \ .$$

Theorem 3.2.11 (Mayer-Vietoris sequence)**.** *Suppose the space X is covered by the union* $\operatorname{int} A \cup \operatorname{int} B$ *for some subspaces $A, B \subset X$. Then there is associated a long exact sequence of cohomology groups:*

$$\dots \longleftarrow H^n(A \cap B; G) \xleftarrow{\Phi} H^n(A; G) \oplus H^n(B; G) \xleftarrow{\Psi} H^n(X; G) \xleftarrow{\partial} H^{n-1}(A \cap B; G) \longleftarrow \dots .$$

This comes from the exactness of

$$0 \longleftarrow C^*(A \cap B) \xleftarrow{\phi} C_*(A) \oplus C_*(B) \xleftarrow{\psi} C_*(A + B) \longleftarrow 0 \ ,$$

similarly to the Mayer-Vietoris sequence of homology (1.4.9). As the barycentric subdivision theorem and its corollary also directly dualize, we acquire the Mayer-Vietoris sequence of cohomology.

About the naturality of the sequences:

Theorem 3.2.12. *All previously mentioned long exact sequences are natural:*

- *The LES of a pair of spaces (X, A).*
- *The reduced LES of a pair of nonempty spaces (X, A).*
- *The LES of a triple of spaces (X, A, B).*
- *The Mayer-Vietoris sequence of a decomposition $X = \operatorname{int} A \cup \operatorname{int} B$.*

3.2.4 A connection with homotopy

While previous subsections were concerned with tools which are nearly as important for cohomology as the definition itself, this one introduces a significantly less central theorem. Nevertheless, it is still a topic of great interest and will be of use later in chapter §6. Covering the proof and all of the related concepts is outside the scope of this thesis, so fairly little will be actually mentioned.

In homotopy theory, two maps are not distinguished if they are homotopic. In other words, only homotopy classes of maps are considered. This, among many things, means that homotopy equivalent spaces are indeed equivalent in the eyes of homotopy theory. Another feature of this theory is that the set of homotopy classes of maps between given spaces X and Y frequently appears. This set is typically denoted $[X, Y]$, and in the category of pointed spaces is restricted to only allow homotopies which send the basepoint x_0 to the basepoint y_0 at all times. For instance, the set $[(S^1, s_0), (X, x_0)]_*$ is the well-known fundamental group $\pi_1(X, x_0)$. While the based version $[X, Y]_*$ will be important in (§6.2), in this section the unbased variant is of interest.

This section is concerned with one of the multiple connections between the world of homotopy theory and algebraic topology, formulated by the following theorem.

Theorem 3.2.13 (`Thm[Hat02]:4.57`). *There are natural bijections $T : [X, K(G, n)] \to H^n(X; G)$ for any CW complex X, abelian group G, and $n > 0$. Such a T has the form[3] $T([f]) = f^*(\alpha)$ for a certain distinguished class $\alpha \in H^n(K(G, n); G)$.*

Remark 3.2.13.1. The naturality of these bijections means that if there is a map $f : X \to Y$, then the square on the left of figure 3.2 commutes, where the left vertical map is just the composition

$$\begin{tikzcd}
{[X, K(G,n)]} \arrow[r, "T"] & H^n(X;G) \\
{[Y, K(G,n)]} \arrow[r, "T"] \arrow[u, "\circ[f]"] & H^n(Y;G) \arrow[u, "f^*"']
\end{tikzcd}$$

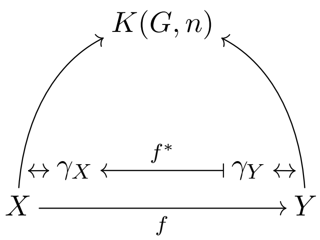


Figure 3.2: Two illustrations of naturality.

by $[f]$ (the homotopy class of f), and the right vertical map is f's induced map on cohomology. Alternatively, this commutativity can be illustrated by the diagram on the right too.

For our purposes it is sufficient to know that the spaces $K(G, n)$ are well-defined – at least up to weak homotopy equivalence. However, we still present a few words about them. For details, see pages 365–366 of [Hat02].

Definition 3.2.14. Let G be a group, $n \geq 1$, and G be abelian if $n > 1$. If for a connected space X we have

$$\pi_k(X) = \begin{cases} 0, & k \neq n \\ G, & k = n \end{cases}$$

then X is said to be an **Eilenberg-MacLane space of type** $K(G, n)$, and is denoted by $K(G, n)$.

For instance:

Example. (a) S^1 is a $K(\mathbb{Z}, 1)$.

(b) $\mathbb{C}\mathrm{P}^\infty$ is a $K(\mathbb{Z}, 2)$.

[3] $[f]$ denotes the homotopy class of f.

The following theorem clarifies why any such space is denoted by the same expression: $K(G,n)$.

Theorem 3.2.15. *For each $n \geq 1$ and group G (G abelian if $n > 1$) there exists a $K(G,n)$ space, unique up to weak homotopy equivalence. There is also a $K(G,n)$ which is a CW complex. Moreover (`prp[Hat02]:4.30`), any two CW complex $K(G,n)$'s are homotopy equivalent.*

Of course, we still have to see what “weak homotopy equivalence” means.

Definition 3.2.16. A continuous mapping $f : X \to Y$ is a weak homotopy equivalence if the induced map on the set of path components is a bijection, while for each $n \geq 1$ and $x \in X$ the following induced homomorphism is an isomorphism:

$$f_* : \pi_n(X,x) \to \pi_n(Y,f(x)).$$

This equivalence is “strong enough” from the perspective of homology and cohomology groups too:

Claim 3.2.17 (`Prp[Hat02]:4.21`)**.** *A weak homotopy equivalence induces isomorphisms of the homology and cohomology groups.*

Moreover, the notions of homotopy equivalence and weak homotopy equivalence coincide for *CW* complexes:

Claim 3.2.18 (`Thm[Hat02]:4.5`)**.** *A weak homotopy equivalence of CW complexes is a homotopy equivalence.*

Chapter 4

Poincaré duality

Sources. *This chapter is based entirely on [Hat02], in particular on section §3.3. To learn more about the direct limits of section §4.2.2, see chapter VIII. of [ES52].*

This section concerns topological manifolds without boundary. The results stated here do not require second-countability (M_2), so throughout this chapter we define manifolds as T_2 spaces locally homeomorphic to $\mathbb{R}^n$. Moreover, coefficients of homology groups will nearly always be omitted from the notation.

The main goal now is to prove (a generalized version) of the following theorem:

Theorem 4.0.1 (Poincaré duality with $R = \mathbb{Z}$ or $\mathbb{Z}_2$; `thm[Hat02]:3.30`). *Let M be a closed n-manifold.*

- *If M is orientable, then there are (natural) isomorphisms $H^k(M) \approx H_{n-k}(M)$.*
- *There are (natural) isomorphisms $H^k(M;\mathbb{Z}_2) \approx H_{n-k}(M;\mathbb{Z}_2)$.*

The proof is constructive.

There are three topics which need to be covered before moving on to this theorem:

- **Orientability of manifolds.** This is of course a prerequisite of stating the theorem above. The definition for *orientability* given here is a generalization of the one for triangulable manifolds.

  We will also examine how an orientation of a manifold M determines an orientation of a given subspace A. In connection with this, we define the *fundamental classes (with coefficients in R)* of a closed connected n-manifold, which are elements of $H_n(M;R)$. A fundamental class can be thought of as "a cycle that is M itself": for a Δ-complex with $R = \mathbb{Z}$ or $\mathbb{Z}_2$ a cycle representing the fundamental class can be given by summing all simplices with appropriate signs.

  Finally, it will be proven that for a closed connected n-manifold M the homology groups of M in dimensions greater than n are all 0.

- **Cohomology with compact supports.** The proof of theorem 4.0.1 stated here uses induction, and the inductive step in turn needs a version of Poincaré duality for open subsets of M. Duality doesn't hold for the usual cohomology groups in this case, but it does when a different type of cohomology, *cohomology with compact supports* is used instead. The next point is concerned with this new duality theorem; in this point, we merely introduce the necessary concepts and the toolkit to deal with them.

  Unlike many other types of (co)homology – such as simplicial, cellular or singular – cohomology with compact supports *does not* define the same groups as before. Moreover, it is no longer a homotopic invariant: $H^0_c(\mathbb{R};\mathbb{Z}) = 0$, while $H^0_c(*;\mathbb{Z}) = \mathbb{Z}$. However, it does coincide with singular cohomology for compact spaces, so it can be deployed in calculations instead of singular cohomology when necessary.

The concept of *direct limits* of (*directed sets*) of groups will also be introduced in this chapter.

- **The cap product.** This is a bilinear map of the form $H_k(M) \times H^l(M) \to H_{k-l}(M)$, which is essential in the construction of the isomorphism in (4.0.1).

Instead of (4.0.1), we will prove a more general statement that holds for compact and noncompact manifolds. This statement is formally the same as the one for compact manifolds, except for the fact that cohomology groups are replaced by cohomology groups with compact supports.

4.1 Orientability of manifolds

In this section we will give an algebraic topological definition of orientations and orientability, and examine how an orientation of a manifold M determines an orientation of a subspace A. As a **notational convention**, let us write $H_n(X, X - A)$ as $H_n(X|A)$, and similarly for cohomology groups.

Definition 4.1.1.
- An orientation of an n-manifold M at a point x is a **generator** of $H_n(M, M-\{x\}) = H_n(M|x) \approx \mathbb{Z}$.
- An orientation of an n-manifold M is a "consistent choice of generators at all points": a function $\mu : x \mapsto \mu_x$ assigning to each point $x \in M$ an orientation at that point $\mu_x \in H_n(M|x)$, satisfying the following local consistency property:

  Each point x has a neighborhood homeomorphic to $\mathbb{R}^n$ with a ball B of finite radius in it ($x \in B$) and a **generator** *$\mu \in H_n(\mathbb{R}^n|B) \approx \mathbb{Z}$ such that for each $y \in B$ μ_y is the image of μ under the canonical isomorphism $H_n(M|y) \approx H_n(\mathbb{R}^n|y) \approx H_n(\mathbb{R}^n|B)$.*
- A manifold M is orientable, if it has an orientation.

The above definition satisfies the following basic property, so it is sensible to define orientations this way:

Claim 4.1.2. *Orientations of $\mathbb{R}^n$ at a point x are preserved by rotations and reversed by reflections.*

In the definition above, we considered groups $H_n(M|x) = H_n(M|x; \mathbb{Z})$. This $\mathbb{Z}$ can be replaced with any commutative ring R with identity, and this creates the notion of *R-orientability*. The two most important cases will be $R = \mathbb{Z}$ and $R = \mathbb{Z}_2$, as any manifold is $\mathbb{Z}_2$-orientable, while any ($\mathbb{Z}$-)orientable manifold will be R-orientable for all R. Using this concept, we can state this section's main theorem:

Theorem 4.1.3 (`Thm[Hat02]:3.26`)**.** *Let M be a closed n-manifold. Then:*

(a) If M is R-orientable, the map $H_n(M; R) \to H_n(M|x; R) \approx R$ is an isomorphism for all $x \in M$.

(b) If M is not R-orientable, the map $H_n(M; R) \to H_n(M|x; R) \approx R$ is injective with image $\{r \in R : 2r = 0\}$.

(c) $H_i(M; R) \approx 0$ for $i > n$.

Using this theorem, we can immediately calculate $H_n(M; \mathbb{Z})$ and $H_n(M; \mathbb{Z}_2)$. We can also deduce, that a fundamental class exists if M is compact and R-orientable, according to the following definition:

Definition 4.1.4. A fundamental class (with respect to R) of an n-manifold M is an element of $H_n(M; R)$ such that its image under the canonical maps $H_n(M; R) \to H_n(M|x; R)$ is a generator for all $x \in R$.

Theorem 4.1.3 follows easily from the following technical lemma. This statement uses the concept of *sections* and a *covering space* M_R, which will be explained later on. For now, we can interpret (a) of the lemma as follows: if for each x we chose a general element of $H_n(M|x;R)$ (not necessarily a generator) in a locally consistent way, then the collection of these choices can be represented by an element of $H_n(M|A;R)$.

Lemma 4.1.5 (`Lem[Hat02]:3.27`)**.** *Let M be an n-manifold and $A \subset M$ a compact subset. Then:*

(a) If $x \mapsto \alpha_x$ is a section of the covering space M_R, then there is a unique class $\alpha_A \in H_n(M|A;R)$ whose image in $H_n(M|x;R)$ is α_x for all $x \in A$.

(b) $H_i(M|A;R) = 0$ for $i > n$.

Next, we will investigate the concepts of orientability and R-orientability, and then cover the proofs of (4.1.3) and (4.1.5) on page 55.

$\sim * \sim$

Now let us go over some details we previously brushed over, namely the topic of sections, R-orientability and M_R. There are covering spaces related to orientations:

Claim 4.1.6. *Every n-manifold M has an orientable*[1] *two-sheeted covering space $\tilde{M}$. The inverse image of a point x are the two local orientations at x. $\tilde{M}$ is topologized using the locally consistent sets of local orientations.*

Claim 4.1.7. *Every n-manifold M has an infinite sheeted cover $M_{\mathbb{Z}} \to M$. The inverse image of a point x are the elements of $H_n(M|x)$. It is topologized similarly to $\tilde{M}$.*

Some of the basic properties of these covers are:

- $\tilde{M}$ has two components iff M is orientable (`Prp[Hat02]:3.25`). In this case $\tilde{M}$ is the disjoint union of two copies of M.
- $\tilde{M}$ can be embedded into $M_{\mathbb{Z}}$ as the set of points corresponding to generators.
- M can be embedded into $M_{\mathbb{Z}}$ as the set of points corresponding to 0 elements.
- $M_{\mathbb{Z}}$ is the disjoint union of a single copy of M and infinitely many copies of $\tilde{M}$. So if M is orientable, $M_{\mathbb{Z}}$ is infinitely many copies of M.
- An orientation of M is a *section* $\mu : M \to M_{\mathbb{Z}}$ – a continuous map that yields the identity when composed with the covering map – such that for each $x \in M$, μ_x is a generator of $H_n(M|x)$.

Let us now consider the case of R-orientability. As previously seen, algebraically speaking there are three things about R which are interesting:

- The subset of R consisting of the invertible elements (*units*).
- The subset of R consisting of elements of order 2: $\{r \in R : 2r = 0\}$
- Whether R has a unit u with $2u = 0$. This property is equivalent to saying that $2 = 0$ in R.

The parts of this chapter stated for $\mathbb{Z}$-orientations generalize to coefficients in R as follows.

- The definition of R-orientation at a point is the same as above. A **generator** in this case is an element $u \in R$ such that $Ru = R$; in other words a *unit*.

[1] It is implicitly stated here that $\tilde{M}$ is a manifold.

- The definition of R-orientations and orientability of a manifold follows directly.
- There is a covering space M_R, which has one sheet for each element of R.
- Each $r \in R$ determines a subcovering space M_r. If $2r = 0$ then M_r is a copy of M, otherwise M_r is homeomorphic to $\tilde{M}$.
- M_R is the disjoint union of M_r's, except for the fact that $M_r = M_{-r}$.

From these, it follows easily that:

Claim 4.1.8. • *An orientable n-manifold M is R-orientable for all R.*

- *A non-orientable n-manifold M is R-orientable iff $2 = 0$ in R.*

Thus every manifold is $\mathbb{Z}_2$-orientable.

$\sim * \sim$

Proof of theorem 4.1.3. Consider $\Gamma_R(M)$, the set of all sections from M to M_R. This is an R-module, and there is a homomorphism $H_n(M;R) \to \Gamma_R(M)$, using the homomorphisms $H_n(M;R) \to H_n(M|x;R)$. Now combining lemma 4.1.5 with our knowledge about the structure of M_R gives us the results of the theorem. □

Proof of lemma 4.1.5. The coefficient ring R will play no role in the proof, so **it will be omitted from the notation**.

Lemma 4.1.9. *If lemma 4.1.5 is true for any section for the compact sets A, B, and $A \cap B$, then it is true for the compact set $A \cup B$.*

Proof. Consider the Mayer-Vietoris sequence:

$$\ldots \longrightarrow H_{n+1}(M|A\cap B) \longrightarrow H_n(M|A\cup B) \xrightarrow{\Phi} H_n(M|A) \oplus H_n(M|B) \xrightarrow{\Psi} H_n(M|A\cap B) \longrightarrow \ldots$$

Next, consider for all $x \in A \cap B$ the following commutative diagram:

$$\begin{tikzcd}
H_n(M|A\cup B) \arrow[r, "a'"] \arrow[d, "b'"] & H_n(M|A) \arrow[d, "a"] \arrow[ddr, bend left] & \\
H_n(M|B) \arrow[r, "b"] \arrow[drr, bend right] & H_n(M|A\cap B) \arrow[dr] & \\
 & & H_n(M|x)
\end{tikzcd}$$

From here on, the proof is purely algebraic. By (a) of lemma 4.1.5 applied to A, B, and $A \cap B$ we get unique elements α_A, α_B, and $\alpha_{A\cap B}$ whose image is α_x in $H_n(M|x)$. Using these elements and the two diagrams above it is easy to prove the existence part of (a). With the diagrams and (a) of lemma 4.1.5 applied to the **zero section** and the compact sets A and B one can derive the uniqueness part of (a) too. (b) comes directly from the application of (b) of lemma 4.1.5 to the compact sets A, B and $A \cap B$. □

Using this lemma, we will prove lemma 4.1.5 in four steps, for increasingly complex A.

1. $A \subset \mathbb{R}^n \subset M$ **is a convex compact set.** The map $H_n(M|A) \to H_n(M|x)$ is equivalent by excision to $H_n(\mathbb{R}^n|A) \to H_n(\mathbb{R}^n|x)$, which is an isomorphism.

2. $A \subset \mathbb{R}^n \subset M$ **is a union of finitely many convex compact sets.** Let A be a union of m convex compact sets: $A = A_1 \cup \ldots A_m$. Induction on m. $m = 1$ is evident by step 1. Apply our inductive lemma (4.1.9) to $A_1 \cup \ldots \cup A_{m-1}$ and A_m. By induction the desired result holds for these two sets. Their intersection is $(A_1 \cap A_m) \cup \ldots (A_{m-1} \cap A_m)$, which is the union of $m-1$ compact convex sets, so by induction the desired result holds here too.

3. $A \subset \mathbb{R}^n \subset M$ **is an arbitrary compact set.** For a given $\alpha \in H_n(M|A)$ we construct a $K \supset A$ from finitely many compact balls, that has a special element α_K such that its image by $H_n(M|K) \to H_n(M|A)$ is α.

   (b) of the desired lemma, and the existence part of (a) come directly from the fact that we already know the lemma for K by step 2. For the uniqueness part of (a), it is sufficient to show that if the image of α is 0 in $H_n(M|x)$ for all $x \in A$, then $\alpha = 0$. This can be proven by considering the following diagram for a compact ball B used in the construction of K and an arbitrary $y \in B$, and applying the result of step 2:

$$\begin{tikzcd}
 & H_n(M|B) \arrow[r, "\approx"] \arrow[dr, "\approx"] & H_n(M|y) \\
H_n(M|K) \arrow[ur] \arrow[r] & H_n(M|A) \arrow[r] & H_n(M|x) \ni \alpha_x = 0
\end{tikzcd}$$

4. $A \subset M$ **is an arbitrary compact set.** Using a compactness argument A can be written as $A = A_1 \cup \ldots \cup A_m$, where A_i has a neighborhood homeomorphic to $\mathbb{R}^n$. Using a similar technique as in step 2, we get the desired result.

□

4.2 Cohomology with compact supports

For certain directed acyclic graphs of groups where each edge is associated with a homomorphism, a limit group can be assigned called the direct limit of the system. Using this algebraic concept the **cohomology groups with compact supports** can be defined. Due to this definition through group limits the compactly supported cohomology of spaces presented as increasing unions of subspaces can be calculated easily.

4.2.1 Geometry

Here we introduce the concept of cohomology with compact supports, and present some claims about the calculation of direct limits of (co)homology groups, which will come handy later.

Definition 4.2.1. Let $C_c^i(X;G)$ be the subcomplex[2] consisting of chains $\psi \in C^i(X;G)$ for which there exists a compact set $K \subset X$ such that ψ is 0 on singular simplices in $X - K$. The cohomology groups $H_c^i(X;G)$ of this subcomplex are called the cohomology groups with compact supports.

Note that if we defined $C_c^i(X;G)$ as the group of chains for which only singular simplices in some compact set K can take non-zero values, then these groups would not form a chain complex.

Claim 4.2.2. *For each compact set $K \subset X$ take the cohomology group $H^i(X|K;G) = H^i(X, X - K;G)$, and for compact sets $K \subset L \subset X$ associate the homomorphisms induced by inclusion $H^i(X|K;G) \to H^i(X|L;G)$. These form a directed system of groups. The direct limit of this system is the group $H_c^i(X;G)$.*

[2] It is easy to check that this is indeed a subcomplex, so δ takes $C_c^i(X;G)$ to $C_c^{i+1}(X;G)$.

This can be easily checked on the level of cocycles and coboundaries.

Corollary 4.2.2.1. *For compact spaces $H^i_c(X;G) = \varinjlim H^i(X|K;G) = H^i(X;G)$ by property 4.2.6.1 of direct limits.*

A direct limit statement for homologies will also be useful later on. To understand this, it should be noted that if there is a homomorphism from each group in a direct limit to some other group (and these homomorphisms are compatible with the directed system structure) then there is a limit homomorphism from the limit group too (see statement 4.2.10.1).

Claim 4.2.3 (`Prp[Hat02]:3.33`)**.** *If a space X is the union of a directed set of subspaces X_α (with respect to the inclusion relation) with the property that each compact set in X is contained in some set X_α, then the natural map $\varinjlim H_i(X_\alpha;G) \to H_i(X;G)$ is an isomorphism for all i and G.*

This can be easily checked on the level of cycles and boundaries.

4.2.2 Direct limits and directed sets

In this section we list the definitions and statements required elsewhere in the thesis without proofs. The proofs are not discussed in this thesis, but are mostly trivial. For a more detailed treatment of the topic, see chapter VIII. of [ES52].

Definition 4.2.4. A partially ordered set $(I, \leq)$ which has the following property is called a **directed set**:

$$\forall \alpha, \beta \in I \exists \gamma \in I : \alpha \leq \gamma \text{ and } \beta \leq \gamma$$

Definition 4.2.5. Suppose for each element α of a directed set I there is associated an abelian group G_α, and for each pair $\alpha \leq \beta$ of elements in I there is a homomorphism $f_{\alpha,\beta} : G_\alpha \to G_\beta$. Moreover, suppose that $f_{\alpha,\alpha} = \mathrm{id}$ and $f_{\alpha,\gamma} = f_{\beta,\gamma} f_{\alpha,\beta}$ for each $\alpha \leq \beta \leq \gamma$. These groups together with their homomorphisms are called a **directed system of groups**.

A somewhat shorter definition of a directed system of groups would be (the image of) a functor from the category of $(I, \leq)$ to the category of abelian groups.

$$\begin{tikzcd}
& G_\gamma & & \cdots \\
G_\alpha \arrow[ur, "f_{\alpha,\gamma}"] & & G_\beta \arrow[ul, "f_{\beta,\gamma}"'] & \\
\cdots \arrow[u] & G_\delta \arrow[ur] & & G_\varepsilon \arrow[ul] \arrow[uu]
\end{tikzcd}$$

Now for the definition of the direct limit:

Definition 4.2.6. Take a directed system of groups G_α indexed by I. Define the following equivalence relation. For $a \in G_\alpha$ and $b \in G_\beta$ (α and β need not be comparable)

$$a \sim b \iff \exists \gamma \geq \alpha, \beta : f_{\alpha,\gamma}(a) = f_{\beta,\gamma}(b).$$

The set of equivalence classes can be equipped with an abelian group structure. For $a \in G_\alpha$, $b \in G_\beta$ denote their equivalence classes by $[a]$ and $[b]$. Then for arbitrary $\gamma \geq \alpha, \beta$ let

$$[a] + [b] := [f_{\alpha,\gamma}(a) + f_{\beta,\gamma}(b)].$$

It is easy to check this is well-defined. The set of equivalence classes equipped with this group structure is called the **direct limit** group of the directed system, denoted by $\varinjlim G_\alpha$.

This can be thought of as follows. For each $a \in G_\alpha, b \in G_\beta$ such that $\alpha \leq \beta$ and $f_{\alpha,\beta}(a) = b$, draw an arrow from a to b. Then the equivalence classes are merely the connected components of the graph produced this way. To calculate the sum of two components, just take a group that both components are present in (this is guaranteed by the directed system structure), and add any representatives there.

Remark 4.2.6.1. Let G_* be a directed system of groups indexed by a poset[3] I that has a maximal element α. Then $\varinjlim G_\beta \approx G_\alpha$.

Claim 4.2.7. *For a directed system of groups G_α indexed by I:*

$$\varinjlim G_\alpha \approx \left(\bigoplus_{\alpha \in I} G_\alpha \right) / \langle a - f_{\alpha\beta}(a) : a \in G_\alpha, \alpha \leq \beta \in I \rangle$$

This gives an alternative definition of the direct limit. It should also be noted here that there are natural maps $g_\alpha : G_\alpha \to \varinjlim G_\beta$, which can be easily defined using the claim above.

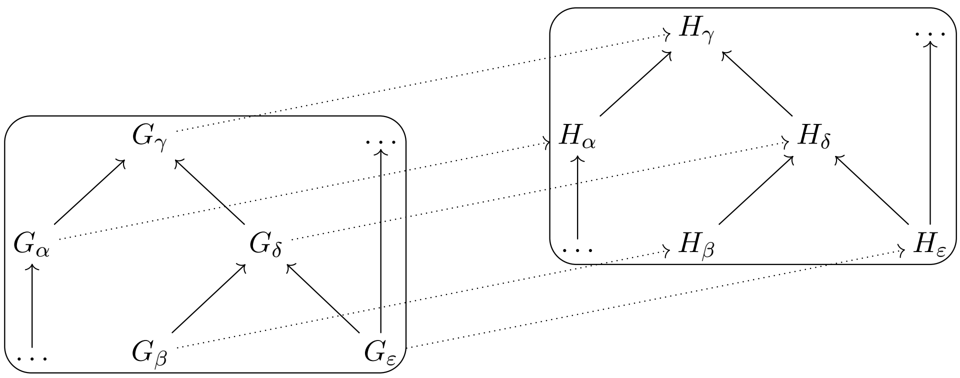
Claim 4.2.8. *Suppose we have the following situation. There are directed sets $I_\mathfrak{a}$ which themselves are indexed by a directed set $\mathcal{I}$ such that $\mathfrak{a} \leq \mathfrak{b} \implies I_\mathfrak{a} \subset I_\mathfrak{b}$. There are corresponding directed systems of groups $\mathcal{G}_\mathfrak{a}$ which are also extensions of each other. This means there is a total directed set I and a total directed system of groups $\mathcal{G}$ constructible as the "union" of the $I_\mathfrak{a}$'s and $\mathcal{G}_\mathfrak{a}$'s. Then:*

$$\varinjlim_{\alpha \in I} G_\alpha = \varinjlim_{\mathfrak{a} \in \mathcal{I}} \varinjlim_{\alpha \in I_\mathfrak{a}} G_\alpha$$

$\sim * \sim$

Now we move on to the topic of homomorphisms defined on directed systems.

Definition 4.2.9. Let G_α with $g_{\alpha,\beta}$ and H_α with $h_{\alpha,\beta}$ be directed systems of abelian groups indexed by the same directed set I. Suppose there are homomorphisms $f_\alpha : G_\alpha \to H_\alpha$ for each $\alpha \in I$ such that the diagram below commutes – or in other words, for each $\alpha \leq \beta : f_\beta g_{\alpha,\beta} = h_{\alpha,\beta} f_\alpha$.



In this case, the collection of homomorphisms f_α – denoted f_* – will be called a **homomorphism** of the directed systems.

Claim 4.2.10. *Let G_α and H_α be directed systems of abelian groups indexed by the same directed set I, and f_* be a homomorphism between them. Then there is a limit homomorphism $\tilde{f} : \varinjlim G_\alpha \to \varinjlim H_\alpha$ such that it "commutes" with the natural[4] homomorphisms $g_\alpha : G_\alpha \to \varinjlim G_\beta$ and $h_\alpha : H_\alpha \to \varinjlim H_\beta$:*

$$\forall \alpha \in I : \tilde{f} g_\alpha = h_\alpha f_\alpha$$

[3]Partially ordered set.

[4]*This* is the naturality property.

Moreover, this limit homomorphism is functorial: if there are directed systems $F_\alpha, G_\alpha, H_\alpha$ indexed by the same directed set I, and there are homomorphisms which fit into the following commutative diagram,

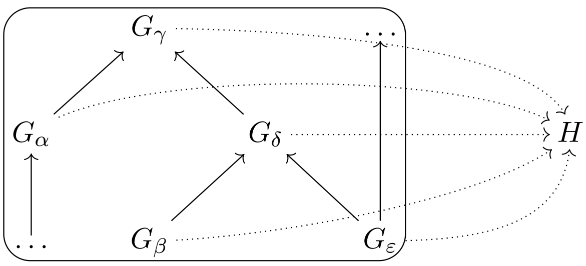
$$\begin{tikzcd} F_* \arrow[rr, "a_*"] \arrow[dr, "b_*"'] & & H_* \\ & G_* \arrow[ur, "c_*"'] & \end{tikzcd}$$

then the limit homomorphisms also commute:

$$\begin{tikzcd} \varinjlim F_\alpha \arrow[rr, "\tilde{a}"] \arrow[dr, "\tilde{b}"'] & & \varinjlim H_\alpha \\ & \varinjlim G_\alpha \arrow[ur, "\tilde{c}"'] & \end{tikzcd}$$

This claim has many handy corollaries:

Corollary 4.2.10.1. *Let G_α with $g_{\alpha,\beta}$ be a directed system of groups and H be another abelian group. Suppose for each α there is a homomorphism $f_\alpha : G_\alpha \to H$, such that the diagram below commutes – or in other words, for each $\alpha \leq \beta : f_\beta g_{\alpha,\beta} = f_\alpha$.*



Then there is a limit homomorphism $f : \varinjlim G_\alpha \to H$ such that it commutes with the natural homomorphisms $g_\alpha : G_\alpha \to \varinjlim G_\beta$; $fg_\alpha = f_\alpha$.

Corollary 4.2.10.2. *Let G_* and H_* be directed systems of abelian groups indexed by the same directed set I, and f_* be a homomorphism between them. Suppose f_α is an isomorphism for all $\alpha \in I$. Then the limit homomorphism $\tilde{f}$ is also an isomorphism.*

Corollary 4.2.10.3. *Suppose there is a commutative diagram of directed systems of abelian groups. Then the limit diagram – composed of the limit groups and limit homomorphisms – is also commutative.*

Finally there is another statement about commutative diagrams:

Claim 4.2.11. *Let F_*, G_*, H_* be directed systems indexed by the same poset I, and $f_* : F_* \to G_*, g_* : G_* \to H_*$ homomorphisms between them, such that for each $\alpha \in I$*

$$F_\alpha \xrightarrow{f_\alpha} G_\alpha \xrightarrow{g_\alpha} H_\alpha$$

is exact. Then

$$\varinjlim F_\alpha \xrightarrow{\tilde{f}} \varinjlim G_\alpha \xrightarrow{\tilde{g}} \varinjlim H_\alpha$$

is also exact.

4.3 The cap product

Let R be an arbitrary ring (although only commutative rings with unity will be utilized later on). **Quick reminder:** usually $R = \mathbb{Z}$ or $\mathbb{Z}_2$.

The isomorphism in the Poincaré duality is constructed using an R-bilinear product defined on the homologies and cohomologies of a space. This product is called the cap product (denoted $\frown$), and has an absolute and numerous relative forms. For $k \geq l$ natural numbers, $X \supset A$ topological spaces, and $U, V \subset X$ open subspaces the cap product has the forms:

$$
\begin{aligned}
H_k(X;R) \times H^l(X;R) &\xrightarrow{\frown} H_{k-l}(X;R)\\
H_k(X,A;R) \times H^l(X;R) &\xrightarrow{\frown} H_{k-l}(X,A;R)\\
H_k(X,A;R) \times H^l(X,A;R) &\xrightarrow{\frown} H_{k-l}(X;R)\\
H_k(X,U\cup V;R) \times H^l(X,U;R) &\xrightarrow{\frown} H_{k-l}(X,V;R)
\end{aligned}
$$

It also satisfies a naturality property illustrated by the following (not commutative) diagram:

$$
\begin{tikzcd}
H_k(X) \arrow[d, "f_*"] & \times & H^l(X) \arrow[r, "\frown"] & H_{k-l}(X) \arrow[d, "f_*"] \\
H_k(X) & \times & H^l(X) \arrow[u, "f^*"] \arrow[r, "\frown"] & H_{k-l}(C)
\end{tikzcd}
$$

This means that for $\alpha \in H_k(X)$ and $\psi \in H^l(X)$:

$$f_*(\alpha) \frown \psi = f_*(\alpha \frown f^*(\psi)) \tag{E4.3.1}$$

$\sim * \sim$

Now let us properly define the cap product. We will first give a formula using singular simplices, then using chains, and finally homologies.

For a singular k-simplex $\sigma : \Delta^k \to X$ and a cochain $\psi : C^l(X;R)$, put

$$\sigma \frown \psi = \psi(\sigma|[v_0, \ldots, v_l])\sigma|[v_l, \ldots, v_k]$$

This can be extended from singular simplices to chains to form an R-bilinear product. To move over to homologies, we should first calculate the boundary of the cap product:

$$\partial(\sigma \frown \psi) = (-1)^l(\partial\sigma \frown \psi - \sigma \frown \delta\psi) \tag{E4.3.2}$$

Of course the same formula for chains instead of simplices can be easily derived from this. (E4.3.2) can be checked by the following calculation:

$$
\begin{aligned}
\partial(\sigma \frown \psi) &= \sum_{i=l}^{k}(-1)^{i-l}\psi(\sigma|[v_0, \ldots, v_l])\sigma|[v_l, \ldots, \hat{v}_i, \ldots, v_k]\\
\partial\sigma \frown \psi &= \sum_{i=0}^{l+1}(-1)^{i}\psi(\sigma|[v_0, \ldots, \hat{v}_i, \ldots, v_{l+1}])\sigma|[v_{l+1}, \ldots, v_k]+\\
&+\sum_{i=l}^{k}(-1)^{i}\psi(\sigma|[v_0, \ldots, v_l])\sigma|[v_l, \ldots, \hat{v}_i, \ldots, v_k]\\
\sigma \frown \delta\psi &= \sum_{i=0}^{l+1}(-1)^{i}\psi(\sigma|[v_0, \ldots, \hat{v}_i, \ldots, v_{l+1}])\sigma|[v_{l+1}, \ldots, v_k]
\end{aligned}
$$

Cycles. To see that there is an induced cap product of homology, we should first check that the product of a cycle and a cocycle is a cycle. This is clear from (E4.3.2): both $\partial\sigma$ and $\delta\psi$ are 0 in this case.

Boundaries. To see that boundaries and coboundaries do not change the value of the cap product, we should prove that *boundary* $\frown$ *cocycle* = *boundary* and that *cycle* $\frown$ *coboundary* = *boundary*. This again is fairly straightforward:

- In the first case let σ be a chain and ψ be a cocycle. Then $\sigma \frown \delta\psi$ will be zero because $\delta\psi$ is zero, hence $\partial(\sigma \frown \psi) = \pm\partial\sigma \frown \psi$, so the product of an arbitrary boundary ($\partial\sigma$) and an arbitrary cocycle (ψ) is a boundary.
- In the second let σ be a cycle and ψ be a cochain. Then $\partial\sigma \frown \delta$ will be zero because $\partial\sigma$ is zero, hence $\partial(\sigma \frown \psi) = \pm\sigma \frown \delta\psi$, so the product of an arbitrary cycle (σ) and an arbitrary coboundary ($\delta\psi$) is a boundary.

To check the existence of the first two relative forms of the cap product, one just needs to check that it is defined correctly on the chain groups. As the formula (E4.3.2) for $\partial(\sigma \frown \psi)$ still holds, we can pass to homology groups.

For the last relative form we have to remember that $H_n(X, U \cup V; R)$ can be computed using the chain groups $C_n(X, U + V; R) = C_n(X; R)/C_n(U + V; R)$.

4.4 The statement of duality

Before we state the main theorems, it is important to remember that in this section about Poincaré duality we omit the requirement of second-countability (M_2) from the definition of a manifold.

Theorem 4.4.1 (Poincaré Duality for compact manifolds; `thm[Hat02]:3.30`)**.** *Let M be a closed R-orientable n-manifold with a fundamental class $[M] \in H_n(M; R)$. Then the map $D_M : H^k(M; R) \to H_{n-k}(M; R)$ defined by $D_M(\alpha) = [M] \frown \alpha$ is an isomorphism for all k.*

There is another version of Poincaré duality for not necessarily compact manifolds; the version for closed manifolds follows directly from it. This however requires the theory of section 4.2.

Theorem 4.4.2 (Poincaré Duality for noncompact manifolds; `thm[Hat02]:3.35`)**.** *Let M be an R-orientable n-manifold. Then there is an isomorphism $D_M : H^k_c(M; R) \to H_{n-k}(M; R)$ for all k.*

The definition of D_M in this case is as follows.

1. **Fix an orientation of M.**
2. **Create a directed system of groups.** For each compact subset $K \subset M$ take the group $H^k(M|K; R)$, and also take the homomorphisms between them induced by inclusion: for compact subsets $K \subset L \subset M$, $H^k(M|K; R) \xrightarrow{i^*} H^k(M|L; R)$. These together form a directed system of groups.
3. **Construct homomorphisms from the directed system to $H_{n-k}(M; R)$.** For each K, take the unique element $\mu_K \in H_n(M|K; R)$ that restricts to the selected orientations of M at points x: this exists thanks to (a) of (4.1.5). Now consider the map $H^k(M|K; R) \to H_{n-k}(M; R)$ defined by $\psi \mapsto \mu_K \frown \psi$; this map is derived from the cap product $H_n(M|K; R) \times H^k(M|K; R) \to H_{n-k}(M; R)$ by fixing the first variable to μ_K.
4. **Take the limits.** The limit of the directed system is $H^k_c(M; R)$ according to (4.2.2). We will check in a minute that the requirements of (4.2.10.1) are satisfied, so a limit homomorphism of the form $H^k_c(M; R) \to H_{n-k}(M; R)$ exists. **Denote this homomorphism by D_M: this will be the duality homomorphism in the theorem.**

It only remains to check that in the last step the requirements of (4.2.10.1) are satisfied. Take compact subsets $K \subset L \subset M$ and the homomorphism between their groups: $H^k(M|K;R) \xrightarrow{i^*} H^k(M|L;R)$. We have to show that the following diagram is commutative:

$$\begin{tikzcd} H^k(M|K;R) \arrow[d, "i^*"] \arrow[dr, "\mu_K \frown"] & \\ & H_{n-k}(M;R) \\ H^k(M|L;R) \arrow[ur, "\mu_L \frown"'] & \end{tikzcd}$$

Thankfully this comes directly from the naturality property of the cap product (E4.3.1) and from the fact that $i_*(\mu_L) = \mu_K$, which is just a consequence of the commutativity of the following diagram.

$$\begin{tikzcd} H_n(M|L;R) \arrow[rr] \arrow[dr, "i_*"'] & & H_n(M|x;R) \\ & H_n(M|K;R) \arrow[ur] & \end{tikzcd}$$

The last sentence expanded: $\mathrm{id}(\mu_L \frown i^*(\psi)) = i_*(\mu_L) \frown \psi$ for each $\psi \in H^k(M|K;R)$; substituting $i_*(\mu_L) = \mu_K$ we get $\mu_L \frown i^*(\psi) = \mu_K \frown \psi$. This is what we had to check to see that (4.2.10.1) can be applied.

4.5 The proof of duality

Reminder: all homology and cohomology groups have coefficients in R in this section, unless explicitly specified otherwise.
The proof of the Poincaré duality (4.4.2) will be done in three steps:

(1) $M = \mathbb{R}^n$

(2) $M \subset \mathbb{R}^n$ open

(3) M is a general R-orientable manifold

Steps (2) and (3) will be proven in an inductive fashion, for which the following two lemmas are required:

Lemma 4.5.1 (Finite inductive lemma). *If an R-orientable manifold M is a union of two open subsets U and V, and the homomorphisms[5] D_U, D_V and $D_{U\cap V}$ are isomorphisms, then D_M is also an isomorphism.*

This is immediate from applying the five lemma to (4.5.3), a technical statement presented later in this section.

Lemma 4.5.2 (Infinite inductive lemma). *If an R-orientable manifold M is a union of open subsets $U_0 \subset U_1 \subset \dots$ (the subsets can be indexed by any totally ordered set, not just $\mathbb{N}$), and each D_{U_i} is an isomorphism, then so is D_M.*

This can be proved by writing $H^k_c(U_i)$ as a direct limit of $H^k(M|K)$'s using (4.2.2), then writing $H^k_c(M)$ both as the direct limit of $H^k(M|K)$'s and the direct limit of $H^k_c(U_i)$'s (the two limit groups coincide because of (4.2.8)). Next, we write $H_{n-k}(M)$ as the direct limit of $H_{n-k}(U_i)$'s by applying (4.2.3). Using all these direct limits, D_M can be written as a direct limit of isomorphisms, so it is one too according to (4.2.10.2).

The proofs of the three steps go as follows.

[5] Here we implicitly use the fact that an open subset of a manifold is a manifold.

(1) $M = \mathbb{R}^n$**:**
Consider $\mathbb{R}^n$ as the interior of Δ. In this case $H_c^k(\mathbb{R}^n) \approx H^k(\Delta, \partial\Delta)$, and $D_M : H^k(\Delta, \partial\Delta) \to H_{n-k}(\Delta)$. This is clearly a $0 \to 0$ isomorphism for $k \neq n$. The $k = n$ case can be proven by checking the definitions of the maps involved.

(2) $M \subset \mathbb{R}^n$ **open:**
Write M as a union of open U_i's homeomorphic to $\mathbb{R}^n$, and let $V_i = \bigcup_{j \leq i} U_j$. We first see by induction on i that D_{V_i} is an isomorphism (using (1) to prove $i = 1$, and that the requirements of the *finite inductive lemma* (4.5.1) are satisfied in the inductive step). From here we get the result by applying the *infinite inductive lemma* (4.5.2) to the V_i's.

(3) M **is a general** R**-orientable manifold:**
Proof by Zorn's lemma. The open subsets $U \subset M$ for which D_U is an isomorphism form a poset under inclusion, and for any totally ordered subset the *infinite inductive lemma* (4.5.2) gives an upper bound from the poset. Zorn's lemma then gives a maximal U, but if $U \neq M$ we can give a larger element of this poset by taking the union of U with a subset $U \not\supset H \subset M$ homeomorphic to $\mathbb{R}^n$ (this is an element of the poset by the *finite inductive lemma* (4.5.1)).

$\sim * \sim$

Finally, the technical result needed for the *finite inductive lemma* (4.5.1):

Lemma 4.5.3 (`Lem[Hat02]:3.36`)**.** *If an R-orientable manifold M is a union of two open subsets U and V, then there is a diagram of Mayer-Vietoris sequences, commutative up to sign:*

$$\begin{tikzcd}
\dots \arrow[r] & H_c^k(U \cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & H_c^k(U) \oplus H_c^k(V) \arrow[r] \arrow[d, "D_U \oplus -D_V"] & H_c^k(M) \arrow[r] \arrow[d, "D_M"] & H_c^{k+1}(U \cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & \dots \\
\dots \arrow[r] & H_{n-k}(U \cap V) \arrow[r] & H_{n-k}(U) \oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r] & H_{n-k-1}(U \cap V) \arrow[r] & \dots
\end{tikzcd}$$

Proof. For compact sets $K \subset U$ and $L \subset V$, take the following diagram, whose top and bottom rows are Mayer-Vietoris sequences:

$$\begin{tikzcd}
\dots \arrow[r] & H^k(M|K \cap L) \arrow[r] \arrow[d, "\approx"] & H^k(M|K) \oplus H^k(M|L) \arrow[r] \arrow[d, "\approx"] & H^k(M|K \cup L) \arrow[r] \arrow[dd, "\mu_{K\cup L}\frown"] & \dots \\
& H^k(U \cap V|K \cap L) \arrow[d, "\mu_{K\cap L}\frown"] & H^k(U|K) \oplus H^k(V|L) \arrow[d, "\mu_K\frown \oplus -\mu_L\frown"] & & \\
\dots \arrow[r] & H_{n-k}(U \cap V) \arrow[r] & H_{n-k}(U) \oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r] & \dots
\end{tikzcd}$$

Assuming commutativity of the diagram, applying a direct limit over compact pairs (K, L) gives the statement of the lemma, thanks to our knowledge about direct limits (statements 4.2.10.3 and 4.2.11).

Unfortunately, this diagram does not commute, but it is easy to see that it suffices to show it commutes up to a sign which only depends on k for each square.

Commutativity of the first two squares can be checked on the level of chains and cochains. The third square can be written as

$$\begin{tikzcd}
H^k(M|K \cup L) \arrow[r, "\delta"] \arrow[d, "\mu_{K\cup L}\frown"] & H^{k+1}(M|K \cap L) \arrow[r, "\approx"] & H^{k+1}(U \cap V, K \cap L) \arrow[d, "\mu_{K\cap L}\frown"] \\
H_{n-k}(M) \arrow[rr, "\partial"] & & H_{n-k-1}(U \cap V)
\end{tikzcd}$$

First we check that δ and ∂ are indeed the connecting homomorphisms in the appropriate Mayer-Vietoris sequences. To check the commutativity of the square, we have to evaluate the homomorphisms on a $\psi \in C^*(M, A \cap B)$ (as per the previous sentence). The following three ideas will be utilized in the proof:

- we write $\psi = \psi_{M-K} - \psi_{M-L}$, the difference of chains inside the given sets
- we write the class $\mu_{K \cup L}$ as the sum of $\alpha_{U-L} + \alpha_{U \cap V} + \alpha_{V-K}$ using barycentric subdivision
- we recall the formula for the ∂ or a cap product (E4.3.2): $\partial(\sigma \frown \psi) = (-1)^l(\partial\sigma \frown \psi - \sigma \frown \delta\psi)$, if ψ is l-dimensional

From here on, it is just manual labor to finish the proof of commutativity. $\square$

Chapter 5

Singular bordism

Sources. *This chapter follows the exposition of [CF64], in particular sections I.4., I.5. and I.6.*

After the singular homology and cohomology groups, in this rather short chapter we will introduce the singular *bordism* groups of (pairs of) spaces: $\Omega_n(X, A)$. We check that they form a generalized homology theory, so the toolkit detailed in (§2.2) and (§2.3) of singular homology can also be applied here. Using these, we will be able to prove in the next chapter that any homology class of a manifold with coefficients in $\mathbb{Q}$ can be represented by a manifold in some sense.

5.1 The oriented case

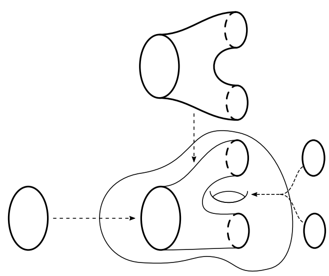
Remark. The term **manifold** in this chapter means compact oriented differentiable manifold (possibly) with a boundary.

Notation. For any manifold B^n, let us denote its boundary by $\dot{B}^n$.

Notation. I denotes the compact interval $[0, 1]$.

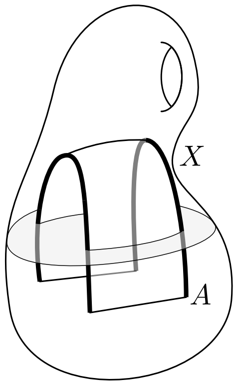
Similarly to singular homology, singular bordism is also a generalized homology theory – in particular, a covariant functor. The construction itself also resembles that of homology: instead of mapping "simplicial complexes" into our space X (with the boundary mapping to some subspace A), we take all compact oriented manifolds and their continuous maps into X (where their boundary is mapped entirely into A). Consequently, the elements of the oriented bordism group $\Omega_n(X, A)$ will be (represented by) *singular manifolds*, or in other words pairs (B^n, f) where B^n is a compact n-dimensional oriented manifold, and f is a continuous map[1] $f : (B^n, \dot{B}^n) \to (X, A)$.

As the addition of homology classes corresponds to taking the disjoint union of the "simplicial complexes" and joining their maps, it is not surprising that the addition in the bordism group will be defined analogously: the sum of the bordism classes represented by singular manifolds (B_1^n, f_1) and (B_2^n, f_2) will be the one represented by $(B_1^n \sqcup B_2^n, f_1 \cup f_2)$.



Only one question remains: *what exactly we mean by* the words "bordism class"? Let us first discuss **the absolute case**. After remembering the construction of absolute homology groups, it is natural to define a boundary operator ∂ which assigns to singular manifold (B^n, f) – where B^n is a compact oriented n-manifold with boundary – the map $(\dot{B}^n, f|_{\dot{B}^n})$, the restriction to the boundary with orientation induced by the orientation of B^n. This way we can form a chain complex from all the possible pairs (B^n, f) with n variable, and take its homology groups to be $\Omega_n(X)$.

[1]The notation $f : (B^n, \dot{B}^n) \to (X, A)$ means that $f : B^n \to X$ and $f(\dot{B}^n) \subset A$.



However, for *pairs* of spaces **the relative case** is a bit more complicated. Recall that for relative homology, two chains were said to represent the same class if their difference was the boundary of a higher dimensional chain, *plus some error inside the subspace A*. The necessity of this additional error forces us to leave the realm of manifolds and boundaries and inspect the geometry in greater detail. Returning to the absolute case, it is easy to guess what "difference" will mean here: $(B_1^n, f_1) - (B_2^n, f_2) = (B_1^n \sqcup -B_2^n, f_1 \cup f_2)$, where $-B_2^n$ is just B_2^n with the opposite orientation. However, this poses a problem: while the boundary of a manifold had no boundary in the absolute case – a desired quality for generalized homology theories: $\partial\partial = 0$ – a difference in the form above (with B_1^n and B_2^n possibly having boundaries) nearly always contains a boundary, so it cannot be the boundary of a higher dimensional manifold. This remark points us towards solving the problem of the definition of the "additional error" from before. Instead of requiring $B_1^n \sqcup -B_2^n$ to be the entire boundary of a manifold C^{n+1}, we only ask it to be a *regular submanifold* of $\dot{C}^{n+1}$: a subspace which is also a manifold. As the rest of $\dot{C}^{n+1}$ now obviously plays the role of the "additional error", we also require it to be mapped into A. The definition is now complete.

Remark. Note that in the relative case we did not use a boundary operator on the "chains formed by singular manifolds (B^n, f)". Defining the equivalence of singular manifolds using *regular submanifolds* prevents us from doing so. Consequently, in the precise definition below this operator will be omitted.

$\sim * \sim$

Let us now move on to a more precise definition.

Definition 5.1.1 (Relative oriented bordism group)**.** Fix an arbitrary pair of spaces (X, A) and a natural number $n \in \mathbb{N}$.

- A(n oriented) **singular manifold** of dimension n in (X, A) is a pair (B^n, f) with B^n being a compact oriented n-manifold with boundary, and $f : (B^n, \dot{B}^n) \to (X, A)$ is a map of pairs of spaces.
- A singular manifold (B^n, f) **bords**, if there is a singular $n+1$-manifold (C^{n+1}, F) where
  - $B^n \subset \dot{C}^{n+1}$ is a regular submanifold of the boundary of C^{n+1}, whose orientation is induced by the orientation of C^{n+1},
  - $F|_{B^n} = f$, and
  - $F(\dot{C}^{n+1} - B^n) \subset A$.
- Two singular manifolds (B_1^n, f_1) and (B_2^n, f_2) are **bordant** if $(B_1^n \sqcup -B_2^n, f_1 \cup f_2)$ bords.
- This relation is an equivalence relation. Let the set of equivalence classes be $\Omega_n(X, A)$, **the n-dimensional oriented singular bordism group**, and let us denote the equivalence class of (B^n, f) by $[B^n, f]$.
- The **addition** on $\Omega_n(X, A)$ is defined as $[B_1^n, f_1] + [B_2^n, f_2] = [B_1^n \sqcup B_2^n, f_1 \cup f_2]$.

0-dimensional compact manifolds are just finite sets of points with the discrete topology. An orientation of such a manifold corresponds to an abstract choice of "sign" for each point. $\Omega_n(X, A) = 0$ for $n < 0$ by definition.

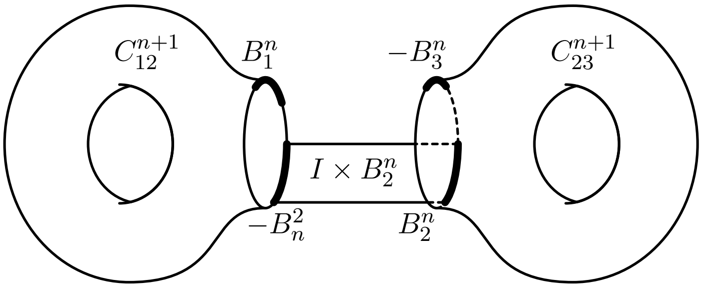


Figure 5.1: The construction of P^{n+1}

Before we move on to statements and proofs, it is advised to check out (§B), the section of the appendix about topology.

Remark 5.1.1.1. It is easy to see that the inverse of $[B^n, f] \in \Omega_n(X, A)$ is $[-B^n, f]$, as $(I \times B^n)^\cdot = (\dot{I} \times B^n) \cup (I \times -\dot{B}^n) = (\{1\} \times B^n) \cup (-\{0\} \times B^n) \cup (I \times -\dot{B}^n)$ by the more general formula $(X \times Y)^\cdot = (\dot{X} \times Y) \cup (X \times -\dot{Y})$.

Claim 5.1.2. *The bordism relation of* (5.1.1) *is indeed an equivalence relation.*

Proof. Reflexivity is clear from the remark above (5.1.1.1), while symmetry comes from the fact that $-(B_2^n \sqcup -B_1^n) = B_1^n \sqcup -B_2^n$.

Transitivity is a bit trickier. Suppose $(B_1^n \sqcup -B_2^n, f_1 \cup f_2)$ and $(B_2^n \sqcup -B_3^n, f_2 \cup f_3)$ bord, and (C_{12}^{n+1}, F_{12}) and (C_{23}^{n+1}, F_{23}) are witnesses of this. We have to show that $(B_1^n \sqcup -B_3^n, f_1 \cup f_3)$ bords, so informally, we would like to sew together C_{12}^{n+1} and C_{23}^{n+1} along B_2^n, see that the result is a differentiable manifold which induces the appropriate structures on each half, and it has an orientation that does the same. First, let us attach along B_2^n a copy of $I \times B_2^n$ onto C_{12}^{n+1} and C_{23}^{n+1}, and check that the results are also witnesses of the two bordism relations with the appropriate extensions of the F's (a good diffeomorphism is easy to provide). Next, we notice that the two resulting manifolds can be sewn together by identifying their $I \times B_2^n$'s (as both larger manifolds induce the same differentiable structure and orientation on it), resulting in a manifold P^{n+1} (see figure 5.1) which proves our statement. ☐

Claim 5.1.3. *Suppose there are maps $f, g : (B^n, \dot{B}^n) \to (X, A)$ from a manifold to a pair of spaces connected by a homotopy H such that $H(I \times \dot{B}^n) \subset A$. Then $[B^n, f] = [B^n, g] \in \Omega_n(X, A)$.*

The proof is immediate from the definition of when two singular manifolds are bordant and the fact that $(I \times B^n)^\cdot = (\dot{I} \times B^n) \cup (I \times -\dot{B}^n)$.

Definition 5.1.4. There is a homomorphism $\mu : \Omega_n(X, A) \to H_n(X, A; \mathbb{Z})$ defined as follows. Given $[B^n, f] \in \Omega_n(X, A)$, denote by $\sigma \in H_n(B^n, \dot{B}^n, \mathbb{Z})$ its orientation class[2], and define $\mu([B^n, f])$ to be $f_*(\sigma)$.

Claim 5.1.5. *This is a well defined homomorphism.*

Definition 5.1.6. The image of μ (that is, $\operatorname{Im} \mu \subset H_n(X, A; \mathbb{Z})$) is the subgroup of integral homology classes representable in the sense of Steenrod.

If we have a closed oriented topological n-manifold[3] P, we can also represent some of its *cohomology* classes in $H^k(P; \mathbb{Z})$ in the following way.

[2] In other words, *fundamental class*.
[3] Not necessarily second-countable, so we use the notion of "manifold" present in chapter §4.

Definition 5.1.7. There is a homomorphism $\Omega_{n-k}(P) \to H^k(P;\mathbb{Z})$ for any closed oriented topological n-manifold P defined as follows. For an element $[B^{n-k}, f] \in \Omega_{n-k}(P)$ consider $\mu([B^{n-k}, f]) \in H_{n-k}(P;\mathbb{Z})$, and take its Poincaré-dual (4.4.2) $\gamma \in H^k(P;\mathbb{Z})$. The image of this homomorphism will be called the **subgroup of representable cohomology classes in** $H^k(P;\mathbb{Z})$.

It is obvious this is a well defined homomorphism in light of claim 5.1.4 about μ being properly defined.

5.2 The unoriented case

If we drop all references to orientations from the definition of the oriented bordism group $\Omega_n(X, A)$, we obtain a simpler generalized homology theory: the *unoriented bordism group* $\mathfrak{N}_n(X, A)$ (typeset as `\mathfrak{N}_n(X,A)`). This topic is discussed in section I.8. of [CF64].

Definition 5.2.1 (Relative unoriented bordism group)**.** Fix an arbitrary pair of spaces (X, A) and a natural number $n \in \mathbb{N}$.

- A(n unoriented) **singular manifold** of dimension n in (X, A) is a pair (B^n, f) with B^n being a compact (not necessarily orientable) n-manifold with boundary, and $f : (B^n, \dot{B}^n) \to (X, A)$ is a map of pairs of spaces.
- A singular manifold (B^n, f) **bords**, if there is a(n unoriented) singular $n+1$-manifold (C^{n+1}, F) where
  - $B^n \subset \dot{C}^{n+1}$ is a regular submanifold of the boundary of C^{n+1},
  - $F|_{B^n} = f$, and
  - $F(\dot{C}^{n+1} - B^n) \subset A$.
- Two singular manifolds (B_1^n, f_1) and (B_2^n, f_2) are **bordant** if $(B_1^n \sqcup B_2^n, f_1 \cup f_2)$ bords.
- This relation is an equivalence relation. Let the set of equivalence classes be $\mathfrak{N}_n(X, A)$, **the** n**-dimensional unoriented singular bordism group**, and let us denote the equivalence class of (B^n, f) by $[B^n, f]$.
- The **addition** on $\mathfrak{N}_n(X, A)$ is defined as $[B_1^n, f_1] + [B_2^n, f_2] = [B_1^n \sqcup B_2^n, f_1 \cup f_2]$.

0-dimensional compact manifolds are just finite sets of points with the discrete topology. $\mathfrak{N}_n(X, A) = 0$ for $n < 0$ by definition.

Remark 5.2.1.1. All nonzero elements of $\mathfrak{N}_n(X, A)$ have order 2.

Claim 5.2.2. *The bordism relation of* (5.2.1) *is an equivalence relation.*

This can be proven analogously to its oriented version (5.1.2). μ and Steenrod-representability can be defined similarly to the oriented case:

Definition 5.2.3. There is a homomorphism $\mu : \mathfrak{N}_n(X, A) \to H_n(X, A; \mathbb{Z}_2)$ defined as follows. Given $[B^n, f] \in \mathfrak{N}_n(X, A)$, denote by $\sigma \in H_n(B^n, \dot{B}^n, \mathbb{Z}_2)$ its mod 2 fundamental class, and define $\mu([B^n, f])$ to be $f_*(\sigma)$.

Claim 5.2.4. *This is a well defined homomorphism.*

Definition 5.2.5. The image of μ (that is, $\operatorname{Im}\mu \subset H_n(X, A; \mathbb{Z}_2)$) is the subgroup of mod 2 homology classes representable in the sense of Steenrod.

If we have a closed topological n-manifold[4] P, we can also represent some of its *cohomology* classes in $H^k(P;\mathbb{Z}_2)$ in the following way.

Definition 5.2.6. There is a homomorphism $\mathfrak{N}_{n-k}(P) \to H^k(P;\mathbb{Z}_2)$ for any closed topological n-manifold P defined as follows. For an element $[B^{n-k}, f] \in \mathfrak{N}_{n-k}(P)$ consider $\mu([B^{n-k}, f]) \in H_{n-k}(P;\mathbb{Z}_2)$, and take its Poincaré-dual (4.4.2) $\gamma \in H^k(P;\mathbb{Z}_2)$. The image of this homomorphism will be called the **subgroup of representable cohomology classes in** $H^k(P;\mathbb{Z}_2)$.

It is obvious this is a well defined homomorphism in light of claim 5.2.5 about μ being properly defined.

5.3 The Eilenberg-Steenrod axioms

The purpose of this section is to show that the bordism functor satisfies the first six Eilenberg-Steenrod axioms stated in section §2.1. Consequently, the results listed in section §2.2 can be directly applied to the bordism groups $\Omega_n(X,A)$ (and $\mathfrak{N}_n(X,A)$). The axioms guarantee the existence of a spectral sequence for a CW pair (X,A), as presented in section §2.3. We also show that the two additional criteria (stated in theorem 2.3.18) necessary for the convergence of the spectral sequence are satisfied, the convergence result can be applied.

The proofs here build on the following lemma, (B.0.4.1) and (B.0.6) (sewing together and taking the product of manifolds with boundary), calculating the boundary of a product, and the separability of closed subspaces according to lemma B.0.1. The axioms will only be proved for the oriented bordism group $\Omega_n(X,A)$, but proofs are analogous for the unoriented bordism groups $\mathfrak{N}_n(X,A)$.

Lemma 5.3.1 (`See pages 12-13 of [CF64]`)**.** *Let V^n be a regular submanifold with boundary in a manifold B^n. If $f : B^n \to X$ is a map with $f(B^n - \operatorname{int} V^n) \cup f(\dot{B}^n) \subset A$, then $[B^n, f] = [V^n, f|_{V^n}]$ in $\Omega(X,A)$. We may even take $V^n = \emptyset$; in this case the lemma states $[B^n, f] = 0$.*

Proof. This can be verified using the definitions. We know by (B.0.6) that the product of two manifolds (differentiable, oriented, with boundary) is a manifold (differentiable, oriented, with boundary), whose "boundary behaves nicely". Applying this to $I\times B^n$ we get that it is a manifold and its boundary is $(I\times B^n)^{\cdot} = (\dot{I}\times B^n)\cup(I\times -\dot{B}^n) = (\{1\}\times B^n)\cup(\{0\}\times -B^n)\cup(I\times -\dot{B}^n)$. So $(\{1\}\times \boxed{V^n})\cup(\{0\}\times -B^n)$ is a regular submanifold of the boundary of $I\times B^n$, and the remainder of the boundary goes to A under the map $F(t,x) = f(x) : I\times B^n \to X$. Moreover, F on $(\{1\}\times V^n)\cup(\{0\}\times -B^n)$ is just the disjoint union of the singular manifolds (B^n, f) and $(V^n, -f|_{V^n})$. This gives the statement of the lemma by definition. □

The axioms are stated here using the Ω notation, and verified one-by-one, following the proof of theorem (5.1) of [CF64]. The first three are trivial, thus proofs are omitted.

(ES.1) *If φ is the identity, then φ_* is the identity also.*

(ES.2) $(\psi\varphi)_* = \psi_*\varphi_*$.

(ES.3) *If $\varphi|_A$ is the restriction of $\varphi : (X,A) \to (Y,B)$ to A, then $\partial\varphi_* = (\varphi|_A)_*\partial$. Namely, the following diagram commutes for any $n \in \mathbb{Z}$:*

$$\begin{array}{ccc} \Omega_n(X,A) & \xrightarrow{\varphi_*} & \Omega_n(Y,B) \\ \downarrow \partial & & \downarrow \partial \\ \Omega_{n-1}(A) & \xrightarrow{(\varphi|_A)_*} & \Omega_{n-1}(B) \end{array}$$

[4] Still not necessarily second-countable, so we use the notion of "manifold" present in chapter §4.

(ES.4) *If $i : A \to X$ and $j : X \to (X, A)$ are inclusions, then the following sequence is exact:*

$$\ldots \longrightarrow \Omega_n(A) \xrightarrow{i_*} \Omega_n(X) \xrightarrow{j_*} \Omega_n(X, A) \xrightarrow{\partial} \Omega_{n-1}(A) \longrightarrow \ldots$$

Proof. There are 6 claims we need to check. The first three state that the sequence is a chain complex:

- $\boxed{\partial j_* = 0}$ This follows from the fact that $[M^n, f] \in \Omega_n(X)$ implies that M^n has no boundary.
- $\boxed{i_*\partial = 0}$ If we take a representative of a given element $[B^n, f] \in \Omega_n(X, A)$, it shows that $(\dot{B}^n, f|_{\dot{B}^n})$ bords in $\Omega_{n-1}(X)$.
- $\boxed{j_*i_* = 0}$ Take a representative $[M^n, f] \in \Omega_n(A)$, and apply lemma 5.3.1 from earlier (M^n is closed, so the lemma can be used!) with $V^n = \emptyset, B^n = M^n$.
- $\boxed{\operatorname{Ker}\partial \subset \operatorname{Im} j_*}$ Take an arbitrary element $[C^n, f] \in \Omega_n(X, A)$ that is in the kernel of ∂. As it is in the kernel of ∂, there is a manifold B^n with $\dot{B}^n = \dot{C}^n$[5], and there is a map $g : B^n \to A$ with $g|_{\dot{B}^n} = f|_{\dot{C}^n}$. Sew together C^n and $-B^n$ by their common boundary using (B.0.4.1): this produces a closed manifold M^n. The maps can also be joined thanks to $g|_{\dot{B}^n} = f|_{\dot{C}^n}$; this gives a map $F : M^n \to X$ – so $[M^n, F] \in \Omega_n(X)$ – which restricts to f and g in the appropriate submanifolds. Applying lemma 5.3.1 from earlier again with $V^n = C^n, B^n = M^n$, we get $j_*[M^n, F] = [C^n, f]$.
- $\boxed{\operatorname{Ker} i_* \subset \operatorname{Im}\partial}$ Take an element $[M^n, f] \in \Omega_{n-1}(A)$. If it is 0 in $\Omega_{n-1}(X)$, then the singular manifold in X showing this also forms an element of $\Omega_n(X, A)$ which maps to $[M^n, f]$.
- $\boxed{\operatorname{Ker} j_* \subset \operatorname{Im} i_*}$ Take an element $[M^n, f] \in \Omega_n(X)$. The manifold which shows that this element is 0 in $\Omega_n(X, A)$ proves that (M^n, f) has the same bordism class as some $(N^n, g : N^n \to A)$ (N^n is closed!).

□

(ES.5) *If $\varphi, \psi : (X, A) \to (Y, B)$ are homotopic, then $\varphi_* = \psi_*$.*

Proof. For an arbitrary singular manifold $[C^n, f]$ we consider the sequence of maps

$$\begin{tikzcd} (C^n, \dot{C}^n) \arrow[r, "f"] & (X, A) \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & (Y, B), \end{tikzcd}$$

and notice that it can be extended into the following sequence using the homotopy H between φ and ψ:

$$(I \times C^n, I \times \dot{C}^n) \xrightarrow{(\mathrm{id}, f)} (I \times X, I \times A) \xrightarrow{H} (Y, B) \tag{E5.3.1}$$

We know by (B.0.6) that the product of two manifolds (differentiable, oriented, with boundary) is a manifold (differentiable, oriented, with boundary), whose "boundary behaves nicely". Applying this to $I \times C^n$ we get that it is a manifold and its boundary is $(I \times C^n)^{\cdot} = (\dot{I} \times C^n) \cup (I \times -\dot{C}^n) = (\{1\} \times C^n) \cup (\{0\} \times -C^n) \cup (I \times -\dot{C}^n)$. So $(\{1\} \times C^n) \cup (\{0\} \times -C^n)$ is a regular submanifold of the boundary of $I \times C^n$, and the remainder of the boundary $(I \times -\dot{C}^n)$ maps into B in (Y, B) by (E5.3.1), so we arrive at $[C^n, \varphi f] = [C^n, \psi f]$ using the definition. □

[5] As $\partial[C^n, f] \in \Omega_{n-1}(A)$, no boundary of B^n should be outside of $\dot{C}^n$, as all boundary outside it should be mapped to $\emptyset$ of $(A, \emptyset)$.

(ES.6) *If U is an open subset of X such that $\overline{U} \subset \operatorname{int} A$, then the inclusion map $i : (X - U, A - U) \to (X, A)$ induces an isomorphism (for each $n \in \mathbb{Z}$):*

$$i_* : \Omega_n(X - U, A - U) \xrightarrow{\approx} \Omega_n(X, A)$$

Proof. First we show that i_* is **surjective**. Let (B^n, f) be a singular manifold in $\Omega_n(X, A)$, and $P = f^{-1}(X - \operatorname{int} A), Q = f^{-1}(\overline{U})$. Take the differentiable submanifold B_1^n provided by (B.0.1): $P \subset B_1^n$, Q is disjoint from B_1^n, B_1^n is closed in B^n. Now $[B_1^n, f|_{B_1^n}] \in \Omega_n(X - U, A - U)$, and $[B_1^n, f|_{B_1^n}] = [B^n, f]$ in $\Omega_n(X, A)$ using lemma 5.3.1 with $V^n = B_1^n$.

Injectivity is similar. Take a $[B^n, f] \in \Omega_n(X - U, A - U)$ which maps to 0 in $\Omega_n(X, A)$. This is shown by a manifold C^{n+1} where B^n is a regular submanifold of $\dot{C}^{n+1}$ and the remainder of $\dot{C}^{n+1}$ maps to A under a map F (for which $F|_{B^n} = f$). Next we take $P = F^{-1}(X - \operatorname{int} A) \cup B^n, Q = F^{-1}(\overline{U})$, and the differentiable submanifold separating these: C_1^{n+1}. Now B^n is a regular submanifold of $\dot{C}_1^{n+1}$, and $(C_1^{n+1}, F|_{C_1^{n+1}})$ shows that $[B^n, f]$ is 0 in $\Omega_n(X-U, A-U)$. □

Now let us move on to proving the additional criteria posed by (2.3.18), the theorem providing the convergent spectral sequence of a CW pair (X, A):

Claim 5.3.2. *(a) $\Omega_n(X) = 0$ for an arbitrary space X and integer $n < 0$.*

(b) $\Omega_n(X, \operatorname{sk}_n(X) \cup A) = 0$ for $n \in \mathbb{Z}$ and a CW pair[6] (X, A).

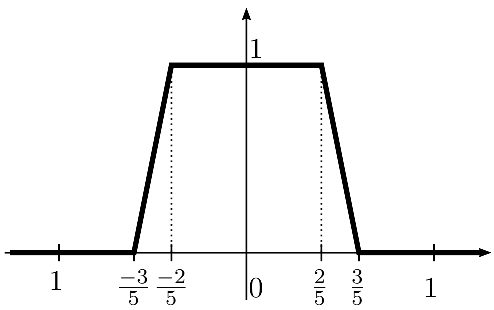
Proof. (a) By definition $\Omega_n(X) = 0$ for $n < 0$.

(b) Take an arbitrary singular manifold (B^n, f) in (X, A). If $f(B^n) \subset \operatorname{sk}_k(X)$ where $k > n$, then we can change f by a homotopy in each k-cell (which is not a part of A) to an $f'(B^n) \subset \operatorname{sk}_{k-1}(X)$: we take a smooth approximation inside each k-cell, and then blow it out to $\operatorname{sk}_{k-1}(X)$. Applying this trick for decreasing k until $k = n$ we get a g that is homotopic to f and shows that $[B^n, g] = 0 \in \Omega_n(X, A)$. As $[B^n, f] = [B^n, g] \in \Omega_n(X, A)$ by (5.1.3), we have finished the proof.

Of course there is a problem with this sketch beside it being highly imprecise: it is unclear what happens for infinite-dimensional CW complexes, as we have no immediate starting k in this case. First, let us make the method above precise, then we will deal with the infinite-dimensional case.

Supposing $f(B^n) \subset \operatorname{sk}_k(X)$, fix an open k-cell $U \cong B_1(0) = \{x \in \mathbb{R}^k : |x| < 1\}$, and let $h : B_1(0) \to \mathbb{R}$,

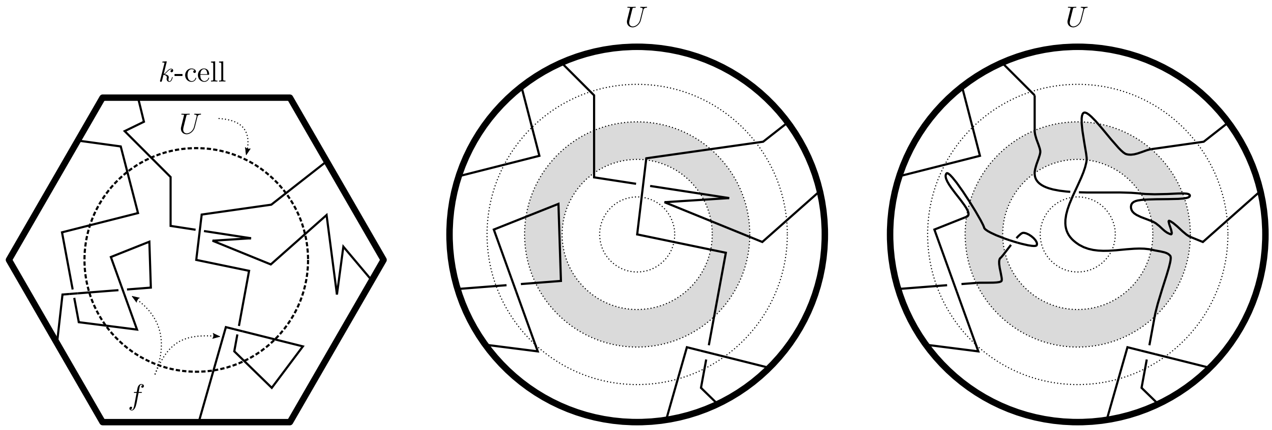
$$h(x) = \begin{cases} 1, & |x| \le \frac{2}{5} \\ 3 - 5|x|, & \frac{2}{5} \le |x| \le \frac{3}{5} \\ 0, & \frac{3}{5} \le |x| \end{cases}$$



and $\hat{f}$ be a smooth approximation of f on $f^{-1}(U) \subset B^n$ such that $|f(x) - \hat{f}(x)| \le 1/5$. Let $\bar{f} : f^{-1}(U) \subset B^n \to U$,

$$\bar{f}(x) = h(f(x))\hat{f}(x) + (1 - h(f(x)))f(x).$$

[6]This means that A is a closed subcomplex.



(a) The k-cell, and U inside it.

(b) f and the boundaries of $B_{n/5}(0)$ for $n = 1, \ldots, 5$. The interesting ring $\frac{2}{5} \leq |x| \leq \frac{3}{5}$ is highlighted.

(c) $\bar{f}$. Note that parts of $\bar{f}$ may spill into different rings then it was originally in.

Then $\bar{f}$ is well-defined, continuous, homotopic to f in U, identical to f on the complement of $f^{-1}(B_{3/5}(0))$, smooth on the inverse image of $B_{2/5}(0)$ by f, and smooth on $\bar{f}^{-1}(B_{1/5}(0))$. As $k > n$, we know that $\bar{f}|_{\bar{f}^{-1}(B_{1/5}(0))} : \bar{f}^{-1}(B_{1/5}(0)) \to B_{1/5}(0)$ is not surjective, there is a $y \in B_{1/5}(0) - \operatorname{Im} \bar{f}$. From this point y we can blow $\bar{f}$ using a homotopy into $\mathrm{sk}_{k-1}(X)$. As $\bar{f}$ and f are identical on the complement of $f^{-1}(\bar{B}_{4/5}(0)) = \bar{f}^{-1}(\bar{B}_{4/5}(0))$, we can interpret this "blowing" homotopy as a homotopy of f. Joining these homotopies for each k-cell we get a homotopy that connects f with the promised f'.

Settling the infinite dimensional case is fairly straightforward. By (B.0.8), a compact set – such as $f(B^n)$ – in a CW complex only intersects finitely many open cells. Corollary B.0.8.1 of this claim states that any compact set of a CW complex is contained in a finite-dimensional skeleton $\mathrm{sk}_k(X)$. Applying this to $f(B^n)$, we get a k which was necessary for finishing the proof.

□

Chapter 6

Classifying bordism

6.1 The statement

Sources. *The main result of this chapter is a paraphrased version of a theorem of Thom (`thm[Tho54]:IV.6`). Most of the section about the Eckmann-Hilton duality (§6.2) are based on [Hat02], while the notation P_+ is borrowed from [CF64]. (§6.3), (§6.4) and (§6.5) are based on [CF64] and [Tho54].*

Remark. The term *manifold* means *(possibly nonorientable) Hausdorff, second-countable, differentiable manifold with boundary* in this chapter.

The aim of this section is to present a theorem concerning a bijection between homotopy classes of maps into a fixed "classifying" space and the (unoriented) bordism groups. This theorem will be both formally extremely similar and also connected to theorem 3.2.13 of (§3.2.4) and theorem 6.5.1 of (§6.5).

Theorem 6.1.1. *(a) For each $n \geq 0$ for any sufficiently large N we have a space called $\Omega^N MO(k+N)$. This space classifies the k codimensional unoriented bordism group for any closed n-manifold P. That is, there is a bijection of the form*

$$\mathfrak{N}_{n-k}(P) \xrightarrow{\sim} [P, \Omega^N MO(k+N)],$$

where $[X, Y]$ denotes the set of homotopy classes of maps between spaces X and Y.

(b) There is a weak homotopy equivalence (3.2.16) *$Q \to \Omega^N MO(k+N)$ only dependent on n and N, such that the homotopy class of any map of any closed n-manifold P into $\Omega^N MO(k+N)$ can be factored through it.*

(c) There is a map $p : Q \to K(\mathbb{Z}_2, k)$ that has the following property.

The cohomology class represented by an element $[B^{n-k}, f] \in \mathfrak{N}_{n-k}(P)$ (defined in (5.2.6)*) can be calculated as follows: take the corresponding homotopy class $[h]$ with $h : P \to \Omega^N MO(k+N)$ given by (a), then factor it through $Q \to \Omega^N MO(k+N)$ to get a homotopy class $[\overline{h}]$ with $\overline{h} : P \to Q$ using (b); finally, compose $[\overline{h}]$ with $[p]$. The result is a homotopy class of the form $P \to K(\mathbb{Z}_2, k)$, so by* (3.2.13) *there is a corresponding element of $\gamma \in H^k(P; \mathbb{Z}_2)$.* ***This γ is the cohomology class associated to $[B^{n-k}, f]$.***

This space $\Omega^N MO(k+N)$ is constructed by applying the loop space functor Ω to the Thom space $MO(k+N)$. Of course, none of these has been defined so far. They will be covered later in this chapter, whose structure – along with the proof of (6.1.1) – is discussed below.

$\sim * \sim$

$$\begin{tikzcd}
P_+ \arrow[rr, "\overline{h}"'] \arrow[rrrr, bend left, "h"] & & Q \arrow[rr, "\varphi"] \arrow[dr, "p"] & & \Omega^N MO(k+N) \\
 & & & K(\mathbb{Z}_2,k) & \\
B \arrow[uu, "f"] \arrow[rr, dashed, "j"] \arrow[dd, hook, "\widehat{f}"] & & A \arrow[uu, "g"] \arrow[dd, hook, "\widehat{g}"] & \text{E-H} & BO(k+N) \arrow[dd, hook, "i"] \\
 & \Sigma^N & & & \\
\Sigma^N P_+ \arrow[rr, dashed, "\widehat{\overline{h}}"] \arrow[rrrr, bend right, "\widehat{h}"'] & & \Sigma^N Q \arrow[rr, "\widehat{\varphi}"] & & MO(k+N)
\end{tikzcd}$$

Figure 6.1: Overview of the proof of (6.1.1)

Figure 6.1 displays the main objects and maps which will be used in the proof, and also some of the ways these maps will be constructed from each other. Normal arrows $\longrightarrow$ indicate continuous maps (only relevant up to homotopy), while dotted ones $\dashrightarrow$ show how different maps can be constructed from each other. This figure is clearly too complicated to aid understanding, so it will return in many different simplified forms throughout the proof. Nevertheless, it is a useful tool to plot all the different topics we have to cover before moving on to actually proving our main theorem (6.1.1). These are:

- The reduced suspension functor Σ, the loop space functor Ω and the Eckmann-Hilton duality: (§6.2)
- Transverse regularity and "pulling back intersections": (§6.4)
- Approximating spaces with manifolds: (§6.3)
- The MO spectrum and the classification of embedded submanifolds up to embedded bordism: (§6.5)

After these topics are covered we may return in section §6.6 to the proof of our central theorem.

6.2 Eckmann-Hilton duality

In this section we will define three functors from the category of pointed spaces to itself: the (unreduced) suspension[1] SX, the reduced suspension ΣX, and the loop space ΩX of a space X: the first two of these are topological spaces containing a subspace identified with X (see definitions 6.2.3, 6.2.4, 6.2.5). Please note that the suspension S will not be used in the thesis, and is only here to make understanding the reduced suspension Σ easier. We will show that

Theorem 6.2.1 (Eckmann-Hilton duality; `see page 395 of [Hat02]`). *For arbitrary pointed spaces X and Y we have a bijection*

$$[\Sigma X, Y]_* \approx [X, \Omega Y]_*.$$

[1]This is actually a functor on the category of topological spaces and continuous maps.

Remark 6.2.1.1. Eckmann-Hilton duality can refer to a much more general concept about reversing the directions of all arrows in a commutative diagram. This generalized duality is not covered in this thesis.

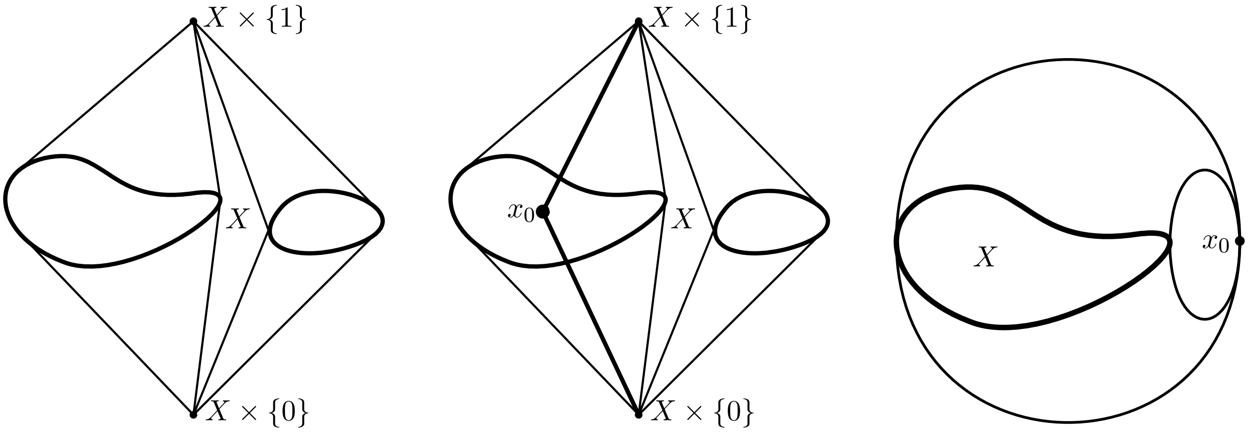
The symbol $[A, B]_*$ denotes in this thesis the set of homotopy classes of maps between pointed spaces A and B (a homotopy between maps should at all times send x_0 to y_0).

It is also important to note that the duality "commutes" with the reduced suspension functor Σ and the loop space functor Ω:

Claim 6.2.2. *Denote temporarily the bijection in the Eckmann-Hilton duality* (6.2.1) *of the last claim by putting a hat* $\widehat{\cdot}$ *on the homotopy class. Then:*

(a) For any maps $g : X \to Y, f : Y \to \Omega Z$ *we have* $\widehat{[fg]} = \widehat{[f]}[\Sigma g]$.

(b) For any maps $g : \Sigma X \to Y, f : Y \to Z$ *we have* $\widehat{[fg]} = [\Omega f]\widehat{[g]}$.



(a) The suspension SX.

(b) The reduced suspension ΣX. The line going through x_0 is collapsed into a point.

(c) ΣX_+.

Figure 6.2: Visualizing suspension.

Before we move on to prove (6.2.1), let us define the three functors from earlier:

Definition 6.2.3. For topological space X, its suspension SX is the space

$$X \times [0, 1]/((X \times \{0\}) \cup (X \times \{1\})).$$

If $X = \emptyset$ then SX is a discrete space with two points instead.

The suspension SX can be thought of as two cones being attached to the space X.

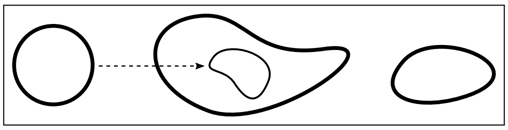
Definition 6.2.4. For a pointed space X with basepoint x_0, its reduced suspension ΣX is the space

$$X \times [0, 1]/((X \times \{0\}) \cup (X \times \{1\}) \cup (\{x_0\} \times [0, 1])),$$

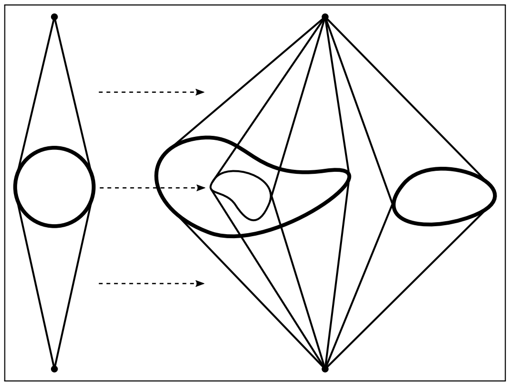
with basepoint the image of $\{x_0\} \times [0, 1]$.

The reduced suspension ΣX can be thought of as starting from the suspension SX, and then taking the quotient by the "vertical" line connecting the tips of the cones and going through x_0.

Definition 6.2.5. For a pointed space X with basepoint x_0, its loop space ΩX is the set of based loops (maps from the pointed S^1) in X equipped with the compact-open topology. The basepoint is the constant loop at x_0.



(a) h



(b) $\widehat{h}$

Figure 6.3: Suspension of a map.

Of course, to make these functors we have to define how morphisms are associated to maps of spaces.

Definition 6.2.6. (a) The suspension (associated map) $\widehat{h} : SX \to SY$ of a map $h : X \to Y$ is obtained by taking $(h, \mathrm{id}) : X \times [0,1] \to Y \times [0,1]$ and then passing to the quotient spaces. *That is, the tips of the cones are mapped into tips, and otherwise the "horizontal slices" homeomorphic to X keep their "height" (their second coordinate $t \in [0,1]$).*

(b) The associated map $\widehat{h} : \Sigma X \to \Sigma Y$ of $h : (X, x_0) \to (Y, y_0)$ is obtained similarly, only that we have to quotient out by a larger subspace.

(c) The associated map $\widehat{h} : \Omega X \to \Omega Y$ of $h : (X, x_0) \to (Y, y_0)$ is obtained by composing any given point of ΩX (its points are based loops in X) with the map h to get a point of ΩY (a based loop in Y).

$$\begin{tikzcd}
(S^1, s_0) \arrow[r, "\gamma\in\Omega X"'] \arrow[rr, bend left, dashed, "h\gamma\in\Omega Y"] & (X, x_0) \arrow[r, "h"'] & (Y, y_0)
\end{tikzcd}$$

A chain of remarks follow:

Remark 6.2.7. S is defined as a functor on the category of topological spaces and not pointed spaces in (6.2.3). When used for pointed spaces in this thesis, the basepoint will be irrelevant, so this distinction can be dismissed.

Remark 6.2.8. The iterated suspension $S^N X$, $N \geq 1$, can be written as a union of two disjoint subsets homeomorphic to S^{N-1} and $X \times \mathbb{R}^N$ (although it is not a "disjoint union" in the topological sense).

Remark 6.2.9. *Let X be a pointed CW complex. Then SX and ΣX are homotopy equivalent.* This can be proved by creating a *CW* complex which is homeomorphic to X but whose basepoint is inside its 0-skeleton (we can do this by subdividing the cell of the basepoint), then using the fact that *CW* pairs are cofibrations and that the part factored in SX to get ΣX is a contractible subcomplex (a compact segment).

Remark 6.2.10. The path components of the loop space ΩX form the fundamental group $\pi_1(X, x_0)$. This is a consequence of the fact that a homotopy of some map $f : X \to Y$ can be regarded as a path in the space of maps from X to Y equipped with the compact-open topology, and vice versa.

Remark 6.2.11. The iterated suspension $\Sigma^N X$ (of a pointed space (X, x_0)) is homeomorphic to $(X \times I^N)/\big((X \times (I^N)^{\cdot}) \cup (\{x_0\} \times I^N)\big)$, or in other words with $(X \times D^N)/\big((X \times S^N) \cup (\{x_0\} \times D^N)\big)$, with basepoint the image of the subspace we take the quotient by. In particular, $\Sigma^N X$ can be written as the union of $(X - \{x_0\}) \times \mathbb{R}^N$ and a point. For an X that is compact and Hausdorff, $\Sigma^N X$ is the one-point compactification of $(X - \{x_0\}) \times \mathbb{R}^N$. This holds in particular for compact manifolds.

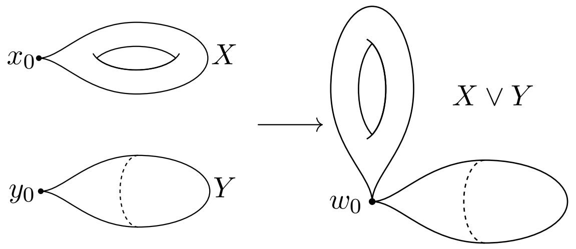
Remark 6.2.12. There are two notable operations on pointed spaces: the wedge sum ($\vee$) and the smash product ($\wedge$). Using these, the reduced suspension ΣX can be written as $X \wedge S^1$.

The definition of these operations are as follows (based on section I.11. of [CF64]):

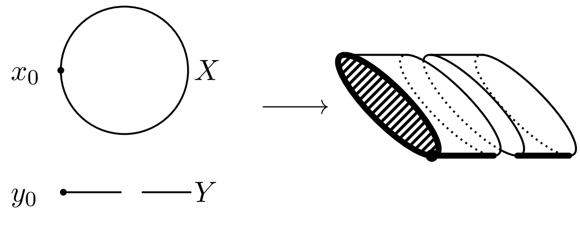
Definition 6.2.13. The wedge sum of two pointed spaces (X, x_0) and (Y, y_0) is the space $X \vee Y$ produced by taking the quotient of the disjoint union of X and Y by the relation $x_0 \sim y_0$.

Definition 6.2.14. The smash product of two pointed spaces X and Y is the space $X \wedge Y$ produced by taking the quotient $(X \times Y)/(X \vee Y)$ and taking the point corresponding to $X \vee Y$ as the basepoint.

Remark. The LaTeXcommand `\vee` produces the symbol of the wedge sum $\vee$, and the command `\wedge` produces the symbol of the smash product $\wedge$.



The wedge sum $X \vee Y$. The basepoint is w_0.



The smash product $X \wedge Y$. The basepoint is the image of the darkened areas under the quotient map.

$\sim * \sim$

Now let us introduce a simple, technical construction which will make our lives significantly easier in the long run.

Definition 6.2.15. For a (not pointed) topological space X, let X_+ denote the disjoint union $X \sqcup \{x_0\}$ of X with a new point. This space can be considered a pointed space, with basepoint x_0.

Claim 6.2.16. *The iterated reduced suspension $\Sigma^N X_+$ ($N \geq 1$) is homeomorphic to $(X \times I^N)/(X \times (I^N)^{\cdot})$, or in other words with $(X \times D^N)/(X \times S^N)$. The basepoint is the point associated to the subspace we take the quotient by. In particular, $\Sigma^N X_+$ can be written as the union of $X \times \mathbb{R}^N$ and a point: if X is compact and Hausdorff, then in this decomposition, $\Sigma^N X_+$ has the topology of the one-point compactification of $X \times \mathbb{R}^N$. In particular, this last claim holds if X is a closed manifold.*

Claim 6.2.17. *There is a bijection between the homotopy classes of* unbased *maps of the form $X \to Y$ and the homotopy classes*[2] *of based maps $X_+ \to (Y, y_0)$ (with $y_0 \in Y$ arbitrary).*

This last claim provides a tool to transition between unbased and based homotopy classes of maps easily. It is particularly useful, as our main theorem (6.1.1) is concerned with unbased homotopy classes, while the tools developed in this section are clearly concerned with the category of pointed spaces.

$\sim * \sim$

Instead of the Eckmann-Hilton duality (theorem 6.2.1), it is easier to prove this more general theorem:

[2] For these classes, only homotopies which fix the basepoint over time are allowed.

Theorem 6.2.18. *Suppose X, Y and A are pointed spaces. Then we have the following homeomorphism:*

$$\mathrm{Maps}_*(X \wedge A, Y) \cong \mathrm{Maps}_*(X, \mathrm{Maps}_*(A, Y)),$$

where $\mathrm{Maps}_*(U, V)$ *are the set of pointed maps $U \to V$ equipped with the compact-open topology.*

Claim 6.2.19. *Suppose (X, x_0) and (Y, y_0) are pointed spaces. Then:*

(a) There is a bijection between paths in $\mathrm{Maps}_*(X, Y)$ *and homotopies $H : X \times [0, 1] \to Y, \forall t : H(x_0, t) = y_0$.*

(b) The path-components of $\mathrm{Maps}_*(X, Y)$ *are the set of homotopy classes $[X, Y]_*$.*

Combining this claim with the previous theorem, putting $A = S^1$, and recalling remark 6.2.12 we get $[\Sigma X, Y]_* \approx [X, \Omega Y]_*$, the Eckmann-Hilton duality (6.2.1).

6.3 Approximating spaces

Some spaces are easier to deal with than others. For example, the $(n + 1)$-skeleton of a *CW* complex determines its first n homology groups, which simplifies algebraic calculation. Moreover, their nearly-combinatorial structures makes them manageable. On the other hand there is the world of (differentiable) manifolds, which are harder to describe, but provide a setting where thanks to smooth approximations any map can be considered "sufficiently nice". Furthermore, the availability of Poincaré duality lets us connect calculations involving homology and cohomology groups.

Thus it would be a great aid if we only had to consider *CW* complexes/(differentiable) manifolds for certain problems. This is especially true in homotopy theory, as there only the homotopy classes of maps are relevant, so any map of manifolds "is" smooth, and any map of *CW* complexes "is" cellular (`thm[Hat02]:4.8`). Thankfully, homotopy theory doesn't distinguish spaces which are homotopy equivalent, and oftentimes only weak homotopy equivalence (3.2.16) is enough to translate between spaces. This can – and does – give us tools to translate certain problems in homotopy theory to only a single kind of spaces. Here we list two of these.

Theorem 6.3.1 (`Prp[Hat02]:4.13`)**.** *For any space X there is a CW complex Z and a weak homotopy equivalence* (3.2.16) *$f : Z \to X$.*

Remark 6.3.1.1. Of course, this Z may have infinitely many cells in a given dimension, and also be infinite dimensional.

Definition 6.3.2. Z of the theorem above (6.3.1) is said to be a *CW* approximation to X.

For the convenience of the reader, let us recall the definition of weak homotopy equivalence:

Definition (See (3.2.16))**.** A continuous mapping $f : X \to Y$ is a weak homotopy equivalence if the induced map on the set of path components is a bijection, while for each $n \geq 0$ and $x \in X$ the following induced homomorphism is an isomorphism:

$$f_* : \pi_n(X, x) \to \pi_n(Y, f(x)).$$

This approximation will be used in the setting where the following property would be desirable: using the notation of the theorem above (6.3.1), *any homotopy class $[h]$, where $h : Y \to X$, can be written as the composition of homotopy classes $[f]$ and $[h_0]$ for some $h_0 : Y \to Z$.* For practical reasons, we will only consider the m-skeleton of Z for some large $m \in \mathbb{N}$. This still means that the induced homomorphisms on the homotopy groups are isomorphisms up to a large dimension, but the restricted map $f|_{\mathrm{sk}_m(Z)}$ will no longer be a weak homotopy equivalence. Nevertheless, the property above will still hold for any low-dimensional CW complex Y, which is going to be sufficient for our purposes.

Now let us turn to the world of manifolds. The following claim states that transitioning from CW complexes to manifolds can be performed systematically:

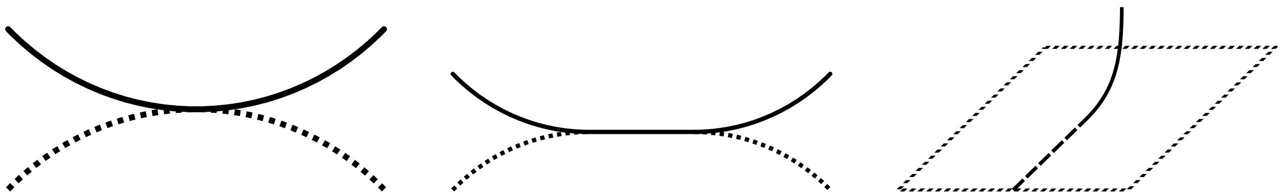
Theorem 6.3.3. *Any finite CW complex is homotopy equivalent to a manifold with boundary.*

This theorem can be proved by embedding the manifold in some euclidean space $\mathbb{R}^N$, and taking a tubular neighborhood (which exists).

6.4 Transverse regularity

As we are going to be dealing with singular manifolds in the proof of our main theorem (6.1.1), and previous chapters already hinted towards a heavy use of homotopies, it is only logical at this point to consider ways in which we can change singular manifolds by a homotopy to get "nicer" singular manifolds (still equivalent to the original). Thankfully, we will be concerned with bordism groups of *manifolds*, so the tools of differential topology will all be available to us. Using these, one obvious way we can pick simpler representatives of a bordism class is by taking a smooth approximation to any given singular manifold. If ε is sufficiently small, this approximation is homotopic to the original map, so we indeed get what we wanted. For more on this topic, see §I.9. of [CF64].

Now let us introduce another concept of "regularity" involving maps between manifolds. In this case, we are interested in not one, but two maps to the same manifold, both of which we immediately assume to be smooth. This may not be surprising to the reader: where there is one smooth map, there are others. In particular, we are interested in the *intersection* of the images of the two maps. While generally we get another (smooth) singular manifold, there can be cases where the two images just "touch" each other, or "coincide" in a significant patch, leading to non-manifold intersections. Thom's concept of *transverse regularity* is introduced to formulate a general setting where taking



(a) Two lines meeting in a single point.

(b) Two lines coinciding in a segment.

(c) A 2-dimensional example.

Figure 6.5: Non-manifold intersections.

the intersection is a sensible operation, while its accompanying theorem states that this regularity is achievable using only a small homotopy. For the following definitions and claims see page 21 of [CF64].

Definition 6.4.1. Suppose N is a manifold without boundary, and N_1 is a regularly embedded submanifold. For any $x \in N_1$, the tangent space T_xN_1 can be regarded as a linear subspace of T_xN. The space of **normal vectors** to N_1 is by definition the vectorspace T_xN/T_xN_1.

Definition 6.4.2. Suppose N, M are manifolds without boundary, $f : M \to N$ is smooth, and N_1 is a regularly embedded submanifold of N. f is said to be **transverse regular** on N_1 if for any $x \in N_1, y \in f^{-1}(x) \subset M$ the composition

$$T_yM \xrightarrow{df} T_xN \longrightarrow T_xN/T_xN_1$$

is surjective.

Claim 6.4.3. *If $f : M \to N$ is transverse regular on $N_1 \subset N$, then $f^{-1}(N_1)$ is a regularly embedded submanifold of M where $f^{-1}(N_1)$ has the same codimension as N_1, or in other words:*

$$\dim M - \dim f^{-1}(N_1) = \dim N - \dim N_1.$$

Remark 6.4.3.1. Let M_1 be $f^{-1}(N_1)$. There is then an induced map $\widetilde{f} : M_1 \to N_1$ – the restriction of f to M_1 – which makes the square below commute.

$$\begin{tikzcd} M_1 \arrow[r, dashed, "\widetilde{f}"] \arrow[d, hook] & N_1 \arrow[d, hook] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Definition 6.4.4. In the setting of the claim above (6.4.3), we will call the embedded submanifold $f^{-1}(N_1) \subset M$ (together with its embedding) the **pullback** of N_1.

Claim 6.4.5. *"The pullback of a pullback is the pullback through the composition". Suppose we are in the following situation: $f : M \to N$ is transverse regular on $N_1 \subset N$, N_1's pullback through f is $M_1 \subset M$, and $g : L \to M$ is transverse regular on M_1. Then fg is transverse regular on N_1, and N_1's pullback though fg is the same as M_1's pullback through g.*

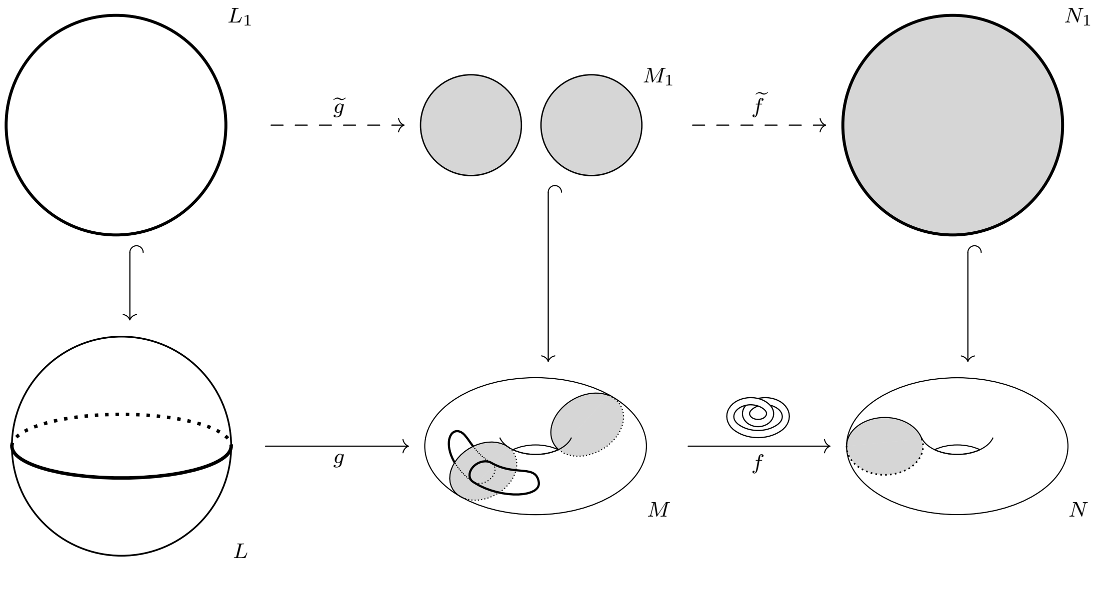


Figure 6.6: Two pullbacks; an illustration for claim 6.4.5.

Claim 6.4.6. *"The cohomology class represented by the pullback is the pullback of the cohomology class represented by the original embedded submanifold". Suppose $f : M^m \to N^n$ is transverse regular on $N_1^p \overset{i}{\hookrightarrow} N^n$ and that the cohomology class represented by $[N_1^p, i] \in \mathfrak{N}_p(N^n)$ is $\gamma \in H^{n-p}(N^n)$. Let the pullback of N_1^p through f be $M_1^{m-(n-p)} \overset{j}{\hookrightarrow} M^m$. Then the cohomology class represented by $[M_1^{m-(n-p)}, j] \in \mathfrak{N}_{m-(n-p)}(M^m)$ is $f^*(\gamma) \in H^{n-p}(M^m)$.*

The following theorem of Thom ensures that we can transform any smooth f to be transverse regular on any N_1.

Theorem 6.4.7 (Thom's transversality theorem; `Thm[CF64]:I.10.4`). *Let $f : M^n \to N^p$ be a smooth map, N_1^{p-q} be a closed submanifold*[3] *of N^p, and ε be a positive real-valued continuous function on M^n. Let A be a closed (possibly empty) subset of M^n such that such that transverse regularity holds for any $y \in A \cap f^{-1}(N_1^{p-q})$. Then there exists a differentiable map $g : M^n \to N^p$ such that:*

1. *g is an ε-approximation of f,*
2. *g is transverse regular on N_1^{p-q}, and*
3. *$g|A = f|A$.*

Remark 6.4.8. While all of the above was stated for finite dimensional spaces, an infinite dimensional variant of this machinery also holds. If N is "infinite dimensional", and N_1 is an "infinite dimensional but finite codimensional" regularly embedded submanifold, then all definitions and theorems (clearly!) generalize. We don't give a formal description of this however, as this is only used to give an intuitive understanding of what happens in the MO classification theorem (6.5.1).

Remark 6.4.9. A similar concept of transverse regularity can be formulated when N_1 is not an *embedded* but an *immersed* submanifold. Of course, in this case we cannot identify the points of N_1 with the point of its image as we did in the definition of normal vectors (6.4.1) and transverse regularity (6.4.2). The claim about receiving a pullback (6.4.3) still holds, with the exception that now we get an *immersed* submanifold "$f^{-1}(N_1)$" instead of an *embedded* one. Moreover, claim 6.4.6 can also be stated for this version of pullback too. Thom's transversality theorem (6.4.7) also generalizes to this case: roughly speaking, we first make f transverse regular on the self intersections, then on the remainder of the image using the form of the theorem discussed above.

Remark 6.4.9.1. This generalized pullback can be explained with a commutative square, similarly to remark 6.4.3.1. Suppose the immersion of N_1 into N is called i and the pullback "$f^{-1}(N_1)$" is the immersion $j : M_1 \to M$. Then there is a map $g : M_1 \to N_1$ which makes the square below commute.

$$\begin{array}{ccc} M_1 & \xrightarrow{g} & N_1 \\ \big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle i} \\ M & \xrightarrow{f} & N \end{array}$$

6.5 Thom spaces

Section §3.2.4 introduced a bijection between the kth cohomology group $H^k(X;\mathbb{Z})$ of a space X and the homotopy classes of maps $X \to K(\mathbb{Z}_2, k)$ (denoted $[X, K(\mathbb{Z}_2, k)]$), using some fixed space $K(\mathbb{Z}_2, k)$ whose definition is not of particular interest now. In this section, we present a similar bijection, which utilizes the spaces denoted $MO(k)$. Similarly to the previous construction, this one also concerns the set of homotopy classes $[X, MO(k)]$ of maps into $MO(k)$, and the exact definition of $MO(k)$ won't be used later in this thesis. Conceptually the bijection builds upon the concept of pullbacks of transverse regular maps (6.4.4), although it uses a generalization.

Theorem 6.5.1 (`Thm[Tho54]:IV.6`[4]). *Let P be a closed manifold*[5]*. Then there is a bijection from the homotopy classes of maps of the form $h : P \to MO(k)$ to the embedded bordism classes of closed embedded*[6] *submanifolds into P of codimension k. That is,*

[3]Manifold – and thus submanifold – still means differentiable manifold.

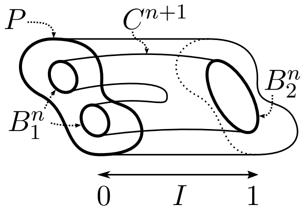
[4]In Thom's original work *embedded bordism* is referred to by the name *L-equivalence*, especially with coefficients in $\mathbb{Z}_2$. For the definitions, see page 71 of [Tho54].

[5]This still means closed *differentiable manifold*.

[6]This means *proper smooth embedding*.

- *for each homotopy class* $[h]$ *with* $h : P \to MO(k)$ *there is a corresponding closed embedded submanifold* $B \hookrightarrow P$ *with* $\dim B = \dim P - k$*, unique up to embedded bordism, and*
- *for each embedded submanifold* $B \hookrightarrow P$ *with* $\dim B = \dim P - k$ *there is associated a unique homotopy class of the form* $h : P \to MO(k)$ *(in other words, an element of* $[P, MO(k)]$*).*

The theorem also holds for the one-point compactification of a (not necessarily compact) manifold P*, and the homotopy classes of based*[7] *maps of the form* $h : P \to MO(k)$*. In this case, the compact embedded submanifolds of* P *are classified up to embedded bordism in* P*.*



So on one hand there are closed embedded submanifolds $B \hookrightarrow P$ of codimension k – up to some equivalence relation – while on the other hand there are elements of $[P, MO(k)]$ – or alternatively $[P, MO(k)]_*$. In many ways, this theorem is very similar to the main theorem we aim to prove (6.1.1), but instead of bordism classes we classify embedded bordism classes, and using homotopy classes of maps to $MO(k)$ rather than $\Omega^N MO(k+N)$. The equivalence relation of (6.5.1) comes from the following definition:

Definition 6.5.2. Two embedded submanifolds $i_1 : B_1^n \hookrightarrow P, i_2 : B_2^n \hookrightarrow P$ into the manifold P are **embedded bordant**, if there is an embedded submanifold $j : C^{n+1} \hookrightarrow P \times I$ with $B_1^n \sqcup B_2^n = \dot{C}^{n+1}$, $(i_1, 0) = j|_{B_1^n}, (i_2, 1) = j|_{B_2^n}$.

Claim 6.5.3. *This is an equivalence relation.*

Remark 6.5.4. The bijection of the MO classification theorem (6.5.1) can be described in the light of remark 6.4.8 about generalizing transverse regularity to "infinite dimensional spaces". For this, all we have to know is that $MO(k)$ is "infinite dimensional", and has a "smoothly" embedded subspace $BO(k)$ of "codimension k". To find the embedded bordism class of a homotopy class $[h]$ of the form $h : P \to MO(k)$, simply take a representative of $[h]$ that is "transverse regular" on $BO(k)$ – let's say h_1. The embedded submanifold it represents can be obtained by taking $h_1^{-1}(BO(k))$ as usual. If h_1 and h_2 are homotopic and "transverse regular" on $BO(k)$ then a "smooth" homotopy of the form $H : P \times I \to MO(k)$ can be chosen which connects them, and is also "transverse regular" on $BO(k)$. The embedded submanifold associated to H shows that the embedded submanifolds B_1, B_2 associated to h_1 and h_2 are indeed embedded bordant.

Also note that $MO(k)$ is a pointed space, with basepoint "far away" from (not inside of) $BO(k)$. As a consequence, the MO classification theorem (6.5.1) works even when P is not a manifold, but instead is a pointed space and looks like a manifold at all points except the basepoint. In particular, if P *is* a manifold, this theorem can be applied to $\Sigma^N(P, p_0)$ or $\Sigma^N P_+$.

$\sim * \sim$

Now we move on to define $MO(k)$ and $BO(k)$, although understanding its construction is not necessary for this thesis.

Definition 6.5.5. $MO(k)$ is the Thom space associated to $\gamma^n \to BO(n)$, the universal vector bundle of rank n.

Of course, at this point this is not really enlightening. There are three things we must cover for this definition to make sense:

- What is a Thom space? (6.5.6)
- What is $BO(n)$? (6.5.7)

[7]The basepoint of P is ∞, while $MO(k)$ is a pointed space, as discussed later on.

- What is a universal vector bundle? (6.5.10)

Definition 6.5.6 (Thom space; `see (§I.11) of [CF64]`)**.** Suppose there is given a vector bundle $\xi : E(\xi) \to B(\xi)$, with a smoothly varying inner product on the fibers[8]. As we have a smoothly varying inner product, for each fiber $\mathbb{R}^n$ we can take the closed unit ball around the origin; let the union of these balls for each fiber be the space $D(\xi)$, and the union of the unit spheres be $S(\xi)$. The **Thom space** $T(\xi)$ associated to the vector bundle ξ is by definition the pointed space $D(\xi)/S(\xi)$, with the basepoint being the point associated to $S(\xi)$.

Remark 6.5.6.1*.* In this thesis, all $B(\xi)$'s can be equipped with a smoothly varying inner product.

Remark 6.5.6.2 (See page 25 of [CF64])*.* T is a functor from the category of vector bundles and bundle maps to the category of pointed spaces.

Definition 6.5.7 ($BO(k)$)**.** Let $BO(k)$ be the Grassmannian $\mathrm{Gr}(k, \mathbb{R}^\infty)$. That is, the points of $BO(k)$ are the k-dimensional linear subspaces of $\mathbb{R}^\infty$.

Equivalently, it can be defined as the direct limit of the Grassmannians $\mathrm{Gr}(k, \mathbb{R}^{k+n})$ with $n \to \infty$.

The following two remarks and one definition are interesting, though highly irrelevant:

Remark 6.5.7.1*.* Usually, $BO(k)$ is not defined to be unique, and the definition above is just a single construction. Following the logic of the next remark, it is usually introduced as a classifying space (6.5.8) for $O(k)$.

Remark 6.5.7.2*.* There is a *weakly contractible* space $EO(k)$, and a *proper, free* action of $O(k)$ on it, such that the quotient by this action is $O(k)$. The definitions of the concepts in this last sentence are:

- *weakly contractible*: a space X such that $\pi_i(X) = 0$ for all $i \geq 0$
- *proper action*: G is a topological group and the map $G \times X \to X \times X, (g, x) \mapsto (gx, x)$ is proper
- *free action*: $gx = hx \implies g = h$ for any $g, h \in G$ and $x \in X$

The quotient map $p : EO(k) \to BO(k)$ is a fiber bundle with $O(k)$ as a fiber. This makes $BO(k)$ a classifying space (6.5.8) for $O(k)$.

Definition 6.5.8. Suppose there is given a topological group G. A **classifying space** for G is a space which is a quotient of a weakly contractible space by a proper free action of G. It is typically denoted BG, while the weakly contractible space is denoted EG.

Before we move on to defining the universal vector bundle, it is customary to present a sketched definition of the Grassmannians.

Definition 6.5.9. The **Grassmannian** $\mathrm{Gr}(k, V)$, with $k > 0$ and V a real vector space, is the topological space whose points are the k-dimensional linear subspaces of V. The topology is as one would expect.

Remark 6.5.9.1*.* For finite dimensional vector spaces, the Grassmannians are compact smooth manifolds.

Finally, let us focus on universal vector bundles.

Definition 6.5.10 (Universal vector bundle)**.** Let B be a Grassmannian of the form $\mathrm{Gr}(k, V)$. The **universal/tautological vector bundle** $\gamma^k \to B$ is defined as follows: for each $W \in B$ and $x \in W$ (W, x) (W is a linear subspace!) is a point of γ^k, and its image in the bundle map is W. The topology is as one would expect.

In the infinite dimensional case, γ^k can be defined as the direct limit of the γ^k_n's, with $\gamma^k_n \to \mathrm{Gr}(k, \mathbb{R}^{n+k})$ being the finite dimensional universal vector bundle.

[8] A fiber is the inverse image of a point in $B(\xi)$.

Remark 6.5.10.1. Combining all these definitions, we can see that $BO(k)$ is "smoothly" embedded into $MO(k)$ in a natural way, and that it is "k-codimensional".

$\sim * \sim$

The sequence $(MO(k))_{k\geq 1}$ – sometimes simply denoted as MO – is sometimes referred to as a (Thom) *spectrum*. This is due to the following definition, similar to those used in stable homotopy theory and important in generalized cohomology theories:

Definition 6.5.11. A spectrum is a sequence of spaces $(X_n)_{n\in\mathbb{N}}$ and maps $\Sigma X_n \to X_{n+1}$.

It is frequent that other criteria are included in the definition. This definition implies that there are maps $\pi_k(X_n) \to \pi_{k+1}(X_{n+1})$ induced by the composition $\pi_k(X_n) \to \pi_{k+1}(\Sigma X_n) \to \pi_{k+1}(X_{n+1})$. We do not detail here how the $MO(k)$'s form a spectrum.

6.6 The proof

Defining everything in figure 6.1, which includes the definitions of both the bijection and its inverse in (a) of theorem 6.1.1, makes up the bulk of the proof: after we are finished with them, the actual proof is fairly straightforward.

The variable N will be increased at a few places throughout the proof, but an exact value for it could be computed. First, let N be large enough that any closed n-manifold can be embedded into $\mathbb{R}^N$ ($2n+1$ is enough by the Whitney embedding theorem (B.0.2)). We only brought this subject forward to somewhat motivate the presence of N in the later parts.

It should also be noted that all horizontal arrows of figure 6.1 will mean pointed maps, and we always consider their based homotopy classes. This is why we wrote P_+ in the top-left corner instead of P. Thankfully, the unbased homotopy classes of maps of the form $P \to X$ are in a one-to-one correspondence with the based homotopy classes of pointed maps $P_+ \to (X, x_0)$ (6.2.17), so it suffices to show a bijection between the latter set and $\mathfrak{N}_{n-k}(P)$.

6.6.1 The static parts

Before we move on to the definition of the bijection, let us establish the "static" right-hand side of figure 6.1. This will be aided by the less cluttered figure of 6.7.

Suppose n and N is already determined. Then as stated in theorem 6.1.1, we will be studying elements of $\mathfrak{N}_{n-k}(P)$ and maps of the form $P \to \Omega^N MO(k+N)$. Thankfully after sections §6.2 and §6.5, we already understand what $\Omega^N MO(k+N)$ is.

Of course, we will want to use the classifying property (6.5.1) of $MO(k+N)$ in the proof, so we keep in mind the inclusion map $i : BO(k+N) \hookrightarrow MO(k+N)$ of this theorem. This gives us the bottom-right corner of figure 6.7.

$$\begin{tikzcd}
Q \arrow[rr, "\varphi"] \arrow[dr, "p"] & & \Omega^N MO(k+N) \arrow[ddd, dotted] \arrow[ddd, phantom, "\text{(S1)}", near start] \\
\arrow[r, phantom, "\text{(S4)}"] & K(\mathbb{Z}_2, k) & \\
A \arrow[uu, "g"] \arrow[d, hook, "\widehat{g}"] \arrow[uu, phantom, bend right=60, "\text{(S3)}"'] & & BO(k+N) \arrow[d, hook, "i"] \\
\Sigma^N Q \arrow[rr, "\widehat{\varphi}"] & \text{(S2)} & MO(k+N)
\end{tikzcd}$$

Figure 6.7: The objects and maps independent of the choice of $[B, f]$ or $[h]$.

Where does Q come from? It is connected to one of the main ideas of the proof, which will be that we want to apply theorem 6.5.1 to the (homotopy class of the) map id : $MO(k+N) \to MO(k+N)$ to get a "universal" embedded submanifold in $MO(k+N)$. Moreover, we would also like to have a similar "universal object" in $\Omega^N MO(k+N)$, corresponding to its identity under our main theorem[9] (6.1.1). However, neither $MO(k+N)$ nor $\Omega^N MO(k+N)$ is a manifold, so we have to approximate them with manifolds first: Q will be the approximation of $\Omega^N MO(k+N)$, while $\Sigma^N Q$ will act as the approximation of $MO(k+N)$.

How do we construct Q? Let X be the CW complex and $\varphi_0 : X \to \Omega^N MO(k+N)$ be the weak homotopy equivalence provided by our CW approximation theorem (6.3.1). Let Y be the skeleton of X up to a large dimension, so that $\varphi_0|_Y : Y \to \Omega^N MO(k+N)$ induces isomorphisms on the homotopy groups up to some dimension greater than $n+1$. This ensures that any map $h : P_+ \to \Omega^N MO(k+N)$ of an n-manifold P can be factored through $\varphi_0|_Y$, up to homotopy at least. Now let Q be a manifold homotopy equivalent to Y (6.3.3)[10], and $\varphi : Q \to \Omega^N MO(k+N)$ be the composition of this homotopy equivalence and $\varphi_0|_Y$. Choose a basepoint $q_0 \in Q$ for Q such that φ becomes a basepoint-preserving map. We will denote the dimension of Q by m.

Now that we have defined our basic objects, let us move over to defining the maps of the diagram in the order indicated by the dotted arrows.

(S0) φ is defined in the paragraph above as an analogue of the identity map id : $\Omega^N MO(k+N) \to \Omega^N MO(k+N)$. We should not forget about the fact that any map $h : P_+ \to \Omega^N MO(k+N)$ of an n-manifold P can be factored through $\varphi : Q \to \Omega^N MO(k+N)$ up to homotopy.

(S1) Applying the Eckmann-Hilton duality (6.2.1) N times gives us a homotopy class of maps $[\widehat{\varphi}]$ of the form $\widehat{\varphi} : \Sigma^N Q \to MO(k+N)$ corresponding to $[\varphi]$. We may assume $\widehat{\varphi}$ to be "smooth".

(S2) By the classification theorem regarding the MO spectrum (6.5.1), there is associated[11] an embedding $\widehat{g} : A \hookrightarrow \Sigma^N Q$ to the homotopy class $[\widehat{\varphi}]$. *Reminder of* (6.5.4)*:* roughly speaking, we simply make $\widehat{\varphi}$ transverse regular on $BO(k+N) \subset MO(k+N)$ using a homotopy, and take the pullback (6.4.4) $A := \widehat{\varphi}^{-1}(BO(k+N) \cap \operatorname{Im} \widehat{\varphi})$.

(S3) A is only determined up to embedded bordism in the $(Q - \{q_0\}) \times \mathbb{R}^N$ part of $\Sigma^N Q$. Whichever representative we take of the equivalence class to be A, it can then be projected to the first coordinate. Composing the embedding $\widehat{g}$ with the projection $(Q - \{q_0\}) \times \mathbb{R}^N \to Q$ gives the differentiable[12] map $g : A \to Q$.

(S4) Take the cohomology class in $H^k(Q; \mathbb{Z}_2)$ represented by $[A, g] \in \mathfrak{N}_{m-k}(Q)$[13]. According to the bijection (3.2.13) between $H^k(X; G)$ and $[X, K(G,k)]$, we have a map

$$p : Q \to K(\mathbb{Z}_2, k)$$

associated to this cohomology class. We will accept the following without proof (for details, see §III of [Tho54]). $\Omega^N MO(k+N)$ is homotopy equivalent to a product of copies of Eilenberg-MacLane spaces, among which is a $K(\mathbb{Z}_2, k)$. This map p corresponds to the projection of $\Omega^N MO(k+N)$ onto this component.

[9] Of course, we can't apply this theorem before we prove it. Instead, we first create the "universal object" through other means, and then use it to obtain the proof.

[10] One can construct such a manifold by embedding Y in some euclidean space and taking a tubular neighborhood (for their definition and existence, see pages 21–22 of [CF64]). This of course yields a manifold *with boundary*.

[11] $\Sigma^N Q$ is the one-point compactification of $(Q - \{q_0\}) \times \mathbb{R}^N$ by (6.2.11), so the pointed version of the MO classification theorem holds. This version of the theorem then ensures that $\widehat{g}$ maps into this direct product part.

[12] The MO classification theorem (6.5.1) gives a smooth embedding $\widehat{g}$, and the projection keeps this smoothness.

[13] $MO(k+N)$ classifies embedded manifolds of codimension $k+N$, and $Q \times \mathbb{R}^N \subset \Sigma^N Q$ is $N+m$ dimensional.

6.6.2 Assigning bordism classes to homotopy classes

We want to define the bijection

$$\mathfrak{N}_{n-k}(P) \xrightarrow{\sim} [P_+, \Omega^N MO(k+N)]_*,$$

however, it is easier to first define its inverse. In this, figure 6.8 will be helpful to us. Suppose there

$$\begin{tikzcd}
P_+ \arrow[r, "\overline{h}"'] \arrow[r, phantom, shift left=3, "\text{(h1)}"] \arrow[rr, bend left, "h"] & Q \arrow[r, "\varphi"] & \Omega^N MO(k+N) \\
B \arrow[u, "f"] \arrow[r, dashed, "j"'] \arrow[ur, phantom, "\text{(h2)}"] & A \arrow[u, "g"'] &
\end{tikzcd}$$

Figure 6.8: How to assign a bordism class $[B, f]$ to the homotopy class represented by h.

is given a homotopy class of pointed maps $[h]$ represented by $h : P_+ \to \Omega^N MO(k+N)$. To this, we want to assign an element $[B, f] \in \mathfrak{N}_{n-k}(P)$. This is done in two steps:

(h1) Change h by a homotopy to get a representative of the homotopy class $[h]$ that can be factored through $\varphi : Q \to \Omega^N MO(k+N)$. This can be done as noted in (S0), and it gives a map (more precisely, a homotopy class of maps $[\overline{h}]$ of the form) $\overline{h} : P_+ \to Q$.

(h2) We take the pullback (6.4.9) of $g : A \to Q$ through $\overline{h}$[14]: first, we make $\overline{h}$ transverse to g, then take the "inverse image" of the "intersection" of $\operatorname{Im} \overline{h} \cap \operatorname{Im} g$ by $\overline{h}$. This gives us the singular manifold (B, f) in P. **Its bordism class** will be the one associated to the homotopy class h. As a matter of fact, because this is a transverse pullback, (B, f) is only well-defined up to unoriented bordism anyway.

The map $j : B \to A$ is constructed in (h2), and is the one that makes the square below commute. It exists due to remark 6.4.9.1.

$$\begin{tikzcd}
P_+ \arrow[r, "\overline{h}"'] & Q \\
B \arrow[u, "f"] \arrow[r, dashed, "j"'] & A \arrow[u, "g"']
\end{tikzcd}$$

Of course, (A, g) is also only well-defined up to unoriented bordism, as $(A, \widehat{g})$ is well-defined up to embedded bordism. It is easy to check however, that changing the choice of $(A, \widehat{g})$ does not change the bordism class $[B, f]$.

6.6.3 Assigning homotopy classes to bordism classes

Now let us move on to defining the bijection $\mathfrak{N}_{n-k}(P) \xrightarrow{\sim} [P_+, \Omega^N MO(k+N)]_*$. In this, figure 6.9 will be of use. Suppose there is given a bordism class $[B, f] \in \mathfrak{N}_{n-k}(P)$. We then want to assign an element of $[P_+, \Omega^N MO(k+N)]_*$ to this class. For this, we fix a singular manifold (B, f).

[14] More precisely, the pullback through the restriction of $\overline{h}$ to P.

$$
\begin{tikzcd}
 & P_+ \arrow[rr, bend left, "h"] & Q & \Omega^N MO(k+N) \\
(B1) & B \arrow[u, "f"'] \arrow[d, hook, "\widehat{f}"] & (B3) & \\
 & \Sigma^N P_+ \arrow[r, phantom, "(B2)"] \arrow[rr, bend right, "\widehat{h}"'] & \Sigma^N Q & MO(k+N)
\end{tikzcd}
$$

Figure 6.9: How to assign a homotopy class $[h]$ to a bordism class $[B, f]$.

(B1) First, we create an injective map $\widehat{f} : B \hookrightarrow P \times \mathbb{R}^N (\subset \Sigma^N P_+)$ that when composed with the projection $P \times \mathbb{R}^N \to P$ yields f. This is easy to achieve using only the second set of coordinates in $P \times \mathbb{R}^N$ if N is large enough[15]: just take any embedding $\iota : B \hookrightarrow \mathbb{R}^N$, and consider the composition $\widehat{f} :\ B \xrightarrow{(f,\iota)} P \times \mathbb{R}^N \hookrightarrow \Sigma^N P$.

The map f of the first set of coordinates may not be smooth, but by smoothing out $\widehat{f}$ and taking its projection onto P we remain in the bordism class $[B, f]$. That is, we may assume that f is smooth and $\widehat{f}$ is a smooth embedding.

(B2) As $\widehat{f}$ is a smooth embedding into $\Sigma^N P_+$ (which is the one-point compactification of $P \times \mathbb{R}^N$), the (pointed) classifying property of the MO spectrum (6.5.1) associates to its embedded bordism class a homotopy class of pointed maps $[\widehat{h}]$ of the form $\widehat{h} : \Sigma^N P_+ \to MO(k + N)$.

(B3) Using the Eckmann-Hilton duality (6.2.1) N times we get a homotopy class $[h]$ of the form $h : P_+ \to \Omega^N MO(k + N)$ associated to this homotopy class $[\widehat{h}]$. **This** will be the homotopy class associated to the bordism class $[B, f]$.

Now only one thing remains to check in this subsection: is this a well-defined correspondence? For this, we only have to show that $\widehat{h}$ introduced in step (B2) is well-defined, that is, any two $(B, \widehat{f})$'s are embedded bordant.

Suppose the singular manifolds (B_1, f_1) and (B_2, f_2) represent the same bordism class in $\mathfrak{N}_{n-k}(P)$. We may take f_1, f_2 and F of the connecting singular $(n - k + 1)$-manifold (C, F) to be smooth (if f_1 and f_2 are smooth, F can be chosen to be too). If we apply (B1) to the singular manifold (C, F) we get a singular manifold $(C, \widehat{F})$ which shows that $(B_1, \widehat{f}_1)$ and $(B_2, \widehat{f}_2)$ are bordant in $P \times \mathbb{R}^N$ – for certain fixed maps $\widehat{f}_1$ and $\widehat{f}_2$. Moreover, it is easy to see that $(C, \widehat{F})$ also proves that these two are *embedded* bordant[16]. Now we only have to prove that if we choose $\widehat{f}_1$ in two different ways – both corresponding to the same map f_1 – then $(B_1, \widehat{f}_{11})$ and $(B_2, \widehat{f}_{12})$ are embedded bordant in $P \times \mathbb{R}^N$. This once again is fairly simple. Clearly $\widehat{f}_{11}$ and $\widehat{f}_{12}$ are homotopic: just linearly interpolate the second coordinate. Change this homotopy so that it becomes constant in time near $t = 0$ and $t = 1$. Using

[15] We could increase N if we wanted to, but our initial value for it is already sufficiently large. However, we may have to increase it to make everything well-defined for bordism classes.

[16] Take a collaring neighborhood of B_1 in C using (B.0.4). To define the second coordinate of the map $C \to (P \times \mathbb{R}^N) \times I$, simply send everything to 1 outside this neighborhood, and send the point (b_1, t) of the collar $B_1 \times I$ to $(\widehat{F}(b_1), \theta(t))$ with an appropriately picked $\theta : I \to I$.

the Whitney embedding theorem (B.0.2), approximate this homotopy with an embedding, keeping it unchanged near $t = 0$ and $t = 1$. The result shows that $(B_1, \widehat{f}_{11})$ and $(B_2, \widehat{f}_{22})$ are embedded bordant.

6.6.4 The inverse relation

As a preparation for the proof that the assignments defined in the last two subsections (§6.6.2 and §6.6.3) are indeed inverses of each other, we finish the figure at the beginning of this section (6.1) by introducing the map $\widehat{\overline{h}} : \Sigma^N P_+ \to \Sigma^N Q$ associated to some map $\overline{h} : P_+ \to Q$. $\widehat{\overline{h}}$ is simply the N-time reduced suspension (6.2.6) of $\overline{h}$, which was in turn defined as an element of the homotopy class $[\overline{h}]$ that when composed with $[\varphi]$ yields $[h]$. Now let us finally move on to the actual proof. Figure 6.11,

$$\begin{tikzcd}
P_+ \arrow[r, "\overline{h}"'] \arrow[rr, bend left, "h"] & Q \arrow[r, "\varphi"] & \Omega^N MO(k+N) \\
\Sigma^N P_+ \arrow[r, dashed, "\widehat{\overline{h}}"] & \Sigma^N Q &
\end{tikzcd}$$

Figure 6.10: The definition of $\widehat{\overline{h}}$ as the suspension of $\overline{h}$.

a diagram of all the morphisms and objects defined so far is presented on page 90 to make following the proof easier.

Applying (§6.6.3) to the result of (§6.6.2) is the identity

We want to show that the assignment $[h] \mapsto [B, f] \mapsto [h']$ is the identity. Let us go over what these assignments mean. The crucial step is going to be how we perform (B1), the others are just applications of previous results.

(h1) A $\overline{h} : P_+ \to Q$ is picked such that $[h] = [\varphi][\overline{h}]$. This exists due to the definition of Q.

(h2) $\overline{h}$ is changed by a homotopy such that it becomes transverse regular (6.4.2) on "$g(A)$". Next, we take (A, g)'s pullback (6.4.9), which is basically "$\overline{h}^{-1}(\operatorname{Im} g)$". This gives us the singular manifold $B \xrightarrow{f} P$ and the map j which makes the square below commute.

$$\begin{tikzcd}
P_+ \arrow[r, "\overline{h}"'] & Q \\
B \arrow[u, "f"'] \arrow[r, dashed, "j"'] & A \arrow[u, "g"']
\end{tikzcd}$$

(B1) f can be considered smooth, as it is a pullback of the smooth embedding g through $\widehat{h}$, which is chosen to be smooth.

Now we pick $\widehat{f}$ in such a way that the square below (with $\widehat{\overline{h}}$ the suspension (6.2.6) of $\overline{h}$) becomes commutative. As $\overline{h}$ was transverse regular (6.4.2) on (A, g), $\widehat{\overline{h}}$ is transverse regular on $\widehat{g}(A)$, so this commutativity implies that $(B, \widehat{f})$ is the pullback (6.4.4) of $(A, \widehat{g})$.

$$\begin{tikzcd}
B \arrow[r, dashed, "j"] \arrow[d, hook, "\widehat{f}"] & A \arrow[d, hook, "\widehat{g}"] \\
\Sigma^N P_+ \arrow[r, dashed, "\widehat{\widehat{h}}"] & \Sigma^N Q
\end{tikzcd} \tag{E6.6.1}$$

We construct the appropriate $\widehat{f}$ using the diagram below. As $\widehat{g}$ maps A into $Q \times \mathbb{R}^N \subset \Sigma^N Q$, we can take its "second coordinate function" (its composition with the projection $Q \times \mathbb{R}^N \to \mathbb{R}^N$).

$$\begin{tikzcd}
P & B \arrow[l, "f"'] \arrow[r, "j"] \arrow[rrd, dashed, "\iota"] \arrow[d, dashed] \arrow[dd, dashed, bend right, "\widehat{f}"'] & A \arrow[r] \arrow[rr, bend left, "\widehat{g}"] & Q \times \mathbb{R}^N \arrow[r, hook] \arrow[d] & \Sigma^N Q \\
 & (f,\iota) \arrow[rd, dashed] & & \mathbb{R}^N & \\
 & \Sigma^N P_+ & P \times \mathbb{R}^N \arrow[l, hook'] & &
\end{tikzcd}$$

By taking the composition of this with j as the ι of the original description of (B1), then defining $\widehat{f}$ to be the composition $\widehat{f}:\ B \xrightarrow{(f,\iota)} P \times \mathbb{R}^N \hookrightarrow \Sigma^N P_+$ we obtain the desired $\widehat{f}$ which makes the square (E6.6.1) commute.

(B2) As noted earlier, the commutativity of (E6.6.1) implies that $(B, \widehat{f})$ is the pullback (6.4.4) of $(A, \widehat{g})$. As $(A, \widehat{g})$ was defined as a generalized pullback (6.4.8) of $BO(k+N) \subset MO(k+N)$ through $\widehat{\varphi}$ by using the MO classification theorem (6.5.1) (see (6.5.4)), we can deduce that $[\widehat{\varphi}\widehat{\widehat{h}}]$ is associated to $(B, \widehat{f})$ by the same theorem. This is because of a similar claim to (6.4.5): $(B, \widehat{f})$ is a pullback of $(A, \widehat{g})$ through $\widehat{\widehat{h}}$, which is a pullback of $(BO(k+N), i)$ through $\widehat{\varphi}$, so $(B, \widehat{f})$ is a pullback of $(BO(k+N), i)$ through $\widehat{\varphi}\widehat{\widehat{h}}$. Thus $\widehat{h} = \widehat{\varphi}\widehat{\widehat{h}}$.

(B3) This step simply consists of applying the Eckmann-Hilton duality (6.2.1) to the homotopy class $[\widehat{h}]$ defined by $\widehat{h} : \Sigma^N P_+ \to MO(k+N)$ a total of N times. Instead, we first apply the duality to the composition $[h] = [\varphi\overline{h}]$ $(h : P_+ \to \Omega^N MO(k+N))$ N times. By an iterated version of (a) of (6.2.2), we can calculate this dual by taking the Nth reduced suspension of $[\overline{h}]$ and the Nth dual of $[\varphi]$ and composing the two. According to the last step, this is none other than $[\widehat{h}] = [\widehat{\varphi}][\widehat{\widehat{h}}]$. So conversely, the Nth dual to $[\widehat{h}]$ is indeed $[h]$, proving this half of our main theorem (6.1.1).

Applying (§6.6.2) to the result of (§6.6.3) is the identity

We want to show that the assignment $[B, f] \mapsto [h] \mapsto [B', f']$ is the identity. The idea is something along these lines: after executing all five steps (B1)-(h2) we do not necessarily get the same singular manifold as we started with. However, we may still continue repeating the steps in a cyclic manner, going over (B2) and (B3) again starting with the singular manifold (B', f') now. In (B2)[17] we note how $(B', \widehat{f'})$ is the embedded manifold associated to the homotopy class $[\widehat{\varphi}][\widehat{\widehat{h}}]$. Applying a similar reasoning to the one that appeared in (B3) of the last subsection (§6.6.4) tells us that $(B, \widehat{f})$ is also

[17] Similarly to what happened in this step in the last subsection (§6.6.4).

$$
\begin{tikzcd}
P_+ \arrow[r, "\overline{h}"'] \arrow[rrr, bend left, "h"] & Q \arrow[rr, "\varphi"] \arrow[dr, "p"] & & \Omega^N MO(k+N) \\
B \arrow[u, "f"] \arrow[r, dashed, "j"] \arrow[d, hook, "\widehat{f}"] & A \arrow[u, "g"] \arrow[d, hook, "\widehat{g}"] & K(\mathbb{Z}_2, k) & BO(k+N) \arrow[d, hook, "i"] \\
\Sigma^N P_+ \arrow[r, dashed, "\widehat{\overline{h}}"] \arrow[rrr, bend right, "\widehat{h}"'] & \Sigma^N Q \arrow[rr, "\widehat{\varphi}"] & & MO(k+N)
\end{tikzcd}
$$

Figure 6.11: All the spaces and maps of the proof.

the embedded submanifold associated to the homotopy class $[\widehat{\varphi}][\widehat{\overline{h}}]$, so they must be of the same embedded bordism class, and thus their projections – (B, f) and (B', f') – must be in the same bordism class, which was our goal to prove.

Let us once again go over how this assignment works, and explain the reasoning above in detail.

(B1) We copy all five steps word by word from their original discussion. That is, in this step we define an $\widehat{f} : B \to \Sigma^N P_+$ where actually $\widehat{f}(B) \subset P \times \mathbb{R}^N$, and the composition of the projection to P and $\widehat{f}$ gives f (and all involved maps are smooth).

(B2) We take an $\widehat{h} : \Sigma^N P_+ \to MO(k+N)$ such that $[\widehat{h}]$ corresponds to the embedded submanifold $B \overset{\widehat{f}}{\hookrightarrow} \Sigma^N P_+$ according to the MO classification theorem (6.5.1).

(B3) Applying the Eckmann-Hilton duality (6.2.1) N times gives a homotopy class $[h]$ with $h : P_+ \to \Omega^N MO(k+N)$ corresponding to $[\widehat{h}]$.

(h1) There is a homotopy class $[\overline{h}]$ with $\overline{h} : P_+ \to Q$ such that $[h] = [\varphi][\overline{h}]$, by the definition of Q. However, using $\overline{h}$, we can further study $[\widehat{h}]$: similar to what we did in step (B3) in the proof of the other direction (§6.6.4), we calculate the Nth dual of $[\varphi\overline{h}]$ using (a) of (6.2.2). That is, by taking the Nth reduced suspension of $\overline{h}$ and the Nth dual of φ, and composing them: $[\widehat{\varphi}][\widehat{\overline{h}}]$. As this is equal to the dual of $[h]$, we get that $[\widehat{\varphi}][\widehat{\overline{h}}] = [\widehat{h}]$.

(h2)-(B2) Now repeat everything we did in the last subsection (§6.6.4) in steps (h2) through (B2). This gives us a singular manifold (B', f') in P, and maps $j' : B' \to A$, $\widehat{f}' : B \to \Sigma^N P_+$ which among other things make the following square commute.

$$
\begin{tikzcd}
B' \arrow[r, dashed, "j'"'] \arrow[d, hook, "\widehat{f'}"] & A \arrow[d, hook, "\widehat{g}"] \\
\Sigma^N_+ P \arrow[r, dashed, "\widehat{\overline{h}}"] & \Sigma^N Q
\end{tikzcd}
\tag{E6.6.2}
$$

Furthermore, the commutativity of this square (in a similar way to what happened in (B2)) implies that the homotopy class $[\widehat{h'}]$ associated to the embedded submanifold $B' \overset{\widehat{f'}}{\hookrightarrow} \Sigma^N P_+$ by the MO classification theorem (6.5.1) is the homotopy class $[\widehat{\varphi}][\widehat{\overline{h}}]$. But just in the last step (h1), we calculated that the homotopy class associated to the embedded submanifold $B \overset{\widehat{f}}{\hookrightarrow} \Sigma^N P_+$ is also $[\widehat{\varphi}][\widehat{\overline{h}}]$. As the MO classification theorem gives a bijection, in particular gives an injection, we know that $B \overset{\widehat{f}}{\hookrightarrow} \Sigma^N P_+$ and $B' \overset{\widehat{f'}}{\hookrightarrow} \Sigma^N P_+$ are embedded bordant. This of course actually means that they are embedded bordant in $P \times \mathbb{R}^N$, as stated in the MO classification theorem (6.5.1).

Now all we have to do is to compose $\widehat{f}$ and $\widehat{f'}$ with the projection $P \times \mathbb{R}^N \to P$ to get two elements of the same[18] bordism class of $\mathfrak{N}_{n-k}(P)$: (B, f) and (B', f') by the constructions of $\widehat{f}$ and $\widehat{f'}$. As $[B', f'] \in \mathfrak{N}_{n-k}(P)$ was the bordism class assigned to the homotopy class assigned to the bordism class $[B, f]$, we have shown that this composition is too the identity.

6.6.5 Cohomology

We still have some words to say about the cohomology classes represented by bordism classes (to prove (c) of 6.1.1). While in (S0) we stated that any map of the form $h_0 : P_+ \to \Omega^N MO(k+N)$ can be factored through $\varphi : Q \to \Omega^N MO(k+N)$, the same is true for maps of the form $h : P \to \Omega^N MO(k+N)$. This way, we can pick $\overline{h} : P \to Q$ corresponding to $[B, f]$ such that (B, f) is the pullback of (A, g) (see (§6.6.4) for the details of the construction). After this, the proof is just an application of the fact that the cohomology class represented by a pullback is the pullback of the cohomology represented by the original embedded/immersed manifold (see (6.4.6), and the mention of its generalization in (6.4.9)).

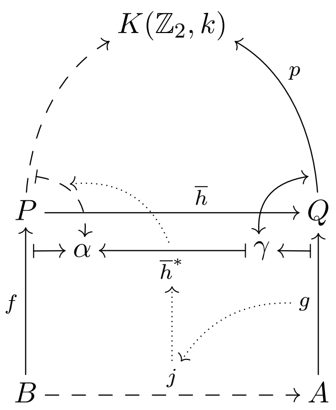


Figure 6.12: The proof

The deduction is as follows (illustrated by figure 6.12): the cohomology class $\alpha \in H^k(P; \mathbb{Z}_2)$ represented by $[B, f] \in \mathfrak{N}_{n-k}(P)$ is the pullback $\alpha = \overline{h}^*(\gamma)$ of the cohomology class $\gamma \in H^k(Q; \mathbb{Z}_2)$ represented by (A, g), because (B, f) is a pullback of (A, g). On the other hand, by the definition of $p : Q \to K(\mathbb{Z}_2, k)$ (see (S4)) the cohomology class represented by (A, g) is the one corresponding to $p : Q \to K(\mathbb{Z}_2, k)$ under the natural correspondence $H^k(Q; \mathbb{Z}_2) \longleftrightarrow [Q, K(\mathbb{Z}_2, k)]$ established in theorem 3.2.13. According to remark 3.2.13.1, the naturality of this bijection means that if there is a map $f_0 : X \to Y$, then the following square commutes:

$$\begin{tikzcd}
{[X, K(\mathbb{Z}_2, k)]} \arrow[r] & H^k(X; \mathbb{Z}_2) \\
{[Y, K(\mathbb{Z}_2, k)]} \arrow[r] \arrow[u, "{\circ[f_0]}"] & H^k(Y; \mathbb{Z}_2) \arrow[u, "f_0^*"']
\end{tikzcd}$$

where the left vertical map is just the composition by $[f_0]$ (the homotopy class of f_0), and the right vertical map is f_0's induced map on cohomology. Putting $X = P$, $Y = Q$, and $f_0 = \overline{h}$ tells us that $\alpha = \overline{h}^*(\gamma)$ corresponds to the composition $p\overline{h}$, which was just the statement of (c) of our main theorem (6.1.1).

With this, the entire proof of (6.1.1) is finished.

[18] The projections are indeed in the same class: the projection of the embedded bordism shows this.

Appendix A

Algebraic preliminaries

A.1 Homological algebra

The contents of this section are covered in compulsory algebra courses here at ELTE, but a quick refresher might come in handy for the reader. Everything here is stated for abelian groups, but it all generalizes to arbitrary R-modules (with R being a commutative ring with unity).

Definition A.1.1. (a) A surjective homomorphism of abelian groups is said to be an epimorphism.

(b) An injective homomorphism of abelian groups is said to be a monomorphism.

Definition A.1.2. (a) Suppose there are homomorphisms $f : B \to C$ and $g : A \to B$ of abelian groups, illustrated with a diagram as

$$A \xrightarrow{g} B \xrightarrow{f} C.$$

The pair of homomorphisms is said to be *exact* (at B) iff $\operatorname{Ker} f = \operatorname{Im} g$, or in other words if the kernel of f is the same as the range of g.

(b) A longer sequence of abelian groups and homomorphisms connecting the adjacent ones is said to be *exact* (at all groups), if all consecutive pairs of homomorphisms are exact.

(c) An exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is said to be a *short exact sequence*.

(d) An infinite sequence of abelian groups (infinite in at least one direction) which is exact at all groups is said to be a *long exact sequence*[1].

Remark A.1.2.1.

- $A \xrightarrow{f} B \longrightarrow 0$ is exact iff f is surjective (epi).
- $0 \longrightarrow A \xrightarrow{f} B$ is exact iff f is injective (mono).
- $0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$ is exact iff f is an isomorphism between A and B.

[1] Sometimes any exact sequence which is not a short exact sequence is called a *long* exact sequence.

Definition A.1.3. A short exact sequence

$$0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$$

is said to *split*, if any of the equivalent statements are satisfied:

(a) There is a homomorphism $p : B \to A$ such that $\mathrm{id}_A = pg$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "g"'] & B \arrow[l, bend right, dotted, "p"'] \arrow[r, "f"] & C \arrow[r] & 0 \end{tikzcd}$$

(b) There is a homomorphism $q : C \to B$ such that $\mathrm{id}_C = fq$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "g"] & B \arrow[r, "f"'] & C \arrow[l, bend right, dotted, "q"'] \arrow[r] & 0 \end{tikzcd}$$

(c) There is an isomorphism $h : B \to A \oplus C$ such that hg is the injection $i : A \hookrightarrow A \oplus C, i(a) = (a, 0)$, and fh^{-1} is the projection $j : A \oplus C \to C, j(a, c) = c$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "g"] \arrow[dr, dotted, "i"'] & B \arrow[r, "f"] \arrow[d, "h"] & C \arrow[r] & 0 \\ & & A \oplus C \arrow[ur, dotted, "j"'] & & \end{tikzcd}$$

Lemma A.1.4 (Splitting lemma)**.** *The three statements of the definition above are indeed equivalent.*

$\sim * \sim$

Definition A.1.5. An abelian group P is called *projective* if for any abelian groups A, B, any homomorphism $g : P \to B$ and any epimorphism $f : A \to B$ there exists a homomorphism $h : P \to A$ such that $fh = g$. In other words, there exists a homomorphism h that makes the diagram below commutative.

$$\begin{tikzcd} & P \arrow[d, "g"] \arrow[dl, dotted, "h"'] \\ A \arrow[r, two heads, "f"'] & B \end{tikzcd}$$

Claim A.1.6. *An abelian group P is projective iff it is isomorphic to a direct summand of a free abelian group F.*

Corollary A.1.6.1. *Free abelian groups are projective.*

Claim A.1.7. *An abelian group P is projective iff every short exact sequence of the form below splits.*

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$$

Corollary A.1.7.1. *Suppose there is a short exact sequence of the form below, with F being a free abelian group. Then this short exact sequence splits.*

$$0 \longrightarrow A \longrightarrow B \longrightarrow F \longrightarrow 0$$

Claim A.1.8. *Any subgroup of a free abelian group is free.*

$\sim * \sim$

Definition A.1.9. $\mathrm{Hom}(A, B)$ is the set of all homomorphisms between the abelian groups A and B. It can easily be given an abelian group structure.

Claim A.1.10. *$\mathrm{Hom}(-, G)$ is a contravariant functor. That is, to every homomorphism $f : A \to B$ there is associated a homomorphism $\mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$ (this is the composition with f), such that:*

(a) The identity homomorphism is associated to the identity homomorphism.

(b) The composition of associated homomorphisms is the associated homomorphism of the (reversed) composition.

Applying the $\mathrm{Hom}(-, G)$ functor will be called *dualization* in this thesis.

Claim A.1.11. *Suppose there is an exact sequence of the form given below.*

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Then the associated sequence below is exact:

$$\mathrm{Hom}(A, G) \longleftarrow \mathrm{Hom}(B, G) \longleftarrow \mathrm{Hom}(C, G) \longleftarrow 0$$

Remark A.1.11.1. It is not true that the dual of a short exact sequence is exact.

Claim A.1.12. *The dualization*

$$\ldots \longleftarrow \mathrm{Hom}(C_{n+1}, G) \xleftarrow{f_{n+1}^*} \mathrm{Hom}(C_n, G) \xleftarrow{f_n^*} \mathrm{Hom}(C_{n-1}, G) \longleftarrow \ldots$$

of the chain complex

$$\ldots \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} C_{n-1} \longrightarrow \ldots$$

is a chain complex itself.

Claim A.1.13. *Suppose there is given a split short exact sequence.*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Then its dual is also a split exact sequence.

$$0 \longleftarrow \mathrm{Hom}(A, G) \longleftarrow \mathrm{Hom}(B, G) \longleftarrow \mathrm{Hom}(C, G) \longleftarrow 0$$

A.2 Diagram chasing

This section is for properly stating all of the diagram chasing lemmas used in this thesis. The five lemma is also proved as an illustration. All others can be done in a similar manner.

Theorem A.2.1 (Five lemma)**.**

$$\begin{tikzcd}
A_1 \arrow[r, dashed] \arrow[d, "\sim"] & B_1 \arrow[r, dashed] \arrow[d, "\sim"] & C_1 \arrow[r, dashed] \arrow[d] & D_1 \arrow[r, dashed] \arrow[d, "\sim"] & E_1 \arrow[d, "\sim"] \\
A_2 \arrow[r, dashed] & B_2 \arrow[r, dashed] & C_2 \arrow[r, dashed] & D_2 \arrow[r, dashed] & E_2
\end{tikzcd}$$

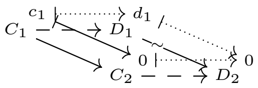
Suppose that the vertical $\xrightarrow{\sim}$ arrows are isomorphisms in the commutative diagram above. Then the homomorphism $C_1 \to C_2$ is also an isomorphism.

Proof.

Injectivity.

$$\begin{tikzcd} c_1 \arrow[r, mapsto] & 0 \\ C_1 \arrow[r] & C_2 \end{tikzcd}$$

Suppose $c_1 \in C_1$ maps to $0 \in C_2$. We will prove that $c_1 = 0$.



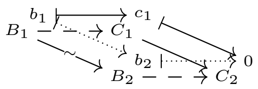
By commutativity, c_1 maps to some $d_1 \in D_1$ which maps to $0 \in D_2$.

$$\begin{tikzcd} d_1 & 0 \arrow[l, dotted, mapsto] \\ D_1 \arrow[r, "\sim"] & D_2 \end{tikzcd}$$

As $D_1 \xrightarrow{\sim} D_2$ is an isomorphism, $d_1 = 0$.

$$\begin{tikzcd} b_1 \arrow[r, dotted, mapsto] & c_1 \arrow[r, mapsto] & 0 \\ B_1 \arrow[r, dashed] & C_1 \arrow[r, dashed] & D_1 \end{tikzcd}$$

As $c_1 \mapsto 0 \in D_1$, by exactness it has an inverse image $b_1 \in B_1$.



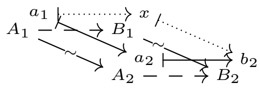
By commutativity b_1 maps to some $b_2 \in B_2$ which maps to $0 \in C_2$.

$$\begin{tikzcd} a_2 \arrow[r, dotted, mapsto] & b_2 \arrow[r, mapsto] & 0 \\ A_2 \arrow[r, dashed] & B_2 \arrow[r, dashed] & C_2 \end{tikzcd}$$

As $b_2 \mapsto 0 \in C_2$, by exactness it has an inverse image $a_2 \in A_2$.

$$\begin{tikzcd} a_1 & a_2 \arrow[l, dotted, mapsto] \\ A_1 \arrow[r, "\sim"] & A_2 \end{tikzcd}$$

As $A_1 \xrightarrow{\sim} A_2$ is an isomorphism, a_2 has an inverse image $a_1 \in A_1$.



By commutativity a_1 maps to some $x \in B_1$ which maps to $b_2 \in B_2$.

$$\begin{tikzcd} x \arrow[dr, mapsto] \arrow[d, phantom, "\vdots"] & \\ b_1 \arrow[r, mapsto] & b_2 \\ B_1 \arrow[r, "\sim"] & B_2 \end{tikzcd}$$

As $B_1 \xrightarrow{\sim} B_2$ is an isomorphism and x and b_1 both map to b_2, they must be equal.

$$\begin{tikzcd} & & 0 \\ a_1 \arrow[r, mapsto] & b_1 \arrow[r, mapsto] \arrow[ur, dotted, mapsto] & c_1 \\ A_1 \arrow[r, dashed] & B_1 \arrow[r, dashed] & C_1 \end{tikzcd}$$

As $a_1 \mapsto b_1 \mapsto c_1$, by exactness $c_1 = 0$. This is what we had to prove.

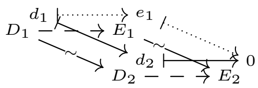
Surjectivity.
Remove any meaning previously assigned to lowercase letters. Assume that $c_2 \in C_2$. We will prove that it has an inverse image $c_1 \in C_1$ which maps to c_2.

$$\begin{tikzcd} c_2 \arrow[r, dotted, mapsto] & d_2 \arrow[r, dotted, mapsto] & 0 \\ C_2 \arrow[r, dashed] & D_2 \arrow[r, dashed] & E_2 \end{tikzcd}$$

c_2 maps to some $d_2 \in D_2$ which maps to $0 \in E_2$ by exactness at D_2.

$$\begin{tikzcd} d_1 & d_2 \arrow[l, dotted, mapsto] \\ D_1 \arrow[r, "\sim"] & D_2 \end{tikzcd}$$

As $D_1 \xrightarrow{\sim} D_2$ is an isomorphism, d_2 has an inverse image $d_1 \in D_1$.



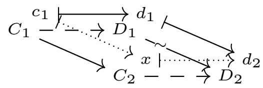
By commutativity d_1 maps to some $e_1 \in E_1$ which maps to $0 \in E_2$.

$$\begin{tikzcd} e_1 = 0 & 0 \arrow[l, dotted, mapsto] \\ E_1 \arrow[r, "\sim"] & E_2 \end{tikzcd}$$

As $E_1 \xrightarrow{\sim} E_2$ is an isomorphism, 0's inverse image $e_1 \in E_1$ is 0.

$$\begin{tikzcd} c_1 \arrow[r, dotted, mapsto] & d_1 \arrow[r, mapsto] & 0 \\ C_1 \arrow[r, dashed] & D_1 \arrow[r, dashed] & E_1 \end{tikzcd}$$

As $d_1 \mapsto 0 \in E_1$, by exactness it has an inverse image $c_1 \in C_1$.



By commutativity c_1 maps to some $x \in C_2$ which maps to $d_2 \in D_2$.

$$\begin{tikzcd} x \arrow[dr, mapsto] & \\ c_2 \arrow[r, mapsto] & d_2 \\ c_2 - x \arrow[r, dotted, mapsto] & 0 \\ C_2 \arrow[r, dashed] & D_2 \end{tikzcd}$$

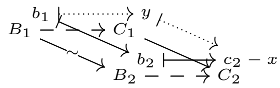
As $c_2, x \mapsto d_2 \in D_2$, we have $c_2 - x \mapsto 0 \in D_2$.

$$\begin{tikzcd} b_2 \arrow[r, dotted, mapsto] & c_2 - x \arrow[r, mapsto] & 0 \\ B_2 \arrow[r, dashed] & C_2 \arrow[r, dashed] & D_2 \end{tikzcd}$$

As $c_2 - x \mapsto 0 \in D_2$, by exactness it has an inverse image $b_2 \in B_2$.

$$\begin{tikzcd} b_1 & b_2 \arrow[l, dotted, mapsto] \\ B_1 \arrow[r, "\sim"] & B_2 \end{tikzcd}$$

As $B_1 \xrightarrow{\sim} B_2$ is an isomorphism, b_2 has an inverse image $b_1 \in B_1$.



By commutativity b_1 maps to some $y \in C_1$ which maps to $c_2 - x \in C_2$.

$$\begin{tikzcd} y \arrow[r, mapsto] & c_2 - x \\ c_1 \arrow[r, mapsto] & x \\ y + c_1 \arrow[r, dotted, mapsto] & c_2 \\ C_1 \arrow[r, dashed] & C_2 \end{tikzcd}$$

As $y \mapsto c_2 - x \in C_2, c_1 \mapsto x \in C_2$, we have $y + c_1 \mapsto c_2 \in C_2$. This is what we had to prove.

$\square$

Lemma A.2.2 (Bow tie lemma)**.** *Suppose the row and column of the commutative diagram below is exact. Then we have* $\operatorname{Im} g_A / \operatorname{Im} f_A \approx \operatorname{Im} g_B / \operatorname{Im} f_B$.

$$\begin{tikzcd} & B_1 \arrow[d] \arrow[dr, "f_B"] & \\ A_1 \arrow[r] \arrow[dr, "f_A"'] & X \arrow[r, "g_B"'] \arrow[d, "g_A"] & B_2 \\ & A_2 & \end{tikzcd}$$

Or in other words:

$$\operatorname{Im}(X \to A_2) / \operatorname{Im}(A_1 \to A_2) \approx \operatorname{Im}(X \to B_2) / \operatorname{Im}(B_1 \to B_2).$$

Theorem A.2.3 (Zig-zag lemma; `thm[Hat02]:2.16`)**.** *Suppose there is given a short exact sequence of chain complexes C_*, D_* and E_* (their maps are the chain maps; see* (1.2.6)*):*

$$0 \longrightarrow C_* \longrightarrow D_* \longrightarrow E_* \longrightarrow 0\ ,$$

that is, in the following commutative diagram the rows are chain complexes and the columns are *exact:*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r] & C_{n+1} \arrow[r] \arrow[d, hook] & C_n \arrow[r] \arrow[d, hook] & C_{n-1} \arrow[r] \arrow[d, hook] & C_{n-2} \arrow[r] \arrow[d, hook] & \dots \\
\dots \arrow[r] & D_{n+1} \arrow[r] \arrow[d, two heads] & D_n \arrow[r] \arrow[d, two heads] & D_{n-1} \arrow[r] \arrow[d, two heads] & D_{n-2} \arrow[r] \arrow[d, two heads] & \dots \\
\dots \arrow[r] & E_{n+1} \arrow[r] \arrow[d] & E_n \arrow[r] \arrow[d] & E_{n-1} \arrow[r] \arrow[d] & E_{n-2} \arrow[r] \arrow[d] & \dots \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}
$$

Then there is associated a long exact sequence of their homology groups:

$$
\begin{tikzcd}
\dots \arrow[r] & H_n(C_*) \arrow[r] & H_n(D_*) \arrow[r] & H_n(E_*) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- node[above]{∂} (\tikztotarget)}] \\
 & H_{n-1}(C_*) \arrow[r] & H_{n-1}(D_*) \arrow[r] & H_{n-1}(E_*) \arrow[r] & \dots
\end{tikzcd}
$$

All maps are the ones induced by the chain maps, except for the connecting homomorphism ∂, which is constructed in the course of the proof.

Theorem A.2.4 (Braid on four strands)**.** *Suppose the following three sequences of abelian groups are exact (different arrow types are utilized to show which homomorphism belongs to which sequence):*

$$
\begin{tikzcd}
\dots \arrow[r, dashed] & B_n \arrow[r, dashed] & A_n \arrow[r, dashed] & AB_n \arrow[r, dashed] & B_{n-1} \arrow[r, dashed] & \dots
\end{tikzcd}
$$

$$
\begin{tikzcd}
\dots \arrow[r, rightsquigarrow] & B_n \arrow[r, rightsquigarrow] & X_n \arrow[r, rightsquigarrow] & XB_n \arrow[r, rightsquigarrow] & B_{n-1} \arrow[r, rightsquigarrow] & \dots
\end{tikzcd}
$$

$$
\begin{tikzcd}
\dots \arrow[r, Rightarrow] & A_n \arrow[r, Rightarrow] & X_n \arrow[r, Rightarrow] & XA_n \arrow[r, Rightarrow] & A_{n-1} \arrow[r, Rightarrow] & \dots
\end{tikzcd}
$$

and the following diagram of "a braid on four strands"[2] *commutes:*

$$
\begin{tikzcd}[column sep=small]
\cdots X_n \arrow[rr, Rightarrow, bend left] \arrow[dr, rightsquigarrow] & & XA_n \arrow[rr, bend left] \arrow[dr, Rightarrow] & & AB_{n-1} \arrow[rr, dashed, bend left] \arrow[dr] & & B_{n-2} \cdots \\
 & XB_n \arrow[ur] \arrow[dr, rightsquigarrow] & & A_{n-1} \arrow[ur, dashed] \arrow[dr, Rightarrow] & & XB_{n-1} \arrow[ur, rightsquigarrow] \arrow[dr] & \\
\cdots AB_n \arrow[ur] \arrow[rr, dashed, bend right] & & B_{n-1} \arrow[ur, dashed] \arrow[rr, rightsquigarrow, bend right] & & X_{n-1} \arrow[ur, rightsquigarrow] \arrow[rr, Rightarrow, bend right] & & XA_{n-1} \cdots
\end{tikzcd}
$$

Then the fourth strand is also exact, except at $(XB_n)_{n\in\mathbb{Z}}$, where only Ker $\subset$ Im *is known.*

[2] The design of this diagram was inspired by the one on page 206 of [Wen19].

A.3 Additional algebra

The following additional algebraic tools are utilized in this thesis:

- **Chain complexes and chain maps**, introduced in section §1.2. Specifically, see definitions 1.2.5, 1.2.6 and 1.2.10, and statements 1.2.7, 1.2.11, 1.4.11 and 3.2.1.
- **The** Tor **functor** – or at least its basics – in section §1.4.2.
- **Spectral sequences**, introduced in section §2.3.
- **Free resolutions** of abelian groups and **the** Ext **functor** in section §3.2.1.
- **Directed sets and direct limits**, introduced in section §4.2.2

Appendix B

Topology

The following statements about topology will be of use in this thesis. Most of these are not proved in the primary and secondary sources, but instead are also referenced there.

$\sim * \sim$

First, a lemma about differentiable manifolds. "Manifold" in this section means *compact differentiable manifold with boundary*, unless otherwise stated.

Claim B.0.1 (`Thm[CF64]:I.3.1`). *Suppose P and Q are closed disjoint subsets of the compact n-manifold B^n. Then there exists a topological manifold $B_1^n \subset B^n$ with $P \subset B_1^n$, Q disjoint from B_1^n and B_1^n closed in B^n. Moreover, B_1^n can be given a differentiable structure by straightening the angle at the corners (see §I.3 of [CF64]).*

We also have the Whitney embedding theorem:

Theorem B.0.2 (Whitney embedding theorem; `thm[CF64]:I.10.2`). *Suppose $p > 2n$. Then any map f of the differentiable n-manifold M^n into $\mathbb{R}^p$ can be ε-approximated by an embedding g. If f is already an embedding on some neighborhood of the closed set $A \subset M^n$, we can choose $g|_A = f|_A$.*

$\sim * \sim$

From here on, we also assume all differentiable manifolds to be oriented.

Theorem B.0.3 (`Thm[CF64]:1.1`). *Suppose U_1 and U_2 are open subsets of the topological n-manifold B^n, which have differentiable structures that induce the same differentiable structure on $U_1 \cap U_2$, and which cover B^n. Then there exists a unique differentiable structure on B^n which induces the differentiable structure of U_1 and U_2.*

Theorem B.0.4 (`Lem[Mil56]:3`). *For any (not necessarily orientable) differentiable manifold B^n there exists an open set $U \supset \dot{B}^n$ and a diffeomorphism $\phi : U \to \dot{B}^n \times [0, 1)$ with $\phi(x) = (x, 0)$ for $x \in \dot{B}^n$.*

Corollary B.0.4.1. *Using* (B.0.3) *and* (B.0.4) *we can glue together (not necessarily orientable, differentiable) n-manifolds B_1^n and B_2^n if $\dot{B}_1^n$ and $\dot{B}_2^n$ are diffeomorphic, to create a manifold $C^n = B_1^n \underset{\dot{B}_*^n}{\cup} B_2^n$ which induces the differentiable structure of the two halves. If both manifolds are oriented, we may choose to glue them together "preserving orientation" or "reversing orientation".*

Now let us move on to theorems about the product of (differentiable) manifolds.

Theorem B.0.5 (`See pages 6-7 of [CF64]`)**.** *Suppose B^n and C^m are manifolds. Then $(B^n \times C^m) - (\dot{B}^n \times \dot{C}^m)$ has a natural differentiable structure that induces the usual differentiable structure on each section of the product, and is thus a manifold.*

Theorem B.0.6 (`See (§I.3.) of [CF64]`)**.** *Suppose B^n and C^m are manifolds. Then $B^n \times C^m$ has a natural differentiable structure which induces the usual differentiable structure on each section of the product, and is thus a manifold.*

$\sim * \sim$

Finally, a few words about CW complexes. Before our main theorem, let us refresh the definition of a good pair according to (1.4.5): a pair of spaces (X, A) is *good*, if A is a nonempty closed subspace of X such that it has an open neighborhood V which deformation retracts onto A.

Theorem B.0.7 (`Prp[Hat02]:Appendix/A.5`)**.** *CW pairs – that is, pairs (X, A) where X is a CW complex and A is a closed subcomplex – are good pairs.*

The following is another fairly standard claim about CW complexes:

Claim B.0.8. *A compact subspace Z (not necessarily a subcomplex) of a CW complex X has nonempty intersections with only finitely many open cells.*

As each point of a CW complex is contained in exactly one open cell, we have the following corollary:

Corollary B.0.8.1. *A compact subspace Z (not necessarily a subcomplex) of a CW complex X is contained in $\mathrm{sk}_k(X)$ for some $k \in \mathbb{N}$.*

Bibliography

Primary references

[CF64] P. E. Conner and E. E. Floyd. *Differentiable Periodic Maps*. Springer-Verlag Berlin Heidelberg GmbH, 1964. ISBN: 978-3-662-41635-8. URL: `https://www.maths.ed.ac.uk/~v1ranick/papers/cf1.pdf`.

[ES52] Samuel Eilenberg and Norman E. Steenrod. *Foundations of Algebraic Topology*. Princeton Mathematical Series. Princeton, New Jersey: Princeton University Press, 1952. ISBN: 978-0691079653. URL: `https://www.maths.ed.ac.uk/~v1ranick/papers/eilestee.pdf`.

[Hat02] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002. ISBN: 0-521-79540-0. URL: `http://pi.math.cornell.edu/~hatcher/AT/ATpage.html`.

[SJ] András Szűcs and András Juhász. *Sáv 4. félév*. URL: `http://bolyai.cs.elte.hu/~szucs/topol4/Binder4.pdf`.

[Tho54] Réné Thom. “Quelques propriétés globales des variétés différentiables.” In: *Commentarii mathematici Helvetici* 28 (1954), pp. 17–86. URL: `http://eudml.org/doc/139072`.

Secondary references

[Lyc16] J. T. Lyczak. *Spectral sequences*. 2016. URL: `https://www.julianlyczak.nl/notes/specseq.pdf`.

[Mil56] John Milnor. *On the relationship between differential manifolds and combinatorial manifolds*. 1956. URL: `https://faculty.tcu.edu/gfriedman/notes/milnor3.pdf`.

[Wen19] Chris Wendl. *Topology I and II*. HU Berlin, 2018–2019. URL: `https://www.mathematik.hu-berlin.de/~wendl/Winter2018/Topologie2/lecturenotes.pdf`.

Reading recommendations

[Eil51] Samuel Eilenberg. *La suite spectrale. I: construction générale*. 1951. URL: `http://www.numdam.org/item?id=SHC_1950-1951__3__A8_0`.

[McC01] John McCleary. *A User's Guide to Spectral Sequences*. Second edition. Cambridge University Press, 2001. ISBN: 978-0521567596. URL: `https://people.math.rochester.edu/faculty/doug/otherpapers/McCleary-UGSS.pdf`.

[Swi75] Robert M. Switzer. *Algebraic Topology – Homotopy and Homology*. New York: Springer-Verlag Berlin Heidelberg, 1975. ISBN: 3-540-42750-3.