# Triangles and Quadrilaterals Inscribed in Jordan Curves

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# 1 Introduction

A lot of research has been done on the question of whether a triangle or a quadrilateral of a given shape can be inscribed in a Jordan curve.

One of the most famous problems in this field is Toeplitz' conjecture[1] which we will call *the inscribed square problem*. It states that every Jordan curve contains the four vertices of a square and was proposed by Otto Toeplitz in 1911. Since then numerous methods have been used to tackle the problem, but the general case remains unsolved.

The aim of this thesis is to review the literature on the inscribed square problem as well as variants regarding other quadrilaterals and triangles, showcasing various different ways of approaching such problems.

We start the thesis by inspecting equilateral triangles. We showcase some exemplary and easy-to-understand proofs, which serve as a motivation for our next section on triangles with arbitrary side lengths. The inscribability of triangles was not as historically well-discussed as the inscribed square problem, or inscribed quadrilaterals in general. Because of this, most of the results we discuss come from M. D. Meyerson [2] and M J. Nielsen [3]. We end our discussion on triangles with some discussion on higher-dimensional curves and showcase a fairly new result by A. Gupta and S. Rubinstein-Salzedo [4].

The second part of the thesis discusses quadrilaterals. We again start with special classes of curves, for which, as the results of V. Klee, M. Wagon [5] and M. J. Nielsen and S. E. Wright [6] showcase, an even stronger version of the inscribe square conjecture holds. We then take a look at generalizations of the conjecture to rectangles and rhombi. H. Vaughan showcased a solution to the rectangular variant in a lecture, which was later summarized by M. D. Meyerson [7]. The question of whether every Jordan curve inscribes a rhombus was answered by M. J. Nielsen [6] based on the polygonal version of the mountain climbing theorem, first stated by T. Homma [8]. Nielsen also presented some remarkable statements about inscribed rhombi, based on the works of A. Emch [9] [10].

Finally, we take a look at the inscribed square problem itself. Although the general problem is still unsolved, it was shown to be true for $C^2$ curves by L. G. Schnirelmann [11], for analytic curves by A. Emch [9] and for locally monotone curves by W. Stromquist [12].

Most of the proofs mentioned in the thesis have their foundation in topology, such as proving intersection by the Jordan curve theorem. However, the research conducted by Terrence Tao [13] indicates that a complete proof of the inscribed square problem might require a combination of different, more advanced techniques. Nevertheless, the problem and its variants are still actively researched and the prevailing belief is that the conjecture will likely be solved in the foreseeable future.

# 2 Preliminaries and notation

As the thesis revolves around polygons inscribed in Jordan curves, we need to clarify what we mean by *inscribed* and *Jordan curve*.

Definition 2.1 (Jordan curve). A Jordan curve is a simple closed curve in the plane, or more formally, the image of an injective continuous map of a circle into the plane.

Alternatively:

Definition 2.2 ((parametrized) Jordan curve). A (parametrized) Jordan curve is the image of a continuous map $\varphi : [0,1] \to \mathbb{R}^2$ such that $\varphi[0] = \varphi[1]$ and $\varphi$ is injective on $[0,1)$.

The former is usually denoted as $J$, while the latter is denoted as $\varphi(t)$. Although Jordan curves are only defined in the plane, for sake of clarity we will refer to planar simple closed curves as *planar Jordan curves* and use the term *simple closed curve* in higher dimensions.

Most of the proofs in the thesis heavily rely on the Jordan Curve theorem:

Theorem 2.3 (Jordan curve theorem). *Let $J$ be a Jordan curve. Then $\mathbb{R}^2 - J$ consists of two components. One of these components is bounded and the other is unbounded, and the curve $J$ is the boundary of each component.*

The Jordan curve theorem is our most powerful tool for proving inscribability. Unfortunately, it cannot be used when working with higher dimensional curves, as it is not true that a smooth curve in $\mathbb{R}^n$ divides the space into two components.

Theorems from analysis often require compactness. We will denote the closure of a set $X$ as cl($X$).

We will refer to the bounded connected component of the complement of a Jordan curve as the *interior* of $J$, denoted as In($J$), and refer to the unbounded component as the *exterior* of $J$, denoted as Ex($J$). These are not to be confused with int($J$) and ext($J$), the topological interior and exterior of $J$.

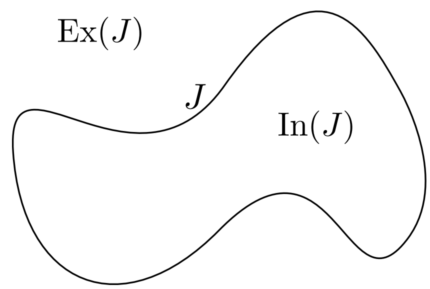



Figure 2.1

Definition 2.4. We say that the polygon $P$ is inscribed in the Jordan curve $J$ if $J$ contains all vertices of $P$.

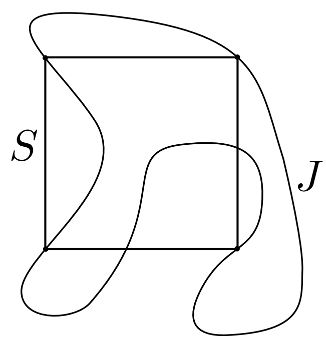



Figure 2.2: The square $S$ inscribed in $J$

As we can see, a polygon does not have to lie inside the curve to be inscribed in $J$. It can cross $J$ infinitely many times as long as its vertices remain on $J$.

Several problems discussed in the thesis mention some condition of smoothness. We say that a curve is *smooth* if it has a regular parametrization of class $\mathcal{C}^1$.

# 3 Equilateral triangles

Theorem 3.1. *Let $J$ be a simple closed curve in the plane. Then $J$ contains the vertices of an equilateral triangle.*

*Proof.* Let $c$ be a point in the interior of $J$. Let $C$ be the circle with the smallest radius that has center $c$ and meets $J$ at some point $x$. Let $y$ and $z$ be points on $C$ such that $\triangle xyz$ is an equilateral triangle.

Let $y$ and $z$ move away from $x$ until $y$ or $z$ meets $J$ (without loss of generality, we can assume $y$ is on $J$ at this point).

Now let $y$ move continuously on $J$ until the distance between $x$ and $y$ is maximal, while letting $z$ move in a way that keeps $\triangle xyz$ equilateral. This way the curve described by $z$ is a $60°$ rotation of the curve described by $y$, meaning it is also a continuous simple arc. Equality $xy = xz$ means that at the end $z$ is either on $J$ or outside $J$, while at the beginning $z$ lies inside $J$, thus, at some moment $x, y$ and $z$ are the vertices of an equilateral triangle inscribed in $J$. $\square$

Although the proof is more complicated, A. N. Milgram [14] showed that the result of Meyerson's theorem holds even if $J$ does not lie in the plane, but is embedded in an $n$-dimensional metric space instead, where $\triangle xyz$ is equilateral if and only if $d(x, y) = d(y, z) = d(z, x)$.

## 3.1 Triods

Definition 3.2 (Triod). A triod $T$ is the union of three arcs, $L_1, L_2, L_3$, which all share a common endpoint.

We will refer to the common endpoint as the *juncture* of $T$. $L_1, L_2, L_3$ are also called the legs of $T$ and any two of the legs only meet at the juncture ($L_1 \cap L_2 \cap L_3 = z$). The other three endpoints of the legs: $e_1, e_2, e_3$ are referred to as the *endpoints* of $T$. A triod can also be thought of as any homeomorphic image of the letter "Y".

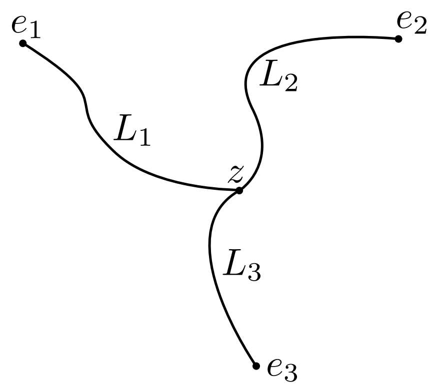



Figure 3.1

Theorem 3.3. *Every triod $T$ in the plane contains the vertices of an equilateral triangle with $e_1, e_2$ or $e_3$ as a vertex.*

Meyerson's proof of this theorem is rooted in complex geometry, however, Richard E. Schwarz [15] gave a proof that is simpler and more elegant. We start by taking a look at polygonal triods (a triod is polygonal if it is a finite union of line segments).

Lemma 3.4. *Every polygonal triod contains the vertices of an equilateral triangle with $e_1, e_2$ or $e_3$ as a vertex.*

*Proof.* The proof is by contradiction. Assume triod $T$ does not satisfy the Lemma. Let $A$ be the union of the legs with endpoints $e_1$ and $e_2$. For any $x \in T$ define $A_x$ as $A$ rotated $60°$ about $x$. First, we take a look at the case $x \in T - \partial A$. We then have $A \cap \partial A_x = \partial A \cap A_x = \emptyset$, otherwise the point of intersection, its inverse image and the point $x$ would form an equilateral triangle. From this we can conclude that the number of intersections between $A$ and $A_x$ is constant mod 2.

The same reasoning follows for $A \cap A_{e_1}$, meaning we can assume $A$ intersects $A_{e_1}$ only at $e_1$. By compactness $A$ intersects $A_x$ only at $x$ if $x \in T$ is su□ciently close to $e_1$. But then the number of intersections is odd for every $x \in T - \partial A$, meaning $A_{e_3} \cap A \neq \emptyset$, therefore $e_3$ is the vertex of an equilateral triangle inscribed in $T$, which we assumed is false. $\square$

R. E. Schwarz showed that this proof can be extended to arbitrary triods by first showing that it is true for triods that have line segments of length $\frac{1}{n}$ at their endpoints and then taking the limit as $n \to \infty$.

Theorem 3.5. *Every triod $T$ has a leg $L_i$, so that every point $x \in L_i - z$ is a vertex of some equilateral triangle on $T$.*

*Proof.* Suppose the theorem is false, thus all three legs have some point $a_i$ that is not a vertex point ($i = 1, 2, 3$). Consider the triod $T' \subseteq T$ with endpoints $a_1, a_2, a_3$. By Theorem 1.2, one of the endpoints is the vertex point of some equilateral triangle in $T'$ and by extension $T$. This contradicts our assumption. $\square$

## 3.2 Jordan curves

Theorem 3.6. *Let $J$ be a Jordan curve in the plane. Then at most two points of J are not vertex points of some equilateral triangle on $J$.*

*Proof.* Suppose there are three points $x_1, x_2, x_3$ on $J$ that are not vertex points of some equilateral on $J$. Let $T$ be any triod with endpoints $x_1, x_2, x_3$ such that $T \in J \cup$ In$(J)$:

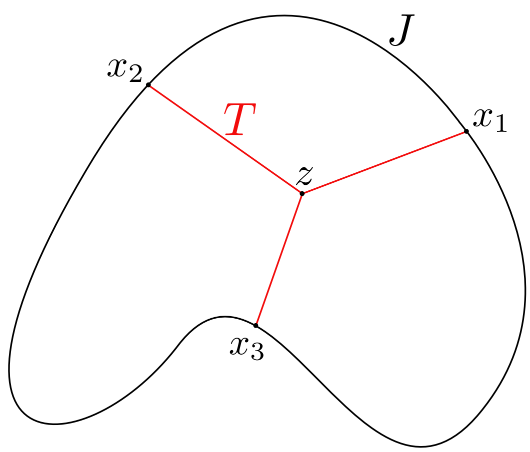



Figure 3.2

By Theorem 1.4, we should have an endpoint $x_i$, which is a vertex of some equilateral triangle on $T$. From here we can use the method from Theorem 1.1 to get an equilateral on $J$ with $x_i$ as a vertex, thus we have our contradiction.

□

## 4 Arbitrary triangles

In this section, $\triangle abc$ will denote an arbitrary triangle with vertices $a, b, c$ and smallest angle $\theta$. The next logical question to ask is whether the theorems above apply to arbitrary triangles: Do planar Jordan curves have points that form a triangle similar to $\triangle abc$?

M. D. Meyerson [2] showed that most of the analysis generalizes well to arbitrary triangles with minor changes.

Theorem 4.1 (Meyerson's theorem). *Let $J$ be a planar Jordan curve. Then $J$ contains the vertices of a triangle similar to $\triangle abc$.*

*Proof.* Define the circle $C$ as in Theorem 1.1. Let $y$ and $z$ be points on $C$ such that $\triangle xyz$ is similar to $\triangle abc$, where the point $x$ on $J$ corresponding to the maximal angle of $\triangle abc$. Let $y$ and $z$ move outward from $x$ until $y$ or $z$ meets $J$ (we can again assume that $y$ is on $J$ at this point).

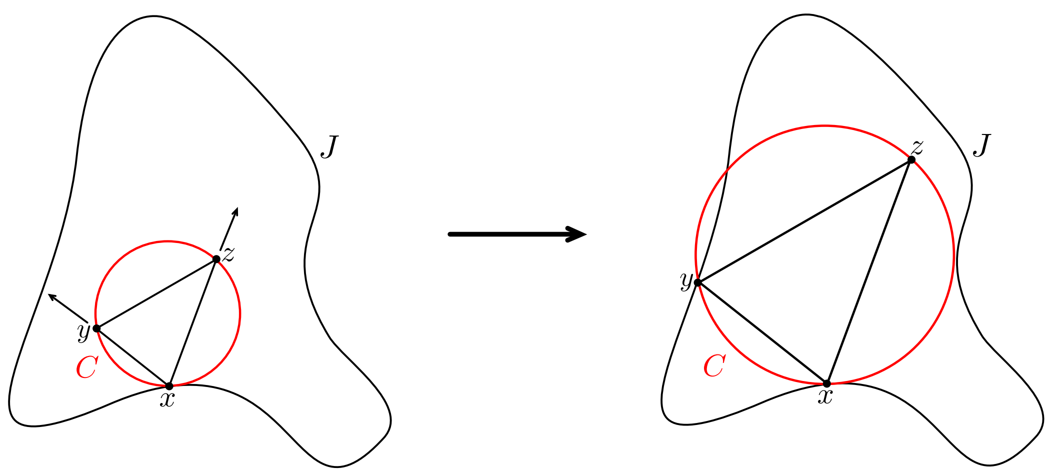



Figure 4.1

Let $p$ and $q$ be maximally distant points on $J$. We first move $x$ to $p$ continuously, while keeping $y$ in place and moving $z$ in a way that keeps $\triangle xyz$ similar to $\triangle abc$. We then move $y$ to $q$ similarly, keeping $x = p$ fixed.

At this point, $z$ must lie on or outside $J$, since $x = p$ and $y = q$ are maximally distant points on $J$ and $yz$ is the longest side of the triangle, thus the argument from the proof of Theorem 1.1 follows.

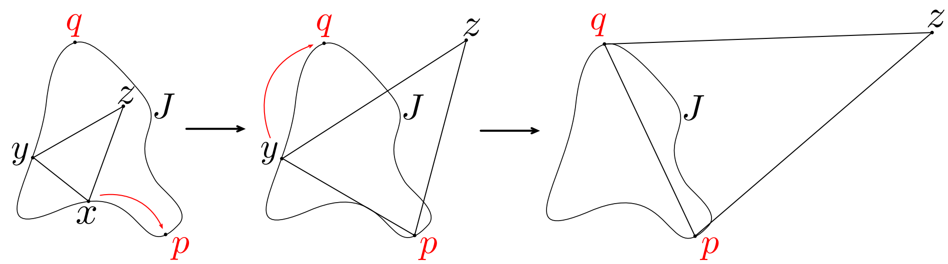



Figure 4.2

□

## 4.1 Nielsen's theorem

Although the proof is a lot more complex, Mark J. Nielsen [3] showed that a much stronger version of Meyerson's theorem is true.

Theorem 4.2 (Nielsen's theorem). *Let $J$ be a planar Jordan curve. Let $V$ be the set of points on $J$ that correspond to angle $\theta$ on a triangle similar to $\triangle abc$ inscribed in $J$. Then $V$ is dense in $J$.*

*Proof.* Before we can prove the theorem, we shall introduce some notations. Let $\delta \geq 1$ be the ratio of the sides of $\triangle abc$ that are adjacent to the angle $\theta$. Let $\varphi_x : \mathbb{R}^2 \to \mathbb{R}^2$ be a family of similarity transformations (scaling rotations about $x$) that are a composition of a rotation with angle $\theta$ about $x$ and a magnification by $\delta$ such that $\varphi_x(x) = x$ for every $x \in \mathbb{R}^2$.

Assume $x$ lies on $J$, then it is easily verified that for every $y \in J - x$ : $\triangle xy\varphi_x(y)$ is similar to $\triangle abc$ with the angle at $x$ corresponding to angle $\theta$.

We say that $x \in J$ is a flat point of $J$ if and only if there exists a bounding pair $(D, L)$, where $D$ is a nontrivial disk and $L$ is an open line segment such that $x \in D \cap L$, $D \subset$ cl$(U)$ and $L \cap U = \emptyset$, Where $U$ is a component of $\mathbb{R}^2 - J$ and cl$(U)$ is the closure of $U$.

Lemma 4.3. *If $x$ is a flat point of $J$ then $x \in V$.*

*Proof.* Assume $x$ is a flat point of $J$. Let $(D, L)$ be its bounding pair on $J$, thus $(\varphi_x(D), \varphi_x(L))$ is a bounding pair of $x$ on $\varphi_x(J)$. Let $S$ be a square with one side on $\varphi_x(L)$, such that $S$ meets the interior of $\varphi_x(D)$, one side of $S$ is in $\varphi_x(L)$ with $x$ being the midpoint of this side, and $S$ is small enough, that $S - \varphi_x(D)$ has two components: one inside $J$ and the other outside $J$.

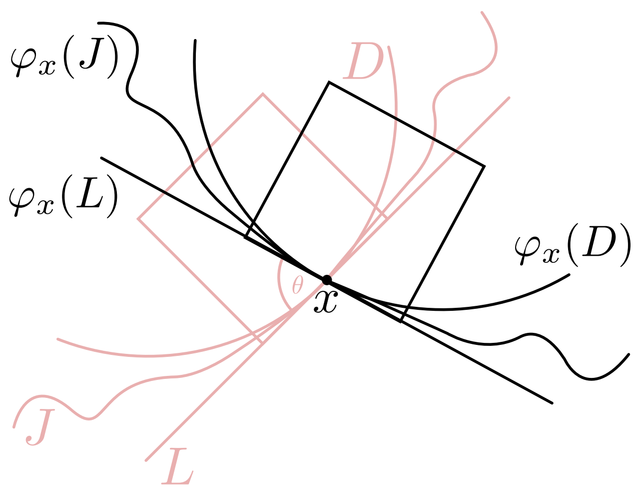



Figure 4.3

We can see that $\varphi_x(J)$ meets both components of $S - \varphi_x(D)$, meaning it meets both the inside and outside of $J$, thus it intersects $J$ at some point $y \neq x$. But then $\triangle x\varphi_x^{-1}(y)y$ is a triangle inscribed in $J$ similar to $\triangle abc$, with angle $\theta$ at $x$, so by definition $x \in V$. □

For our second lemma, we will define the set of accessible points:
Let $A = \{x \in J$ : there is a nontrivial disk $D$ in cl(In($J$)), such that $x \in D \cap J\}$. Here In(J) denotes the interior of $J$, similarly Ex($J$) will denote the exterior of $J$.

Lemma 4.4. *$A$ is dense in $J$.*

*Proof.* Let $D$ be an arbitrarily small open disk with center $x \in J$. Let $D'$ be a closed disk such that $D' \subset \text{In}(J) \cap D$. Move $D'$ toward $x$ in a straight line until it meets $J$ at some point $y$. Then by definition $y \in A$, meaning there are points of $A$ arbitrarily close to $x$. This is true for all $x \in J$, thus $A$ is in fact dense in $J$. □

It is easy to see that if a point $x$ is accessible, meaning we can place a disk in cl(In($J$)) meeting $J$ at $x$, then we can do the same with triangles of any shape. We will use this fact in our third lemma and the rest of the proof.

Lemma 4.5. *Let $[x, y]$ be a line segment in $cl(In(J))$ with $x, y \in J$ and at least one of which is in $A$. Then at least one of $\{x, y\}$ is in $V$.*

*Proof.* Assume $x \in A$. We then have three cases:

Case 1: $\varphi_x(y) \in J$.
In this case, we have $\triangle x\varphi_x(y)y$ similar to $\triangle abc$ inscribed in $J$.

Case 2: $\varphi_x(y) \in$ In($J$).
Let $z$ be a point on $J$ at maximal distance from $x$, then:

$$\|\varphi_x(z) - x\| = \delta\|z - x\| \geq \|z - x\|$$

Because of this, $\varphi_x(z)$ must lie on the exterior of $J$, but $\varphi_x(y) \in$ In($J$). In this case, there exists an arc of $\varphi_x(J)$ from $\varphi_x(y)$ to $\varphi_x(z)$ that intersects $J$ at some point $y'$ other than $x$. But then, similar to Lemma 3.3, we have $\triangle x\varphi_x^{-1}(y')y'$ inscribed in $J$ similar to $\triangle abc$, with angle $\theta$ at $x$. This implies $x \in V$.

Case 3: $\varphi_x(y) \in$ Ex($J$).
We will assume $y \notin V$, and show that this implies $x \in V$. Similar to case 2, $\varphi_y(J)$ intersects Ex($J$), thus $\varphi_y$(cl(In($J$)))$\cap$ Ex($J$) $\neq \emptyset$. But by the assumption $y \notin V$ we know that the set $\varphi_y(J) - y$ cannot intersect $J$, meaning it lies entirely in Ex($J$) and $J - y$ lies in Ex($\varphi_y(J)$).

Consider the similar triangles $T_1 = \triangle yx\varphi_y(x)$ and $T_2 = \triangle xy\varphi_x(y)$. We have $[x, y] \subset$ cl(In($J$)) and [$\varphi_y(x), y) = \varphi_y$[$x, y$) $\subset$ Ex($J$) meaning these two segments are separated in $T_1$ by $J$. But $J$ does not cross these segments, so there must be an arc $R \subset T_1$ of $J$ joining $y$ to a point on [$x, \varphi_y(x)$].

We now apply $\varphi_x$ to $T_1$. Note that the angle of $\varphi_x(T_1)$ at $\varphi_x(y)$ is not greater than the angle of $T_2$ at $\varphi_x(y)$, since it is equal to $\theta$, the smallest angle of the triangle.

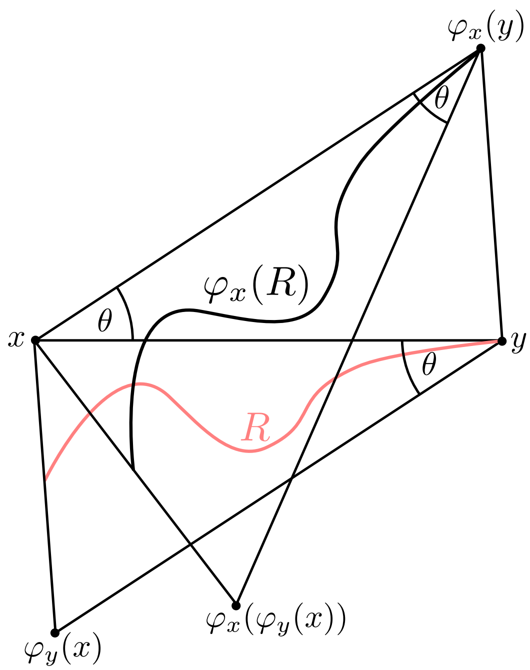



Figure 4.4

Note that $\|\varphi_x(\varphi_y(x)) - \varphi_x(y)\| = \delta^2\|x-y\| \geq \|x-y\|$. From this we can conclude that the side $[\varphi_x(\varphi_y(x)), \varphi_x(y)]$ of $\varphi_x(T_1)$ intersects $[x, y]$.

Our assumption was $\varphi_x(y) \in \text{Ex}(J)$, meaning $J$ must separate $[x, y]$ and $\varphi_x(y)$ in $T_2$. But as seen on figure 4.4, some subarc of $\varphi_x(R) \cap T_2$ connects $[x, y]$ and $\varphi_x(y)$, implying $J \cap \varphi_x(R) \cap T_2 \neq \emptyset$.

By the assumption $x \in A$ it can easily be shown that $J \cap \varphi_x(R) \cap T_2$ cannot be the singleton set $x$, meaning $J \cap \varphi_x(J)$ contains a point other than $x$, thus for the same reason as in case 2, this implies $x \in V$.

$\square$

We can now prove the theorem, by showing that any arbitrary open set $N$ that intersects $J$ also intersects $V$. Since $J$ is locally connected, we can assume $N \cap J$ is also connected. Let $D$ be a closed disk in $N$, with its interior intersecting $J$. Let $H = D \cap$ cl(Ex($J$)). We now have two cases:

Case 1: $H$ is convex.
If $H$ is convex, then a portion of its boundary is an arc of $J$, meaning it would contain flat points of $J$, which by Lemma 3.3 are in $V$, thus we have $N \cap V \neq \emptyset$.

Case 2: $H$ is not convex.
If $H$ is not convex, then there are points $z_1, z_2 \in H$ such that $[z_1, z_2]$ intersects In($J$). Let $x_1, x_2$ be points in $[z_1, z_2] \cap J$ such that the line segment $(x_1, x_2)$ is in the interior of $J$. Let $R \subset N$ be an arc of $J$ from $x_1$ to $x_2$ (such arc exists by the assumption that $N \cap J$ is connected). We now have the Jordan curve $J' = R \cup [x_1, x_2]$ with In($J'$) $\subset$ In($J$). Let $B \in$ In($J'$) be a closed nontrivial disk. Move $B$ parallel to $[x_1, x_2]$ in one direction until it meets $R$ at some point $a$. Note that $a \in A$ by definition. Now move $B$ parallel to $[x_1, x_2]$ in the other direction, until the point corresponding to $a$ meets $J$ at some point $b$. We now have a line segment $[a, b] \in$ cl(In($J'$)) $\subset$ cl(In($J$)), with $a \in A$, so by Lemma 3.5 at least one of $a$ and $b$ is in $V$, meaning $N \cap V \neq \emptyset$.

$\square$

Nielsen noted that something like Theorem 3.6 is likely true for arbitrary triangles. Proving this would strengthen the theorem considerably.

# 5 Triangles inscribed in higher dimensional curves

At first look, one might hope that our results so far would generalize well to simple closed curves embedded in $n$-dimensional Euclidean spaces. This is not the case though, since most of our proofs so far rely heavily on the Jordan curve theorem, which we cannot utilize here, as it is not true that a simple closed curve divides the space into two components. This makes achieving meaningful results considerably harder, and in many instances, it can only be accomplished for special classes of curves.

## 5.1 Some notes on triods

Recall our main result from section 3.1:

> *Every triod $T$ has a leg $L_i$, so that every point $x \in L_i - z$ is a vertex of some equilateral triangle on $T$.*

The proof of this theorem does not use the Jordan curve theorem, nor does it rely at all on the triod lying in the plane, meaning it is true for triods in higher dimensional Euclidean spaces. Despite this, the theorem does not seem to be useful when it comes to simple closed curves. It does however give us a somewhat trivial but interesting result when it comes to higher dimensional manifolds.[2]

**Theorem 5.1.** *Let $M$ be a connected $k$-manifold embedded in $n$-dimensional Euclidean space. If $2 \leq k < n$ then at most two points of $M$ are not the vertices of some equilateral triangle inscribed in $M$.*

*Proof.* Assume $x_1, x_2, x_3 \in M$ are not vertices of some equilateral triangle inscribed in $M$. Since $M$ is a connected $k$-manifold with $k \geq 2$, it is easy to show that there exists some triod $T \subset M$ with endpoints $x_1, x_2, x_3$. But our main theorem on triods implies that one of the endpoints has to be a vertex point, thus we have our contradiction. $\square$

## 5.2 Triangles in $\mathbb{R}^n$

All of our previous discussions have centered around findings from the 20th century. However, in this section, we will look at a fairly recent result from 2021 by A. Gupta and S. Rubinstein-Salzedo [16].

Let $\gamma : [0,1] \to \mathbb{R}^n$ be a parametrization of the simple closed curve $J$.

The functions

$$F_\delta^x(y,z) : (x, x+\delta) \times (x, x+\delta) \to [0, \infty)$$

and

$$G_\delta^x(y,z) : (x-\delta, x) \times (x, x+\delta) \to [0, \infty)$$

will denote the functions that measure the angles between $\gamma(x)\gamma(y)$ and $\gamma(x)\gamma(z)$ respectively.

$\triangle abc$ will denote an arbitrary triangle with vertices $a, b, c$ and angles $\theta_1 \leq \theta_2 \leq \theta_3$.

Theorem 5.2. *If* $\limsup_{\delta \to 0^+} F_\delta^x < \theta_i < \liminf_{\delta \to 0^+} G_\delta^x$ *for some* $i \in \{1,2,3\}$ *and* $x \in (0,1)$ *then* $\gamma(x)$ *is a vertex of some triangle similar to* $\triangle abc$ *inscribed in* $J$ *with angle* $\theta_i$ *at* $\gamma(x)$*.*

Although the proof of this theorem is not within the realm of this thesis, assuming smoothness greatly simplifies the proof while still retaining the fundamental concepts.

Theorem 5.3. *Let* $J$ *be a smooth simple closed curve in* $\mathbb{R}^n$*. Then every point of* $J$ *is the vertex of a triangle similar to* $\triangle abc$*.*

*Proof.* We define $T(x,y)$ as the set of points $z$ for which $\triangle xyz$ is similar to $\triangle abc$ with angles at $x, y, z$ corresponding to angles at $a, b, c$ respectively.

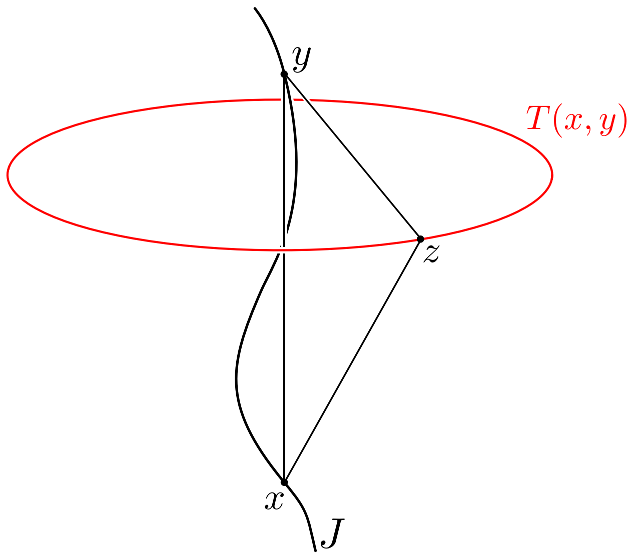



Figure 5.1: $T(x, y)$ in $\mathbb{R}^3$

With $x = \gamma(s)$, for every $t \in [0, 1]$ we will define the scaled isometry $I_t$ that maps $T(\gamma(s), \gamma(t))$ to the $(n-2)$-sphere $S = \{z \in \mathbb{R}^{n-1} : \|z\| = 1\}$.

To prove the theorem we need to show that for any $x \in J$ there exists a point $y \in J - x$ such that $T(x, y) \cap J \neq \emptyset$. Equivalently, for any $s \in [0, 1]$ there exists $t \in [0, 1] - s$ such that $I_t(\gamma) \cap S \neq \emptyset$.

Lemma 5.4. *Assume there is a point $\gamma(s)$ for which no such $t$ exists, then for any $t_1, t_2 \in [0, 1] - s$ we have $I_{t_1}(\gamma) \simeq I_{t_2}(\gamma)$ in $\mathbb{R}^n - S$.*

*Proof.* This can be verified with the homotopy $H_{t_1,t_2}(\cdot, T) = I_{(1-T)t_2+(T)t_1}(\gamma)$. □

We now assume there exists some $s \in [0, 1)$ that satisfies the conditions of lemma 5.4. We choose $t_1$ such that $\|\gamma(t_1) - \gamma(s)\|$ is maximal. Then $I_{t_2}(\gamma)$ is homotopic in $\mathbb{R}^n - S$ to the constant path at $\gamma(s)$ by straight-line homotopy.

Since $J$ is smooth, there exists some $\delta > 0$ such that $|\mathbb{R}^{n-1} \cap I_{s+\delta}(\gamma)| \geq 2$, but the only point in int($S$)$\cap I_{s+\delta}(\gamma)$ is $I_{s+\delta}(\gamma(s + \delta))$, where int($S$) is the interior of $S$ when taking $\mathbb{R}^{n-1}$ as its ambient space. This means there are parts of $I_{s+\delta}(\gamma)$ in both the interior and the exterior of $S$, thus $I_{s+\delta}(\gamma)$ is not homotopic in $\mathbb{R}^n - S$ to the constant path at $\gamma(s)$, meaning $I_{s+\delta}(\gamma) \not\simeq I_{t_1}(\gamma)$ in $\mathbb{R}^n - S$. This contradicts Lemma 5.4. $\square$

# 6 Quadrilaterals

While the inscribed square problem in its general form remains unsolved, there are certain special cases that can be solved with relative ease. In this section, we will explore these particular cases.

## 6.1 Symmetric curves

V. Klee and M. Wagon [5] showed that if a Jordan curve is symmetric about a point, then it inscribes rectangles of all types.

Here $Q$ will denote an arbitrary rectangle with $\theta$ denoting the greater angle between the diagonals.

Theorem 6.1. *Let $J$ be a Jordan curve in the plane that is symmetric about a point $z$. Then $J$ contains the vertices of some rectangle similar to $Q$.*

*Proof.* We can assume $z$ is the origin. Let $x$ be a point closest to $0$. Let $L$ be the line segment $[w, -w]$ that separates $J$ into two arcs: $A$ and $A'$. Consider the Jordan curve $J' = A \cup L$. Let $\varphi$ be a rotation about the origin with angle $\vartheta$. Since $L$ and $\varphi(L)$ only meet at the origin, the Jordan curves $J'$ and $\varphi(J')$ must intersect at some other point $y \in A \cap \varphi(A)$ and so we have our rectangle with vertices $\{y, \varphi^{-1}(y), -y, \varphi^{-1}(-y)\} \in J$. $\square$

M. J. Nielsen and S. E. Wright showed that a similar argument applies to Jordan curves with line symmetry. [6]

Theorem 6.2. *Let $J$ be a Jordan curve in the plane that is symmetric through a line. Let $Q$ be any rectangle. Then $J$ contains the vertices of some rectangle similar to $Q$.*

*Proof.* Without loss of generality, we can make the assumption that $J$ is symmetric through the $x$-axis, intersecting it at $(-1, 0)$ and $(1, 0)$. Then $(x, y) \in J$ implies $\pi(x, y) = (x, -y) \in J$. The origin lies in the inside of $J$, so there is a rectangle similar to $Q$ with vertices $\{(-a,0),(a,0),(-a,b),(a,b)\}$. Let $L$ be the line containing $(-a,b)$ and $(a,b)$, then $L$ intersects $J$ at two points: $p_1=(x_1,b)$ and $p_2=(x_2,b)$. Assume $x_1<0$ and $x_2>0$. Let $V$ be the union of the line segments connecting the origin to $p_1$ and $p_2$. Let $J'$ be the Jordan curve consisting of $V$ and the arc $A$ of $J$ from $p_1$ to $p_2$. Let $\varphi$ be the homeomorphism defined by $\varphi(x,y)=(x+y(\frac{2a}{b}),y)$. It is easy to see that $V$ and $\varphi(V)$ only meet at the origin, thus by the Jordan curve theorem $J'$ must meet $\varphi(J')$ at some point $z\in\varphi(A)$. The nature of $\varphi$ is such that the rectangle with vertices $\{z,\varphi^{-1}(z),\pi(z),\pi(\varphi^{-1}(z))\}\subset J$ is similar to $Q$. $\square$

## 6.2 Inscribed rectangles

The most natural way to relax the inscribed square problem is to ask whether every Jordan curve inscribes a rectangle. H. Vaughan proved this to be true in his lecture *Rectangles and simple closed curves* which M. D. Meyerson [7] later summarized.

Theorem 6.3. *Let $J$ be a Jordan curve in the plane. Then $J$ contains the vertices of a rectangle.*

*Proof.* In order to prove the theorem, we will prove that there exist two distinct pairs of points on $J$ which are the same distance apart from each other pairwise and their midpoint (which corresponds to the midpoint of the rectangle) is the same.

Notice that since $J$ is topologically equivalent to a circle, the set of pairs of points on our curve: $J\times J$ is topologically equivalent to a torus. To not allow degenerate rectangles, we must associate pairs $(x,y)\in J\times J$ with $(y,x)$. With some topological arguing, we can see that the surface $S$ we get is a Möbius strip.

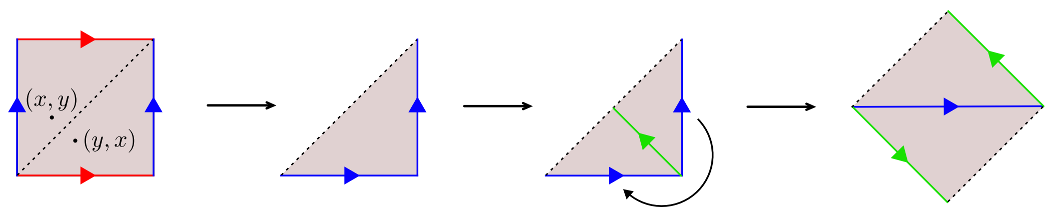



Figure 6.1

We now define the function $f : S \to \mathbb{R}^3$ which maps each pair of points on $J$ above the midpoint of the two points with the $z$-coordinate equal to the distance between them. Notice that $f$ is continuous and for all $x \in J$ we have $f(x, x) = x$.

Assume $f$ is an injection. This means that $f(S)$ is a Möbius strip in the half-space $x \geq 0$ and meets the plane $x = 0$ at its boundary. $f(S)\cup$In($J$) is then the topological equivalent of the projective plane $\mathbb{R}P^2$ embedded in $\mathbb{R}^3$, which is a contradiction, thus there must be two distinct pairs of points on $J$ that map to the same point.

□

Unfortunately, the same cannot be said about simple closed curves in higher dimensional curves, since as Meyerson noted, it can easily be shown that there are simple closed curves in $\mathbb{R}^3$ that do not have an inscribed rectangle.

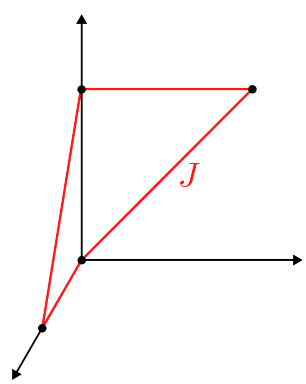



Figure 6.2: $J$ does not contain the vertices of a rectangle

## 6.3 Inscribed rhombi

Our next question is whether every Jordan curve inscribes a rhombus. M. J. Nielsen [17] showed this to be the case using the well-known theorem known as the mountain climbing problem, first proved by T. Homma [8], which states that two climbers can scale a two-dimensional mountain starting at opposite sides while always staying at the same height.

For the sake of our theorem, it is su□cient to consider the piecewise linear version of the problem.

Theorem 6.4 (Polygonal mountain climbing problem [18]). *Let* $f, g\colon [0,1] \to [0,1]$ *be piecewise linear continuous functions with* $f(0) = g(0) = 0$ *and* $f(1) = g(1) = 1$*. Then there exist piecewise linear continuous functions* $r, s\colon [0,1] \to [0,1]$ *with* $r(0) = s(0) = 0$ *and* $r(1) = s(1) = 1$ *such that* $f \circ r = g \circ s$*.*

This powerful tool gives us a much stronger result compared to what our original question proposes:

Theorem 6.5. *Let* $J$ *be a polygonal Jordan curve and* $L$ *be any line in the plane. Then* $J$ *inscribes a rhombus with two sides parallel to* $L$*.*

*Proof.* Without loss of generality, we can assume $L$ is parallel to the $x$-axis and $z_{min}$, $z_{max}$ are points on $J$ with minimal and maximal $y$-coordinates $0$ and $1$ respectively. Then $z_{min}$ and $z_{max}$ split $J$ into two arcs: $A_1$ and $A_2$. The theorem can be restated as follows:

*There exists piecewise-linear continuous functions* $x_1$*,* $x_2$*,* $y : [0,1] \to \mathbb{R}$ *such that* $(x_i(0), y(0)) = z_{min}$*,* $(x_i(1), y(1)) = z_{max}$ *and* $(x_i(t), y(t)) \in A_i$ *for all* $t \in [0,1]$ *and* $i \in \{1,2\}$*.*

Let $\gamma_i(t) = (u_i(t), v_i(t))$ be parametrizations of $A_i$ from $z_{min}$ to $z_{max}$ for $i = 1, 2$. Since $v_1$ and $v_2$ are piecewise linear and continuous, by theorem 6.4 there exists $r, s\colon [0,1] \to \mathbb{R}$ piecewise-linear continuous functions $r, s$ with

$r(0) = s(0) = 0$ and $r(1) = s(1) = 1$ such that $v_1 \circ r = v_2 \circ s$. And thus we have our functions:

$$x_1(t) = u_1(r(t)),$$
$$x_2(t) = u_2(s(t)),$$
$$y(t) = v_1(r(t)).$$

This essentially concludes our proof, although some explanation may be needed.

Let $\delta(t)$ denote the distance between $x_1(t)$ and $x_2(t)$ and let $t^* \in (0,1)$ be such that $\delta(t^*)$ is maximal. Then $\delta(0) = \delta(1) = 0$ and since $x_1(t)$ and $x_2(t)$ are continuous, $\delta$ is also continuous, thus for all $d \in (0, \delta(t^*))$ there exists $0 < t_1^d < t^* < t_2^d < 1$ such that $\delta(t_1^d) = \delta(t_2^d) = d$. Let $R(d)$ denote the ratio between the sides of the parallelogram with vertices: $\{(x_1(t_1^d), y(t_1^d)),$ $(x_2(t_1^d), y(t_1^d)), (x_1(t_2^d), y(t_2^d)), (x_2(t_2^d), y(t_2^d))\}$. Note that $R$ maps $(0, \delta(t^*))$ to $(0, \infty)$ continuously, thus for some $d' \in (0, \delta(t^*))$ we have $R(d') = 1$. And so we have our rhombus. $\square$

M. J. Nielsen showed that with some analysis it is possible to extend this result to arbitrary Jordan curves by approximating them with polygonal curves and taking the limit. Additionally, he theorized that since the orientation of the line $L$ is arbitrary, there could be a way to use this degree of freedom to find inscribed squares.

# 7 The inscribed square problem

In this section, we take a look at our main conjecture first formulated by O. Toeplitz [1]:

Conjecture 7.1 (The inscribed square problem). *Every planar Jordan curve $J$ contains the vertices of a square.*

The problem has been proven for certain special classes of curves, however, as of writing this thesis, the main conjecture remains unsolved.

Due to the technical nature of several proofs, this section shall serve as a compilation of results regarding the square peg problem.

## 7.1 Convex curves

A. Emch was among the first to study the inscribed square problem and solved it for convex curves. [9]

Theorem 7.2. *Let $J$ be a convex Jordan curve in the plane. Then $J$ contains the vertices of a square.*

Although his proof preceded theorem 6.5 and the formulation of the mountain climbing problem, he used a similar technique of examining an infinite family of rhombi to show that at least one of them is a square inscribed in $J$.

He later examined relaxing the criteria of convexity and obtained the following more general result [10]:

Theorem 7.3. *Let $J$ be a closed curve formed by a finite number of analytic arcs with a finite number of inflections and other singularities. Then $J$ contains the vertices of a square.*

## 7.2 Smooth curves

L. G. Schnirelmann proved the theorem to be true for $C^2$ curves [11]:

Theorem 7.4. *Let $J$ be a $C^2$-continuous closed curve. Then $J$ contains the vertices of a square.*

W. Stromquist [12] strengthened this theorem considerably, showing that it su□ces that $J$ is "locally monotone":

Theorem 7.5. *Let $J$ be a Jordan curve on which every point $x$ has a neighborhood $N_x$ and a direction $d_x$ such that no chord of $J$ is contained in $N_x$ and parallel to $d_x$. Then $J$ contains the vertices of a square.*

This covers $C^1$ curves, convex curves and polygons, as they each satisfy the above condition.

W. Stromquist also showed that theorem 7.5 can be extended to smooth curves in $\mathbb{R}^n$ if we do not require the vertices to lie in the same plane.

Theorem 7.6. *Let $J$ be a smooth simple closed curve in $\mathbb{R}^n$. Then $J$ contains the vertices of a quadrilateral with equal sides and equal diagonals.*

# 8 A different perspective on squares

Our last section is dedicated to Terrence Tao's 2017 paper [13] on the inscribed square problem, in which he theorized that counting intersections and other homological arguments (such as the ones explored in this thesis so far) are not sufficient to solve the inscribed square problem. He did not manage to solve the conjecture, however, his work included various new results and provided new ways of looking at the problem.

## 8.1 The periodic inscribed square problem

The main area of Tao's interests were some periodic versions of the inscribed square problem.

Conjecture 8.1 ((polygonal) periodic inscribed square problem). *Let $\gamma_1, \gamma_2$ be two disjoint (piecewise linear) simple closed curves in the cylinder $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$ which have a winding number of $\pm 1$. Then $\gamma_1 \cup \gamma_2$ contains the vertices of a square.*

Here winding number of $\pm 1$ means the curves can travel back and forth through periods as long as the difference between the number of travels in each direction is 1.

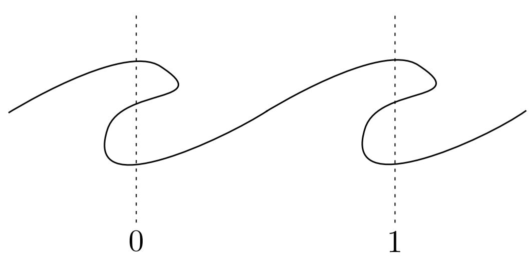



Figure 8.1: an example of a periodic curve with winding number 1

Tao's conjecture is open for both smooth and polygonal curves. However, proving any version of it would not directly resolve the inscribed square problem. Nevertheless, Tao did show however, that the inscribed square problem does imply the polygonal version of conjecture 8.1. He did this by transforming $\gamma_1 \cup \gamma_2$ into a Jordan curve $\gamma$ in a way such that for any square inscribed in $\gamma$, a corresponding square can be found inscribed in $\gamma_1 \cup \gamma_2$.

This explains Tao's interests, as finding a counterexample for the polygonal version of conjecture 8.1 would disprove the inscribed square problem. Tao noted that the chances of this happening are low, as he believes both conjectures are likely to be true.

## 8.2 Finding squares via integration

We start by defining the set of (non-degenerate) squares in the plane by their vertices.

Definition 8.2.

$$S=\{((x,y),(x+a,y+b),(x+a-b,y+a+b),(x-b,y+a))\in(\mathbb{R}^2)^4 : (a,b)\neq(0,0)\}$$

Using this notation we can see that a parametrized Jordan curve $\gamma$ inscribes a square if and only if $\gamma^4 \cap S \neq \emptyset$.

In order to make the problem more analytically approachable, we define the closure cl($S$), which includes degenerate squares (with $(a,b)=(0,0)$).

Tao's approach was to approximate $\gamma$ by a polygonal curve and then take limits. This proved di□cult in the general case, as the squares on polygonal approximations of $\gamma$ could shrink to a point when taking the limit. However, under certain assumptions, it does solve some cases not covered by previous results.

Theorem 8.3. *Let $\gamma$ be the union of two curves which are graphs of two Lipschitz functions $f,g:[t_0,t_1]\to\mathbb{R}$ that have Lipschitz constants less than one. Then $\gamma$ contains the vertices of a square.*

This case is not covered by the locally monotone theorem, as $\gamma$ is not monotone at the endpoints of $\gamma_1$ and $\gamma_2$.

*Proof.* Let $\gamma_1(t) = (x(t), y(t))$ parametrize the graph of $f$. By methods similar to those seen in theorem 3.1 we can show that for any point $(x(t), y(t))$ on $\gamma_1$ there is a unique pair $(a(t), b(t))$ for which $\gamma_2(t) = (x(t) + a(t), y(t) + b(t))$ lies in the graph of $f$ and $\gamma_4(t) = (x(t) - b(t), y(t) + a(t))$ lies in the graph of $g$.

We define the simple curve $\gamma_3(t) = (x(t) + a(t) - b(t), y(t) + a(t) + b(t))$. Notice that $\gamma_3$ has the same endpoints as the other three curves. If for some $t^* \in (t_0, t_1)$ the point $\gamma_3(t^*)$ lies on the graph of $g$ then we have our square: $(\gamma_1(t^*), \gamma_2(t^*), \gamma_3(t^*), \gamma_4(t^*)) \in \gamma^4 \cap S$.

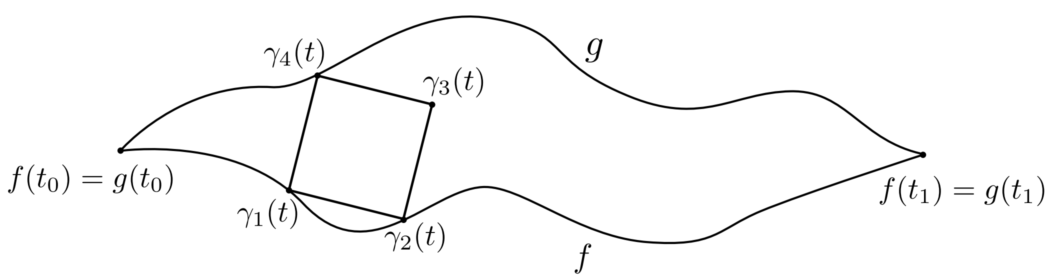



Figure 8.2

Assume no such $t^*$ exists. We now examine the following expression:

$$\int_{\gamma_1} y\,dx - \int_{\gamma_2} y\,dx + \int_{\gamma_3} y\,dx - \int_{\gamma_4} y\,dx$$

These integrals should be interpreted as Riemann–Stieltjes integrals:

$$\int_{\gamma_1} y\,dx = \int_{t_0}^{t_1} y(t)\,dx(t)$$

By using our parametrization, with some calculation almost everything cancels out and we can simplify our expression:

$$\int_{\gamma_1} y\,dx - \int_{\gamma_2} y\,dx + \int_{\gamma_3} y\,dx - \int_{\gamma_4} y\,dx =$$

$$= \int_{t_0}^{t_1} y(t)\,dx(t) - \int_{t_0}^{t_1} (y(t)+b(t))\,(dx(t)+da(t)) +$$

$$+ \int_{t_0}^{t_1} (y(t)+a(t)+b(t))\,(dx(t)+da(t)-db(t)) - \int_{t_0}^{t_1} (y(t)+a(t))\,(dx(t)-db(t)) =$$

$$= \int_{t_0}^{t_1} a(t)\,da(t) - \int_{t_0}^{t_1} b(t)\,db(t).$$

From the nature of the parametrization, we can conclude that $a$ and $b$ are Lipschitz continuous, thus the fundamental theorem of calculus applies:

$$\int_{t_0}^{t_1} a(t)\,da(t) - \int_{t_0}^{t_1} b(t)\,db(t) = \left(\frac{1}{2}a^2(t_1) - \frac{1}{2}a^2(t_0)\right) - \left(\frac{1}{2}b^2(t_1) - \frac{1}{2}b^2(t_0)\right).$$

It does not take much convincing to see that the square $(\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_4(t))$ shrinks to a point near $t_0$ and $t_1$, thus $a(t_0) = b(t_0) = a(t_1) = b(t_1) = 0$ and our expression must evaluate to zero.

Observe that $\gamma_1$ and $\gamma_2$ are different parametrizations of the same curve. This means the first two terms of our four-term expression must be equal. We now have:

$$\int_{\gamma_1} y\,dx - \int_{\gamma_2} y\,dx + \int_{\gamma_3} y\,dx - \int_{\gamma_4} y\,dx = 0$$

and

$$\int_{\gamma_1} y\,dx - \int_{\gamma_2} y\,dx = \int_{t_0}^{t_1} f(t)\,dt - \int_{t_0}^{t_1} f(t)\,dt = 0.$$

By subtraction we get

$$\int_{\gamma_3} y\,dx - \int_{\gamma_4} y\,dx = \int_{\gamma_3} y\,dx - \int_{t_0}^{t_1} g(t)\,dt = 0.$$

Since we assumed that $\gamma_3$ does not cross the graph of $g$, by the Jordan curve theorem the curve $\gamma_3 \cup \gamma_4$ must enclose some non-empty bounded region $\Omega$. By Stokes' theorem we have

$$\int_{\Omega} y\,dx = \int_{\gamma_3} y\,dx - \int_{\gamma_4} y\,dx = 0,$$

meaning the area of $\Omega$ is 0, which is a contradiction. $\square$

Remark. The proof relies on $\gamma_3$ being simple and the functions $a, b$ being Lipschitz continuous. Both of which can be proven with some analysis.

It is worth mentioning that the same method can be used to prove the periodic inscribed square problem in the case of $\gamma_1$ and $\gamma_2$ being graphs of Lipschitz functions with Lipschitz constant less than one.

## References


[1] Otto Toeplitz. “Über einige Aufgaben der Analysis situs”. German. In: *Verhandlungen der Schweizerischen Naturforschenden Gesellschaft in Solothurn* 4.197 (1911), pp. 29–30.

[2] Mark D. Meyerson. “Equilateral triangles and continuous curves”. English. In: *Fundam. Math.* 110 (1980), pp. 1–9. issn: 0016-2736. doi: 10.4064/fm-110-1-1-9.

[3] Mark J. Nielsen. “Triangles inscribed in simple closed curves”. English. In: *Geom. Dedicata* 43.3 (1992), pp. 291–297. issn: 0046-5755. doi: 10.1007/BF00151519.

[4] Aryaman Gupta and Simon Rubinstein-Salzedo. *Inscribed triangles of Jordan curves in* $\mathbb{R}^n$. 2021. arXiv: 2102.03953 [math.MG].

[5] Victor Klee and Stan Wagon. *Old and new unsolved problems in plane geometry and number theory*. English. Vol. 11. Dolciani Math. Expo. Washington, DC: Mathematical Association of America, 1991. isbn: 0-88385-315-9.

[6] Mark J. Nielsen and S. E. Wright. “Rectangles inscribed in symmetric continua”. English. In: *Geom. Dedicata* 56.3 (1995), pp. 285–297. issn: 0046-5755. doi: 10.1007/BF01263570.

[7] Mark D. Meyerson. “Balancing acts”. English. In: *Topol. Proc.* 6.1 (1982), pp. 59–75. issn: 0146-4124.

[8] Tatsuo Homma. “A theorem on continuous functions”. In: *Kodai Mathematical Seminar Reports* 4.1 (1952), pp. 13–16. doi: 10.2996/kmj/1138843207. url: https://doi.org/10.2996/kmj/1138843207.

[9] A. Emch. “Some properties of closed convex curves in a plane.” English. In: *Am. J. Math.* 35 (1913), pp. 407–412. issn: 0002-9327. doi: 10.2307/2370404.

[10] A. Emch. "On some proprieties of the medians of closed continuous curves formed by analytic arcs." English. In: *Am. J. Math.* 38 (1916), pp. 6–18. issn: 0002-9327. doi: 10.2307/2370541.

[11] Л. Г. Шнирельман. "О некоторых геометрических свойствах замкнутых кривых". Russian. In: *Усп. Мат. Наук* 10 (1944), pp. 34–44.

[12] Walter Stromquist. "Inscribed squares and square-like quadrilaterals in closed curves". In: *Mathematika* 36.2 (1989), pp. 187–197. doi: https://doi.org/10.1112/S0025579300013061. url: https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/S0025579300013061.

[13] Terence Tao. *An integration approach to the Toeplitz square peg problem*. 2017. arXiv: 1611.07441 [math.GN].

[14] Arthur N. Milgram. *Some metric topological invariants*. English. Rep. Math. Colloq., II. Ser. 5/6, 25-33 (1944). 1944.

[15] Richard Evan Schwartz. "Rectangles, curves, and Klein bottles". English. In: *Bull. Am. Math. Soc., New Ser.* 59.1 (2022), pp. 1–17. issn: 0273-0979. doi: 10.1090/bull/1755.

[16] Aryaman Gupta and Simon Rubinstein-Salzedo. *Inscribed triangles of Jordan curves in* $\mathbb{R}^n$. 2021. arXiv: 2102.03953 [math.MG].

[17] Mark J. Nielsen. "Rhombi inscribed in simple closed curves". English. In: *Geom. Dedicata* 54.3 (1995), pp. 245–254. issn: 0046-5755. doi: 10.1007/BF01265340.

[18] J. E. Goodman, J. Pach, and C. K. Yap. "Mountain Climbing, Ladder Moving, and the Ring-Width of a Polygon". In: *Am. Math. Monthly* 96.6 (June 1989), pp. 494–510. issn: 0002-9890. doi: 10.2307/2323971. url: https://doi.org/10.2307/2323971.

# NYILATKOZAT

**Név:** Apró János

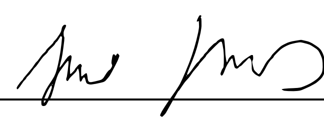
**ELTE Természettudományi Kar, szak:** Matematika BSc

**NEPTUN azonosító:** FJABH3

**Szakdolgozat címe:**
Triangles and Quadrilaterals Inscribed in Jordan Curves

A **szakdolgozat** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

Budapest, 2023. 06.04.



*a hallgató aláírása*