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Simulating Isonormal Processes and Fractal Noise Driven Stochastic Processes

Thesis

Applied Mathematics MSc

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Budapest, 2022

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ELTE Természettudományi Kar, szak: Alkalmazott matematikus Msc

NEPTUN azonosító: MBT92H

Diplomamunka címe:
Simulating Isonormal Processes and Fractal Noise Driven Stochastic Processes

A **diplomamunka** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

Budapest, 20 22.05.30.

a hallgató aláírása

Acknowledgements

First of all, I thank to my family to all their love and encouragement, which let me finish this chapter of my life enjoying every minute that I spent with mathematics and I also thank them to let me focus only to my studies during this long time. I am grateful to my grandmother, to my mother and to my girlfriend to tolerate all my inconvenient habits I am doing during I am deeply charmed by some mathematical problems.

I am also grateful to András Lukács for supervising me and I really appreciate all his advices during our corporation. His endless support over the last four years helped me to develop and to understand the methodology of the research job. I am sure that without him I would not look at the parts of machine learning with the excitement and pleasure, the way I do now.

Let me thank to András Zempléni that he trusted and encouraged me in so many situations. I am also thankful for his support during my studies and for the conversations planning my future together.

I am grateful for Vilmos Prokaj for showing me the beautiful parts of the theory of stochastic processes I could charm about in many nights. I would like to thank him his patience and the oral consultations about my questions. It helped me a lot that the stochastic analysis has been introduced to me according to his perspective.

I thank to Balázs Csanád Csáji for developing my perspective in statistical machine learning and leading me to many exciting topics, e.g. Reproducing Kernel Hilbert Spaces. I have been learnt a lot from him, especially he has shown me the proper work ethic and the level of preciseness needed for high quality works.

Contents

Chapter 1

Introduction

There is now a great deal of interest in parameter estimation of Ornstein-Uhlenbeck processes with fractional driving noise in finance because of its modelling capability (see e.g. [7, 1, 14]). That is why the ELTE AI research group, which I am a member of, aims at investigating the prediction of the unknown parameters of certain transformed fractional Ornstein-Uhlenbeck processes, e.g. Stochastic Correlation Processes, with neural networks in the hope that there will be obtained more accurate estimators than the ones using classical statistic methods, e.g. [9], without any assumptions on the Hurst exponent. As it will be precisely introduced, the published parameter estimators based on classical statistics possess such large asymptotic variance, which reduces their applicability, especially for the parameters chosen close to zero. The importance of an efficient data generator system for each analysed process rises high in this case, since if huge and complex neural network structures are applied in the learning procedure, then one needs a big amount of data for a good performance.

If one aims at simulating stochastic integrals with respect to fractional Wiener process, first of all an efficient fractal noise generator has to be developed. Several exact methods have been published about simulating fractional Wiener processes, e.g. the Cholesky and the Hosking method [8], which will be precisely determined in the third section in the framework of isonormal processes. It will be shown that the circulant embedding based algorithms, such as [6, 3, 10], perform the most efficiently among the exact methods according to the main advantage derived from their complexity of order $\mathcal{O}(N\log N)$, where N denotes the number of grid points used in generation. All of the mentioned methods focus on calculating the square root of the covariance matrix, in the least computionally demanding way, to obtain a realization of fractional Wiener increments. Because of its efficiency, the idea developed by Davies and Harte [5] will be generalised for simulating isonormal integrals.

As the introduced issue has been solved, i.e. there has been investigated and implemented an efficient generator system for simulating fractional Ornstein-Uhlenbeck processes over both equidistant and non-equidistant time grid, see in section four. An idea

has led me to simulate fractional Wiener integrals of λ-Hölder continuous functions, where λ depends on the Hurst exponent of the driving noise as $\lambda > 1 - H$, which assumption is necessary to handle the matter in hand integrals as pathwise Riemann-Stieltjes integrals instead of Skorohod-integrals. Moreover, I was able to come up with an observation that by investigating a generator for the class below, certain multiple fractional Wiener integrals can be simulated with the order of computional demanding $p\mathcal{O}(N\log N)$

$$\mathcal{M}_p^H \doteq \{f : [0,T]^p \to \mathbb{R} : \exists \phi \in \mathcal{L}^2([0,T],\mu), \lambda - \text{Hölder continuous}$$
$$\text{and } \eta : f = \eta^{\otimes p} \text{ such that } \exists (\phi_n)_n \subset \varepsilon_1 : \phi_n \xrightarrow{\mathcal{L}^2} \phi, \eta \in (\phi_n)_n\},$$

where ε_1 denotes the class of elementary functions, which will be defined precisely in chapter two. Now, by determining a generator system for certain multiple fractional Wiener integrals, I aimed at investigating a similar theory to the one, which will be introduced in the first chapter, i.e. the Wiener-Ito chaos decomposition of the $\mathcal{L}^2(\Omega)$ space. The Wiener-Ito chaos decomposition is based on such orthogonal subspaces, which can be determined by Wiener integrals. That is why, I am interested in developing a generator procedure for the fractional case. Note that I was not able the prove any property about the class $\mathcal{M}_p^H$ being dense in some sense. So it is a work in progress to characterise the theory needed to develop such a generator system for this type of decomposition.

Since there has been implemented many generator procedures, I aimed at endowing the classes with some additional methods, e.g. the Malliavin-derivative and its adjoint operator the Skorohod-integral. After the many oral consultations with Vilmos Prokaj, the main concept has been generalised to the framework of isonormal processes, since the mentioned operators can be defined in the most general way for isonormal processes as D. Nualart presented in [12]. One can observe that the mentioned stochastic processes are isonormal processes, except the fractional Ornstein-Uhlenbeck process in case of arbitrary initial value. In chapter four, there has been precisely determined the corresponding kernel spaces with the proper inner product structure to obtain each stochastic process derived from the framework of isonormal processes. The structure of the implemented classes had to be changed because of the paradigm shift, i.e. a meta-class has been implemented for isonormal processes with arbitrary kernel space and inner product structure and the generator of each process can be deduced from the meta-class. For the most general case only the Cholesky decomposition based method can be applied to simulate isonormal processes, but with additional assumptions, e.g. defining the stationary property for arbitrary kernel space in certain sense, more efficient procedures can be applied, which procedures have been introduced in chapter four and have been implemented for the meta-class also. My main goal is adding the Malliavin-derivative operator to the meta-class as a subroutine, which can be deduced to all the classes derived from the

meta-class, where this investigation is work in progress.

As I mentioned before, a Python library is under development for simulating several stochastic processes, especially the ones which can be deduced from isonormal processes. There has been implemented a meta-class for isonormal processes determined by arbitrary kernel spaces and inner product structures. For isonormal processes assumed to be indexed stationary, another meta-class has been implemented, where some calculations of the procedures have been changed to be able to cache as much computations as possible into the memory, e.g. for the applied circulant matrix embedding based procedure more than the half of the computational steps can be cached. This idea gives a huge boost to the execution time of the generator class besides some other implementation tricks. Some examples have been derived from the isonormal process framework to be simulated efficiently, e.g. fractional Ornstein-Uhlenbeck processes, which processes can be obtained by determining the kernel space and the inner product structure according to the way it will be presented in chapter four. Note that the task of simulating 10000 fractional Ornstein-Uhlenbeck sequences over 1500 grid-points needs eight-times less execution time than the procedure can be derived from the published *fBM* package, where the initial value of the process is assumed to be zero.

Chapter 2

Stochastic calculus on the Wiener space

The focus of this chapter is mainly on introducing integrals with respect to the Wiener process and the Wiener-Ito chaos decomposition of the $\mathcal{L}^2(\Omega)$ space, where the concept, the characterisations and the proves are based on the many oral consultations with Vilmos Prokaj and also based on my main perspective, which is formed and developed deeply by Vilmos Prokaj. The precisely introduced analysis is necessary for the calculations and tricks will be applied at the investigation of the several simulation procedures. The characterisation and the build-up of the Wiener integrals step by step, which will be presented in this chapter, is aimed to be generalised for the fractal noise case in chapter four. The Wiener-Ito chaos decomposition of the square-integrable random variables is also important, since as it has been mention in the Introduction one of my goals is generalising the decomposition for fractional Wiener integrals, where a dense kernel set has to be determined according to the fact that the discretization of the processes can be simulated. Note that this idea is under investigation.

2.1 Analysis on one-dimensional Gaussian space

Consider the probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma_1)$, where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra and γ_1 is the standard Gaussian measure on $\mathbb{R}$, i.e. $\gamma_1(H) \doteq \int_H \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{t^2}{2}} dt$. Let me introduce the following Hilbert space

$$\mathcal{L}^2(\gamma_1) = \{h \text{ Borel measurable} : \int_{\mathbb{R}} |h|^2 d\gamma_1 < \infty\},$$

where for a function $h \in \mathcal{L}^2(\gamma_1)$ the integral $\int_{\mathbb{R}} h d\gamma_1 = \mathbb{E}(h(Z))$, where $Z \sim N(0,1)$. One can observe that space of polynomial function is a subspace of $\mathcal{L}^2(\gamma_1)$, moreover the following statement holds true.

Lemma 2.1.1. The polynomials are dense in $\mathcal{L}^2(\gamma_1)$.

Proof. The proof contains the common steps of showing a set is dense in a space, i.e. in this case the statement holds true only if an element of the space is orthogonal to every element of the matter in hand space then it follows that the square of this element has to be zero γ_1-almost surely. Let me denote the polynomials with $\mathcal{P}$ for simplifying the calculations.

Assume that $\mathcal{P}$ is dense in $\mathcal{L}^2(\gamma_1)$ and $h \in \mathcal{L}^2(\gamma_1)$ is orthogonal to $\mathcal{P}$, i.e. $\forall \varepsilon > 0\ \exists p \in \mathcal{P} : \|h-p\|^2_{\mathcal{L}^2(\gamma_1)} < \varepsilon$ and since $h \perp p$ the norm can be bounded as $\|h\|^2 + \|p\|^2 < \varepsilon$. Now we have that $\|h\|^2 < \varepsilon,\ \forall \varepsilon > 0$ that is $\|h\|^2_{\mathcal{L}^2(\gamma_1)} = 0$. Let $(h_n)_n$ be a sequence from the subspace $\overline{\mathcal{P}} \subset \mathcal{L}^2(\gamma_1)$, where the closure has taken in $\mathcal{L}^2(\gamma_1)$ sense and let $h \in \overline{\mathcal{P}}$ be defined as the limit $h_n \to h$ in $\mathcal{L}^2(\gamma_1)$ sense. For all $\varepsilon > 0$ there exists p_n such that $\|h_n - p_n\| < \varepsilon$ then set $\varepsilon \doteq \frac{1}{2^n}$, which leads to the fact that there exists a p_n polynomial satisfying $\|h_n - p_n\| < \frac{1}{2^n}$. Now it can be claimed that the space $\overline{\mathcal{P}}$ is a closed subspace of $\mathcal{L}^2(\gamma_1)$ by the inequality

$$\|h - p_n\| \leq \|h - h_n\| + \|h - p_n\| \to 0.$$

Now we have that if $h \in \mathcal{L}^2(\gamma_1)$ is orthogonal to the polynomials then $h = 0$ and let me denote with $\mathcal{C}$ the orthogonal complement of $\mathcal{L}^2(\gamma_1)$. Then $\mathcal{C}$ consists of orthogonal functions to $\mathcal{L}^2(\gamma_1)$, i.e. $\mathcal{C}^\perp$ is a subset of the space of polynomials. Then one can claim that $\mathcal{C} = \{0\}$.

As a sub-statement it will be shown that for a μ σ-finite measure on the Borel sets of $\mathbb{R}$, $\mathcal{B}(\mathbb{R})$ if there exists a $c > 0$ such that

$$\int_{\mathbb{R}} \mathrm{e}^{cx} d\mu(x) < \infty \tag{2.1}$$

then the polynomials form a dense set in $\mathcal{L}^2(\mu)$, i.e. the orthogonal complement of the polynomials consists of only the functions with the property that its square is μ-almost surely 0 function. Let h be an element from the space $\mathcal{L}^2(\mu)$ which is orthogonal to the polynomials and let me consider $x \to \mathrm{e}^{\frac{c}{2}|x|} \in \mathcal{L}^2(\mu)$ and then $h\mathrm{e}^{\frac{c}{2}|x|} \in \mathcal{L}^1(\mu)$. Thus the integral in equation 2.1 can be written in this case as following

$$\int_{\mathbb{R}} |h| \mathrm{e}^{\frac{c}{2}|x|} d\mu(x) = \int_{\mathbb{R}} \sum_{n=0}^{\infty} \left(\frac{c}{2}\right)^n \frac{|x|^n}{n!} |h| d\mu(x),$$

which is finite according to the condition of the statement. Now if $|\lambda| < c/2$ then the equation can be expressed as

$$\int_{\mathbb{R}} h(x) \sum_{n=0}^{\infty} (\lambda)^n \frac{|x|^n}{n!} d\mu(x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_{\mathbb{R}} h(x) x^n d\mu(x)$$

and the fact that for arbitrary n for which $< h, x^n >_{\mathcal{L}^2(\mu)} = 0$ implies the orthogonal

property of h to e^{λ} for $|\lambda| < c/2$, i.e.

$$\int_{\mathbb{R}} h(x)e^{(itx)}d\mu(x) = 0 \ \forall t \in \mathbb{R}.$$

Let me denote the Fourier transform of the signed measure with ν and for a Borel set $\mathcal{A} \in \mathcal{B}(\mathbb{R})$ it is defined as

$$\nu(\mathcal{A}) \doteq \int_{\mathcal{A}} h(x)d\mu(x),$$

which is identically zero for all $\mathcal{A} \in \mathcal{B}(\mathbb{R})$ thus $\nu = 0$ and as a result h is a μ-almost surely zero function in $\mathcal{L}^2(\mu)$. By proving the substatement above for any σ-finite measure the proof of the lemma can be claimed as a consequence.

□

Let me introduce the following operator defined on the space $\mathcal{C}^1(\mathbb{R})$:

$$(\partial^* f)(x) \doteq xf(x) - \partial f(x). \tag{2.2}$$

Lemma 2.1.2. The derivative operator ∂ and operator defined above 2.2 are adjoint with respect to the measure γ_1, i.e. for any $f, g \in \mathcal{L}^2(\gamma_1)$ such that $\exists \partial f, \partial g \in \mathcal{L}^2(\gamma_1)$ satisfying the following equation

$$\langle \partial f, g \rangle_{\mathcal{L}^2(\gamma_1)} = \langle f, \partial^* g \rangle_{\mathcal{L}^2(\gamma_1)}.$$

Proof. Let me denote the density function of the standard Gaussian measure by p and the derivation operator acts on p as $(\partial p)(x) = -xp(x)$. Then the following equation holds true by applying integration by parts and the properties of the standard normal density function

$$\begin{aligned}
\langle \partial f, g \rangle_{\mathcal{L}^2(\gamma_1)} &= \langle \partial f, gp \rangle_{\mathcal{L}^2(Leb)} = -\langle f, \partial(gp) \rangle_{\mathcal{L}^2(Leb)} \\
&= -\langle \partial g, fp \rangle_{\mathcal{L}^2(Leb)} + \langle f, gpI \rangle_{\mathcal{L}^2(Leb)} \\
&= \langle f, (\partial^* g)p \rangle_{\mathcal{L}^2(Leb)} = \langle f, \partial^* g \rangle_{\mathcal{L}^2(\gamma_1)},
\end{aligned}$$

where I denotes the identity function and $\mathcal{L}^2(Leb)$ denotes the square-integrable functions with respect to the Lebesgue measure. □

Lemma 2.1.3. If $f \in \mathcal{C}^n(\mathbb{R})$ then

$$(\partial(\partial^*)^n - (\partial^*)^n\partial)f = n(\partial^*)^{n-1}f.$$

Note that in case of $n = 1$ the statement is called Heisenberg's commutation relation.

Proof. For $n = 1$ one can write

$$\begin{aligned}(\partial\partial^* f)(x) &= (\partial(fI - \partial f))(x) = f(x) + x(\partial f)(x) - (\partial^2 f)(x)\\ (\partial^*\partial f)(x) &= x(\partial f)(x) - (\partial^2 f)(x)\\ ((\partial\partial^* - \partial^*\partial)f)(x) &= f(x)\end{aligned}$$

Now for $n \geq 2$ apply induction on n

$$\begin{aligned}(\partial(\partial^*)^n f)(x) &= (\partial\partial^*((\partial^*)^{n-1})f)(x) = (\partial^*\partial((\partial^*)^{n-1})f)(x) + ((\partial^*)^{n-1}f)(x)\\ &= \Big(\partial^*\Big((\partial^*)^{n-1} + (n-1)(\partial^*)^{n-2}\Big)f\Big)(x) + ((\partial^*)^{n-1}f)(x)\\ &= ((\partial^*)^n\partial f)(x) + n((\partial^*)^{n-1}f)(x)\end{aligned}$$

□

Let me introduce the $\{H_n\}_n$ Hermite polynomials, where the Nth Hermite polynomial can be defined as the image of ∂^* operator composed N-times and applied on the unit element of the $\mathcal{L}^2(\gamma_1)$ space, i.e.

$$H_N \doteq (\partial^*)^N \mathbb{1}, \tag{2.3}$$

where H_0 is initialized as the unit function of the matter in hand Hilbert space. To simplify the upcoming computations let me calculate how the derivative operator acts on the Hermite polynomials:

$$\begin{aligned}\partial H_n &= \partial\partial^* H_{n-1} = (\mathrm{Id} + \partial^*\partial)H_{n-1} = H_{n-1} + \partial^*\partial H_{n-1}\\ &= H_{n-1} + \partial^* H_{n-2} + (\partial^*)^2\partial H_{n-2}\\ &= kH_{n-1} + (\partial^*)^k\partial H_{n-k}.\end{aligned}$$

In case of $n = k$ the image of the derivative operator can be obtained as $\partial H_n = nH_{n-1} + (\partial^*)^n\partial H_0 = nH_{n-1}$, since H_0 is the unit element. Now let me investigate the inner product structure of $\{H_n\}_n$ with respect to the standard Gaussian measure, i.e. let me consider the indices $m \leq n$ and

$$\big\langle H_n, H_m\big\rangle_{\mathcal{L}^2(\gamma_1)} = \big\langle H_n, (\partial^*)^m\mathbb{1}\big\rangle_{\mathcal{L}^2(\gamma_1)} = \big\langle \partial^m H_n, \mathbb{1}\big\rangle_{\mathcal{L}^2(\gamma_1)} = \mathbb{E}(\partial^m H_n(Z)),$$

where Z is a standard normal variable. To finish the characterization of the inner product structure the following formula has to be proven.

Lemma 2.1.4. Let X and Y be standard normal variables. If (X, Y) is also a normal variable then

$$\mathbb{E}(H_n(X)H_m(Y)) = n!\mathrm{cov}(X, Y)^n\chi_{\{n=m\}},$$

where χ denotes the indicator function.

Proof. Let Y be defined as $Y \doteq \rho X + \sqrt{1-\rho^2}Z$, where X, Z are independent standard normal variables. Then Y is also a standard normal variable with the property that the covariance function with respect to X being $\mathrm{cov}(X, Y) = \rho$.

$$\mathbb{E}(H_n(X)H_m(Y)) = \mathbb{E}(\mathbb{E}(H_n(X)H_m(\rho X + \sqrt{1-\rho^2}Z)|\sqrt{1-\rho^2}Z)),$$

where X as the argument of H_n and ρX as the argument of H_m are independent from the condition and $\sqrt{1-\rho^2}Z$ as the argument of H_m is measurable to the condition. Thus the conditional expected value can be written as following

$$\mathbb{E}(H_n(X)H_m(Y)) = \mathbb{E}(\mathbb{E}(H_n(X)H_m(\rho X + z)|_{z=\sqrt{1-\rho^2}Z})).$$

Now $\mathbb{E}(\mathbb{E}(H_n(X)H_m(\rho X + z)|_{z=\sqrt{1-\rho^2}Z}))$ can be calculated by applying the previously introduced properties of the Hermite polynomials, which are $H_n = (\partial^*)^n \mathbb{1}$ and $\partial H_n = nH_{n-1}$.

$$\begin{aligned}\mathbb{E}(H_n(X)H_m(\rho X + z)|_{z=\sqrt{1-\rho^2}Z}) &= \left\langle H_n, H_m \circ (\rho x + y)\right\rangle_{\mathcal{L}^2(\gamma_1)} = \\ &= \left\langle (\partial^*)^n \mathbb{1}, H_m \circ (\rho x + y)\right\rangle_{\mathcal{L}^2(\gamma_1)} = \mathbb{E}(\partial^n (H_m \circ (\rho x + y))).\end{aligned}$$

The image of $H_n \circ (\rho x + y)$ with respect to the derivative operator can be determined as

$$\begin{aligned}\partial(H_n \circ (\rho x + y)) &= nH_{n-1} \circ (\rho x + y)\partial(\rho x + y) = n\rho H_{n-1} \\ \partial^n(H_n \circ (\rho x + y)) &= n!\rho^n H_0 = n!\rho^n.\end{aligned}$$

It follows that $\mathbb{E}(H_n(X)H_m(Y)) = \mathbb{E}(\mathbb{E}(H_n(X)H_m(\rho X + z)|_{z=\sqrt{1-\rho^2}Z})) = n!\rho^n$, which completes the proof. □

Now the characterization of the inner product structure of the Hermite polynomials with respect to the standard Gaussian measure can be finished by applying the result formalised in the previous lemma on 2.1, i.e.

$$\begin{aligned}\left\langle H_n, H_m\right\rangle_{\mathcal{L}^2(\gamma_1)} &= \mathbb{E}(\partial^m H_n(Z)) = n!\chi_{\{n=m\}} \\ \|H_n\|^2_{\mathcal{L}^2(\gamma_1)} &= n!\end{aligned}$$

Thus it has been obtained that the Hermite polynomials are orthogonal in the space $\mathcal{L}^2(\gamma_1)$ and the nth Hermite polynomial has norm $\sqrt{n!}$ in $\mathcal{L}^2$ sense with respect to the standard Gaussian measure.

Now it will be shown that if a function $f \in \mathcal{L}^2(\gamma_1)$ is orthogonal to all the elements of the sequence $\left\{H_n, n \geq 0\right\}$ then the square of f has to be γ_1-almost surely zero function.

Consider the following inner product for an arbitrary $t \in \mathbb{R}$

$$\left\langle f, \mathrm{e}^{itx} \right\rangle_{\mathcal{L}^2(\gamma_1)} = \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \left\langle f, x^j \right\rangle_{\mathcal{L}^2(\gamma_1)} = 0,$$

since if a function is orthogonal to the sequence of Hermite polynomials, which is a sequence of polynomials with leading coefficient 1, then the function is orthogonal to the all monomials in $\mathcal{L}^2(\gamma_1)$. The integral and the sum could be commuted according to Fubini's theorem if $\sum_{j=0}^{\infty} \int_{\mathbb{R}} \frac{|xt|^j}{j!} |f(x)| d\mu_1(x) < \infty$, which can be written as following

$$\left\langle \mathrm{e}^{|tx|}, f \right\rangle_{\mathcal{L}^2(\gamma_1)} \leq \|\mathrm{e}^{|tx|}\|_{\mathcal{L}^2(\gamma_1)} \|f\|_{\mathcal{L}^2(\gamma_1)} < \infty.$$

Now, it has been obtained that the Fourier transform of f multiplied by λ is zero Lebesgue almost everywhere, where λ is the Radon-Nikodym derivative of the standard Gaussian measure with respect to the Lebesgue measure. Thus f is zero γ_1-almost surely.

Corollary 2.1.1. $\left\{ \frac{H_n}{\sqrt{n!}}, n \geq 0 \right\}$ forms a complete orthonormal basis in $\mathcal{L}^2(\gamma_1)$ and any $h \in \mathcal{L}^2(\gamma_1)$ has the following expansion

$$h = \sum_{j=0}^{\infty} \left\langle h, H_n \right\rangle_{\mathcal{L}^2(\gamma_1)} \frac{H_n}{n!} = \sum_{j=0}^{\infty} \mathbb{E}(\partial^n h(Z)) \frac{H_n}{n!},$$

where Z is a standard normal variable and the second equation holds true because of the following computation

$$\begin{aligned} \left\langle h, H_n \right\rangle_{\mathcal{L}^2(\gamma_1)} &= \left\langle h, (\partial^*)^n \mathbb{1} \right\rangle_{\mathcal{L}^2(\gamma_1)} = \left\langle \partial h, (\partial^*)^{n-1} \mathbb{1} \right\rangle_{\mathcal{L}^2(\gamma_1)} \\ &= \left\langle \partial^n h, \mathbb{1} \right\rangle_{\mathcal{L}^2(\gamma_1)} = \mathbb{E}(\partial^n h(Z)). \end{aligned}$$

The corollary above can be restated as it is formalised in the theorem below, which formalism will be important for simplifying some relations between the introduced analysis of the Gaussian space and the subspaces obtained by multidimensional Wiener-Ito integrals.

Theorem 2.1.1. Let $\mathcal{P}_j$ denote the space generated by the jth Hermite polynomial, i.e. $\mathcal{P}_j \doteq \mathrm{span}\{H_j(x)\}$ with the following properties

$$\mathcal{L}^2(\gamma_1) = \bigoplus_{j=0}^{\infty} \mathcal{P}_j,$$

where $\mathcal{P}_k$ and $\mathcal{P}_l$ are orthogonal subspaces if $k \neq l$.

2.2 The Wiener-Ito chaos decomposition

In this section a decomposition of the space of square-integrable stochastic processes, i.e. $\mathcal{L}^2(\Omega) \otimes \mathcal{L}^2([0,T])$, will be introduced by defining the multiple Wiener-Ito integrals and the corresponding generated subspaces. It has been shown by D. Nualart that the Mailliavin calculus can be developed in the framework of isonormal processes [12], thus there will be a short introduction into that part of analysis. In the sections related to simulation tasks the isonormal processes will play an important part.

As it has been investigated before, consider a separable Hilbert space $\mathcal{G}$ provided with the inner product structure $\langle .,. \rangle_{\mathcal{G}}$ and the norm of an element induced by the inner product will be denoted by $\|.\|_{\mathcal{G}}$.

Definition 2.2.1. A stochastic process $\zeta = \{\zeta(g), g \in \mathcal{G}\}$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called isonormal process on $\mathcal{G}$ if ζ is a centered Gaussian family of random variables with the following inner-product isomorphism between the kernel space and the image space

$$\langle \zeta(h), \zeta(g) \rangle_{\mathcal{L}^2(\Omega)} = \langle h, g \rangle_{\mathcal{G}}.$$

Let me use the notation $\mathcal{Z}$ for the σ-algebra generated by the family of random variables $\{\zeta(g), h \in \mathcal{G}\}$. One can observe that the mapping, corresponding to an isonormal process, from the kernel space to the space of square-integrable random variables is linear. Moreover, the following shows that the the matter in hand mapping is linear isometric between the kernel space and the image space

$$\zeta : \mathcal{G} \to \mathcal{H}_1, \tag{2.4}$$

where $\mathcal{H}_1$ is a closed subspace of $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. For any λ, μ scalar numbers and for any h, g two elements of the kernel space it can be shown using the isomorphism between the two spaces that

$$\|\zeta(\lambda h + \mu g) - \lambda\zeta(h) - \mu\zeta(g)\|^2_{\mathcal{L}^2(\Omega)} = \|\lambda h + \mu g\|^2_{\mathcal{G}}.$$

Lemma 2.2.1. An isonormal process transformed by the exponential function forms a total subset of $\mathcal{L}^2(\Omega, \mathcal{Z}, \mathbb{P})$.

Proof. Let $X \in \mathcal{L}^2(\Omega, \mathcal{Z}, \mathbb{P})$ be orthogonal to all the elements of the image space obtained by the transformed isonormal process, i.e.

$$\langle X, \mathrm{e}^{\zeta(g)} \rangle_{\mathcal{L}^2(\Omega)} = 0, \quad \forall g \in \mathcal{G}.$$

For any $g_1, ..., g_m \in \mathcal{G}$ and $\lambda_1, ..., \lambda_m$ scalars the following holds true because of the mapping

2.4 being a linear isometry

$$\left\langle X, \mathrm{e}^{\Sigma_{j=1}^{m} \lambda_j \zeta(g_j)} \right\rangle_{\mathcal{L}^2(\Omega)} = 0,$$

where for fixed kernel elements it is actually the Laplace transform of the measure ν, i.e. for $B \in \mathcal{B}(\mathbb{R}^m)$ it is defined by the inner product:

$$\nu(B) \doteq \left\langle X, \mathbb{1}_B(\zeta(g_1), ..., \zeta(g_m)) \right\rangle_{\mathcal{L}^2(\Omega)}.$$

Now, we have that the signed measure is zero for any Borel subset, which means that ν can only be the identically zero measure on $\mathbb{R}^m$ and for any B Borel set $\nu(B) = \left\langle X, \mathbb{1}_B(\zeta(g_1), ..., \zeta(g_m)) \right\rangle_{\mathcal{L}^2(\Omega)} = 0$ holds true if and only if X is zero $\mathbb{P}$ almost surely. The fact that a finite measure is uniquely determined by its Laplace transform completes the proof.

□

Let me use the following notations, as Nualart introduced in [12]. Let $\mathcal{H}_n$ denote the closed subspace of $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, which is generated by the random variables obtained by the nth Hermite polynomial applied on the matter in hand isonormal process by assuming that the kernel elements are provided with norm one,

$$\mathcal{H}_n \doteq \{H_n(\zeta(g)), g \in \mathcal{G}, \|g\|_{\mathcal{G}} = 1\},$$

where $\mathcal{H}_n$ is called the Wiener chaos of order n. It can be shown that if $H_k(\xi)$ is an element of the kth Wiener chaos and $H_l(\eta)$ is an element of the lth Wiener chaos with $k \neq l$ then $H_k(\xi)$ and $H_l(\eta)$ are orthogonal in $\mathcal{L}^2(\Omega)$. The property that the kernel elements of ξ and η are equipped with norm one let us conclude that ξ and η are standard normal variables and elements of a Gaussian family, which means that the conditions of the Lemma 2.1.4 is satisfied by the considered variables, i.e.

$$\left\langle H_k(\xi), H_l(\eta) \right\rangle_{\mathcal{L}^2(\Omega,\mathbb{P})} = \left\langle H_k, H_l \right\rangle_{\mathcal{L}^2(\mathbb{R},\gamma_1)} = 0 \quad \text{if } k \neq l.$$

Corollary 2.2.1. The closed linear subspaces $\mathcal{H}_k$ and $\mathcal{H}_l$ of $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ are orthogonal in case of $k \neq l$.

Theorem 2.2.1. The $\mathcal{L}^2(\Omega, \mathcal{Z}, \mathbb{P})$ space can be decomposed according to the introduced orthogonal closed linear subspaces $\{\mathcal{H}_j\}_{j=0}^{\infty}$ as following

$$\mathcal{L}^2(\Omega, \mathcal{Z}, \mathbb{P}) = \bigoplus_{j=0}^{\infty} \mathcal{H}_j.$$

Proof. Let X be an element of the space of square-integrable random variable with respect to the measure $\mathbb{P}$ such that X is orthogonal to the subspaces $\{\mathcal{H}_j\}_{j=0}^{\infty}$ and it will be shown that an element with the previous property can be only a $\mathbb{P}$-almost surely zero element

from the complete probability space $(\Omega, \mathcal{Z}, \mathbb{P})$. Now we have that for any element from the kernel space with unit norm and for subspaces of any order the following equation holds true

$$\left\langle X, H_j(\zeta(g))\right\rangle_{\mathcal{L}^2(\Omega,\mathbb{P})} = 0, \quad \text{for any } \|g\|_{\mathcal{G}} = 1 \text{ and } j \geq 0.$$

Since for any N there exists $\alpha_0, ..., \alpha_N$ such that the monomial x^N can be expressed as the weighted sum of the corresponding Hermite polynomials, i.e. $x^N = \sum_{j=0}^{N} \alpha_j H_j(x)$, which observation let us rewrite the equation above for monomials as below

$$\left\langle X, (\zeta(g))^j\right\rangle_{\mathcal{L}^2(\Omega,\mathbb{P})} = 0, \quad \text{for any } j \geq 0.$$

Now we have that for arbitrary $t \in \mathbb{R}$ and $g \in \mathcal{G}$ with unit norm:

$$0 = \sum_{j=0}^{\infty} \frac{t^j}{j!}\left\langle X, (\zeta(g))^j\right\rangle_{\mathcal{L}^2(\Omega,\mathbb{P})} = \left\langle X, \mathrm{e}^{t\zeta(g)}\right\rangle_{\mathcal{L}^2(\Omega,\mathbb{P})},$$

where $\{\mathrm{e}^{\zeta(g)}\}_{g\in\mathcal{G}}$ forms a total subset of $\mathcal{L}^2(\Omega, \mathcal{Z}, \mathbb{P})$ and the fact that X is zero $\mathbb{P}$ almost everywhere can be deduced from the previous observations, which completes the proof. Note that the space of polynomials with leading order N can be obtained as the direct sum of the spaces $\{\mathcal{H}_j\}_{j=0}^{N}$.

□

As the first step of introducing the multidimensional Wiener-Ito integrals let me define the space of elementary functions as following

$$\varepsilon_p \doteq \{f : [0,T]^p \to \mathbb{R} : \ f = \sum_K a_K \mathbb{1}_{A_{K,1}\times...\times A_{K,p}}(x), \text{ where } A_{K,1}, ..., A_{K,p} \text{ are disjoints } \}. \tag{2.5}$$

Let me define a mapping from the space of elementary functions to the square-integrable random variables as following

$$\begin{aligned}
\mathcal{I}_p : \varepsilon_p &\to \mathcal{L}^2(\Omega, \mathcal{F}_T, \mathbb{P}) \\
\mathcal{I}_p(\mathbb{1}_{A_1\times...\times A_p}) &\doteq \prod_{j=1}^{p} W(\mathbb{1}_{A_j}) \\
\mathcal{I}_p(\sum_K a_K \mathbb{1}_{A_{K,1}\times...\times A_{K,p}}) &\doteq \sum_K a_K \mathcal{I}_p(\mathbb{1}_{A_{K,1}\times...\times A_{K,p}}),
\end{aligned}$$

where $A_1 \times ... \times A_p$ and $\sum_K a_K \mathbb{1}_{A_{K,1}\times...\times A_{K,p}}$ are elements of ε_p, $\mathcal{F}_T$ is the natural filtration at time T of the Wiener process denoted by W and $W(\mathbb{1}_{A_j})$ denotes the Wiener increment over the interval A_j. Note that the disjointness of the intervals is needed to obtain the image of an indicator $\mathbb{1}_{A_1\times...\times A_p}$ as a product according to the property that the increments of the Wiener process are independent.

Now the inner product of $\mathcal{I}_p(f)$ and $\mathcal{I}_q(g)$ will be calculated for any p, q and any

$f \in \varepsilon_p, g \in \varepsilon_q$. It is enough to reduce the problem for indicator functions because of the linearity of the introduced mapping, so let the kernel functions be $f \doteq \mathbb{1}_{A_1 \times ... \times A_p}$ and $g \doteq \mathbb{1}_{B_1 \times ... \times B_q}$. For the sequence of arbitrary intervals $\{A_j\}_{j=1}^p$ and $\{B_j\}_{j=1}^q$ there exists a sequence of disjoint intervals $\{C_j\}_{j=1}^r$ and mappings $\sigma_1 : \{1, ..., p\} \to \{1, ..., r\}, \sigma_2 : \{1, ..., q\} \to \{1, ..., r\}$ such that for all j :

$$A_j = \bigcup_{C_k \in A_j} C_k, \quad B_j = \bigcup_{C_k \in B_j} C_k$$
$$\mathbb{1}_{A_j} = \sum_{C_k \in A_j} \mathbb{1}_{C_k}, \quad \mathbb{1}_{B_j} = \sum_{C_k \in B_j} \mathbb{1}_{C_k}$$
$$\prod_{j=1}^{p} \mathbb{1}_{A_j} = \sum_{i=1}^{r} \prod_{j=1}^{p} \mathbb{1}_{C_{\sigma_1(i)}}, \quad \prod_{j=1}^{q} \mathbb{1}_{B_j} = \sum_{i=1}^{r} \prod_{j=1}^{q} \mathbb{1}_{C_{\sigma_2(i)}},$$

where $C_{\sigma_1(i)} \subset A_j$ and $C_{\sigma_2(i)} \subset B_j$. So it is enough to determine the inner product for the kernel functions f, g in the form $f \doteq \prod_{j=1}^p \mathbb{1}_{C_{\sigma_1(j)}}, g \doteq \prod_{j=1}^q \mathbb{1}_{C_{\sigma_2(j)}}$ with the images respectively

$$\mathcal{I}_p(f) = \prod_{j=1}^{p} W(\mathbb{1}_{C_{\sigma_1(j)}}), \quad \mathcal{I}_q(g) = \prod_{j=1}^{q} W(\mathbb{1}_{C_{\sigma_2(j)}}).$$

Now, to determine the inner product structure with respect to the measure $\mathbb{P}$ of the space $\mathcal{L}^2(\Omega)$ we can apply the following result, which is that the product of the corresponding integrals can be computed in the reduced case as

$$\mathcal{I}_p(f)\mathcal{I}_q(g) = \prod_{j=1}^{r} W(\mathbb{1}_{C_j})^{\alpha_j' + \alpha_j''},$$

where the exponents are indicator functions depending on the relation of the disjoint interval sequence and the intervals occurred in the definition of the kernel functions, to be more precise $\alpha_j' \doteq \chi_{\{C_j \in \{C_{\sigma_1(k)}\}_{k=1}^p\}}$ and $\alpha_j'' \doteq \chi_{\{C_j \in \{C_{\sigma_2(k)}\}_{k=1}^q\}}$. The increments of the Wiener process have Gaussian distribution, i.e. $W(\mathbb{1}_{C_j}) \sim N(0, |C_j|)$, moreover the sequence obtained as the image of any linear combination of the introduced kernel functions $\{\mathbb{1}_{C_j}\}_{j=1}^r$ is jointly Gaussian. Applying the isometry between the two spaces $\mathcal{L}^2(\Omega, \mathbb{P})$ and $\mathcal{L}^2([0, T], Leb)$ as

$$\left\langle W(\mathbb{1}_{C_j}), W(\mathbb{1}_{C_k}) \right\rangle_{\mathcal{L}^2(\Omega,\mathbb{P})} = \left\langle \mathbb{1}_{C_j}, \mathbb{1}_{C_k} \right\rangle_{\mathcal{L}^2([0,T],Leb)} = |C_j| \chi_{\{C_j = C_k\}}$$

leads us to the following result

$$\left\langle \mathcal{I}_p(f), \mathcal{I}_q(g) \right\rangle_{\mathcal{L}^2(\Omega,\mathbb{P})} = \prod_{j=1}^{r} \mathbb{E}\left(W(\mathbb{1}_{C_j})^{\alpha_j' + \alpha_j''} \right) = \prod_{j=1}^{p} |C_{\sigma_1(j)}| \chi_{\{p=q\}}.$$

So the inner product of two random variables defined as the image of two indicator functions has been determined above. Now, to calculate the inner product of two random

variables obtained as applying the $\mathcal{I}$ operator on two arbitrary functions from the kernel space determined by the elementary functions, the symmetrization of multivariate functions will be introduced.

Definition 2.2.2. The simmetrized of the function $f : [0,T]^p \to \mathbb{R}$ is denoted by $\tilde{f}$ and is defined as

$$\tilde{f}(\underline{t}) \doteq \frac{1}{p!} \sum_{\pi \in S_p} f(t_{\pi(1)}, ..., t_{\pi(p)}),$$

where S_p denotes the group of all permutations of the set $\{1, ..., p\}$.

Consider the problem investigated above in case of $p = q$ and let the kernel elements be $f = \prod_{j=1}^{p} \mathbb{1}_{C_{\sigma_1(j)}}$ and $g = \prod_{j=1}^{p} \mathbb{1}_{C_{\sigma_2(j)}}$, which leads to the following inner product

$$\begin{aligned}
\left\langle \tilde{f}, g \right\rangle_{\mathcal{L}^2([0,T],Leb)} &= \frac{1}{p!} \sum_{\pi \in S_p} \prod_{j=1}^{p} \left\langle \mathbb{1}_{C_{\sigma_1(\pi(j))}}, \mathbb{1}_{C_{\sigma_2(j)}} \right\rangle_{\mathcal{L}^2([0,T],Leb)} \\
&= \frac{1}{p!} \sum_{\pi \in S_p} \prod_{j=1}^{p} |C_{\sigma_1(\pi(j))} \cap C_{\sigma_2(j)}| \\
&= \frac{1}{p!} \prod_{j=1}^{p} |C_{\sigma_1(j)}| \chi_{\{\{\sigma_1(j):j=1,...,p\}=\{\sigma_2(j):j=1,...,p\}\}}.
\end{aligned}$$

Corollary 2.2.2. For $f \in \varepsilon_p$ and $g \in \varepsilon_q$ the inner product structure between the images can be obtained as

$$\left\langle \mathcal{I}_p(f), \mathcal{I}_q(g) \right\rangle_{\mathcal{L}^2(\Omega,\mathbb{P})} = p! \left\langle \tilde{f}, g \right\rangle_{\mathcal{L}^2([0,T]^p,Leb)} \chi_{\{p=q\}} = p! \left\langle \tilde{f}, \tilde{g} \right\rangle_{\mathcal{L}^2([0,T]^p,Leb)} \chi_{\{p=q\}}.$$

The symmetrization can be considered as the orthogonal projection onto the space of the symmetric functions, which mapping is conctraction because of the inequality below $\|\tilde{f}\|_{\mathcal{L}^2([0,T]^p,Leb)} \leq \|f\|_{\mathcal{L}^2([0,T]^p,Leb)}$ and it leads us to the following upper bound

$$\|\mathcal{I}_p(f)\|^2_{\mathcal{L}^2(\Omega)} = p! \|\tilde{f}\|^2_{\mathcal{L}^2([0,T]^p)} \leq p! \|f\|^2_{\mathcal{L}^2([0,T]^p)}$$

Corollary 2.2.3. The operator $\mathcal{I}_p : \varepsilon_p \to \mathcal{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$ is continuous and linear.

The first step to extend the operator $\mathcal{I}_p$ for the whole $\mathcal{L}^2([0,T]^p)$ with satisfying the linearity, the property that the norm of the image depends only on the symmetrized of the kernel function and the condition that the inner product structure determined for kernel functions from ε_p holds the same for the extension, is showing that the introduced space ε_p is dense in $\mathcal{L}^2([0,T]^p)$.

Lemma 2.2.2. ε_p is dense in $\mathcal{L}^2([0,T]^p)$.

Proof. The proof will be based on Dynkin's $\pi - \lambda$ theorem, where the role of the λ- and

π-system will be played by the following sets, respectively

$$\mathcal{D} \doteq \{H \in \mathcal{B}([0,T]^p),\ \mathbb{1}_H \in \overline{\varepsilon}_p\}$$
$$\mathcal{C} \doteq \{A_1 \times ... \times A_p,\ A_i \subset [0,T] \text{ and } A_i \text{ is interval}\},$$

where $\mathcal{C}$ is closed under intersections. Now let $H_1, H_2 \in \mathcal{D}$ such that $H_1 \subset H_2$, which means that $\mathbb{1}_{H_1}$ and $\mathbb{1}_{H_2}$ are in $\overline{\varepsilon}_p$, but then $\mathbb{1}_{H_1} - \mathbb{1}_{H_2} = \mathbb{1}_{H_2 \backslash H_1}$ leads to the property that $H_2 \backslash H_1 \in \mathcal{D}$. Consider a countable sequence of growing sets $H_n \subset H_{n+1} \in \mathcal{D}$ then the following limit holds almost everywhere and in $\mathcal{L}^2$, which let us conclude the next property to show that $\mathcal{D}$ is a λ-system

$$\lim_{n\to\infty} \mathbb{1}_{H_n} = \mathbb{1}_{\cup H_n} \quad \rightarrow \quad \mathbb{1}_{\cup H_n} \in \overline{\varepsilon}_p \quad \rightarrow \quad \bigcup_n H_n \in \mathcal{D},$$

since $\mathbb{1}_{H_n} \in \overline{\varepsilon}_p$ for every element of the sequence.

If $\{A_1, ..., A_p\}$ are intervals then for the indicator function of a p-dimensional rectangle we have that $\mathbb{1}_{\times_i A_i} \in \overline{\varepsilon}_p$ and furthermore let $\alpha = \{\alpha_1, ..., \alpha_N\}$ be an equidistant partition of $[0,T]$. Let me introduce the following notation to obtain disjoint parts of the intervals $\{A_i\}_i$

$$A_{i,j} \doteq \alpha_j \cap A_i,$$

then the indicator function of the interval A_i can be determined by the sum $\sum_j \mathbb{1}_{A_{i,j}}$. Now the indicator function of the tensor product of the matter in hand interval sequence can be deduced as following

$$\begin{aligned}\mathbb{1}_{\times_i A_i} &= \sum_{\sigma:\{1,...,p\}\to\{1,...,N\}} \mathbb{1}_{\times_{j=1}^p A_{j,\sigma(j)}} \\ &= \sum_{\sigma \text{ injection }:\{1,...,p\}\to\{1,...,N\}} \mathbb{1}_{\times_{j=1}^p A_{j,\sigma(j)}} + \sum_{\sigma \text{ non-injection }:\{1,...,p\}\to\{1,...,N\}} \mathbb{1}_{\times_{j=1}^p A_{j,\sigma(j)}},\end{aligned}$$

where the first part of the right-hand side of the equation determined by the case σ being injection is an element of $\overline{\varepsilon}_p$ unlike the second part. The inequalities below leads us to complete the proof of the necessary condition $\mathcal{C} \subset \mathcal{D}$ to apply Dynkin's theorem

$$\begin{aligned}&\Big\| \sum_{\sigma \text{ non-injection }:\{1,...,p\}\to\{1,...,N\}} \mathbb{1}_{\times_{j=1}^p A_{j,\sigma(j)}} \Big\|^2_{\mathcal{L}^2([0,T]^p, Leb)} \leq \\ &\leq (N^p - N(N-1)...(N-(p-1)))\Big(\frac{T}{N}\Big)^p \leq (1 - \Big(\frac{p}{N}\Big)^p)T^p \xrightarrow{N\to\infty} 0,\end{aligned}$$

where $(N^p - N(N-1)...(N-(p-1)))$ obtained as the number of non-injective σ mappings.

So now we have that $\mathcal{C} \subset \mathcal{D}$, where $\mathcal{D}$ is a λ-system and $\mathcal{C}$ is a π-system and it can be claimed that $\sigma(\mathcal{C}) \subset \mathcal{D}$ according to Dynkin's theorem, where $\mathcal{D} \subset \mathcal{B}([0,T]^p)$. Moreover, the σ-algebra generated by the set $\mathcal{C}$ is the same as the Borel sets of $[0,T]^p$,

i.e. $\sigma(\mathcal{C}) = \mathcal{B}([0,T]^p)$, which leads us to conclude that $\mathcal{D} = \mathcal{B}([0,T]^p)$.

Let f be a function from $\mathcal{L}^2([0,T]^p, Leb)$ and let f_n be a bounded function derived from f as $f_n \doteq (f \wedge n) \vee (-n)$, where $f_n \to f$ in $\mathcal{L}^2$ as $n \to \infty$. By the previous claim for all n and $f_n \in \overline{\varepsilon}_p$ for the function defined as

$$f_{n,k}(x) \doteq \sum_j \frac{j}{k} \mathbb{1}_{\frac{j}{k} < f_n(x) \leq \frac{j+1}{k}}(x) \in \overline{\varepsilon}_p$$

we have that $f_{n,k} \xrightarrow{k\to\infty} f_n$ in $\mathcal{L}^2$, which completes the proof.

□

Definition 2.2.3. Consider $f \in \mathcal{L}^2([0,T]^p), g \in \mathcal{L}^2([0,T]^q)$ then the contractive tensor product of f and g for $l \leq p \wedge q$ is defined as

$$(f \otimes_l g)(t,s) \doteq \int_{[0,T]^l} f(t,r)g(s,r)dr,$$

where $t \in [0,T]^{p-l}, s \in [0,T]^{q-l}$.

Note that the mapping contractive tensor product is a contraction, which can be easily obtained by applying Chebishev's inequality as below

$$\|f \otimes_l g\|^2_{\mathcal{L}^2([0,T]^{p+q-l})} \leq \|f \otimes g\|^2_{\mathcal{L}^2([0,T]^{p+q})} = \|f\|^2_{\mathcal{L}^2([0,T]^p)} \|g\|^2_{\mathcal{L}^2([0,T]^q)},$$

where the notations presented in the definition above have been used.

Lemma 2.2.3. Consider $f \in \mathcal{L}^2([0,T]^p)$ and $g \in \mathcal{L}^2([0,T])$ then

$$\mathcal{I}_p(f)\mathcal{I}_1(g) = \mathcal{I}_{p+1}(f \otimes g) + p\mathcal{I}_{p-1}(\tilde{f} \otimes_1 g).$$

Proof. It is enough to show that the equation holds true above for $f \in \varepsilon_p, g \in \varepsilon_1$, since by taking the limes in $\mathcal{L}^2$ sense the formula can be obtained for $f \in \mathcal{L}^2([0,T]^p)$ and $g \in \mathcal{L}^2([0,T])$ according to the the Lemma 2.2.2, i.e. for $f_n \xrightarrow{\mathcal{L}^2} f, g_n \xrightarrow{\mathcal{L}^2} g$ we have the following limits

$$f_n \otimes g_n \xrightarrow{\mathcal{L}^2} f \otimes g$$
$$\tilde{f}_n \otimes_1 g_n \xrightarrow{\mathcal{L}^2} \tilde{f} \otimes_1 g$$
$$\mathcal{I}_p(f_n)\mathcal{I}_1(g_n) \xrightarrow{\mathcal{L}^1} \mathcal{I}_p(f)\mathcal{I}_1(g).$$

The remaining part of the proof can be reduced for $f \doteq \mathbb{1}_{A_1 \times ... \times A_p}, g \doteq \mathbb{1}_B$ functions written in the form because of the operator $\mathcal{I}_p : \varepsilon_p \to \mathcal{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$ being linear, moreover it is enough to examine the two cases $B = A_j$ for an index $j \in \{1, ..., p\}$ or B is disjoint from $\{A_j\}_{j=1}^p$. Let B be equivalent with the set A_1 and let me consider a disjoint partition of

A_1, e.g. $A_1 \doteq \bigcup_i^* A_{1,i}$, now the images of the kernel functions in the discussed form can be written as $\mathcal{I}_p(f) = \prod_j W(\mathbb{1}_{A_j})$ and $\mathcal{I}_1(g) = W(\mathbb{1}_{A_1})$, which observations lead us to

$$\begin{aligned}
\mathcal{I}_p(f)\mathcal{I}_1(g) &= W(\mathbb{1}_{A_1})^2 \prod_{j=2}^{p} W(\mathbb{1}_{A_j}) \\
&= \sum_{i\neq k} W(\mathbb{1}_{A_{1,i}}) W(\mathbb{1}_{A_{1,k}}) \prod_{j=2}^{p} W(\mathbb{1}_{A_j}) + \sum_i W(\mathbb{1}_{A_{1,i}})^2 \prod_{j=2}^{p} W(\mathbb{1}_{A_j}) \\
&= \sum_{i\neq k} \mathcal{I}_{p+1}(\mathbb{1}_{A_{1,i}} \otimes \mathbb{1}_{A_{1,k}} \otimes \mathbb{1}_{\times_{k=2}^{p} A_k}) \\
&+ \sum_i \Big(W(\mathbb{1}_{A_{1,i}})^2 - |A_{1,i}| \Big) \prod_{j=2}^{p} W(\mathbb{1}_{A_j}) + |A_1| \prod_{j=2}^{p} W(\mathbb{1}_{A_j}) \\
&= \mathcal{I}_{p+1}(g \otimes f) + |A_1| \mathcal{I}_{p-1}(\mathbb{1}_{A_2 \times \ldots \times A_p}),
\end{aligned}$$

where $(W(\mathbb{1}_{A_{1,i}})^2 - |A_{1,i}|) \xrightarrow{\mathcal{L}^2} 0$ and it has been also applied that A_1 has been determined by the disjoint union of intervals $\{A_{1,j}\}_j$.

Consider a function written in the form $f \doteq \mathbb{1}_{A_1 \times \ldots \times A_p}$, where its projection onto the space of symmetric functions can be obtained as $\tilde{f} = \frac{1}{p!} \sum_{\pi \in S_p} \mathbb{1}_{A_{\pi(1)} \times \ldots \times A_{\pi(p)}}$ and by now its contractive tensor product with the function $g = \mathbb{1}_{A_1}$ can be calculated as

$$\begin{aligned}
\tilde{f} \otimes_1 g(t) &= \int_{[0,T]} \tilde{f}(t,s) g(s) ds = \frac{1}{p!} \sum_{\pi \in S_p} \mathbb{1}_{A_{\pi(1)} \times \ldots \times A_{\pi(p-1)}}(t) |A_{\pi(p)} \cap A_1| \\
&= |A_1| \frac{1}{p!} \sum_{\pi \in S_{p-1}(2,\ldots,p)} \mathbb{1}_{A_{\pi(1)} \times \ldots \times A_{\pi(p-1)}}(t),
\end{aligned}$$

where $S_{p-1}(2, \ldots, p)$ denotes the group of all permutations of the set $\{2, \ldots, p\}$. Now applying the operator $\mathcal{I}_{p-1}$ on the calculated contractive tensor product leads to the following random variable, which completes the proof for the case of B being the same set as A_1

$$\mathcal{I}_{p-1}(\tilde{f} \otimes_1 g) = |A_1| \frac{1}{p!} (p-1)! \prod_{k=2}^{p} W(\mathbb{1}_{A_k}).$$

In case of B not equals to any A_i the product of the images can be computed as

$$\mathcal{I}_p(f)\mathcal{I}_1(g) = W(\mathbb{1}_B) \prod_{k=1}^{p} W(\mathbb{1}_{A_k}) = \mathcal{I}_{p+1}(g \otimes f),$$

since the contractive tensor product of f and g is zero if the corresponding intervals are disjoint, which observations let us complete the proof.

□

Corollary 2.2.4. For $h \in \mathcal{L}^2([0,T])$ provided with unit norm the following equation shows the relation between the investigated operator $\mathcal{I}_p$ and the introduced analysis on

the Gaussian space

$$\mathcal{I}_p(h^{\otimes p}) = H_p(W(h)).$$

Proof. The proof is based on induction according to the order of the subspace containing the matter in hand square integrable random variable. For $\mathcal{H}_0$ and $\mathcal{H}_1$ we have $\mathcal{I}_0(1) = 1$ and $\mathcal{I}_1(h) = W(h) = H_1(W(h))$. Assume that for p the equation $\mathcal{I}_p(h^{\otimes p}) = H_p(W(h))$ holds true and consider the calculation for $p+1$ using applying the formula proven as Lemma 2.2.3

$$\begin{aligned}\mathcal{I}_{p+1}(h^{\otimes p+1}) &= \mathcal{I}_p(h^{\otimes p})\mathcal{I}_1(h) - p\mathcal{I}_{p-1}(h^{\otimes p} \otimes_1 h) \\ &= \mathcal{I}_p(h^{\otimes p})\mathcal{I}_1(h) - p\mathcal{I}_{p-1}(h^{\otimes p-1}),\end{aligned}$$

where the corresponding contractive tensor product can be determined by the following equation using the fact that the kernel function is provided with unit norm

$$(h^{\otimes p} \otimes_1 h)(t) = \int_{[0,T]} h^{\otimes p-1}(t)h(s)h(s)ds = h^{\otimes p-1}(t)\|h\|^2_{\mathcal{L}^2([0,T],Leb)} = h^{\otimes p-1}(t).$$

Let me recall that the operator ∂^* acts on a function as $(\partial^* f)(x) = xf(x) - \partial f(x)$ and the Hermite polynomial with order n has been defined as $H_n = (\partial^*)^n \mathbb{1}$. Moreover, the definition of the Hermite polynomials led to the property $\partial H_n = nH_{n-1} + (\partial^*)^n \partial H_0 = nH_{n-1}$, which property can be observed for the sequence of operators $\{\mathcal{I}_n\}_n \geq 0$, i.e.

$$\mathcal{I}_N(h^{\otimes N}) = \mathcal{I}_{N-1}(h^{\otimes N-1})\mathcal{I}_1(h) - (N-1)\mathcal{I}_{N-2}(h^{\otimes N-2}).$$

Defining the ∂^* operator for random variables in form $\mathcal{I}_p(h^{\otimes p})$ with $\|h\|^2_{\mathcal{L}^2([0,T],Leb)} = 1$ as

$$(\partial^* \mathcal{I}_{p+1})(h^{\otimes p+1}) \doteq \mathcal{I}_1(h)\mathcal{I}_p(h^{\otimes p}) - p\mathcal{I}_{p-1}(h^{\otimes p-1}),$$

which leads us to another definition of the operator $\mathcal{I}_N$, which is similar to the previously presented one and for any $h \in \mathcal{L}^2([0,T])$ provided with unit norm it can be determined by applying ∂^* as

$$\mathcal{I}_N(h^{\otimes N}) \doteq (\partial^*)^N h.$$

It can be concluded according to the previous analysis about the equation $\mathcal{I}_N(h^{\otimes N}) = H_N(W(h))$ if $\|h\|^2_{\mathcal{L}^2([0,T],Leb)} = 1$.

□

Lemma 2.2.4. For $\mathcal{F}_T \doteq \sigma(W_u : u \leq T)$ the set defined as

$$\mathcal{S} \doteq \{\sin(W(h)), \cos(W(h)), h \in \mathcal{L}^2([0,T], Leb)\}$$

is a total set in $\mathcal{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$.

Proof. Recall that in Lemma 2.2.1 it has been proven that the isonormal processes transformed by the exponential function is dense in the space of square integrable random variables. Now let the corresponding separable kernel Hilbert space denoted by $\mathcal{G}$ be the space $\mathcal{L}^2([0,T],\mathcal{B}([0,T]),Leb)$ and let me define the inner product structure of the kernel elements as the corresponding $\mathcal{L}^2$ inner product, i.e.

$$\langle f,g\rangle_{\mathcal{G}} \doteq \langle \tilde{f},g\rangle_{\mathcal{L}^2([0,T],Leb)}.$$

Let the isonormal process denoted by $\{\zeta(g): g\in\mathcal{G}\}$ be in this case $\{W(h): h\in\mathcal{L}^2([0,T])\}$ and by definition the inner product structure of the image elements is defined as

$$\langle W(f),W(g)\rangle_{\mathcal{L}^2(\Omega)} \doteq \langle \tilde{f},g\rangle_{\mathcal{L}^2([0,T],Leb)},$$

which let us consider the white noise integrals as an isonormal process determined as above. We have the fact that if a random variable X is orthogonal to $\mathrm{e}^{W(h)}$ for any $h\in\mathcal{L}^2([0,T])$ then X has to be zero in $\mathcal{L}^2$ sense. Now it can be written that

$$0=\langle X,\mathrm{e}^{iW(h)}\rangle_{\mathcal{L}^2(\Omega)}=\langle X,\sin(W(h))+i\cos(W(h))\rangle_{\mathcal{L}^2(\Omega)},$$

which let us conclude that the set denoted by $\mathcal{S}$ is a total set in $\mathcal{L}^2(\Omega,\mathcal{F}_T,\mathbb{P})$ according to the Lemma 2.2.1.

□

For $\|h\|^2=1$ the random variable $W(h)$ has standard normal distribution, which let us apply the introduced Hermite polynomial based expansion for a $\phi\in\mathcal{L}^2(\gamma_1)$ as

$$\phi(W(h))=\sum_{p=0}^{\infty}\langle\phi,H_p\rangle_{\mathcal{L}^2(\gamma_1)}\frac{H_p(W(h))}{p!}=\sum_{p=0}^{\infty}\mathbb{E}((\partial^p\phi)(W(h)))\frac{H_p(W(h))}{p!},$$

where the choice $\phi(.)\doteq\sin(.)$ provides the $\mathcal{L}^2$-convergent series for random variables, which form a total set in the space of square-integrable random variables, where the corresponding series can be obtained by applying the previous memorandum as

$$\sin(W(h))=\sum_{p=0}^{\infty}\alpha_p\frac{H_p(W(h))}{p!}\quad\text{for }\|h\|^2=1.$$

Analogue to the Hermite polynomial case define the linear subspace of $\mathcal{L}^2(\Omega,\mathcal{F}_T,\mathbb{P})$ of order p as

$$\mathcal{H}_p\doteq\{\mathcal{I}_p(f):\ f\in\mathcal{L}^2([0,T]^p,Leb)\}.$$

It can be concluded that the $\mathcal{L}^2(\Omega)$-closure of the space obtained as the direct sum of the introduced subspaces includes $\sin(W(h))$ and $\cos(W(h))$ for kernel elements provided

with norm one in $\mathcal{L}^2([0,T], Leb)$ sense, i.e.

$$\sin\Big(\|h\|W\Big(\frac{h}{\|h\|}\Big)\Big), \cos\Big(\|h\|W\Big(\frac{h}{\|h\|}\Big)\Big) \in \overline{\bigoplus_{p=0}^{\infty} \mathcal{H}_p} \quad \text{for any } h \in \mathcal{L}^2([0,T]).$$

Which means that the set of random variables determined by the direct sum of the $\mathcal{H}_p$ subspaces is a dense set in $\mathcal{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, moreover it can be claimed that the subspaces are closed orhtogonal subspaces by the following inner product structure between subspaces with different orders

$$\big\langle \mathcal{I}_p(f), \mathcal{I}_q(g) \big\rangle_{\mathcal{L}^2(\Omega)} = p! \int_{[0,T]^p} \tilde{f} g \chi_{p=q},$$

where $f \in \mathcal{L}^2([0,T]^p)$ and $g \in \mathcal{L}^2([0,T]^q)$. The property of the subspaces $\mathcal{H}_p$ being closed can be shown as following: consider a Cauchy sequence $\tilde{f}_n \subset \mathcal{L}^2([0,T]^p)$ and denote its $\mathcal{L}^2$ limit by f and let the images of the sequence be $F_n \doteq \mathcal{I}_p(\tilde{f}_n)$ then the limit in the image space can be deduced as

$$\text{if } \tilde{f}_n \xrightarrow{\mathcal{L}^2([0,T]^p)} f, \text{ then } F_n \xrightarrow{\mathcal{L}^2(\Omega)} F,$$

where $F = \mathcal{I}_p(f)$ completes the proof of $\mathcal{H}_p$ being a closed subset of $\mathcal{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$ and being orthogonal to the the subspaces with different order to p.

Corollary 2.2.5. An orthogonal decomposition of the $\mathcal{L}^2(\Omega, \mathcal{F}_T)$ space has obtained according to the previous calculations and observations, which is called Wiener-Ito chaos decomposition and is written as

$$\mathcal{L}^2(\Omega, \mathcal{F}_T) = \bigoplus_{p=0}^{\infty} \mathcal{H}_p,$$

where the closed orthogonal subspaces denoted by $\mathcal{H}_p$ is called the pth Wiener-Ito chaos. Moreover, for $F \in \mathcal{L}^2(\Omega, \mathcal{F}_T)$ there exists a sequence $(f_p)_p \subset \mathcal{L}^2([0,T]^p)$ such that

$$F = \sum_{p=0}^{\infty} \mathcal{I}_p(f_p)$$
$$\|F\|^2_{\mathcal{L}^2(\Omega)} = \sum_{p=0}^{\infty} p! \|f_p\|^2_{\mathcal{L}^2([0,T]^p)}.$$

Chapter 3

Generator system for isonormal processes

In this chapter the task of simulating isonormal processes will be investigated, where its importance rises high based on the introduced analysis in the previous sections. Moreover, my main goal is implementing Malliavin calculus for the matter in hand generator system, where it has been shown by D. Nualart that the operator and its adjoint, i.e. Malliavin-derivative and Skorohod-integral respectively, can be defined for isonormal processes in the most general case.

I will come up with several examples which are isonormal processes with additional assumptions, e.g. fractional Wiener process, fractional Ornstein-Uhlenbeck process in case of zero initial value. Note that the assumptions needed to obtain a special case let us fasten the simulation procedure. The generator systems and the subroutines, methods and operators have been implemented in Python and form a library, where several investigated stochastic process generators have not been implmented in Python or in any other language before, e.g. arbitrary fractional Wiener integral for Hölder-continuous kernel functions.

In the first section of this chapter I will introduce the general task of simulating an isonormal process defined by the elements of the kernel space and by the inner product structure the kernel space is provided with. It will be shown that for this case without any simplifying assumptions the only way to generate an isonormal sequence is taking the Cholesky decomposition of the covariance matrix, where this matrix contains actually the inner products of the kernel elements by definition.

There has to be two simulation tasks solved, i.e. generating trajectories of certain discretised stochastic processes over equidistant and over arbitrary non-equidistant gird. Note that assuming that the time grid is determined by a sequence of equidistant intervals can simplify the problem, e.g. it can lead to a stationary sequence in some cases, which let us handle the covariance matrix in a more efficient way. For the two tasks I will

introduce two operators to obtain a discretized stochastic process over the given time grid to make the upcoming analysis more simple.

Let me define the following *discretization* operator for a function denoted by $\zeta(.)$, for a time interval $[0,\tau]$ and for the number of data points $\mathcal{N}$ of the equidistant time grid as following, using the notation $\mathcal{T}:\mathbb{R}\times\mathbb{N}\times\mathcal{C}^1([0,\tau])\to\mathbb{R}^{\mathcal{N}+1}$

$$\mathcal{T}(\tau,\mathcal{N},\zeta(.)) \doteq \left\{\zeta(t)\mathbb{1}_{t=\frac{k\tau}{\mathcal{N}}}\right\}_{k=0}^{\mathcal{N}}, \tag{3.1}$$

so by applying the operator $\mathcal{T}$ with the given attributes a discretized sequence can be obtained of the corresponding function.

By generalizing the previous task and formalizing the introduced operator let me define a similar operator for discretizing a function over an arbitrary time grid. Let $\tilde{\mathcal{T}}$ be defined for a function $\zeta(.)$ and for a disjoint partition $\{[\tau_k,\tau_{k-1}]\}_{k=1}^{\mathcal{N}}$ of a subinterval $[0,\tau]$ of $\mathbb{R}$ as following, using the notation $\tilde{\mathcal{T}}:\mathbb{R}^{\mathcal{N}+1}\times\mathcal{C}^1([0,\tau])\to\mathbb{R}^{\mathcal{N}+1}$

$$\tilde{\mathcal{T}}(\{\tau_k\}_{k=0}^{\mathcal{N}},\zeta(.)) \doteq \left\{\zeta(t)\mathbb{1}_{t=\tau_k}\right\}_{k=0}^{\mathcal{N}}, \tag{3.2}$$

where in case of $\tau_k \doteq \frac{k\tau}{\mathcal{N}}$ a relation between the two introduced operators can be obtained as

$$\tilde{\mathcal{T}}(\{\tau_k\}_{k=0}^{\mathcal{N}},\zeta(.)) = \mathcal{T}(\tau,\mathcal{N},\zeta(.)).$$

3.1 Isonormal processes in case of arbitrary inner product structure

In this section I will investigate the simulation of isonormal processes with respect to an arbitrary kernel space, which can be applied for several special cases, e.g. for generating certain integrals with respect to fractional Wiener process. Note that in the most general case there cannot be applied any additional complex procedures, which can make the simulation much faster, but under some conditions a really fast generator system can be implemented. This additional assumption will be the stationary property, which let us handle the covariance matrix, determined by the inner product structure of the kernel elements, in a much more efficient way.

Recall that a centered Gaussian process $\eta(.)$ defined on a Hilbert space $\mathcal{G}$ provided with the inner product $\langle,\rangle_{\mathcal{G}}$ and the induced norm $\|.\|_{\mathcal{G}}$ is called isonormal process if there is an isomorphism between the kernel space and the image space of the mapping η, i.e.

$$\langle\eta(g),\eta(h)\rangle_{\mathcal{L}^2(\Omega)} = \langle g,h\rangle_{\mathcal{G}} \quad \text{for any } g,h\in\mathcal{G}.$$

Now let me store the covariance structure of the isonormal process $\{\eta(g), g\in\mathcal{G}\}$ in the

matrix denoted by Ψ_η and let me index the elements of the kernel space $\mathcal{G}$, i.e. $\mathcal{G} \doteq \{g_k\}_{k\geq 1}$. The elements of the matrix representing the covariance structure will be denoted by $\Psi_\eta^{(i,k)}$ according to the indices of the corresponding kernel functions and the elements for any $i, k \geq 1$ will be defined as following

$$\Psi_\eta^{(i,k)} \doteq \mathrm{cov}(\eta(g_i), \eta(g_k)) = \left\langle g_i, g_k \right\rangle_{\mathcal{G}},$$

where the second equation can be claimed by definition. Note that indexing the elements of the space does not require any additional conditions at this point, it just simplifies the upcoming introduction of several notations and definitions. Since $\{\eta(g), g \in \mathcal{G}\}$ is a Gaussian sequence and the covariance matrix is positive definite, (η_g) can be written as

$$\eta = \psi_\eta^{\frac{1}{2}} \varepsilon, \tag{3.3}$$

where ε is an independently sampled vector with standard normal distribution.

Corollary 3.1.1. For the task of simulating trajectories of an isonormal process denoted by $\{\eta(g), g \in \mathcal{G}\}$ the inner product structure depending on each element of the kernel space is the only input needed, i.e. by determining the values below a sequence of the isonormal process η can be generated according to 3.3

$$\left\langle g, h \right\rangle_{\mathcal{G}} \quad \text{for all } g, h \in \mathcal{G}.$$

So let me consider an isonormal process determined by its kernel space provided with an inner product structure, $\{\eta(g), g \in \mathcal{G}\}$, and consider its covariance matrix Ψ_η according to the indexed sequence of the elements of the Hilbert space $\mathcal{G} \doteq \{g_1, g_2, ..., g_N\}$ the isonormal process is defined on, where it has been assumed that the kernel space consists of finite number of elements. The previous assumption is necessary to be able to model the simulation of the obtained stochastic process. The following calculations aim at finding the square root of the covariance matrix in the fastest way in the given framework to obtain the realisations of the matter in hand stochastic process. Note that the square root of the matrix has to be calculated once for each kernel space with the corresponding inner product structures, i.e. for a given isonormal process after the determination of the square root of the covariance matrix to obtain realisations it is enough to multiply the square root matrix with a sequence of independent standard normal variables.

Since Ψ_η is a symmetric positive definite matrix, it admits the Cholesky decomposition $\Psi_\eta = LL^T$, where $L = (l_{ij})_{i,j=1,...,N}$ is a lower triangular matrix, which decomposition is unique in this case, since the covariance matrix is real and positive definite. So rewrite

the condition in the coordinate-wise form:

$$\sum_{k=1}^{i\wedge j} l_{ik}l_{jk} = \Psi_\eta^{(i,j)} = \langle g_i, g_j \rangle_{\mathcal{G}}. \tag{3.4}$$

The elements of L can be computed recursively based on the previously calculated elements according to the (3.4) equation, i.e. l_{ij} can be determined for $i \le j$.

For the first step the value of l_{11} can be determined since it is actually the norm of the first element, which is $\|g_1\|_{\mathcal{G}}$. It can be shown easily that for an arbitrary $j \ge 2$ and $i < j$ the corresponding elements can be computed based on the previously calculated values of the lower triangular matrix, which elements can be determined as following:

$$l_{ij} \doteq \frac{1}{l_{ii}}\Big(\langle g_i, g_j \rangle_{\mathcal{G}} - \sum_{k=1}^{i-1} l_{ik}l_{jk}\Big),$$

where all values being represented in the sum above has been calculated before according to the presented recursive scheme. In case of the indices being the same the corresponding matrix elements can be computed as following, i.e. for $j \ge 1$

$$l_{jj} \doteq \sqrt{\|g_j\|_{\mathcal{G}}^2 - \sum_{k=1}^{j-1}(l_{jk})^2}.$$

As soon as the matrix L has been determined, the isonormal process, $\{\eta(g), g \in \mathcal{G}\}$, can be simulated according to the equation (3.3) by adding indices to the elements of the Hilbert space as $\mathcal{G} \doteq \{g_1, ..., g_N\}$, i.e.

$$\eta(g_j) = \sum_{k=1}^{j} \varepsilon_k l_{jk},$$

where $(\varepsilon_k)_k$ is a vector of independent standard normal variables. Note that the simulation procedure does not depend on the choice of indexing of the kernel elements in this case.

3.2 Stationary isonormal processes

In this section there will be introduced additional assumptions about the class of isonormal processes to apply some more efficient methods with many applications such as fractional Wiener process simulation. So at the point of characterising the additional assumptions it has to be attended that the conditions have to be strict enough to let me apply Hosking's and Kroese's methods for the framework of isonormal processes, but not too strict to determine a specific class of stochastic processes without any applications. So the matter in hand assumption will be quite similar to the weak-stationary property

except that it will be formalised for processes defined over an arbitrary Hilbert space.

Definition 3.2.1. Let $\mathcal{M}$ be a separable Hilbert space and consider the stochastic process $\{\xi(m) : m \in \mathcal{M}\}$, which is called *indexed stationary process* if there exists an indexation of the Hilbert space $\mathcal{M} \doteq \{m_1, m_2, ...\}$ such that

$$\begin{aligned} \mathbb{E}(\xi(m)) &= \mu \quad \forall m \in \mathcal{M} \\ \mathbb{E}(\xi(m)^2) &< \infty \quad \forall m \in \mathcal{M} \\ R_\xi(m_j, m_k) &= R_\xi(m_{j+\tau}, m_{k+\tau}) \quad \forall \tau \in \mathbb{R} \text{ and for any } j, k. \end{aligned}$$

where $R_\xi(.,.)$ denotes the auto-covariance function of the stochastic process $\xi(.)$.

Note that the definition above has been introduced only for simplifying the characterisation of the class of the stochastic processes which will be investigated in the simulation task. Moreover one can observe that the definition of indexed stationary processes is a generalisation of the weak stationary property for processes defined on arbitrary separable Hilbert space in certain sense.

I aim at investigating an efficient generator procedure for isonormal processes satisfying the indexed stationary property, which has several special cases, e.g. fractional Wiener integrals. Let $\{\eta(g) : g \in \mathcal{G}\}$ be an isonormal process then for any $g \in \mathcal{G}$ the expected value of $\eta(g)$ is zero and for any kernel element g the following holds true

$$\mathbb{E}(\eta(g)^2) = \|g\|_{\mathcal{G}}^2 < \infty,$$

since $\eta(.)$ is a centered Gaussian process determined by the introduced isomorphism. Thus the conditions of an isonormal process being indexed stationary can be reduced to the property: there exists an indexed sequence of the kernel space $\mathcal{G} \doteq \{g_{t_1}, g_{t_2}, ..\}$ such that for any $\tau \in \mathbb{R}$ and any $0 \leq t_j, t_k$

$$\begin{aligned} R_\eta(g_{t_j}, g_{t_k}) &= \left\langle \eta(g_{t_j}), \eta(g_{t_k}) \right\rangle_{\mathcal{L}^2(\Omega)} = \left\langle g_{t_j}, g_{t_k} \right\rangle_{\mathcal{G}} \\ &= \left\langle \eta(g_{t_j+\tau}), \eta(g_{t_k+\tau}) \right\rangle_{\mathcal{L}^2(\Omega)} = \left\langle g_{t_j+\tau}, g_{t_k+\tau} \right\rangle_{\mathcal{G}} = R_\eta(g_{t_j+\tau}, g_{t_k+\tau}) \\ R_\eta(t_j - t_k) &\doteq \left\langle g_{t_j}, g_{t_k} \right\rangle_{\mathcal{G}} = \left\langle g_{t_j+\tau}, g_{t_k+\tau} \right\rangle_{\mathcal{G}}, \end{aligned}$$

where the isomorphism between the kernel space and the $\mathcal{L}^2(\Omega)$ space has been applied.

Recall that a centered Gaussian process is uniquely determined by its covariance structure, so an isonormal process is uniquely determined by the kernel elements and by the inner product structure of the kernel space. Thus for any Hilbert space playing the role of the kernel space of the corresponding isonormal process if there exists such an indexing needed the corresponding isonormal process to be indexed stationary then the obtained indexing satisfying the introduced conditions is unique. The previous observation will be

an important segment in some simulation tasks to model the random variables belonging to the observations of a stochastic processes according to certain time grid, which will be introduced in the next sections.

Example 3.2.1. Consider a separable Hilbert space with the natural indexation of the elements $\mathcal{M} \doteq \{m_t : t \in [0,T]\}$ and equip the space with the inner product structure

$$\left\langle m_{t_j}, m_{t_k} \right\rangle_{\mathcal{M}} \doteq t_j \wedge t_k \quad \text{for any } t_j, t_k \in [0,T].$$

Note, that $\mathcal{M}$ is isometrically isomorphic to a complete subspace of $\mathcal{L}^2([0,T])$, i.e. the space formed by the indicator functions of all subintervals starting with the point zero of $[0,T]$ and it is equipped with the $\mathcal{L}^2([0,T], Leb)$ inner product. So let me use the notation for the matter in hand subspace $\mathcal{U} \doteq \{\mathbb{1}_{[0,s]} : s \in [0,T]\}$ and let the corresponding inner product structure be defined for any $v, w \in \mathcal{U}$ as

$$\left\langle v, w \right\rangle_{\mathcal{U}} \doteq \left\langle v, w \right\rangle_{\mathcal{L}^2([0,T])}.$$

So a natural indexation of the space $\mathcal{U}$ can be determined as following: let the element indexed by $s \in [0,T]$ be the indicator function of the $[0,s]$ interval. One can observe that the isonormal process defined on the inner product space $\mathcal{U}$ is the standard Wiener process, since let the isonormal process be denoted as $\{\eta(u) : u \in \mathcal{U}\}$, where $\left\langle v, u_0 \right\rangle_{\mathcal{U}} = 0$ and $\|v\|^2_{\mathcal{U}} = \tau$ for index τ belonging to the element v. Thus the stochastic process $\eta(.)$ is actually the Wiener process according to the following equations for any element $v \in \mathcal{U}$ and for the corresponding index τ

$$\eta(u) = W(u) = W(\mathbb{1}_{[0,\tau]}) = W_\tau - W_0 = W_\tau.$$

So now there has been shown an example how apparent the indexation of the kernel space can be and if an indexation has been defined then according to the definition of the corresponding stochastic process it is unique.

Example 3.2.2. Let me define the separable Hilbert space $\mathcal{W}^{(\alpha,\sigma,T)}$ for any given positive parameters α, σ, T as it consists of elements which can be obtained in the form

$$\mathrm{e}^{-\alpha(\tau - \mathrm{Id}(.))}\mathbb{1}_{[0,\tau]}(.) \quad \text{for any } \tau \in [0,T]$$

and let me equip the space with the inner product of $\mathcal{L}^2([0,T], \mathcal{B}([0,T]), \nu)$, where the measure ν is defined as

$$\nu(B) \doteq \sigma Leb(B) \quad \text{for any } B \in \mathcal{B}([0,T]).$$

Now, analogous to the previous example an indexation of the space $\mathcal{W}^{(\alpha,\sigma,T)}$ can be

determined as following: label an element with index $s \in [0,T]$ if it can be written as

$$w_s \doteq \mathrm{e}^{-\alpha(s-\mathrm{Id}(.))}\mathbb{1}_{[0,s]}(.).$$

Thus, by defining the space with the indexation of the elements and the inner product the space is equipped with, the inner product can be calculated between two arbitrary elements, i.e. for any $u, v \in [0,T]$

$$\begin{aligned}
\langle w_u, w_v \rangle_{\mathcal{W}} &= \langle \mathrm{e}^{-\alpha(u-\mathrm{Id}(.))}\mathbb{1}_{[0,u]}(.), \mathrm{e}^{-\alpha(v-\mathrm{Id}(.))}\mathbb{1}_{[0,v]}(.)\rangle_{\mathcal{W}} \\
&= \mathrm{e}^{-\alpha(u+v)}\langle \mathrm{e}^{\alpha\mathrm{Id}(.)}\mathbb{1}_{[0,u]}(.), \mathrm{e}^{\alpha\mathrm{Id}(.)}\mathbb{1}_{[0,v]}(.)\rangle_{\mathcal{W}} \\
&= \mathrm{e}^{-\alpha(u+v)}\sigma^2\langle \mathrm{e}^{\alpha\mathrm{Id}(.)}\mathbb{1}_{[0,u]}(.), \mathrm{e}^{\alpha\mathrm{Id}(.)}\mathbb{1}_{[0,v]}(.)\rangle_{\mathcal{L}^2([0,T],Leb)} \\
&= \mathrm{e}^{-\alpha(u+v)}\sigma^2\int_0^{u\wedge v}\mathrm{e}^{2\alpha x}dLeb(x) \\
&= \frac{\sigma^2}{2\alpha}\mathrm{e}^{-\alpha(u+v)}\big(\mathrm{e}^{2\alpha(u\wedge v)}-1\big).
\end{aligned}$$

Let me denote the isonormal process determined by the Hilbert space $\mathcal{W}^{(\alpha,\sigma,T)}$ and by the calculated inner product as $\{\eta(w) : w \in \mathcal{W}\}$. One can observe that the stochastic process $\eta(.)$ is the pathwise unique solution of the following stochastic differential equation by assuming that the initial value is zero

$$d\xi_t = -\alpha\xi_t dt + \sigma dW_t,$$

where W_t denotes the driving Wiener noise process and the matter in hand solution is called *Ornstein-Uhlenbeck* process. In this case the Ornstein-Uhlenbeck process can be written as a Wiener-integral, i.e.

$$\xi_t = \sigma\mathrm{e}^{-\alpha_t}\int_0^t \mathrm{e}^{\alpha_s}dW_s = \eta(w_t),$$

which equation let us conclude that the isonormal process defined on the introduced Hilbert space is the Ornstein-Uhlenbeck process with the respect to the corresponding parameters and the zero initial value.

Example 3.2.3. For any $\tau \in \mathbb{R}$ and $H \in [0,1]$ let me consider the Hilbert space $\mathcal{U}^{\tau,H}$ with elements written in the form $\mathbb{1}_{[0,t]} - \mathbb{1}_{[0,t-\tau]}$ for $t-\tau$ being non-negative and consider the indexation as in the previous examples, i.e. label the element $\mathbb{1}_{[0,s]} - \mathbb{1}_{[0,s-\tau]}$ by s. Now, equip the space $\mathcal{U}^{\tau,H}$ with the inner product for any $s,t \geq \tau$

$$\langle u_t, u_s \rangle_{\mathcal{U}^{\tau,H}} \doteq \frac{1}{2}(|t-s+\tau|^{2H} - 2|t-s|^{2H} + |t-s-\tau|^{2H}),$$

which induces the norm for each element $||u||^2_{\mathcal{U}^{\tau,H}} = \tau$ for all $u \in \mathcal{U}^{\tau,H}$. Recall that for isonormal processes the indexed stationary property leads to restrictions only for the

elements and the inner product structure of the kernel space. So before defining the isonormal process on $\mathcal{U}^{\tau,H}$ let me verify that

$$R_\eta(u_s, u_t) = R_\eta(u_{s+z}, u_{t+z}) \quad \text{for any } -((s \wedge t) - \tau) \leq z \in \mathbb{R}.$$

Now for any $-((s \wedge t) - \tau) \leq z$ consider the following inner product structure

$$\begin{aligned}\left\langle u_{t+z}, u_{s+z} \right\rangle_{\mathcal{U}^{\tau,H}} &= \frac{1}{2}(|(t+z) - (s+z) + \tau|^{2H} - 2|(t+z) - (s+z)|^{2H} + \\ &+ |(t+z) - (s+z) - \tau|^{2H}) \\ &= \left\langle u_t, u_s \right\rangle_{\mathcal{U}^{\tau,H}},\end{aligned}$$

which let us conclude that an isonormal process defined on the space $\mathcal{U}^{\tau,H}$ has to be indexed stationary too. So let $\{\eta(u) : u \in \mathcal{U}^{\tau,H}\}$ be an isonormal process with the covariance structure for kernel functions $u = \mathbb{1}_{[0,t]} - \mathbb{1}_{[0,t-\tau]}$ and $v = \mathbb{1}_{[0,s]} - \mathbb{1}_{[0,s-\tau]}$

$$\left\langle \eta(u), \eta(v) \right\rangle_{\mathcal{L}^2(\Omega)} \doteq \frac{1}{2}(|t-s+\tau|^{2H} - 2|t-s|^{2H} + |t-s-\tau|^{2H}) = \left\langle u, v \right\rangle_{\mathcal{U}^{\tau,H}},$$

which leads us to the conclusion that the stochastic process $\{\eta(u) : u \in \mathcal{U}^{\tau,H}\}$ is stationary. One can observe that $\eta(.)$ is actually the increment process of the fractional Wiener process with respect to the time gap τ and the Hurst exponent $H \in [0,1]$, i.e.

$$\eta(u_t) = W^H(\mathbb{1}_{[0,t]} - \mathbb{1}_{[0,t-\tau]}) = W_t^H - W_{t-\tau}^H.$$

Corollary 3.2.1. If an isonormal process is an indexed stationary process then it is strictly stationary and the indexation is uniquely defined based on the observation time of the process.

Hosking method

Recall that to simulate isonormal processes one has to find the square root of the covariance matrix as it has been presented in the previous section. Now, consider an indexed stationary isonormal process $\{\eta(g_i), i = 1, ..., N\}$ defined on the Hilbert space $\mathcal{G}$ equipped with the inner product $\left\langle ., . \right\rangle_{\mathcal{G}}$, which sequence forms a stationary Gaussian sequence, therefore it can be computed more efficiently than in the first case based on the fact that the covariance matrix forms a Toeplitz matrix. In this subsection Hosking's method will be introduced, which was investigated for simulating stationary Gaussian sequences in a less computationally demanding way than the standard Cholesky method [8]. Recall that since $\eta(.)$ is an isonormal process satisfying the indexed stationary property it is strictly stationary, so its auto-covariance structure depends only on the time gap between the

two observations, i.e.

$$R_\eta(g_i, g_j) = \langle g_i, g_j \rangle_{\mathcal{G}} = \langle g_{i+\tau}, g_{j+\tau} \rangle_{\mathcal{G}} \quad \text{for any } -(i \wedge j) \leq \tau \leq N - (i \vee j)$$
$$\Psi_\eta^{(i,j)} \doteq R_\eta(g_i, g_j) = R_\eta(i-j) \quad \text{for any } 0 \leq i, j \leq N$$

Now, let me introduce the following notations and note that the description of the matter in hand procedure is based on the paper [10] :

$$\gamma_n \doteq \begin{pmatrix} \Psi_\eta^{(1,2)} \\ \Psi_\eta^{(1,3)} \\ \vdots \\ \Psi_\eta^{(1,n+1)} \end{pmatrix}, \quad J_n \doteq \mathbb{I}_n^T,$$

where $\mathbb{I}_n$ denotes the $n \times n$ dimensional identity matrix. In case of the matrix $\{\Psi_{\eta(i,j)}\}_{i,j=1}^m$ is given for $m < N$, the $\{\Psi_{\eta(i,j)}\}_{i,j=1}^{m+1}$ covariance matrix of the sequence $\{\eta(g_i), i = 1, ..., m+1\}$ can be written in two forms based on the previously introduced notations using the simplification that for $\Psi_{\eta,m} \doteq \{\Psi_{\eta(i,j)}\}_{i,j=1}^m$:

$$\Psi_{\eta,m+1} = \begin{pmatrix} 1 & \gamma_m^T \\ \gamma_m & \Psi_{\eta,m} \end{pmatrix} = \begin{pmatrix} \Psi_{\eta,m} & J_m\gamma_m \\ \gamma_m^T J_m & 1 \end{pmatrix}. \tag{3.5}$$

The conditional distribution of $\eta(g_{m+1})$ given $\eta(g_1), ..., \eta(g_m)$ can be obtained by applying the normal correlation theorem, i.e.

$$\mu_m = \mathbb{E}[\eta(g_{m+1})|\eta(g_1), ..., \eta(g_m)] = \gamma_m^T \Psi_{\eta,m}^{-1} \begin{pmatrix} \eta(g_1) \\ \eta(g_2) \\ \vdots \\ \eta(g_m) \end{pmatrix}$$
$$\sigma_m^2 = \mathbb{D}^2[\eta(g_{m+1})|\eta(g_1), ..., \eta(g_m)] = 1 - \gamma_m^T \Psi_{\eta,m}^{-1} \gamma_m.$$

However, calculating the inverse of $\Psi_{\eta,m}$ is a computationally expensive procedure, the result obtained above makes it possible to simulate $\eta(g_1), \eta(g_2), ..., \eta(g_m)$ subsequently.

As an efficient stationary Gaussian sequence simulation, taking advantage of calculating $\Psi_{\eta,m}^{-1}$ recursively, Hosking's method will be described above in a slightly different form published [6]. In order to simplify the notations in the presentation of the method, let me denote $d_m \doteq \Psi_{\eta,m}^{-1}\gamma_m$. The second representation of $\Psi_{\eta,m+1}$ formalised in (3.5) leads

us to the following equation by applying block matrix inversion:

$$\Psi_{\eta,m+1}^{-1} = \frac{1}{\sigma_m^2}\begin{pmatrix} \sigma_m^2\Psi_{\eta,m}^{-1} + J_m d_m d_m^T J_m & -J_m d_m \\ -d_m^T J_m & 1 \end{pmatrix}$$

The following recursive forms can be obtained for d_m and σ_m^2 by applying block matrix multiplication:

$$d_{m+1} = \begin{pmatrix} d_m - \phi_m J_m d_n \\ \phi_m \end{pmatrix}, \quad \sigma_{m+1}^2 = \sigma_m^2 - \frac{\left(\phi_m\right)^2}{\sigma_m^2}, \tag{3.6}$$

where

$$t_m \doteq d_m^T J_m \gamma_m, \quad \phi_m \doteq \frac{\Psi_\eta^{(1,m+1)} - t_m}{\sigma_m^2},$$

where the calculation steps not presented here can be found with description in [2].

The result formalised in (3.6) allows us to generate $\eta(g_1), ..., \eta(g_N)$ according to the method described above. For the first step, one simulates a standard normal variable to declare the first element of the sequence, $\eta(g_1)$, then the previously introduced corresponding variables have to be calculated as follows

$$\mu_1 \doteq \left\langle g_1, g_2 \right\rangle_{\mathcal{G}} \eta(g_1); \quad \sigma_1^2 \doteq 1 - \left(\left\langle g_1, g_2 \right\rangle_{\mathcal{G}}\right)^2;$$
$$t_1 \doteq \left(\left\langle g_1, g_2 \right\rangle_{\mathcal{G}}\right)^2; \quad d_1 \doteq \left\langle g_1, g_2 \right\rangle_{\mathcal{G}}.$$

Suppose that the mth values have been determined, then the $(m+1)$th element of the sequence can be obtained as $\eta = (g_m) \sim N(\mu_m, \sigma_m^2)$, while the t_m, ϕ_m and d_{m+1} auxiliary variables can be computed as described in (3.6). The last step of the algorithm consists of calculating the variance σ_{m+1}^2 and the mean value $\mu_{m+1} \doteq d_{m+1}^T(\eta(g_{m+1}), ..., \eta(g_1))^T$ of the next element in the sequence with respect to the previously calculated auxiliary variables. As in the method based on the Cholesky decomposition, the isonormal process can be simulated just by determining the inner product structure of the given kernel elements according to the presented procedure.

Davies-Harte method

As the previous methods, the procedure developed by Davies and Harte [5] also focuses on finding the square root of Ψ_η by taking circulant matrix embedding into account. Note that this algorithm was proposed by Davies and Harte and was generalized by Wood and Chan [15] and later by Dietrich and Newsam [6] according to [2]. The main idea of the method is embedding the $\Psi_{\eta,N}$ covariance matrix in the circulant covariance matrix, $\mathcal{C}$, which algorithm is based on the following theorem [2]. The whole procedure will be introduced later, including the details for fractional Wiener integrals of Hölder continuous integrands, which will be investigated in the next section.

Theorem 3.2.1. The $\mathcal{C}$ circulant matrix with $M \times M$ dimensions has a representation $\mathcal{C} = \mathcal{Q}\Lambda\mathcal{Q}^*$, where

$$\Lambda = \mathrm{diag}(\lambda_0, \lambda_1, ..., \lambda_M - 1), \quad \lambda_k = \sum_{j=0}^{M-1} c_j \exp\{-2\pi i \frac{jk}{M}\}$$

is the diagonal matrix of eigenvalues of $\mathcal{C}$, and the matrix $\mathcal{Q}$ is defined as

$$\mathcal{Q} = (q_{jk})_{j,k=0}^{M-1}, \quad q_{jk} = \frac{1}{\sqrt{M}} \exp\{-2\pi i \frac{jk}{M}\},$$

and $\mathcal{Q}^*$ denotes the conjugate transpose of $\mathcal{Q}$.

Recall, that in order to simulate Gaussian processes, one need to find the square root matrix of the covariance matrix, i.e. $\psi\psi^T = \Psi_\eta$, then the matter in hand isonormal process can be determined by multiplying the matrix ψ with a standard normal vector $(\varepsilon_1, ..., \varepsilon_N)^T$. The main advantage of this method can be derived from its complexity of order $\mathcal{O}(N \log N)$, which made it probably the most efficient among the exact methods [13]. Note that several circulant embedding methods have already been published, see e.g. [6, 3, 10].

Chapter 4

Generator system for fractal noise driven stochastic processes

In this section there will be presented several methods for isonormal processes determined by certain kernel space and inner product structure, e.g. fractional Wiener process, fractional Ornstein-Uhlenbeck process. It will be shown how can be derived the matter in hand stochastic processes from the introduced isonormal process framework and how can be applied the investigated procedures for certain processes. Moreover, there will be presented a method based on circulant matrix embedding for simulating fractional Wiener integrals of Hölder-continuous integrands. An additional interesting point is that there will be investigated simulation procedures not only for the case of the time grid is assumed to be equidistant, but for any arbitrary time grid, which generalization can be done for some processes without causing too much increase in the execution time, while in some cases it needs to be applied other more computational demanding algorithms to overcome this issue.

4.1 Fractional Wiener processes

The Fractional Wiener process, which was developed by Mandelbrot [11], can be interpreted as the generalisation of the standard Wiener process in the sense that the long- and short-term dependency of the process can be adjusted by adding the Hurst exponent to the concept. The definition and some basic properties of fractional Wiener processes, needed to the investigation of simulating fractal noise, will be introduced below.

A $\{W_t^H, t \geq 0\}$ *fractional Wiener process* is defined as a centered Gaussian process with the following covariance structure

$$\left\langle W_s^H, W_t^H \right\rangle_{\mathcal{L}^2(\Omega)} = \frac{1}{2}\Big(t^{2H} + s^{2H} - |t-s|^{2H}\Big),$$

where the H *Hurst exponent* has to be an element of the $(0,1)$ interval.

Remark. Note that determining the mean value and the covariance structure is enough to specify the distribution of a Gaussian process, which means in this case that the distribution of a fractional Wiener process is unique for fixed Hurst exponent.

Lemma 4.1.1. As a corollary of the Kolmogorov-Chentsov continuity theorem, fractional Wiener processes have continuous modification with probability 1. Moreover, for any $\gamma \in (0, h)$ this modification is γ-Hölder continuous.

Equidistant time grid

Consider the task of generating discrete trajectories of fractional Wiener processes, $\{W_t^H : t \in [0,T]\}$, for a given Hurst exponent, $H \in [0,1]$, over equidistant time grid. Therefore, the discretized noise process, $\{\hat{W}_t^H\}_{t\in[0,T]}$, with respect to the given Hurst exponent and time interval, can be obtained by taking the equidistant partition of the given time scale, $[0,T]$, and the fractional Wiener process by applying the previously defined $\mathcal{T}$ operator, i.e.

$$\{\hat{W}_t^H(\omega)\}_{t\in[0,t]} \doteq \mathcal{T}(T, N, W^H(\omega, .)).$$

Since the values $\{\hat{W}_t^H\}_{t\in[0,T]}$ form a Gaussian vector with a certain covariance matrix, they can be simulated by applying the corresponding linear transform onto a sequence of independently sampled standard normal variables.

Definition 4.1.1. A stochastic process $(\xi_t)_{t\geq 0}$ is said to be self-similar if for any $a > 0$ there exists a $b > 0$ such that

$$(\xi_{at})_{t\geq 0} = (b\xi_t)_{t\geq 0}.$$

In case of simulating fractional Wiener process, the H-self-similarity property of the matter in hand process can be taken into account to simplify the generator procedure by reducing an arbitrary time scale to the $[0,1]$ interval, i.e. a simulated sequence over the unit interval can be scaled as following to obtain a sequence over the $[0,T]$ interval

$$\{\hat{W}_t^H\}_{t\in[0,1]} = \Big(\frac{N}{T}\Big)^H \{\hat{W}_t^H\}_{t\in[0,T]},$$

which equation holds in distribution sense. This observation leads us to the amended task of generating fractional Wiener process with respect to the $[0,1]$ time scale. It is also well-known that the increments of W_t^H form a stationary Gaussian process, which leads us to the idea of rather simulating the increments and taking the cumulative sum of them to obtain the process than generating the actual process. Let me introduce an operator for taking the increments of a certain function over the equidistant partition of a given interval, $\mathcal{I} : \mathbb{R} \times \mathbb{N} \times \mathbb{R} \to \mathbb{R}^{\mathcal{N}}$, which can be formalised as

$$\mathcal{I}(\tau, \mathcal{N}, \zeta(.)) \doteq \left\{ \mathcal{T}(\tau, \mathcal{N}, \zeta(.)) \Big(\mathbb{1}_{t=\frac{k\tau}{\mathcal{N}}} - \mathbb{1}_{t=\frac{(k-1)\tau}{\mathcal{N}}} \Big) \right\}_{k=1}^{\mathcal{N}}. \tag{4.1}$$

Remark. The task of simulating discrete fractional Wiener processes with respect to the given arbitrary time scale has been reduced to generating over the unit interval, since the time attribute of the fractional Wiener process can be scaled into the unit interval by a constant, depending only on the ratio of the time interval lengths and on the corresponding Hurst exponent, thus an equality in distribution sense can be obtained between the primary and the scaled time series. To be more precise it is enough to come up with a simulation procedure for $\mathcal{I}(1, N, W^H(\omega, .))$ because of the fact that the following equation holds in distribution sense

$$\left(\frac{N}{T}\right)^H \mathcal{I}(1, N, W^H(\omega, .)) = \mathcal{I}(T, N, W^H(\omega, .)).$$

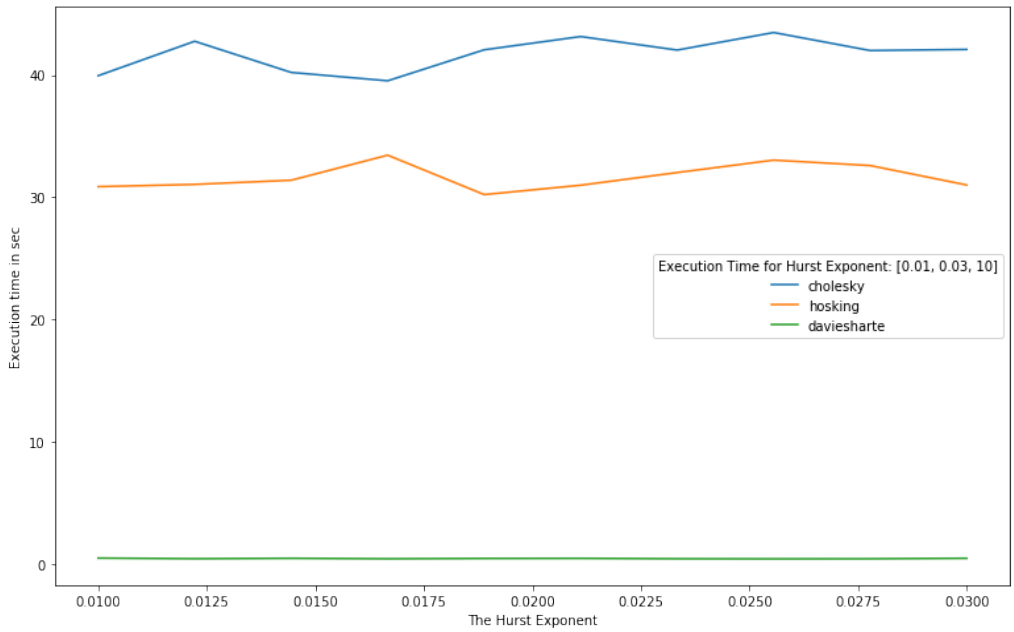
Consider the covariance structure of the matter in hand stationary time series given by $\{\eta_i\}_{i=1}^N \doteq I(1, N, W^H(\omega, .))$, which can be written as following by the definition of the fractional Wiener process:

$$\Psi_{W^H}^{(1,k)} \doteq \left\langle \eta_1; \eta_{k+1} \right\rangle_{\mathcal{L}^2(\Omega)} = \frac{1}{2}\Big((k+1)^{2H} + (k-1)^{2H} - 2k^{2H}\Big).$$

Since η_n is a Gaussian sequence and the covariance matrix is positive definite, (η_n) can be written as

$$\eta = \psi^{\frac{1}{2}}\varepsilon, \tag{4.2}$$

where $\varepsilon \doteq (\varepsilon_1, ..., \varepsilon_N)^T$ is an independently sampled vector with standard normal distribution. This case has been investigated deeply in the (3.2.3) example of the previous section, where it has been shown how the fractional Wiener noise process with arbitrary time gap can be derived as an isonormal. Thus the increment process of the fractional Wiener process can be simulated by applying the generator methods introduced for isonormal processes. Moreover, it has been shown that the kernel space endowed with inner product structure determined by the covariance function of the fractional noise process leads to an indexed stationary process, which means for isonormal processes that the corresponding process is strictly stationary, i.e. the more efficient methods for the stationary case can be applied also. Note that the circulant matrix embedding based method will be precisely described for fractional Wiener integrals, but it is the least computational demanding among the exact methods and it can be obtained from the generator system of fractional Wiener integrals by considering the constant one integrand case.



The chart above shows the result of the execution time simulating discrete fractional Wiener processes with respect to the Hurst parameter interval $[0.01, 0.03]$, i.e. $\mathcal{I}(1, 1500, W^H(\omega, .))$, according to the introduced Cholesky, Hosking and Davies-Harte methods. As mentioned before the $[0.01, 0.03]$ interval for the Hurst exponent of fractal noise is one of the most relevant in mathematical finance, especially in the investigation of the time-dependent correlation of given stock prices. So as expected, the circulant embedding based method, in this case the Davies-Harte, is outstanding in the execution time of simulating 10 sequences with 1500 grid points over the $[0.01, 0.03]$ Hurst interval, compared to the Cholesky and Hosking methods.

Non-equidistant time grid

Consider the task of simulating fractional Wiener process over an arbitrary time grid $0 = t_0 < t_1 < ... < t_N = T$ of the given $[0, T]$ interval. There are two different ways to handle the issue coming up by omitting the equidistant assumption.

A straightforward procedure can be that at first determining an appropriate kernel space and equip it with the corresponding inner product structure then by considering the isonormal process defined over the introduced kernel space can be simulated according to the methods have been presented previously, where the obtained isonormal process is actually the fractional Wiener process. Let the $\mathcal{W}^{T,H}$ kernel space consist of the elements $\{\mathbb{1}_{[0,s]} : s \in [0, T]\}$ which space is endowed with the inner product based on the covariance structure of the fractional Wiener process

$$\left\langle \mathbb{1}_{[0,s]}, \mathbb{1}_{[0,t]} \right\rangle_{\mathcal{W}^{T,H}} \doteq \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right),$$

where note that the way the form of the kernel elements has been determined as indicator

functions of certain intervals is just for simplification, since there could be considered any Hilbert space endowed with the inner product structure similar to the form written above. Now, the uniquely defined isonormal process over the Hilbert space $\mathcal{W}^{T,H}$ is the fractional Wiener process, i.e. choosing the kernel elements with indices according to the given time grid $\{\mathbb{1}_{[0,t_j]}\}_{j=0}^{N}$ leads to the discretized sequence of the fractional Wiener process, which is aimed to be simulated. Since in arbitrary case the matter in hand sequence does not form a stationary sequence, the only method which can be applied under these conditions is the Cholesky decomposition based procedure investigated before.

Another possible procedure can be obtained by introducing auxiliary time points and generating the corresponding auxiliary random variables instead of the straightforward method to simulate over non-equidistant time grid, where it will be shown that this method will be much more efficient than the previously introduced because of the applicability of the circulant matrix embedding based algorithm. Let the given time grid be denoted as $0 = t_0 < t_1 < ... < t_N = T$ as before and let me introduce an auxiliary time grid $0 = t_0' < t_1' < ... < t_N' = T$, where the $t_j' \doteq jT/N$ time point is one of the equidistant time grid over $[0, T]$ including $N + 1$ observation points. Recall that the most efficient exact methods are the ones based on circulant matrix embedding, which procedures can be applied only for stationary sequences. Thus, simulate the increment process of the fractional Wiener process over the auxiliary time grid according to the Davies-Harte method and then get the cumulative sum of the increments to obtain the auxiliary discretized process as before. The trick now is applying the self-similarity property of the fractional Wiener process to determine the values according to the given $\{t_j\}_{j=0}^{N}$ observation points by deriving it from the auxiliary variables. So, we have that the following equation holds true in distribution sense for any $0 \leq j \leq N$

$$W^H(\mathbb{1}_{[0,t_j']}) = \left(\frac{Nt_j}{jT}\right)^H W^H(\mathbb{1}_{[0,t_j]}),$$

which means that to obtain the variables aimed to be simulated it is actually enough to multiply the generated auxiliary random variables by a scalar number, i.e. the computational demanding of simulating fractional Wiener process over non-equidistant time grid according to the presented steps is still $\mathcal{O}(N\log N)$.

4.2 Integrals of Hölder-continuous functions with respect to fractional Wiener process

In this section the task of simulating fractional Wiener integrals of Hölder continuous functions will be investigated according to two different conceptions: a procedure derived from the general isonormal generator by determining the kernel space and the inner

product structure properly and another method can be obtained by simuating the fractional Wierner inrcements efficiently and weighing the increments by the corresponding discretized values of the integrand. Note that this section will include the introduction and the precise description of the previously mentioned circulant matrix based algorithm, which has been developed for simulating stationary Gaussian sequences [10].

Consider the class of stochastic process for any given $H \in [0,1]$ Hurst exponent and for any time interval $[0,T]$

$$\mathcal{S}^{H,T} \doteq \Big\{ \int_0^\tau \phi(s) dW_s^H, \quad \tau \in [0,T] \Big\}. \tag{4.3}$$

Since it is not aimed to simulate Skorohod integrals, additional assumptions are needed to handle the introduced class of stochastic integrals as pathwise Riemann-Stieltjes integrals. Thus let me assume that the kernel functions are λ-Hölder continuous, where λ has to be greater than $1-H$, which condition leads us to the previously mentioned case, see the proof for example in [16].

Non-equdistant time grid

Consider the Hilbert space $\mathcal{K}^{H,T,\phi}$ containing elements indexed as $\{k_\tau : \tau \in [0,T]^2\}$ and endow the space with the inner product determined for any $s \doteq (s_1,s_2), t \doteq (t_1,t_2) \in [0,T]^2$ as following

$$\big\langle k_s, k_t \big\rangle_{\mathcal{K}^{H,T,\phi}} \doteq \frac{1}{2}\phi(s_2)\phi(t_2)\big(|s_1-t_2|^{2H} + |s_2-t_1|^{2H} - |s_1-t_1|^{2H} - |s_2-t_2|^{2H}\big),$$

where $\phi(.)$ is assumed to be a λ-Hölder continuous function with the condition $\lambda > 1-H$. Now let me consider the isonormal process $\{\zeta(k) : k \in \mathcal{K}^{H,T,\phi}\}$ determined by the introduced kernel space, which process does not satisfy the indexed stationary property, i.e. there can be applied only the Cholesky decomposition based algorithm to simulate sequences from the $\zeta(.)$ isonormal process. Now let me calculate the covariance structure of the discretized fractional Wiener integral over the arbitrary time grid $\{t_j\}_{j=0}^N \in [0,T]^{N+1}$ with the Hölder-continuous integrand $\phi(.)$:

$$\begin{aligned}
&\big\langle \phi(t_j)W^H(\mathbb{1}_{[t_{j-1},t_j]}), \phi(t_k)W^H(\mathbb{1}_{[t_{k-1},t_k]}) \big\rangle_{\mathcal{L}^2(\Omega)} = \\
&= \phi(t_j)\phi(t_k)\big(\big\langle W_{t_j}^H, W_{t_k}^H \big\rangle_{\mathcal{L}^2(\Omega)} - \big\langle W_{t_j}^H, W_{t_{k-1}}^H \big\rangle_{\mathcal{L}^2(\Omega)} - \big\langle W_{t_{j-1}}^H, W_{t_k}^H \big\rangle_{\mathcal{L}^2(\Omega)} + \big\langle W_{t_{j-1}}^H, W_{t_{k-1}}^H \big\rangle_{\mathcal{L}^2(\Omega)}\big) \\
&= \frac{1}{2}\phi(t_j)\phi(t_k)\big(|t_j-t_{k-1}|^{2H} - |t_j-t_k|^{2H} - |t_{j-1}-t_{k-1}|^{2H} + |t_{j-1}-t_k|^{2H}\big),
\end{aligned}$$

which means that the discretized sequence of the integral below can be simulated according to the previously described Cholesky decomposition based method

$$\int_0^T \phi(s) dW_s^H \to \{\phi(t_j)W^H(\mathbb{1}_{[t_{j-1},t_j]})\}_{j=1}^N,$$

which sequence contains elements from the introduced space $\mathcal{K}^{H,T,\phi}$.

Equdistant time grid

Now let me consider the task of simulating the matter in hand integrals over equidistant time grid. So the task aimed to be solved has been precisely formalised as simulating the discretized elements of $\mathcal{S}^{H,T}$ with λ-Hölder continuous integrands, i.e. the following sequences have to be generated efficiently

$$\mathcal{I}\Big(T, N, \int_0^T \phi(s) dW^H(\omega)_s\Big),$$

where recall that the $\mathcal{I}$ operator acts as a descretization operator on the determined function over the given time interval and the corresponding time grid, which time grid is assumed to be equidistant for the matter in hand operator.

The first simulate procedure for this case can be easily derived from the one investigated in case of non-equidistant time grid, since replacing t_j with jT/N leads to the following covariance structure

$$\begin{aligned} R_{W^H,\phi}(t_j, t_k) &= \big\langle \phi(t_j) W^H(\mathbb{1}_{[t_{j-1},t_j]}), \phi(t_k) W^H(\mathbb{1}_{[t_{k-1},t_k]}) \big\rangle_{\mathcal{L}^2(\Omega)} \\ &= \frac{1}{2}\Big(\frac{T}{N}\Big)^{2H} \phi\Big(\frac{jT}{N}\Big)\phi\Big(\frac{kT}{N}\Big)\big(|j-k+1|^{2H} + |j-k-1|^{2H} - 2|j-k|^{2H}\big), \end{aligned}$$

which let the sequence, obtained by the discretization of the investigated fractional Wiener integral, to be simulated with the Cholesky algorithm, since the calculation above shows that the elements of the discretization led sequence are elements of the space $\mathcal{K}^{H,T,\phi}$.

Now, another procedure will be presented for simulating the fractional Wiener integrals written in the same form as before and the conditions introduced previously are assumed to hold too, which procedure is based on the circulant matrix embedding and it is considered as the most efficient one among the exact methods.

If one aims at simulating discretized stochastic integrals with respect to an almost surely continuous process, then the issue one has to face is generating discretized pathwise Riemann-Stieltjes integrals, which simplifies the procedure to just calculating the Riemann-Stieltjes sum according to the considered stochastic integral, i.e.

$$\int_0^T h(s) d\zeta(\omega)_s \to \mathcal{I}\Big(T, N, \int_0^T h(s) d\zeta(\omega)_s\Big),$$

where $\mathcal{I}(\tau, \mathcal{N}, h(.))$ operator was defined in (4.1) as

$$\Big\{\mathcal{T}(\tau, \mathcal{N}, h(.))\Big(\mathbb{1}_{t=\frac{k\tau}{\mathcal{N}}} - \mathbb{1}_{t=\frac{(k-1)\tau}{\mathcal{N}}}\Big)\Big\}_{k=1}^{\mathcal{N}}.$$

Therefore, the following sequence is aimed to be simulated efficiently

$$\mathcal{I}\big(T, N, \int_0^T \phi(s) dW^H(\omega)_s\big).$$

Recall, that in case of simulating fractional Wiener processes, the fastest exact methods are the circulant matrix embedding based algorithms. Thus, for discretized fractional Wiener integrals the Fast Fourier Transform and circulant embedding based methods can be applied with some additional steps, so the ideas of Davies and Harte [5], Wood and Chan [15], Dietrich [6] and Kroese [10] will be used as weighing the efficiently simulated increments with the corresponding discretized values of the kernel function. The method will be presented according to Dietrich's generalisation, but note that some tricks can be added, which can give a boost to the execution time. Let me denote the auto-covariance function describing the inner product structure of the stationary driving noise by $\psi(.)$ related to the time series below

$$\mathcal{I}\big(T, N, \int_0^T h(s) dW^H(\omega)_s\big),$$

i.e. the values of $\psi(.)$ for any time gap k can be determined as following based on the calculation made for the case of simulating over non-equidstant time grid

$$\begin{aligned}\psi(k) &\doteq R_{W^H}(1, k+1) \\ &= \frac{1}{2}\big((k-2)^{2H} + k^{2H} - 2(k-1)^{2H}\big).\end{aligned}$$

The main idea of the [5], [15], [6], [10] methods is embedding the covariance matrix in the so-called circulant matrix, i.e.

$$\mathcal{C}(\psi) \doteq \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{M-2} & c_{M-1} \\ c_{M-1} & c_0 & c_1 & \dots & c_{M-3} & c_{M-2} \\ c_{M-2} & c_{M-1} & c_0 & \dots & c_{M-4} & c_{M-3} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ c_1 & c_2 & c_3 & \dots & c_{M-1} & c_0 \end{pmatrix},$$

where $\mathcal{C}$ is a $2(N-1)\times 2(N-1)$ dimensional matrix, $c_0 = 1$ and $c_j \doteq \psi(k)\chi_{\{1\le k\le N-1\}} + \psi(M-k)\chi_{\{N\le k\le M\}}$. Since *Theorem* (3.2.1) the following decomposition holds true $\mathcal{C} = \mathcal{Q}\Lambda\mathcal{Q}^*$, where the matrices Λ and $\mathcal{Q}$ can be written in the following form

$$\Lambda = \mathrm{diag}(\lambda_0, \lambda_1, ..., \lambda_M - 1), \quad \lambda_k = \sum_{j=0}^{M-1} c_j \exp\big\{-2\pi i \frac{jk}{M}\big\}, \quad q_{jk} = \frac{1}{\sqrt{M}} \exp\big\{-2\pi i \frac{jk}{M}\big\}.$$

Since this method is obtained as a generalisation of simulating any fractal noise, the practical realization of this approach requires the computation of discrete Fourier transform,

see in [4], both direct and inverse with regard to [2]. The multiplication by the matrix $\mathcal{Q}$ acts, up to the constant $\frac{1}{\sqrt{M}}$, as taking the discrete Fourier transform and similarly, multiplying by the conjugate transpose of $\mathcal{Q}$, up to the constant $\sqrt{M}$, is the same operation as taking the inverse discrete Fourier transform. Without going into the details of the implementation tricks, an efficient exact method can be built up as following.

The first step is calculating the inner product structure of the driving noise over the the grid one aims at simulating, i.e. $\psi(1), ..., \psi(N-1)$ has to be determined, where $\psi(k)$ is the covariance between the first and the k+1th increment of the fractional Wiener process over equidistant time grid. Now one can fill the elements $c_0, ..., c_{M-1}$ needed for the embedded structure. The next step is acting the discrete Fourier transform on $(c_0, c_1, ..., c_{M-1})$ and taking the fast Fourier transform of the obtained eigenvalue vector, which has to be a real vector theoritically, but numerical imprecision can lead to negligible imaginary parts, so taking the real part of the obtained vector is suggested. In case of simulating several realizations, the steps introduced above have to calculated only once. Note, that the computation needed for generating one sequence can be reduced with some changes on the presented methods according to [10]. The third step is taking the inverse fast Fourier transform of and independent standard normal sequence $(\varepsilon_1, ..., \varepsilon_M)$ and multiplying the obtained $\frac{1}{\sqrt{M}}\mathcal{Q}^*(\varepsilon_1, ..., \varepsilon_M)$ element-wise with the square root of the vector determined in the second step. The fractional Wiener increments can be obtained as the real part of the fast Fourier transform of the result computed in the third step, i.e. $\mathcal{I}(T, N, W^H(\omega, .))$ has been calculated. As the last step one has to multiply the previously determined discrete driving fractional Wiener process element-wise with $\mathcal{T}(T, N, \phi(.))$ and taking the cumulative sum of the result, which leads to $\mathcal{I}\big(T, N, \int_0^T \phi(s) dW^H(\omega)_s\big)$.

4.3 Multiple fractional Wiener integrals

There will be introduced a generator system for simulating multiple fractional Wiener integrals of intgerands, which can be written as tensor product of Hölder continuous functions, which class will be characterized precisely. Since the increments of the fractional Wiener process are correlated and only the integrals of Hölder continuous functions can be handled as Riemann-Stieltjes integrals, such subspaces of the $\mathcal{L}^2(\Omega)$ space can be simulated, where the direct sum of the subspaces form a strict subset of $\mathcal{L}^2(\Omega)$ unlike the Wiener integrals introduced for the Wiener-Ito chaos decomposition.

Let me consider the following subspace of the kernel space $\mathcal{L}^2([0,T]^p, \mu)$:

$$\mathcal{M}_p^H \doteq \{f : [0,T]^p \to \mathbb{R} : \exists \phi \in \mathcal{L}^2([0,T], \mu), \lambda - \text{Hölder continuous}$$
$$\text{and } \eta : f = \eta^{\otimes p} \text{ such that } \exists (\phi_n)_n \subset \varepsilon_1 : \phi_n \xrightarrow{\mathcal{L}^2} \phi, \eta \in (\phi_n)_n\},$$

where $\lambda > 1 - H$ and $\eta^{\otimes p}$ denotes the p-times tensor product of η with itself. Recall that

the class of elementary functions have been determined as following

$$\varepsilon_p \doteq \{f : [0,T]^p \to \mathbb{R} : \ f = \sum_K a_K \mathbb{1}_{A_{K,1}\times\ldots\times A_{K,p}}(x), \text{ where } A_{K,1},\ldots,A_{K,p} \text{ are disjoints } \}.$$

So the simulating task has been formalised as investigating a generator procedure for the multiple fractional Wiener integrals of the integrands belonging to the introduced class $\mathcal{M}_p^H$, where let me denote the p-dimensional fractional Wiener integral operator as $\tilde{\mathcal{I}}_p(.)$. One can observe that generating a discrete-time stochastic process obtained as applying the $\tilde{\mathcal{I}}_p(.)$ operator onto an $\hat{m}^{\otimes p} \in \mathcal{M}_p^H$ function is actually the task arises when one aims at simulating stochastic process written in the from below over a given time grid

$$\int_0^T m^{\otimes p}(\underline{s}) dW_{\underline{s}}^H,$$

where $\hat{m}$ is in fact the corresponding discretized integrand to m. So the class formalised above includes such functions, which are related to Hölder-continuous functions as they are the discretized versions of them with respect to any time grid of the $[0,T]$ interval. Since the class has been determined as it consists of functions derived from considering the multiple tensor product of a one-dimensional function with itself, the simulation of such multidimensional integrals can be handled as multiple one-dimensional integrals, i.e. for this case the computational demanding of generating a fractional Wiener integral of any element of the class $\mathcal{M}_p^H$ belongs only to the order of magnitude $p\mathcal{O}(N\log N)$. Moreover, a condition of the definition of $\mathcal{M}_p^H$ requires the tensor producted intergands to be Hölder-continuous with the corresponding order, which let the integrals be handled as pathwise Riemann-Stieltjes integrals, i.e. the procedures presented before for one-dimensional fractional Wiener integrals can be applied for this case too.

Note that if one of the conditions cannot be satisfied then the generator procedures are getting too complex and complicated to be derived from the previously presented exact methods, e.g. if the tensor product property is violated then the kernel space cannot be considered as the direct product of orthogonal spaces and the increments of the fractional Wiener process are correlated, so the procedures have to be developed further, which is a work in progress for me; in case of the kernel functions cannot be considered as Hölder-continuous functions then the integrals have to be handled as Skorohod-integrals, which is not aimed to be investigated at this point of my work.

4.4 Fractional Ornstein-Uhlenbeck processes

In this section an efficient generator procedure will be investigated in both equidistant and non-equidistant cases and there will be handled differently the cases when the initial value has chosen zero and non-zero. In case of the initial value of the corresponding stochastic

differential equation describing a fractional Ornstein-Uhlenbeck process is chosen zero then the unique pathwise solution of the differential equation can be written as a fractional Wiener integral of an exponential function, which is actually an element of the class of stochastic processes have been simulated previously. In case of non-zero initial value the fractional Wiener increments will be generated and then the corresponding Lebesgue and Riemann-Stieltjes integrals will be simulated, i.e. there will be applied Euler's scheme to generate the solution of the stochatic differential equation.

Let me consider the fractional Ornstein-Uhlenbeck process driven by fractional Wiener process as the unique pathwise solution of the following stochastic differential equation with the initial value $\xi_0 \in \mathbb{R}$

$$d\xi_t = \alpha(\mu - \xi_t)dt + \sigma_t dW_t^H, \tag{4.4}$$

where $\alpha > 0$ is the drift parameter, σ_t is the time-dependent diffusion parameter and the μ constant is called the mean-reverting parameter. By applying the integral operator on the both sides of the 4.4 equation the fractal noise driven Ornstein-Uhlenbeck processes can be obtained as following

$$\xi_t = \xi_0 + \alpha \int_0^t \mu - \xi_s ds + \int_0^t \sigma_s dW_s^H. \tag{4.5}$$

Non-equidistant time grid

Let the time grid be any ordered sequence on the $[0, T]$ interval, i.e. $\{0 = t_0 < t_1 < ... < t_N = T\}$ and the value of the discretized fractional Ornstein-Uhlenbeck process with respect to the given parameters and time grid can be written at the time point t_i as below

$$\xi_{t_i} = \xi_0 + \alpha \sum_{j=0}^{i-1} (\mu - \xi_{t_j})(t_{j+1} - t_j) + \sum_{j=0}^{i-1} \sigma_{t_{j+1}} \big(W^H(\mathbb{1}[t_j, t_{j+1}])\big).$$

Since the Euler's scheme is aim to be applied the increments of the sequence determined previously have to be calculated, so the ith increment of the discretized fractional Ornstein-Uhlenbeck process can be obtained as

$$\xi_{t_i} - \xi_{t_{i-1}} = \alpha(\mu - \xi_{t_{i-1}})(t_i - t_{i-1}) + \sigma_{t_i} \big(W^H(\mathbb{1}_{[t_{i-1}, t_i]})\big).$$

As the first step the $\{W^H(\mathbb{1}_{[t_{i-1}, t_i]})\}_{i=1}^N$ fractional Wiener increments will be simulated, where recall that clearly the most efficient generator method is the circulant matrix embedding based. The mentioned simulating method can be applied only for stationary Gaussian sequences, so there will be simulated auxiliary increments over equidistant time grid and the obtained values will be scaled properly according to the self-similarity property of the fractional Wiener process. Thus let me introduce the ausxiliary time grid

$\{0 = t'_0 < ... < t'_N = T\}$, where t'_j is actually the $j+1$th time point of the equidistant time grid, i.e. $t'_j \doteq jT/N$. Since the following equation holds in distribution sense for any $0 \leq j \leq N$ the previously trick can be applied to obtain the increments over the arbitrary time grid through the properly scaled auxiliary variables

$$W^H(\mathbb{1}_{[0,t'_j]}) = \left(\frac{Nt_j}{jT}\right)^H W^H(\mathbb{1}_{[0,t_j]}),$$

where despite the additional calculations the computational demanding of the simulation of the fractional Wiener increments over arbitrary time grid is still $\mathcal{O}(N\log N)$. Now, initialize the value of the discretized fractional Ornstein-Uhlenbeck process ξ_0 by the given initial value. The next step is calculating the first increment of the discretized stochastic process according to the procedure formalised above with respect to the given parameters, the simulated corresponding fractional Wiener increment and the determined initial value, so $\xi_{t_1} - \xi_0$ has been calculated, i.e. the value of the fractional Ornstein-Uhlenbeck process at the first time point can be obtained as $(\xi_{t_1} - \xi_0) + \xi_0$. Now, in case of any index $1 \leq i \leq N$ the corresponding simulation step is based on the previously determined increment $\xi_{t_{i-1}} - \xi_{t_{i-2}}$, the calculated value at the time point t_{i-2}, the simulated corresponding increment $W^H([\mathbb{1}_{[t_{i-1},t_i]}])$ and the given parameters $\alpha, \mu, \sigma_{t_i}$ and it can be obtained by determining the increment first as $\xi_{t_i} - \xi_{t_{i-1}} = \xi_{t_i} - (\xi_{t_{i-1}} - \xi_{t_{i-2}}) + \xi_{t_{i-2}}$. Note that the most efficient way to handle the calculation presented above is caching in the memory the increments, but the increments only and as the last step the exact values can be given back, where the obtained boost of this concept rises higher in case of simulating longer and longer series, which depends also on the cache memory limit of the used computer.

In case of zero initial value the pathwise unique solution of the stochastic differential equation 4.4 can be written as a fractional Wiener integral in the form below

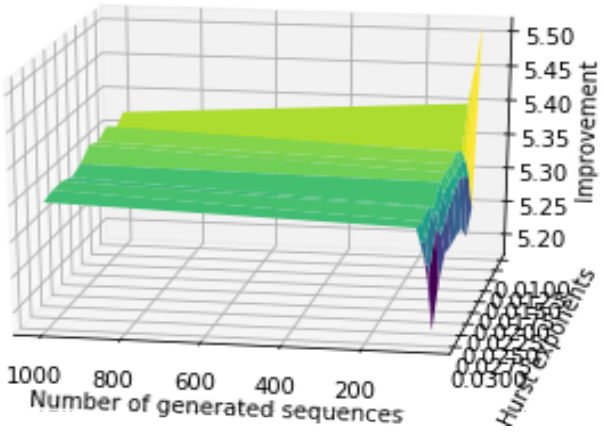
$$\xi_t = -\sigma \int_0^t \mathrm{e}^{-\alpha(t-s)} dW_s^H,$$

where its discretization over the arbitrary time grid $\{0 = t_0 < t_1 < ... < t_N = T\}$ can be obtained as

$$\xi_{t_0} \doteq 0 \quad \xi_{t_i} \doteq -\sigma \sum_{j=0}^{i} \mathrm{e}^{-\alpha(T-t_j)} W^H(\mathbb{1}_{[t_{j-1},t_j]}),$$

which is actually an element of the class of farctional Wiener integrals of Hölder continuous functions, i.e. it can be simulated according to the presented methods for the matter in hand class with the corresponding integrand.

Note that for equidistant time grid the investigated procedures can be applied by omitting the step of scaling the increments according to the self-similarity property.



The chart above has been obtained as testing how many times faster the implemented class of the presented method is able to simulate the given number of fractional Ornstein-Uhlenbeck sequences, with the given Hurst exponent from the investigated $[0.01, 0.03]$ interval over 1500 grid-points in the $[0, 1]$ time interval, than the procedure based on the fact that the driving noise is simulated by the Python package *fBM* and the standard solution method is applied to the given stochastic differential equation. One may notice the improvement can be observed as getting smaller and smaller Hurst exponents, it is caused by the stabilisation of the inner product structure embedding on the edges of the $[0, 1]$ interval, which cannot be reached by the procedures implemented in the *fBM* package.

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