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Set theoretical aspects of Haar null and Haar meager sets

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Contents

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2 Introduction

In almost all branches of mathematics, there are notions of a phenomenon happening very often or very rarely. For example, one can think of the probability method in combinatorics or the notion of arithmetic density in number theory.

When we study subsets of $\mathbb{R}^n$, the generally accepted notions of smallness are sets with Lebesgue measure zero or being of first category in the sense of Baire. For both notions, the small sets form σ-ideals. One often handles these σ-ideals as the duals of each other, see for example the Erdős-Sierpiński duality theorem, which states that assuming the Continuum Hypothesis there is a bijection $f : \mathbb{R} \to \mathbb{R}$ such that X is meager if and only if $f(X)$ has Lebesgue measure zero and X has Lebesgue measure zero if and only if $f(X)$ is meager. Other examples that show the duality of the two systems are Fubini's theorem for measure and Kuratowski-Ulam theorem for category, or the existence of G_δ hull for measure and the existence of F_σ hull for category. These and other statements that show the analogy between category and measure can be found in [20].

The Haar measure is a well-known generalization of the Lebesgue measure for locally compact topological groups. For Polish groups that are not locally compact, there is no Haar measure, but J.P.R. Christensen introduced a generalization of sets with Haar measure zero in [6], namely the so-called *Haar null* sets.

Definition 2.1. Let G be a Polish group. We say that $X \subseteq G$ is *Haar null* if there is a Borel set $B \supseteq X$ and a Borel probability measure μ such that $\mu(gBh) = 0$ for all $g, h \in G$.

J.P.R. Christensen required in his original definition the hull B from the previous definition only to be universally measurable. When twenty years later B. Hunt, T. Sauer, and J. Yorke independently redefined Haar null sets (under the name "shy" sets), they already used the definition with Borel hulls. As it turned out, these notions do not coincide in many cases, so we handle them separately. We call the system of sets with universally measurable hulls *generalized Haar null* sets.

Since then many nice applications of Haar null and generalized Haar null sets have been found. For example, C. Rosendal proved in [23] that every universally measurable homomorphism from a Polish group to a separable topological group (in particular to a Polish group) is continuous. Another example is a generalization of the Rademacher theorem where the set of points where the function is non-differentiable turns out to be generalized Haar null (see [7]).

The notion of a set being meager remains meaningful in non-locally compact Polish groups, however, it is not a very good dual of Haar null sets. For example, there is no Fubini-type theorem for the Haar null sets which could be the dual of Kuratowski-Ulam theorem. Thus U. B. Darji introduced in [8] the concept of Haar meager sets.

The natural next step was to define generalized Haar meager sets. In order to do that, we need a category analog of the notion of universally measurable sets. M. Pálfy introduced the (in fact choose one, as there were many) notion of universally Baire sets.

The goal of the thesis is to examine the system of (generalized) Haar null and (generalized) Haar meager sets. In Sections 3 and 4 we will introduce some notations and preliminaries. Then in Section 5 we define Haar measure in locally compact Polish groups and list its basic properties that we will use later. Haar null and Haar meager sets already have relatively large literature, so in Section 6 first we will state and prove some of the most basic properties of them. Then we turn to more recent, set-theoretical results. It is a natural question whether two σ-ideals are isomorphic to each other. In Section 7 we introduce the definition of cardinal invariants, which are cardinals assigned to isomorphism classes of σ-ideals. Then in Section 8 we calculate these cardinal invariants for the systems of Haar null, generalized Haar null, Haar meager, and generalized Haar meager sets. It turns out that consistently the values of the cardinal invariants differ for these systems, therefore they are not isomorphic to each other. On the other hand, in Section 9 we prove that if we assume the Continuum Hypothesis, then there exists isomorphism between the Haar null and Haar meager sets of uncountable Polish groups, that is, consistently the σ-ideals of Haar null and Haar meager sets are isomorphic. We remark that the theorems from Section 8 related to generalized Haar meager sets and Section 9 are new results of M. Elekes, M. Pálfy and the author of the thesis, and will be published in an upcoming paper.

3 Notation

In this section, we summarize the notations that we will use later.

- for any set $B \subseteq X \times Y$, let B_x denote $\{y \in Y : (x,y) \in B\}$ and respectively, B^y denote $\{x \in X : (x,y) \in B\}$
- $\mathbb{1}_B$ denotes the characteristic function of B
- for any set X, $\overline{X}$ denotes the closure of X
- a system $\mathcal{I} \subseteq \mathcal{P}(X)$ is a σ-ideal if the following conditions hold:
  - $\emptyset \in \mathcal{I}$, $X \notin \mathcal{I}$,
  - if $I \in \mathcal{I}$ and $J \subseteq I$ then $J \in \mathcal{I}$,
  - if $I_1, I_2 \ldots \in \mathcal{I}$ then $\bigcup_{n\in\omega} I_n \in \mathcal{I}$
- we denote the symmetric difference by $\triangle$, that is, $A \triangle B = (A \setminus B) \cup (B \setminus A)$
- for any given set X, we denote with $\mathcal{P}(X)$ the power set of X
- in a metric space (X,d), we denote the ε-ball around the point x with $B(x,\varepsilon)$, that is, $B(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$
- for any topological space X, we denote with $\mathcal{B}(X)$ the Borel sets of X, that is, the σ-algebra generated by the open sets

- for any well-ordered set X, we denote with $\mathrm{otp}(X)$ the order type of X

4 Preliminaries

In this section, we present some well-known results that we will use later. We will not include most of the proofs of the theorems in this section, but we will give references to all of them. We assume that the reader is familiar with basic notions and definitions from topology, analysis, set theory, and group theory.

4.1 Polish groups

Definition 4.1 (Polish space)**.** We call a topological space (X, τ) *Polish* if it is completely metrizable and separable.

We will need some well-known facts about Polish spaces, so we list them here. The proof of them (and a very nice introduction to descriptive set theory) can be found in [17].

Theorem 4.2. *Countable products of Polish spaces (equipped with the product topology) are Polish. E.g.* $\mathbb{Z}^\omega$ *and* 2^ω *are Polish.*

Theorem 4.3. *A subspace of a Polish space is Polish if and only if it is* G_δ*.*

Definition 4.4 (Topological group)**.** We call a group G together with a topology τ a *topological group* if the multiplication $\cdot : G \times G \to G$ and the inverse map $^{-1} : G \to G$ are continuous (with respect to $\tau \times \tau$ and τ).

Definition 4.5 (Polish group)**.** We call a topological group G *Polish* if it is Polish as a topological space.

Most of the results of this paper are about Polish groups. Unless we state otherwise, any group G will always denote an arbitrary Polish group.

Remark 4.6. Polish groups are widely studied objects of descriptive set theory. A very nice introduction to the topic is [4].

An important subclass of Polish groups is the ones with two-sided invariant metrics.

Definition 4.7 (Groups with two-sided, left, right invariant metric)**.** Let G be a Polish group and let d be a compatible metric on it. We say that d is *two-sided invariant* if $d(t_1gt_2, t_1ht_2) = d(g, h)$ for all $g, h, t_1, t_2 \in G$. Respectively, if $d(tg, th) = d(g, h)$ or $d(gt, ht) = d(g, h)$ for all t, g, h then it admits a left or right invariant metric.

Theorem 4.8. *Every Polish group admits a compatible left (respectively, right) invariant metric, but it may not be complete. If it admits a two-sided invariant metric, then it is complete. Thus every abelian Polish group has a two-sided invariant metric.*

4.2 Well-known classes of subsets of Polish spaces

Definition 4.9 (Borel measure, measurable set)**.** Let X be a Polish space. We say that μ is a *Borel measure* if it is a measure that is defined (only) on the Borel subsets of X. We say that $A \subseteq X$ is measurable with respect to the completion of μ if there are Borel sets B, N such that $B \subseteq A$, $\mu(N) = 0$, and $A \setminus B \subseteq N$.

Definition 4.10 (Analytic, coanalytic set)**.** Let X be a Polish space, then we call a set $A \subseteq X$ *analytic* if there is a Polish space Y, a continuous function $f : Y \to X$ and a Borel set $B \subseteq Y$ such that $f(B) = A$. We call a set $A \subseteq X$ *coanalytic* if $X \setminus A$ is analytic.

Notation 4.11. We denote the system of analytic sets in X with $\mathbf{\Sigma}^1_1(X)$ or simply $\mathbf{\Sigma}^1_1$ if there is no risk of misunderstanding. Similarly, we denote the system of coanalytic sets in X with $\mathbf{\Pi}^1_1(X)$ or simply $\mathbf{\Pi}^1_1$.

Definition 4.12 (Meager set, set of first category)**.** Let X be a Polish space. Then a set $M \subseteq X$ is called *meager* or a set of *first category* if it is a countable union of nowhere dense sets. We denote by $\mathcal{M}(X)$ the system of meager sets.

Lemma 4.13. *For every meager set M there is a Borel meager set $B \supseteq M$.*

Proof. Let $M = \bigcup_{n\in\omega} M_n$, where M_n is nowhere dense for every $n \in \omega$. Then $\overline{M}_n$ is closed and nowhere dense, and thus $B = \bigcup_{n\in\omega} \overline{M}_n$ satisfies the lemma. □

Definition 4.14 (Set of second category)**.** Let X be a Polish space. Then a set $S \subseteq X$ is a set of *second category* if it is not meager.

Definition 4.15 (Comeager set)**.** Let X be a Polish space. Then a set $U \subseteq X$ is called *comeager* if $X \setminus U$ is meager.

A very widely used theorem is the following:

Theorem 4.16 (Baire category theorem)**.** *In a Polish space, every nonempty open set is of second category.*

Corollary 4.17. *In a Polish space X, the system of meager sets $\mathcal{M}(X)$ forms a σ-ideal.*

Definition 4.18 (Universally measurable set)**.** Let X be a Polish space and $A \subseteq X$. We call A *universally measurable* if it is measurable with respect to the completion of every Borel probability measure μ.

Definition 4.19 (Universally null set)**.** Let X be a Polish space and $A \subseteq X$. We call A *universally null* if it has zero measure with respect to the completion of every continuous Borel probability measure μ.

Remark 4.20. Every $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ set is universally measurable (see [17, Theorem 21.10]), thus Borel sets are universally measurable. Furthermore, every universally null set is universally measurable.

Proposition 4.21. *Let X, Y be Polish spaces and $f : X \to Y$ a continuous function. If $U \subseteq Y$ is universally measurable, then $f^{-1}(U)$ is universally measurable.*

Proof. Let us take any continuous Borel probability measure μ on X. Let $f_*(\mu)$ be the pushforward measure of μ by f, that is, $f_*(\mu)(B) = \mu(f^{-1}(B))$. It is clearly a Borel probability measure on Y, thus U can be written as a union $B \cup N$, where B is Borel and N has $f_*(\mu)$-measure zero. We can write $f^{-1}(U) = f^{-1}(B) \cup f^{-1}(N)$, where $f^{-1}(B)$ is Borel and $\mu(f^{-1}(N)) = 0$ by definition. Thus $f^{-1}(U)$ is μ-measurable, and the proposition follows. $\square$

The following notion is the category analog of measurability.

Definition 4.22 (Property of Baire)**.** Let X be a Polish space and $A \subseteq X$. We say that A has the *property of Baire* (or the *Baire property*) if it differs from an open set by a meager set. That is, there exists an open set U and a meager set M such that $A = U \triangle M$.

Remark 4.23. It is easy to prove that the sets with the property of Baire form a σ-algebra. Thus B has the property of Baire for every Borel set B. Moreover, every $\boldsymbol{\Sigma}^1_1$ and $\boldsymbol{\Pi}^1_1$ set has the Baire property (see [17, Corollary 29.14]).

We will need the following well-known Kuratowski-Ulam theorem (for a proof see [17, Theorem 8.41]):

Theorem 4.24 (Kuratowski–Ulam)**.** *Let X and Y be Polish spaces, and let $B \subseteq X \times Y$ be a set with the Baire property. Then the following are equivalent:*

- *$B \in \mathcal{M}(X \times Y)$, that is, B is meager in $X \times Y$,*
- *$\{x \in X : B_x$ is meager in $Y\}$ is comeager in X ,*
- *$\{y \in Y : B^y$ is meager in $X\}$ is comeager in Y.*

Notice that the sets with the property of Baire are often handled as the dual of measurable sets. So it is natural to define the dual of universally measurable sets, that is, universally Baire sets. Recently, M. Pálfy in his MSc thesis [21] collected various possible definitions and examined their relations to each other. It turns out that the following is the most natural:

Definition 4.25 (Universally Baire set)**.** Let X be a Polish space and $A \subseteq X$. The set A is called *universally Baire* if $f^{-1}(A)$ has the property of Baire in Y for any Polish space Y and continuous function $f : Y \to X$.

Remark 4.26. Since the preimage of any $\boldsymbol{\Sigma}^1_1$ set is $\boldsymbol{\Sigma}^1_1$, by Remark 4.23 we get that every $\boldsymbol{\Sigma}^1_1$ set is universally Baire (and the same holds for $\boldsymbol{\Pi}^1_1$ sets). Thus every Borel set is universally Baire.

Definition 4.27 (Nowhere locally constant function)**.** Let X and Y be Polish spaces. Then we call a continuous function $f : X \to Y$ *nowhere locally constant*, if for every $y \in Y$ the set $f^{-1}(y)$ has empty interior.

Definition 4.28 (Universally meager set)**.** Let X be a Polish space. Then the set $M \subseteq X$ is called *universally meager* if for every Polish space Y and for every continuous nowhere locally constant function $f : Y \to X$ the set $f^{-1}(M)$ is meager.

Remark 4.29. If we only consider every nowhere locally constant continuous function $f : \mathbb{N}^\omega \to X$ in the previous definition, we get an equivalent notion.

Remark 4.30. Let X be a Polish space. Then every universally meager set $M \subseteq X$ is universally Baire.

5 Haar measure

In this section, we define and list some of the basic properties of the Haar measure that we will use later. An introduction to the topic and proofs of the theorems that we mention can be found in [16, Chapter 6].

The Lebesgue measure is the unique measure defined on the Borel subsets of $\mathbb{R}^n$ that is translation invariant and the unit cube has measure 1. It is natural to generalize it for arbitrary topological groups, thus we get the following definition.

Definition 5.1 (Left Haar measure)**.** Let G be an arbitrary topological group. We call a measure $\mu : \mathcal{B}(G) \to [0, \infty)$ a (left) Haar measure, if the following properties hold:

1) μ is left translation invariant, that is, $\mu(gB) = \mu(B)$ for any $g \in G$ and $B \in \mathcal{B}(G)$,
2) $\mu(C) < \infty$ for every compact set $C \subseteq G$,
3) $\mu(U) > 0$ for every nonempty open set $U \subseteq G$,
4) $\mu(U) = \sup\{\mu(C) : C \subseteq U$, C is compact$\}$ for every nonempty open set $U \subseteq G$,
5) $\mu(B) = \inf\{\mu(U) : B \subseteq U$, U is open$\}$ for every Borel set $B \subseteq G$.

Remark 5.2. The Lebesgue measure is a left Haar measure in $\mathbb{R}^n$. One could also define right Haar measure instead of left Haar measure if in 1) instead of $\mu(gB) = \mu(B)$, we require $\mu(Bg) = \mu(B)$ for all $g \in G$ and $B \subseteq G$ Borel. It is easy to check that if μ is left Haar measure then $\nu(B) := \mu(B^{-1})$ is right Haar measure.

Remark 5.3. Many alternative (and sometimes equivalent) definitions of the left Haar measure emerge in the literature, for example, we get the same definition if we require instead of 3) that there exists a compact set with positive Haar measure.

Theorem 5.4 (Haar, Weil)**.** *Let G be a topological group. Then there exists a left Haar measure (or equivalently, a right Haar measure) on G if and only if G is locally compact, and if there exists one, then it is unique up to a multiplicative constant.*

Theorem 5.5. *Let G be a locally compact topological group and μ and ν be left and respectively, right Haar measures. Then the system of sets with left Haar measure zero is well-defined and coincides with the system of sets with right Haar measure zero. We denote by $\mathcal{N}(G)$ the system of sets with left (or equivalently right) Haar measure zero.*

6 Haar null and Haar meager sets

In this part of the thesis, we will introduce the definition of Haar null, Haar meager, generalized Haar null, and generalized Haar meager sets and prove some basic facts about them. Most of the results of this section can be found in [11], while others in [21].

6.1 Haar null and generalized Haar null sets

Definition 6.1 (Haar null set)**.** Let G be a Polish group. We say that $X \subseteq G$ is *Haar null* if there is a Borel set $B \supseteq X$ and a Borel probability measure μ such that $\mu(gBh) = 0$ for all $g, h \in G$. We denote by $\mathcal{HN}(G)$ the system of Haar null subsets of G.

Remark 6.2. We call the measure μ and B from the previous definition the *witness measure* of X and the *Borel hull* of X. Some authors (including Christensen) require universally measurable (see Definition 4.18) hull instead of Borel hull. Many proofs work for the system of sets with this alternative definition, but the two notions are not equivalent (see Corollary 6.6), so we will handle them separately.

Definition 6.3 (Generalized Haar null set)**.** Let G be a Polish group. We say that $X \subseteq G$ is *generalized Haar null* if there is a universally measurable set $A \supseteq X$ and a Borel probability measure μ such that $\mu(gAh) = 0$ for all $g, h \in G$. We denote by $\mathcal{GHN}(G)$ the system of generalized Haar null subsets of G.

We would like these definitions to extend the definition of Haar measure zero, so we have to examine groups in which Haar measure exists, that is, locally compact groups.

Theorem 6.4. *Let G be a locally compact Polish group. Then $\mathcal{GHN}(G) = \mathcal{HN}(G) = \mathcal{N}(G)$, that is, the systems of generalized Haar null sets, Haar null sets, and sets with Haar measure zero coincide.*

Proof. $\mathcal{N}(G) \subseteq \mathcal{HN}(G)$: As G is locally compact, it follows from Theorem 5.4 that there exists a left and a right Haar measure λ and λ'. Let us fix any set X which has left Haar measure zero. By Theorem 5.5 we know that $\lambda(X) = 0 \iff \lambda'(X) = 0$. As left and right Haar measures are regular (4) and 5) from Definition 5.1), there is a G_δ set B such that $X \subseteq B$, and $\lambda(B) = \lambda'(B) = 0$. Also from the regularity and 2) of Definition 5.1, there is a compact set C such that $0 < \lambda(C) < \infty$. Let us consider $\mu(Y) := \frac{\lambda(Y \cap C)}{\lambda(C)}$. It is clear that μ is a Borel probability measure. As $X \subseteq B$ and B is Borel, it is enough to show that $\mu(gBh) = 0$ for every $g, h \in G$. Using the translation invariance and Theorem 5.5 we get that $0 = \lambda'(B) = \lambda'(Bh) = \lambda(Bh) = \lambda(gBh)$, and thus $\mu(gBh) = \frac{\lambda(gBh \cap C)}{\lambda(C)} \leq \frac{\lambda(gBh)}{\lambda(C)} = 0$.

$\mathcal{HN}(G) \subseteq \mathcal{GHN}(G)$: This follows easily from Remark 4.20.

$\mathcal{GHN}(G) \subseteq \mathcal{N}(G)$: Let us take any $X \in \mathcal{GHN}(G)$. Without loss of generality, we can suppose that X is universally measurable. Let μ be a witness measure of X. We will use a convolution technique that is very common when one examines Haar null sets. Let $B := \{(x, y) \in G \times G : xy \in X\}$. As μ is a probability measure, λ is σ-finite and B is universally measurable (as it is a continuous preimage of a universally measurable set, see Proposition 4.21), we can apply Fubini's theorem:

$$(\mu \times \lambda)(B) = \int_{G\times G} \mathbb{1}_B \, \mathrm{d}(\mu \times \lambda) = \int_G \int_G \mathbb{1}_{B_x} \, \mathrm{d}\lambda(y) \, \mathrm{d}\mu(x) = \int_G \int_G \mathbb{1}_{B^y} \, \mathrm{d}\mu(x) \, \mathrm{d}\lambda(y) =$$

$$= \int_G \mu(\{x \in G : xy \in X\}) \, \mathrm{d}\lambda(y) = \int_G \mu(Xy^{-1}) \, \mathrm{d}\lambda(y) = 0.$$

The last equality follows from the fact that μ is a witness measure for X. But this means that for μ-almost all x the integral $\int_G \mathbb{1}_{B_x} \, \mathrm{d}\lambda(y) = \lambda(\{y \in G : xy \in X\}) = \lambda(x^{-1}X)$ is zero. Since λ is left Haar measure, from the translation invariance we can conclude that $\lambda(X) = 0$ and thus $X \in \mathcal{N}(G)$, which completes the proof. □

Now we have seen that the systems $\mathcal{GHN}(G)$ and $\mathcal{HN}(G)$ coincide in locally compact Polish groups. We also know from Remark 4.20 that $\mathcal{HN}(G) \subseteq \mathcal{GHN}(G)$ is always true. Thus it is natural to ask whether there are groups in which $\mathcal{HN}(G) \subsetneq \mathcal{GHN}(G)$ holds. The following theorem and corollary give an answer to this.

Theorem 6.5. *Let G be an abelian Polish group that is not locally compact. Then there is a coanalytic set $A \subseteq G$ and a Borel probability measure μ on G such that $\mu(gAh) = 0$ for any $g, h \in G$, but A is not Haar null.*

For a full proof see [14]. It is a quite long and complicated proof, therefore we do not reproduce it here. Notice however, that the set A from the previous theorem is generalized Haar null (because every coanalytic set is universally measurable, see Remark 4.20). Thus we have the following:

Corollary 6.6. *Let G be an abelian Polish group that is not locally compact. Then $\mathcal{HN}(G) \subsetneq \mathcal{GHN}(G)$.*

As we would like these definitions to be notions of smallness, it is natural to require that $\mathcal{HN}(G)$ and $\mathcal{GHN}(G)$ form σ-ideals (see Section 3). To prove this, we will need some technical lemmas.

Lemma 6.7. *Let μ be a Borel probability measure on a Polish group G and U a nonempty open set. Then there exists a compact set C and $g \in G$ such that $C \subseteq gU$ and $\mu(C) > 0$.*

Proof. From [17, Theorem 17.11] we know that there is a compact set C' such that $\mu(C') > 0$. Let $V \subseteq U$ be a nonempty open set such that $\overline{V} \subseteq U$. As C' is compact, there are finitely many elements $g_1, g_2 \dots g_n \in G$ such that $C' \subseteq \bigcup_{i=1}^n g_i V$ and thus $\mu(g_i V \cap C') > 0$ for some g_i. Notice that $C := g_i \overline{V} \cap C'$ and $g := g_i$ satisfies the lemma. □

Using Lemma 6.7, we can prove that for any nonempty open set U, the witness measure of any generalized Haar null set can have compact support contained in U.

Corollary 6.8. *Let μ be a Borel probability measure on G and let B be a universally measurable set such that $\mu(gBh) = 0$ for any $g, h \in G$. Then for any nonempty open set U there exists a Borel probability measure μ' such that $\mu'(gBh) = 0$ for any $g, h \in G$, and μ' has compact support contained in U.*

Proof. We use Lemma 6.7 for μ and U, thus we get a compact set C and $g \in G$ such that $C \subseteq gU$ and $0 < \mu(C)$. Let us define $\mu'(X) := \frac{\mu(C \cap gX)}{\mu(C)}$. Then μ' has support contained in $g^{-1}C \subseteq U$ and clearly satisfies the other required properties. □

The proof of Theorem 6.10 would be slightly less technical for groups that admit a complete left-invariant metric (see Definition 4.7). Since we would like to prove it for all Polish groups, we will need the following technical lemma (which is trivial in groups with complete left-invariant metric).

Lemma 6.9. *Let d be a metric on G that is compatible with the topology of G. Then for any compact set $C \subseteq G$ and any $\varepsilon > 0$ there is a nonempty open neighborhood U of 1_G such that $d(x, x \cdot u) < \varepsilon$ for any $x \in C$ and $u \in U$.*

Proof. From the continuity of the multiplication and the distance function, for any $x \in C$ there are nonempty open sets $U_x \ni 1_G, V_x \ni x$ such that $d(y, y \cdot u) < \varepsilon$ for any $u \in U_x$ and $y \in V_x$. Using that C is compact, there are $x_1, x_2 \dots x_n$ such that $C \subseteq \bigcup_{i=1}^n V_{x_i}$. It is easy to check that $U := \bigcap_{i=1}^n U_{x_i}$ satisfies the statement of the lemma. □

Theorem 6.10. *Let G be an arbitrary Polish group. Then the system of Haar null sets $\mathcal{HN}(G)$ and the system of generalized Haar null sets $\mathcal{GHN}(G)$ form σ-ideals.*

Proof. We will prove the theorem for the system $\mathcal{HN}(G)$, but if we replace "Borel set" with "universally measurable set" and "Haar null" with "generalized Haar null", it becomes a proof for the system $\mathcal{GHN}(G)$.

Clearly, the system $\mathcal{HN}(G)$ is closed under taking subsets. That is, if $X \in \mathcal{HN}(G)$ and $Y \subseteq X$ then $Y \in \mathcal{HN}(G)$ since the same Borel hull and witness measure will work. It is also easy to see that $\emptyset \in \mathcal{HN}(G)$ and $G \notin \mathcal{HN}(G)$.

To see that $\mathcal{HN}(G)$ is closed under countable union, let us fix $X_0, X_1 \dots \in \mathcal{HN}(G)$, and a compatible complete metric d on G. As these are Haar null sets, there are Borel hulls $B_0, B_1, \dots$ and witness measures $\mu_0, \mu_1 \dots$ such that $\mu_n(gB_nh) = 0$ for any $g, h \in G$. Clearly, it is enough to prove that $\bigcup_{n\in\omega} B_n \in \mathcal{HN}(G)$, for which it is enough to find a common witness measure μ such that $\mu(gB_nh) = 0$ for any $n \in \omega$ and $g, h \in G$.

The idea is that we will define μ as an infinite convolution. For this, we will define for all $n \in \omega$ a compact set C_n and a measure $\widetilde{\mu}_n$ with support C_n such that the size of C_n is going to decrease "quickly", and we can multiply infinitely many elements of them.

We will do this recursively. For the initial step, let us use Corollary 6.8 for μ_0, B_0, and G. Then we get another witness measure $\widetilde{\mu}_0$ of B_0 with compact support $C_0 \subseteq G$.

Assume that for all $k < n \in \omega$ the compact sets C_k and the witness measures $\widetilde{\mu}_k$ have been defined. Let us use Lemma 6.9 for the compact set $C_0C_1 \dots C_{n-1}$ and for $\varepsilon = 2^{-n}$. Then we get a nonempty open set U_n such that for any $x_0 \in C_0, x_1 \in C_1, \dots, x_{n-1} \in C_{n-1}$ and $x_n \in U_n$ the distance $d(x_0x_1 \dots x_{n-1}, x_0x_1 \dots x_{n-1}x_n) < 2^{-n}$. Now let us use Corollary 6.8 for μ_n, B_n, and U_n and find a measure $\widetilde{\mu}_n$ with compact support $C_n \subseteq U_n$ such that $\widetilde{\mu}_n$ is witness measure of B_n.

Consider any sequence $(c_n)_{n\in\omega}$ such that $c_n \in C_n$ for all $n \in \omega$. Then it is easy to check that $(c_0c_1 \dots c_n)_{n\in\omega}$ is a Cauchy sequence. As the metric d is complete, we can define the infinite product $c_0c_1 \dots c_n \dots$ to be the limit point of the Cauchy sequence. As the function $\varphi : \prod_{n\in\omega} C_n \to G$, $\varphi((c_0, c_1, \dots, c_n, \dots)) = c_0c_1 \dots c_n \dots$ is a uniform limit of continuous functions, it is continuous, too.

Let $\mu^\times$ be the product measure $\widetilde{\mu}_0 \times \widetilde{\mu}_1 \times \dots \widetilde{\mu}_n \dots$ on the product space $C_0 \times C_1 \dots \times C_n \dots$ and let $\mu := \varphi_*(\mu^\times)$ be the pushforward of $\mu^\times$ by φ, that is:

$$\mu(X) := \mu^\times(\varphi^{-1}(X))$$

for any $X \subseteq G$. It is clear that μ is a Borel probability measure. Now we will show that $\mu(gB_nh) = 0$ for any $g, h \in G$ and $n \in \omega$.

$$\mu(gB_nh) = \mu^\times(\varphi^{-1}(gB_nh)) = \mu^\times\left(\{(c_0, c_1 \dots, c_n, \dots) : c_0c_1 \dots c_n \dots \in gB_nh\}\right).$$

We would like to apply Fubini's theorem in the product space $\left(\prod_{j\in\omega, j\neq n} C_j\right) \times C_n$, that is:

$$\mu(gB_nh) = \mu^\times(\varphi^{-1}(gB_nh)) = \int_{\left(\prod_{j\in\omega,j\neq n} C_j\right)} \widetilde{\mu}_n(\varphi^{-1}(gB_nh)_{c_0,c_1\dots c_{n-1},c_{n+1}\dots})\, \mathrm{d} \prod_{j\in\omega,j\neq n} \widetilde{\mu}_j.$$

Thus for fixed $c_0, c_1 \dots c_{n-1}, c_{n+1} \dots$ we have to calculate the $(c_0, c_1 \dots c_{n-1}, c_{n+1} \dots)$-section of $\varphi^{-1}(gB_nh)$, that is:

$$\varphi^{-1}(gB_nh)_{c_0,c_1\dots c_{n-1},c_{n+1}\dots} = \{x \in C_n : c_0c_1 \dots c_{n-1}xc_{n+1} \dots \in gB_nh\} =$$
$$= C_n \cap (c_0c_1 \dots c_{n-1})^{-1}gB_nh(c_{n+1}c_{n+2} \dots)^{-1} = C_n \cap g'B_nh',$$

which has $\widetilde{\mu}_n$ measure zero for every fixed $c_0, c_1 \dots c_{n-1}, c_{n+1} \dots$ by the fact that $\widetilde{\mu}_n$ is a witness measure of B_n. Thus $\mu(gB_nh) = 0$ for every g, h, which completes the proof.

□

6.2 Haar meager and generalized Haar meager sets

Now we turn to the definition of the system of Haar meager and generalized Haar meager sets and prove the same two basic facts about them: they extend the definition of meager sets and they form σ-ideals.

Definition 6.11 (Haar meager set)**.** Let G be a Polish group. We say that $X \subseteq G$ is *Haar meager* if there is a Borel set $B \supseteq X$, a compact metric space C and a continuous function $f : C \to G$ such that $f^{-1}(gBh)$ is meager in C for all $g, h \in G$. We denote by $\mathcal{HM}(G)$ the system of Haar meager subsets of G.

M. Pálfy introduced generalized Haar meager sets in his MSc thesis [21]:

Definition 6.12 (Generalized Haar meager set)**.** Let G be a Polish group. We say that $X \subseteq G$ is *generalized Haar meager* if there is a universally Baire set $A \supseteq X$, a compact metric space C and a continuous function $f : C \to G$ such that $f^{-1}(gAh)$ is meager in C for all $g, h \in G$. We denote by $\mathcal{GHM}(G)$ the system of generalized Haar meager subsets of G.

Remark 6.13. Similarly, as in Remark 6.2, we call the function f and the set B (or A) from the previous definitions the *witness function* of X and the *Borel (or universally Baire) hull* of X.

Unlike in the case of measure, the notion of meagerness remains meaningful in Polish groups which are not locally compact. In the next theorems we will examine the relation of the systems $\mathcal{M}(G), \mathcal{HM}(G)$ and $\mathcal{GHM}(G)$ in locally compact and non-locally compact Polish groups.

Theorem 6.14. *Let G be an arbitrary Polish group. Then $\mathcal{HM}(G) \subseteq \mathcal{GHM}(G) \subseteq \mathcal{M}(G)$, in other words, every Haar meager set is generalized Haar meager, and every generalized Haar meager set is meager.*

Proof. The inclusion $\mathcal{HM}(G) \subseteq \mathcal{GHM}(G)$ is clear from Remark 4.26. For $\mathcal{GHM}(G) \subseteq \mathcal{M}(G)$ let us take any $X \in \mathcal{GHM}(G)$ with a universally Baire hull $A \supseteq X$, a compact metric space C and a continuous witness function $f : C \to G$ with the property that $f^{-1}(gAh)$ is meager in C for any $g, h \in G$. Analogously as in the proof of Theorem 6.4, let $B := \{(x, g) \in C \times G : f(x) \in gA\}$. Notice that B has the Baire property, since it is the preimage of the universally Baire set A by the continuous function $(x, g) \mapsto g^{-1}f(x)$ ($C \times G$ is Polish by Theorem 4.2). As $B^g = \{x \in C : f(x) \in gA\} = f^{-1}(gA)$ is meager in C by definition for all $g \in G$, after applying the Kuratowski-Ulam theorem (see Theorem 4.24) for B we get that it is meager in $C \times G$. After applying it again we can conclude that $B_x = \{g \in G : f(x) \in gA\} = f(x)A^{-1}$ is meager for comeager many $x \in C$. Since the inverse map and the multiplication by a fixed element in a topological group are homeomorphisms, $f(x)A^{-1}$ is meager if and only if A is meager. Thus $X \subseteq A$ is meager, which completes the proof. $\square$

Similarly, as in Theorem 6.4, we would like to prove that in the locally compact case $\mathcal{HM}(G) = \mathcal{GHM}(G) = \mathcal{M}(G)$ holds.

Theorem 6.15. *Let G be a locally compact Polish group. Then $\mathcal{HM}(G) = \mathcal{GHM}(G) = \mathcal{M}(G)$, that is, the system of meager, Haar meager, and generalized Haar meager sets coincide.*

Proof. We have seen in Theorem 6.14 that $\mathcal{HM}(G) \subseteq \mathcal{GHM}(G) \subseteq \mathcal{M}(G)$, so it is enough to prove that $\mathcal{M}(G) \subseteq \mathcal{HM}(G)$ in the locally compact case. Take any meager set $M \subseteq G$, then by Lemma 4.13 there exists a meager Borel set $B \supseteq M$. Since G is locally compact there is a nonempty open set U such that $\overline{U}$ is compact. Let the witness function $f := id : \overline{U} \to G$ be the identity map restricted to $\overline{U}$. Clearly, gBh is meager for any $g, h \in G$ and thus $f^{-1}(gBh) = gBh \cap \overline{U}$ is meager in $\overline{U}$. So $M \in \mathcal{HM}(G)$, which completes the proof. $\square$

As in the Haar null case, we would like to know in which groups the notion of Haar meager, generalized Haar meager, and meager can be separated from each other.

First of all, we will need the following interesting (and pretty complicated) theorem of S. Solecki, the proof of which can be found in [26]. The theorem (together with Lemma 6.17) states that there are continuum many pairwise disjoint closed sets in every Polish group G with a two-sided invariant metric such that none of them are Haar null and none of them are Haar meager. (Notice that this is not the case for the σ-ideals $\mathcal{N}(G)$ and $\mathcal{M}(G)$.)

Theorem 6.16. *Assume that G is a non-locally compact Polish group that admits a two-sided invariant metric. Then there exists a closed set $F \subseteq G$ and a continuous function $\varphi : F \to 2^\omega$ such that for any $x \in 2^\omega$ and any compact set $C \subseteq G$ there is $g \in G$ such that $gC \subseteq \varphi^{-1}(x)$.*

The following lemma gives a sufficient condition for a set being not (generalized) Haar meager and not (generalized) Haar null.

Lemma 6.17. *Let G be a Polish group and let $X \subseteq G$ have the following property: for any compact set C there is $g, h \in G$ such that $C \subseteq gXh$. Then $X \notin \mathcal{HN}(G)$ and $X \notin \mathcal{HM}(G)$.*

Proof. First, we prove that X is not Haar null. Suppose that there is a witness measure μ and a Borel hull $B \supseteq X$. From Lemma 6.7 we may assume that μ has compact support C. Then we can find $g, h \in G$ such that $C \subseteq gXh \subseteq gBh$. Therefore $1 = \mu(C) \leq \mu(gBh)$, contradicting that μ is a witness measure.

Similarly, suppose that X is Haar meager set, then there is a Borel hull B, a compact metric space C and a continuous witness function $f : C \to G$. We know that $f(C)$ is compact in G thus there is $g, h \in G$ such that $f(C) \subseteq gXh \subseteq gBh$. But then $f^{-1}(gBh) = C$, contradicting again that f is a witness function. □

Remark 6.18. We call sets with the property from the previous lemma *compact catcher*. Notice that the same proof shows that compact catcher sets are neither generalized Haar null nor generalized Haar meager.

Theorem 6.19. *Let G be a non-locally compact Polish group that admits a two-sided invariant metric. Then $\mathcal{GHM}(G) \subsetneq \mathcal{M}(G)$ (which implies $\mathcal{HM}(G) \subsetneq \mathcal{M}(G)$).*

Proof. Let φ be the function that we get from Theorem 6.16. Let us pick an element $x \in 2^\omega$ for which $\varphi^{-1}(x)$ has empty interior. (As the inverses are all disjoint and there is a countable basis of G, there is such an x.) As $X := \varphi^{-1}(x)$ is closed with empty interior, we can conclude that it is nowhere dense and thus $X \in \mathcal{M}(G)$. On the other hand, X is compact catcher thus by Lemma 6.17 and Remark 6.18 $X \notin \mathcal{GHN}(G)$, which concludes the proof. □

We also have the dual statement of Theorem 6.5 and Corollary 6.6. The proof of the theorem can be found in [10].

Theorem 6.20. *Let G be an abelian Polish group that is not locally compact. Then there is a coanalytic set $A \subseteq G$, a compact metric space C and a continuous function $f : C \to G$ such that $f^{-1}(gAh)$ is meager for any $g, h \in G$, but A is not Haar meager.*

From the fact that every coanalytic set is universally Baire (see Remark 4.26), we have the following:

Corollary 6.21. *Let G be an abelian Polish group that is not locally compact. Then $\mathcal{HM}(G) \subsetneq \mathcal{GHM}(G)$.*

Now we would like to prove that the systems $\mathcal{HM}(G)$ and $\mathcal{GHM}(G)$ form σ-ideals. As in many cases, when one works with Haar null or Haar meager sets, the proof will be very similar to the proof of Theorem 6.10.

Theorem 6.22. *Let G be an arbitrary Polish group. Then the system of Haar meager sets $\mathcal{HM}(G)$ and the system of generalized Haar meager sets $\mathcal{GHM}(G)$ form σ-ideals.*

Proof. As in Theorem 6.10, this proof will work for Haar meager sets, but if we replace "Borel set" with "universally Baire set" and "Haar meager" with "generalized Haar meager", it becomes a proof for the system $\mathcal{GHM}(G)$.

Clearly, the system $\mathcal{HM}(G)$ is closed under taking subsets, since the same witness function and Borel hull will work. It is also easy to see that $\emptyset \in \mathcal{HM}(G)$ and $G \notin \mathcal{HM}(G)$.

To see that $\mathcal{HM}(G)$ is closed under countable union, let us fix $X_0, X_1 \ldots \in \mathcal{HM}(G)$ and a compatible complete metric d on G. As these are Haar meager sets, there are Borel hulls $B_0, B_1 \ldots$, compact metric spaces $C_0, C_1 \ldots$ and witness functions $f_0, f_1 \ldots$ such that $f_n : C_n \to G$. Clearly, it is enough to prove that $\bigcup_{n\in\omega} B_n \in \mathcal{HM}(G)$, for which it is enough to find a compact metric space C and a common witness function f such that $f : C \to G$ and $f^{-1}(gB_nh)$ is meager in C for every $n \in \omega$ and $g, h \in G$.

We will find compact spaces $\widetilde{C}_n \subseteq C_n$ and witness functions $\widetilde{f}_n$ such that the size of $\widetilde{f}_n(\widetilde{C}_n)$ is going to decrease "quickly", and we can multiply infinitely many elements of them.

Our construction will be recursive. For the initial step, let $\widetilde{C}_0 = C_0$ and $\widetilde{f}_0 = f_0$. Assume that for all $k < n$ the compact metric spaces $\widetilde{C}_k$ and the continuous functions $\widetilde{f}_k : \widetilde{C}_k \to G$ have already been defined. Let us apply Lemma 6.9 for the compact set $\widetilde{f}_0(\widetilde{C}_0)\widetilde{f}_1(\widetilde{C}_1) \ldots \widetilde{f}_{n-1}(\widetilde{C}_{n-1})$ and for $\varepsilon = 2^{-n}$. Then we get a nonempty open set $U_n \ni 1_G$ such that $d(x, x \cdot u) \le 2^{-n}$ for any $x \in \widetilde{f}_0(\widetilde{C}_0)\widetilde{f}_1(\widetilde{C}_1) \ldots \widetilde{f}_{n-1}(\widetilde{C}_{n-1})$ and $u \in U_n$. Let us take any element $x_n \in f_n(C_n)$, and let us define $\widetilde{C}_n$ as $\overline{f_n^{-1}(x_nU_n)}$ (which is a closed subset of a compact set, thus it is compact). Furthermore, let us define the function $\widetilde{f}_n : \widetilde{C}_n \to G$ as $\widetilde{f}_n(c) := x_n^{-1}f_n(c)$, which is clearly continuous.

Now we will prove that the compact metric space $\widetilde{C}_n$ and the function $\widetilde{f}_n$ also witness that B_n is Haar meager for any $n \in \omega$.

Claim 6.23. *For every $n \in \omega$ and $g, h \in G$ the set $\widetilde{f}_n^{-1}(gB_nh)$ is meager in $\widetilde{C}_n$.*

Proof. Fix $n \in \omega$ and $g, h \in G$. It is easy to see that $\widetilde{f}_n^{-1}(gB_nh) = f_n^{-1}(x_ngB_nh) \cap \widetilde{C}_n$. Using that f_n is a witness function, we get that $f_n^{-1}(x_ngB_nh)$ is meager in C_n, thus

$f_n^{-1}(x_n g B_n h) \cap f_n^{-1}(x_n U_n)$ is meager in the open set $f_n^{-1}(x_n U_n)$. By definition, $\widetilde{C}_n$ is the closure of $f_n^{-1}(x_n U_n)$. Each nonempty open set in a Polish space is comeager in its closure, and thus $\widetilde{f}_n^{-1}(gB_n h)$ remains meager in $\widetilde{C}_n$. □

Now let us define the compact metric space $C := \prod_{n\in\omega} \widetilde{C}_n$. Let $(c_n)_{n\in\omega}$ be any element of C, then it is easy to see that $\left(\widetilde{f}_0(c_0)\widetilde{f}_1(c_1)\ldots\widetilde{f}_n(c_n)\right)_{n\in\omega}$ is a Cauchy-sequence. As the metric d fixed in G is complete, this sequence is convergent, so we can define the infinite product $\widetilde{f}_0(c_0)\widetilde{f}_1(c_1)\ldots$ as the limit point of this Cauchy sequence. Let the function $f : C \to G$ be defined as $f((c_0, c_1 \ldots)) := \widetilde{f}_0(c_0)\widetilde{f}_1(c_1)\ldots$. As it is a uniform limit of continuous functions, it is continuous, too.

Fix $g, h \in G$ and $n \in \omega$, then it suffices to prove that $f^{-1}(gB_n h)$ is meager in C. We would like to use the Kuratowski-Ulam theorem (see Theorem 4.24) in the product space $\left(\prod_{j\in\omega, j\neq n} \widetilde{C}_j\right) \times \widetilde{C}_n$ for the set $f^{-1}(gB_n h)$. Let us fix $c_0, c_1 \ldots c_{n-1}, c_{n+1} \ldots$, then the $(c_0, c_1 \ldots c_{n-1}, c_{n+1} \ldots)$-section of $f^{-1}(gB_n h)$ is the following:

$$f^{-1}(gB_n h)_{c_0,c_1\ldots c_{n-1},c_{n+1}\ldots} =$$
$$= \{x \in \widetilde{C}_n : \widetilde{f}_0(c_0)\widetilde{f}_1(c_1)\ldots\widetilde{f}_{n-1}(c_{n-1})\widetilde{f}_n(x)\widetilde{f}_{n+1}(c_{n+1})\ldots \in gB_n h)\} =$$
$$= \left\{x \in \widetilde{C}_n : \widetilde{f}_n(x) \in \left(\widetilde{f}_0(c_0)\ldots\widetilde{f}_{n-1}(c_{n-1})\right)^{-1} gB_n h \left(\widetilde{f}_{n+1}(c_{n+1})\widetilde{f}_{n+2}(c_{n+2})\ldots\right)^{-1}\right\} =$$
$$= \widetilde{f}_n^{-1}(g' B_n h').$$

In Claim 6.23 we proved that $\widetilde{f}_n^{-1}(g' B_n h')$ is meager in $\widetilde{C}_n$ for every $g', h' \in G$, thus the $(c_0, c_1 \ldots c_{n-1}, c_{n+1} \ldots)$-section of $f^{-1}(gB_n h)$ is meager for every $c_0, c_1 \ldots c_{n-1}, c_{n+1} \ldots$. Therefore $f^{-1}(gB_n h)$ is meager in C, which concludes the proof. □

6.3 Alternative definitions

There are plenty of alternative definitions for both Haar null and Haar meager sets. Without claiming to be exhaustive, in this section we list some of these definitions that we will use later.

The first theorem states that we can define Haar null sets with functions, similarly to Haar meager sets. Here $\mathcal{N}(2^\omega)$ denotes sets that have zero measure with respect to the Haar measure on the Cantor space which coincides with the "usual" coin-tossing measure on it.

Theorem 6.24. *A Borel set $B \subseteq G$ is Haar null if and only if there is a continuous injective function $f : 2^\omega \to G$ such that $f^{-1}(gBh) \in \mathcal{N}(2^\omega)$ for every $g, h \in G$.*

The statement of the theorem is essentially [2, Theorem 4.3], the difference is that [2] considers only the case when G is abelian. However, the proof of this theorem remains valid when we consider arbitrary Polish groups.

The following is an equivalent definition of Haar meager sets that considers only functions from the Cantor set.

Theorem 6.25. *A Borel set $B \subseteq G$ is Haar meager if and only if there is a continuous function $f : 2^\omega \to G$ such that $f^{-1}(gBh) \in \mathcal{M}(2^\omega)$ for every $g, h \in G$.*

Remark 6.26. Notice that in Theorem 6.25 we cannot assume the function f to be injective. The sets with an injective witness function f are the so-called *strongly Haar meager* sets. Haar meager and strongly Haar meager sets may (see [2, Theorem 5.13]) or may not (see [12, Theorem 1.8]) coincide, depending on G. It is not known whether strongly Haar meager sets form a σ-ideal.

7 Cardinal invariants

Before we turn to the notion of cardinal invariants, let us recall what we mean under two σ-ideals being isomorphic.

Definition 7.1. Let X and Y be given sets with two σ-ideals $\mathcal{I} \subseteq \mathcal{P}(X)$ and $\mathcal{J} \subseteq \mathcal{P}(Y)$. We call a bijection $f : X \to Y$ an *isomorphism* between the σ-ideals $\mathcal{I}$ and $\mathcal{J}$, if $I \in \mathcal{I}$ if and only if $f(I) \in \mathcal{J}$.

Cardinal invariants (or cardinal characteristics) are infinite cardinal numbers that we assign to isomorphism classes of ideals, that is, if there is an isomorphism between two ideals, then their cardinal invariants are the same. Thus a basic motivation for cardinal characteristics is that we can distinguish ideals from each other if one of their invariants do not coincide. We will include other nice applications of them in Section 7.1. In this thesis we will only focus on the below defined 4 "standard" cardinal characteristics, for a more extensive list (and a brief introduction into the topic) see [5, Chapter 6].

As we will work only with σ-ideals, we define additivity ($\mathrm{add}(\mathcal{I})$), uniformity ($\mathrm{non}(\mathcal{I})$), covering number ($\mathrm{cov}(\mathcal{I})$), and cofinality ($\mathrm{cof}(\mathcal{I})$) only for a σ-ideal $\mathcal{I}$.

Definition 7.2. Let X be an arbitrary set, and $\mathcal{I} \subseteq \mathcal{P}(X)$ be a given σ-ideal on X that contains all the singletons from X. Then

- $\mathrm{add}(\mathcal{I}) := \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} \notin \mathcal{I}\}$
- $\mathrm{non}(\mathcal{I}) := \min\{|A| : A \subseteq X \text{ and } A \notin \mathcal{I}\}$
- $\mathrm{cov}(\mathcal{I}) := \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} = X\}$
- $\mathrm{cof}(\mathcal{I}) := \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } \forall I \in \mathcal{I}\ \exists A \in \mathcal{A} \text{ such that } I \subseteq A\}$.

It is easy to see that these definitions are indeed invariant under isomorphism between σ-ideals.

Example 7.3. Let $X = 2^\omega$ (or the real line $\mathbb{R}$) and $\mathcal{C}$ be the σ-ideal of its countable subsets. Then $\mathrm{add}(\mathcal{C}) = \mathrm{non}(\mathcal{C}) = \omega_1$ and $\mathrm{cov}(\mathcal{C}) = \mathrm{cof}(\mathcal{C}) = 2^\omega$.

Definition 7.4 (dominating number, bounding number)**.** Let $X = \mathbb{Z}^{\mathbb{N}}$, and $\mathcal{K}_\sigma$ be the σ-ideal generated by the compact sets, that is:

$$\mathcal{K}_\sigma := \{X \subseteq \mathbb{Z}^\omega : \exists C_0, C_1, \ldots \text{ compact such that } \bigcup_{n\in\omega} C_n \supseteq X\}.$$

Then we define the *dominating number* as $\mathfrak{d} = \operatorname{cov}(\mathcal{K}_\sigma)$ and the *bounding number* as $\mathfrak{b} = \operatorname{non}(\mathcal{K}_\sigma)$.

For another definition of $\mathfrak{b}$ and $\mathfrak{d}$ (which explains their names), let us introduce the following partial ordering on $\mathbb{Z}^\omega$.

Notation 7.5. Let $f, g \in \mathbb{Z}^\omega$. Then we say that $f \leq^* g$ if and only if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. We say that a set $B \subseteq \mathbb{Z}^\omega$ is *bounded* if there exists $g \in \mathbb{Z}^\omega$ such that $f \leq^* g$ for all $f \in B$. Respectively, we say that a set $D \subseteq \mathbb{Z}^\omega$ is *dominating* if for every $f \in \mathbb{Z}^\omega$ there is $g \in D$ such that $f \leq^* g$.

Proposition 7.6. *Let $\mathcal{K}_\sigma$ be as in Definition 7.4. Then $\operatorname{cof}(\mathcal{K}_\sigma) = \operatorname{cov}(\mathcal{K}_\sigma) = \mathfrak{d} = \min\{|D| : D \subseteq \mathbb{Z}^\omega$ is dominating$\}$ and $\operatorname{add}(\mathcal{K}_\sigma) = \operatorname{non}(\mathcal{K}_\sigma) = \mathfrak{b} = \min\{|B| : B \subseteq \mathbb{Z}^\omega$ is **not** bounded$\}$.*

The following almost trivial lemma will be helpful in situations when a σ-ideal contains another.

Lemma 7.7. *Let $\mathcal{I} \subseteq \mathcal{P}(X)$ and $\mathcal{J} \subseteq \mathcal{P}(X)$ be two σ-ideals on X such that $\mathcal{I} \subseteq \mathcal{J}$. Then $\operatorname{cov}(\mathcal{I}) \geq \operatorname{cov}(\mathcal{J})$ and $\operatorname{non}(\mathcal{I}) \leq \operatorname{non}(\mathcal{J})$.*

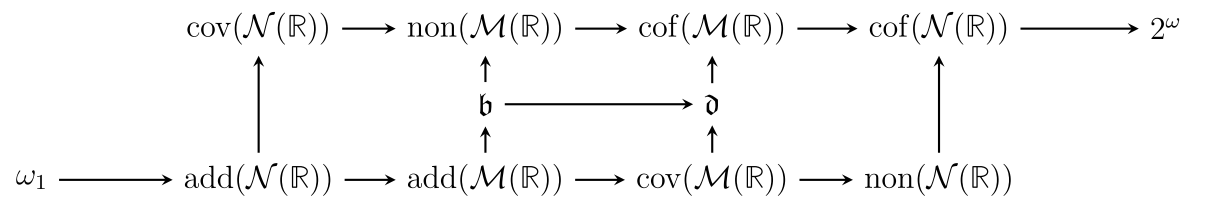
Proof. Assume that $\operatorname{cov}(\mathcal{I}) = \kappa$ and $\{X_\alpha\}_{\alpha<\kappa} \subseteq \mathcal{I}$ is a covering system, that is $\bigcup_{\alpha<\kappa} X_\alpha = X$. Since $\{X_\alpha\}_{\alpha<\kappa} \subseteq \mathcal{J}$, it is a covering system of $\mathcal{J}$ as well and thus $\operatorname{cov}(\mathcal{J}) \leq \kappa$.

For the uniformity assume that $\operatorname{non}(\mathcal{J}) = \kappa$, and there are elements $x_\alpha \in X$ for every $\alpha < \kappa$ such that $\{x_\alpha : \alpha < \kappa\} \notin \mathcal{J}$. But then $\{x_\alpha : \alpha < \kappa\} \notin \mathcal{I}$ either, which shows that $\operatorname{non}(\mathcal{I}) \leq \kappa$. □

Using the notation from the previous sections, we denote with $\mathcal{M}(\mathbb{R})$ the system of meager sets and with $\mathcal{N}(\mathbb{R})$ the system of sets with Lebesgue measure zero.

Cichoń's famous diagram shows some inequalities of the above-mentioned cardinal invariants (see Definition 7.2) of the σ-ideals $\mathcal{C}$ (see Example 7.3), $\mathcal{K}_\sigma$ (see Definition 7.4), $\mathcal{M}(\mathbb{R})$, and $\mathcal{N}(\mathbb{R})$. As the proof of the theorem is quite complicated and requires forcing techniques, we do not reproduce it here. For a full proof see [3].

Theorem 7.8 (Cichoń diagram)**.** *Consider the following diagram. Any arrow in the diagram going from x to y means that $x \leq y$. Furthermore, $\operatorname{add}(\mathcal{M}(\mathbb{R})) = \min\{\operatorname{cov}(\mathcal{M}(\mathbb{R})), \mathfrak{b}\}$ and $\operatorname{cof}(\mathcal{M}(\mathbb{R})) = \max\{\operatorname{non}(\mathcal{M}(\mathbb{R})), \mathfrak{d}\}$.*



The relations from the diagram and the other two mentioned above are the only ones that can be proved in ZFC. That is, let A be an assignment of the cardinals ω_1 and ω_2 to the 12 cardinals that satisfies the relations of the diagram and the additional two relations. Then there exists a model of ZFC in which A is realized.

7.1 Applications of cardinal invariants

In this subsection let us mention two nice applications of the cardinal invariants.

Example 7.9. The first of them is the following question: Does there exist a subgroup G of the real line (with addition as group operation), such that it is not meager but it has Lebesgue measure zero? And one that is meager but its outer Lebesgue measure is positive? In other words does there exists $G \leq \mathbb{R}$ such that $G \in \mathcal{N}(\mathbb{R}) \setminus \mathcal{M}(\mathbb{R})$ or $G \in \mathcal{M}(\mathbb{R}) \setminus \mathcal{N}(\mathbb{R})$?

It is easy to see that we can't expect G to be Borel (not even a set with Baire property in the first case and a Lebesgue measurable set in the second). Let us prove that if a subgroup $G \in \mathcal{N}(\mathbb{R}) \setminus \mathcal{M}(\mathbb{R})$, then it does not have the Baire property. Suppose to the contrary. It is well-known that for every non-meager set A with the Baire property there exists a nonempty open set U such that $A \cap U$ is comeager in U. Thus G is comeager in $B(x, \varepsilon)$, which means that $(G \cap B(x, \varepsilon)) \cap (y + G \cap B(x + y, \varepsilon)) \neq \emptyset$ for any $|y| < \varepsilon$. Therefore $G \cap (y + G) \neq \emptyset$, thus $G = G - G \supseteq B(0, \varepsilon)$. From this, we can conclude that $G = \mathbb{R}$ which has nonzero Lebesgue measure. A very similar argument (that uses the Steinhaus theorem) shows that there is no Lebesgue measurable subgroup $G \in \mathcal{M}(\mathbb{R}) \setminus \mathcal{N}(\mathbb{R})$.

Now we prove that consistently there exists a subgroup G that is not meager but has positive outer Lebesgue measure (the same proof works in the other case). From Theorem 7.8 we know that it is consistent with ZFC that $\mathrm{non}(\mathcal{M}(\mathbb{R})) < \mathrm{non}(\mathcal{N}(\mathbb{R}))$. Then there is a set A, such that it is not meager, and $|A| = \mathrm{non}(\mathcal{M}(\mathbb{R}))$. Let G be the subgroup generated by A, that is, $G := \{n_1 a_1 + n_2 a_2 \dots n_k a_k : n_i \in \mathbb{Z}, a_i \in A \text{ for } i = 1, 2 \dots k\}$. It is easy to see that the $|G| = |A|$, thus $|G| < \mathrm{non}(\mathcal{N}(\mathbb{R}))$, which means that G must have Lebesgue measure zero. On the other hand, it contains A, thus it is not meager, which concludes the proof.

We remark that A. Rosłanowski and S. Shelah proved in [24] the existence of a subgroup $G \in \mathcal{N}(\mathbb{R}) \setminus \mathcal{M}(\mathbb{R})$ in ZFC.

Example 7.10. In [25] J. Shipman proved a consistent form of the Fubini theorem. Let $\mathrm{non}(\mathcal{N}(\mathbb{R})) < \mathrm{cov}(\mathcal{N}(\mathbb{R}))$, which is consistent with ZFC because of Theorem 7.8. Then for any non-negative function $f : \mathbb{R}^2 \to \mathbb{R}$, for which the iterated integrals $\int \int f \, \mathrm{d}x \, \mathrm{d}y$ and $\int \int f \, \mathrm{d}x \, \mathrm{d}y$ exist, they are equal.

8 Cardinal invariants of Haar null and Haar meager sets

In this section we would like to present some very recent results about the cardinal invariants of Haar null, generalized Haar null, Haar meager, and generalized Haar meager sets. Most of the results can be found in [13] and in [1], while the results of Section 8.5 are recent results of M. Elekes, M. Pálfy and the author of the thesis.

Notice that the value of the cardinal invariants (introduced in Section 7) of the σ-ideals $\mathcal{HM}(G), \mathcal{GHM}(G), \mathcal{HN}(G), \mathcal{GHN}(G)$ may depend on the group G. It turns out that in the locally compact case the four invariants do not depend on the underlying group G, we prove this in Section 8.1. When G is not locally compact, there are results only in certain cases, e.g. when G admits a two-sided invariant metric, or when it is the countable product of locally compact groups. Notice that the results always hold for the Baer-Specker group $\mathbb{Z}^\omega$, which is a very widely studied special case, when one examines non-locally compact Polish groups.

8.1 The locally compact case

Theorem 8.1. *Let G be a locally compact non-discrete Polish group. Then the following hold:*

$$\begin{aligned}
\operatorname{add}(\mathcal{HN}(G)) &= \operatorname{add}(\mathcal{GHN}(G)) = \operatorname{add}(\mathcal{N}(\mathbb{R}))\\
\operatorname{non}(\mathcal{HN}(G)) &= \operatorname{non}(\mathcal{GHN}(G)) = \operatorname{non}(\mathcal{N}(\mathbb{R}))\\
\operatorname{cov}(\mathcal{HN}(G)) &= \operatorname{cov}(\mathcal{GHN}(G)) = \operatorname{cov}(\mathcal{N}(\mathbb{R}))\\
\operatorname{cof}(\mathcal{HN}(G)) &= \operatorname{cof}(\mathcal{GHN}(G)) = \operatorname{cof}(\mathcal{N}(\mathbb{R}))
\end{aligned}$$

Proof. In Theorem 6.4 we have seen that $\mathcal{HN}(G) = \mathcal{GHN}(G) = \mathcal{N}(G)$ (where $\mathcal{N}(G)$ is the system of sets with Haar measure zero) if G is locally compact. Let ν denote the left Haar measure on G, and λ denote the Lebesgue measure on $\mathbb{R}$. Using [17, Theorem 17.41], it follows that there is a Borel bijection $f : G \to \mathbb{R}$ such that the pushforward of ν by f, that is, $f_*(\nu) = \lambda$. □

The dual statement for Haar meager sets also holds. To show this, we will prove that the σ-ideals of meager sets are isomorphic in all perfect Polish spaces.

Definition 8.2 (Perfect Polish space)**.** We call a Polish space X *perfect*, if it has no isolated points.

Lemma 8.3. *Let X be a perfect Polish space. Then there is a Borel isomorphism $f : \mathbb{N}^\omega \to X$, such that $M \subseteq \mathbb{N}^\omega$ is meager if and only if $f(M) \subseteq X$ is meager.*

Proof. It is well-known that for every perfect Polish space X there is a dense subset $B \subseteq X$ and a homeomorphism $h : \mathbb{N}^\omega \to B$ (see [17, Theorem 8.38]). The space $\mathbb{N}^\omega$ is Polish by Theorem 4.2, thus from Theorem 4.3 we know that B is a G_δ subset of X, thus

$X \setminus B$ is meager. Let C be an arbitrary compact subset of $\mathbb{N}^\omega$ (which is automatically nowhere dense by [17, Theorem 7.7]) with cardinality 2^ω. Homeomorphisms preserve meagerness, thus $h(C) \cup (X \setminus B)$ is a meager Borel subset of X with cardinality 2^ω. Then from [17, Exercise 15.8] we know that there is a Borel isomorphism $g : C \to h(C) \cup X \setminus B$. Now let us define $f : \mathbb{N}^\omega \to X$ as

$$f(x) = \begin{cases} h(x) & \text{if } x \in \mathbb{N}^\omega \setminus C \\ g(x) & \text{if } x \in C. \end{cases}$$

The function f is clearly a Borel isomorphism. A set M is meager in $\mathbb{N}^\omega \Leftrightarrow$ it is meager in $\mathbb{N}^\omega \setminus C \Leftrightarrow f(M)$ is meager in $B \setminus h(C) \Leftrightarrow f(M)$ is meager in X, which completes the proof of the lemma. □

Corollary 8.4. *Let G be a non-discrete Polish group. Then the following hold:*

$$\begin{aligned} \mathrm{add}(\mathcal{M}(G)) &= \mathrm{add}(\mathcal{M}(\mathbb{R})) \\ \mathrm{non}(\mathcal{M}(G)) &= \mathrm{non}(\mathcal{M}(\mathbb{R})) \\ \mathrm{cov}(\mathcal{M}(G)) &= \mathrm{cov}(\mathcal{M}(\mathbb{R})) \\ \mathrm{cof}(\mathcal{M}(G)) &= \mathrm{cof}(\mathcal{M}(\mathbb{R})) \end{aligned}$$

Proof. It is easy to see that any non-discrete Polish group is a perfect Polish space. Thus we can apply Lemma 8.3 (twice) to get a Borel bijection $f : G \to \mathbb{R}$, for which $M \subseteq G$ is meager if and only if $f(M) \subseteq \mathbb{R}$ is meager. Therefore $\mathcal{M}(G)$ and $\mathcal{M}(\mathbb{R})$ are isomorphic σ-ideals, and their cardinal invariants are the same. □

Corollary 8.5. *Let G be a locally compact non-discrete Polish group. Then the following hold:*

$$\begin{aligned} \mathrm{add}(\mathcal{HM}(G)) &= \mathrm{add}(\mathcal{GHM}(G)) = \mathrm{add}(\mathcal{M}(\mathbb{R})) \\ \mathrm{non}(\mathcal{HM}(G)) &= \mathrm{non}(\mathcal{GHM}(G)) = \mathrm{non}(\mathcal{M}(\mathbb{R})) \\ \mathrm{cov}(\mathcal{HM}(G)) &= \mathrm{cov}(\mathcal{GHM}(G)) = \mathrm{cov}(\mathcal{M}(\mathbb{R})) \\ \mathrm{cof}(\mathcal{HM}(G)) &= \mathrm{cof}(\mathcal{GHM}(G)) = \mathrm{cof}(\mathcal{M}(\mathbb{R})) \end{aligned}$$

Proof. Recall from Theorem 6.15 that in a locally compact group $\mathcal{HM}(G) = \mathcal{GHM}(G) = \mathcal{M}(G)$. □

8.2 Cardinal invariants of Haar null sets

We will need a lemma that will be useful in the Haar meager case as well.

Lemma 8.6. *Let $\varphi : G \to H$ be a continuous surjective homomorphism between Polish groups. Then the preimage of a Haar null (respectively, a Haar meager) set is always Haar null (respectively, Haar meager).*

Proof. Let $B \subseteq H$ be Haar null (respectively, Haar meager). Without loss of generality, we may assume that B is Borel, and applying Theorem 6.24 (Theorem 6.25) we get a witness function $f : 2^\omega \to H$ such that $f^{-1}(gBh) \in \mathcal{N}(2^\omega)$ ($f^{-1}(gBh) \in \mathcal{M}(2^\omega)$) for every $g, h \in G$. Let us consider the following multi-function $\Phi : 2^\omega \to \mathcal{P}(G)$ defined as $\Phi(x) = \varphi^{-1}(f(x))$. We would like to apply the Michael Selection Theorem (see [18, Theorem 2.2]), thus we have to show that Φ is closed-valued, nonempty-valued (which are easy to see) and that it is lower semicontinuous, that is, $\{x : \Phi(x) \cap U \neq \emptyset\}$ is open in 2^ω for every $U \subseteq G$ open. It is easy to see that $\{x : \Phi(x) \cap U \neq \emptyset\} = f^{-1}(\varphi(U))$, which is open by the continuity of f and the fact that every continuous surjective homomorphism between Polish groups is open (see [4, Theorem 1.2.6]). Thus there is a continuous selection $s : 2^\omega \to G$, such that $s(x) \in \Phi(x)$, which means that $\varphi \circ s = f$.

We claim that s will be a witness function for $\varphi^{-1}(B)$. Indeed, as $g\varphi^{-1}(B)h = \varphi^{-1}(\varphi(g)B\varphi(h))$ for any $g, h \in G$, we have the following:

$$s^{-1}(g\varphi^{-1}(B)h) = s^{-1}(\varphi^{-1}(\varphi(g)B\varphi(h))) = f^{-1}((\varphi(g)B\varphi(h))),$$

which is in $\mathcal{N}(2^\omega)$ (respectively, $\mathcal{M}(2^\omega)$), since f was a witness function. □

Corollary 8.7. *Let G be a Polish group that admits a continuous surjective homomorphism φ onto a non-discrete locally compact Polish group H. Then*

$$\begin{aligned} \mathrm{cov}(\mathcal{HN}(G)) &\leq \mathrm{cov}(\mathcal{N}(\mathbb{R})) \\ \mathrm{non}(\mathcal{HN}(G)) &\geq \mathrm{non}(\mathcal{N}(\mathbb{R})) \\ \mathrm{cov}(\mathcal{HM}(G)) &\leq \mathrm{cov}(\mathcal{M}(\mathbb{R})) \\ \mathrm{non}(\mathcal{HM}(G)) &\geq \mathrm{non}(\mathcal{M}(\mathbb{R})) \end{aligned}$$

Proof. We only prove the corollary for the Haar null case, but the same proof works in the Haar meager case. From Theorem 8.1 we know that $\mathrm{cov}(\mathcal{HN}(H)) = \mathrm{cov}(\mathcal{N}(\mathbb{R}))$, so let $\{X_\alpha\}_{\alpha<\kappa}$ be a covering system of H, that is, $\bigcup_{\alpha<\kappa} X_\alpha = H$, where $X_\alpha \in \mathcal{HN}(H)$ for every $\alpha < \kappa$ and $\kappa = \mathrm{cov}(\mathcal{N}(\mathbb{R}))$. Applying Lemma 8.6 for the sets $\{X_\alpha\}_{\alpha<\kappa}$, we get that $\varphi^{-1}(X_\alpha) \in \mathcal{HN}(G)$ for every $\alpha < \kappa$. Notice that the system $\{\varphi^{-1}(X_\alpha)\}_{\alpha<\kappa}$ is a covering system of G, thus $\mathrm{cov}(\mathcal{HN}(G)) \leq \kappa = \mathrm{cov}(\mathcal{N}(\mathbb{R}))$.

For the second inequality let $\mathrm{non}(\mathcal{HN}(G)) = \kappa$, and let $X = \{x_\alpha : \alpha < \kappa\}$ be a non-Haar null subset of G. Then by Lemma 8.6 $\{\varphi(x_\alpha) : \alpha < \kappa\} = \varphi(X) \notin \mathcal{HN}(H)$, therefore applying Theorem 8.1 we get that $\mathrm{non}(\mathcal{HN}(G)) = \kappa \geq \mathrm{non}(\mathcal{HN}(H)) = \mathrm{non}(\mathcal{N}(\mathbb{R}))$. □

Before we state the theorem about $\mathrm{cov}(\mathcal{HN}(G))$ and $\mathrm{non}(\mathcal{HN}(G))$, let us introduce a generalization of compact sets (which will be a notion of smallness), that is, the o-bounded sets.

Definition 8.8 (o-bounded set)**.** Let G be an arbitrary Polish group, then we call a subset $X \subseteq G$ *o-bounded* if for every sequence of open neighborhoods $(U_n)_{n\in\omega}$ of 1_G there is a sequence of finite sets $(F_n)_{n\in\omega}$ such that $X \subseteq \bigcup_{n\in\omega} F_n U_n$. We denote the system of o-bounded sets with $o\mathcal{B}(G)$.

It is clear that $o\mathcal{B}(G)$ is a σ-ideal. We will need some basic lemmas, the proof of which can be found in [1]. Notice that T. Banakh calls "Haar null" the sets that we call "generalized Haar null". But as in the proof of the following lemma he constructs a Borel hull, the result remains true.

Lemma 8.9. *If G admits a two-sided invariant metric, then $o\mathcal{B}(G) \subseteq \mathcal{HN}(G)$.*

Lemma 8.10. *If G is Polish non-locally compact group then* $\mathrm{cov}(o\mathcal{B}(G)) \leq \mathfrak{b}$ *and* $\mathrm{non}(o\mathcal{B}(G)) \geq \mathfrak{d}$.

Theorem 8.11.

$$\mathrm{cov}(\mathcal{HN}(\mathbb{Z}^\omega)) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N}(\mathbb{R}))\},$$
$$\mathrm{non}(\mathcal{HN}(\mathbb{Z}^\omega)) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N}(\mathbb{R}))\}.$$

Proof. By Lemma 8.9 we have $o\mathcal{B}(G) \subseteq \mathcal{HN}(G)$, thus using Lemma 8.10 and Lemma 7.7 we have $\mathrm{cov}(\mathcal{HN}(\mathbb{Z}^\omega)) \leq \mathrm{cov}(o\mathcal{B}(\mathbb{Z}^\omega)) \leq \mathfrak{b}$ and $\mathrm{non}(\mathcal{HN}(\mathbb{Z}^\omega)) \geq \mathrm{non}(o\mathcal{B}(\mathbb{Z}^\omega)) \geq \mathfrak{d}$. On the other hand there is a continuous surjective homomorphism $\varphi : \mathbb{Z}^\omega \to 2^\omega$, namely $\varphi(x)(n) := x(n)(\text{mod } 2)$. Thus we can apply Lemma 8.7, therefore $\mathrm{cov}(\mathcal{HN}(\mathbb{Z}^\omega)) \leq \mathrm{cov}(\mathcal{N}(\mathbb{R}))$ and $\mathrm{non}(\mathcal{HN}(\mathbb{Z}^\omega)) \geq \mathrm{non}(\mathcal{N}(\mathbb{R}))$.

For the other direction of the inequalities notice that $\mathcal{HN}(\mathbb{Z}^\omega) \subseteq \mathcal{GHN}(\mathbb{Z}^\omega)$, and thus by Theorem 8.23 we get that $\mathrm{cov}(\mathcal{HN}(\mathbb{Z}^\omega)) \geq \mathrm{cov}(\mathcal{GHN}(\mathbb{Z}^\omega)) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N}(\mathbb{R}))\}$ and $\mathrm{non}(\mathcal{HN}(\mathbb{Z}^\omega)) \leq \mathrm{non}(\mathcal{GHN}(\mathbb{Z}^\omega)) = \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N}(\mathbb{R}))\}$. □

Let us turn to the cofinality of $\mathcal{HN}(G)$, which was calculated in [13, Theorem 2.7]. As the proof is quite long and complicated, we will only present a sketch of it.

Theorem 8.12. *Let G be a non-locally compact Polish group that admits a two-sided invariant metric. Then* $\mathrm{cof}(\mathcal{HN}(G)) = \mathfrak{c}$.

Sketch of the proof. The main difficulty of the theorem (which we omit) is the following lemma:

Lemma 8.13. *Let G be a non-locally compact Polish group that admits a two-sided invariant metric. Let us denote with $\mathcal{C}(G)$ the set of closed subsets of G, equipped with the Effros Borel structure (see [17, Section 12.C]). Then there is a Borel function $\varphi : 2^\omega \to \mathcal{C}(G)$ with the following properties:*

1) *$\varphi(x) \in \mathcal{HN}(G) \cap \mathcal{HM}(G)$ for every $x \in 2^\omega$,*
2) *if $P \subseteq 2^\omega$ is nonempty perfect then $\bigcup_{x \in P} \varphi(x)$ is compact catcher.*

The other technical lemma that we will need is the following:

Lemma 8.14. *Let $B \subseteq G$ be a Borel set. Then $\{x \in 2^\omega : \varphi(x) \subseteq B\}$ is coanalytic.*

Using these two lemmas we are ready to prove the theorem.

First, it is easy to see that $\mathrm{cof}(\mathcal{HN}(G)) \leq \mathfrak{c}$, since the set of all Borel Haar null sets is a cofinal system of cardinality $\mathfrak{c}$.

For the other direction, let us fix any Borel Haar null set B. Let $X := \{x \in 2^\omega : \varphi(x) \subseteq B\}$ then we prove that $|X| \leq \omega_1$. By the second lemma X is coanalytic, therefore either it has cardinality at most ω_1 or it contains a nonempty perfect set (this is so, since every coanalytic set is the union of ω_1 many Borel sets, and every uncountable Borel set contains a nonempty perfect set). But the second cannot be the case, because then from 2) B would contain a compact catcher set, thus it would not be Haar null by Lemma 6.17.

Clearly, $\omega < \mathrm{cof}(\mathcal{HN}(G))$, thus if the Continuum Hypothesis holds then we are done. Let us suppose to the contrary that there is a cofinal system $\{B_\alpha\}_{\alpha<\kappa} \subseteq \mathcal{HN}(G)$ for some cardinal $\omega < \kappa < \mathfrak{c}$. Without loss of generality we may assume that B_α is Borel for every $\alpha < \kappa$. Then by 1) for every α there are at most ω_1 elements $x \in 2^\omega$ such that $\varphi(x) \subseteq B_\alpha$. Consequently, $|\{x \in 2^\omega : \exists \alpha < \kappa\ \varphi(x) \subseteq B_\alpha\}| \leq \kappa \cdot \omega_1 < \mathfrak{c}$, which means that there is $x \in 2^\omega$ such that $\varphi(x)$ cannot be covered by any set from the cofinal system $\{B_\alpha\}_{\alpha<\kappa}$. But from 1) we know that $\varphi(x)$ is Haar null, thus this is a contradiction. $\square$

For computing the additivity of $\mathcal{HN}(G)$ let us state [14, Theorem 1.4].

Theorem 8.15. *Let G be a non-locally compact abelian Polish group and $1 \leq \xi < \omega_1$ an arbitrary countable ordinal. Then there exists a (Borel) Haar null set $B \subseteq G$ that cannot be covered by a $\mathbf{\Pi}^0_\xi$ set.*

Now we get the additivity of Haar null sets in non-locally compact abelian Polish groups as an easy, but surprising corollary of Theorem 8.15.

Corollary 8.16. *Let G be a Polish group that is not locally compact and abelian. Then we have $\mathrm{add}(\mathcal{HN}(G)) = \omega_1$.*

Proof. Let us use Theorem 8.15 for any countable cardinal $1 \leq \xi < \omega_1$, thus we get Haar null sets B_ξ. We prove that $X := \bigcup_{\xi<\omega_1} B_\xi$ is not Haar null. Suppose to the contrary that there is a Borel hull B of X. Then there is $\xi < \omega_1$, such that $B \in \mathbf{\Pi}^0_\xi$. But then B is a $\mathbf{\Pi}^0_\xi$ hull for the set B_ξ, which gives the contradiction. $\square$

8.3 Cardinal invariants of Haar meager sets

First, we will compute the covering number and the uniformity in Polish groups that admit a continuous surjective homomorphism to locally compact non-discrete Polish groups.

Theorem 8.17. *Let G be a Polish group that admits a continuous surjective homomorphism to a non-discrete locally compact Polish group H. Then $\mathrm{cov}(\mathcal{HM}(G)) = \mathrm{cov}(\mathcal{M}(\mathbb{R}))$ and $\mathrm{non}(\mathcal{HM}(G)) = \mathrm{non}(\mathcal{M}(\mathbb{R}))$.*

Proof. Recall from Theorem 6.14 that $\mathcal{HM}(G) \subseteq \mathcal{M}(G)$, so by Lemma 7.7 we know that $\mathrm{cov}(\mathcal{HM}(G)) \geq \mathrm{cov}(\mathcal{M}(G))$ and $\mathrm{non}(\mathcal{HM}(G)) \leq \mathrm{non}(\mathcal{M}(G))$. In Corollary 8.4 we have seen that $\mathrm{cov}(\mathcal{M}(G)) = \mathrm{cov}(\mathcal{M}(\mathbb{R}))$ and $\mathrm{non}(\mathcal{M}(G)) = \mathrm{non}(\mathcal{M}(\mathbb{R}))$, thus $\mathrm{cov}(\mathcal{HM}(G)) \geq \mathrm{cov}(\mathcal{M}(\mathbb{R}))$ and $\mathrm{non}(\mathcal{HM}(G)) \leq \mathrm{non}(\mathcal{M}(\mathbb{R}))$. On the other hand, we know from Corollary 8.7 that $\mathrm{cov}(\mathcal{HM}(G)) \geq \mathrm{cov}(\mathcal{M}(\mathbb{R}))$ and $\mathrm{non}(\mathcal{HM}(G)) \leq \mathrm{non}(\mathcal{M}(\mathbb{R}))$, which completes the proof. □

The cofinality of Haar meager sets was calculated in [13]. The proof for Haar meager sets is the same as the proof in the Haar null case (which we sketched in Theorem 8.12), so we only state the theorem here.

Theorem 8.18. *Let G be a non-locally compact Polish group that admits a two-sided invariant metric. Then $\mathrm{cof}(\mathcal{HM}(G)) = \mathfrak{c}$.*

Finally, for computing the additivity of $\mathcal{HM}(G)$ let us state the dual of Theorem 8.15, the proof of which can be found in [10].

Theorem 8.19. *Let G be a non-locally compact abelian Polish group and $1 \leq \xi < \omega_1$ an arbitrary countable ordinal. Then there exists a (Borel) Haar meager set $B \subseteq G$ that cannot be covered by a $\mathbf{\Pi}^0_\xi$ set.*

Using Theorem 8.19 we can derive the same corollary as in the Haar null case.

Corollary 8.20. *Let G be a Polish group that is not locally compact and abelian. Then* $\mathrm{add}(\mathcal{HM}(G)) = \omega_1$.

Proof. The same proof work as in Corollary 8.16 just replace the word "Haar null" with "Haar meager." □

8.4 Cardinal invariants of generalized Haar null sets

We will first examine the covering number and the uniformity of $\mathcal{GHN}(G)$. Let us state an equivalent characterization of Haar null sets in $\mathbb{Z}^\omega$ from [19]. Notice that S. Solecki proved in [27, Theorem 4.1] a very similar result for countable products of locally compact amenable groups.

For any sequence of natural numbers $a \in \mathbb{N}^\omega$, let us denote with μ_a the probability measure $\prod_{n=1}^\infty \varrho_{a(n)}$ on $\mathbb{Z}^\omega$ (where $\varrho_{a(n)}$ denotes the uniform probability measure on $[0, a(n)]$).

Theorem 8.21. *A subset X of $\mathbb{Z}^\omega$ is (generalized) Haar null if and only if there is a (universally measurable) Borel set $B \supseteq X$ and a sequence $(a(n))_{n\in\omega} \in \mathbb{N}^\omega$ such that $\mu_a(B + x) = 0$ for every $x \in \mathbb{Z}^\omega$.*

Remark 8.22. From the proof of Theorem 8.21 in [19] it is clear that if B is Haar null, then there is $a \in \mathbb{N}^\omega$ such that μ_b is witness measure for B for every $b \geq^* a$.

The proof of the following theorem can be found in [1]. T. Banakh proved this theorem for countable products of locally compact amenable Polish groups. As the proof is less technical (but still interesting) for $\mathbb{Z}^\omega$, we handle only this case.

Theorem 8.23.

$$\mathrm{cov}(\mathcal{GHN}(\mathbb{Z}^\omega)) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N}(\mathbb{R}))\},$$

$$\mathrm{non}(\mathcal{GHN}(\mathbb{Z}^\omega)) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N}(\mathbb{R}))\}.$$

Proof. Notice that from Lemma 8.9 $o\mathcal{B}(\mathbb{Z}^\omega) \subseteq \mathcal{GHN}(\mathbb{Z}^\omega)$, thus from Lemma 8.10 and Lemma 7.7 $\mathfrak{b} \geq \mathrm{cov}(o\mathcal{B}(\mathbb{Z}^\omega)) \geq \mathrm{cov}(\mathcal{GHN}(\mathbb{Z}^\omega))$ and $\mathfrak{d} \leq \mathrm{non}(o\mathcal{B}(\mathbb{Z}^\omega)) \leq \mathrm{non}(\mathcal{GHN}(\mathbb{Z}^\omega))$. We know that $\mathcal{HN}(\mathbb{Z}^\omega) \subseteq \mathcal{GHN}(\mathbb{Z}^\omega)$, furthermore there is a continuous surjective homomorphism $\varphi : \mathbb{Z}^\omega \to 2^\omega$, namely $\varphi(x)(n) := x(n)(\mathrm{mod}\ 2)$. Thus by Corollary 8.7 and Lemma 7.7 we can conclude that $\mathrm{cov}(\mathcal{N}(\mathbb{R})) \geq \mathrm{cov}(\mathcal{HN}(\mathbb{Z}^\omega)) \geq \mathrm{cov}(\mathcal{GHN}(\mathbb{Z}^\omega))$ and $\mathrm{non}(\mathcal{N}(\mathbb{R})) \leq \mathrm{non}(\mathcal{HN}(\mathbb{Z}^\omega)) \leq \mathrm{non}(\mathcal{GHN}(\mathbb{Z}^\omega))$. So we get that $\mathrm{cov}(\mathcal{GHN}(\mathbb{Z}^\omega)) \leq \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N}(\mathbb{R}))\}$ and $\mathrm{non}(\mathcal{GHN}(\mathbb{Z}^\omega)) \geq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N}(\mathbb{R}))\}$.

For $\mathrm{cov}(\mathcal{GHN}(\mathbb{Z}^\omega)) \geq \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N}(\mathbb{R}))\}$ take any family of universally measurable sets $\mathcal{S} \subseteq \mathcal{GHN}(\mathbb{Z}^\omega)$ such that $|\mathcal{S}| < \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N}(\mathbb{R})\}$. Then from Theorem 8.21 and Remark 8.22 we know that for every $S \in \mathcal{S}$ there exists $a_S \in \mathbb{N}^\omega$ such that μ_{a_S} is witness measure for S, moreover μ_b is witness measure for S for every $b \geq^* a_S$. Using that $|\mathcal{S}| < \mathfrak{b}$, there is a function $b \in \mathbb{N}^\omega$ such that $a_S \leq^* b$ for every $S \in \mathcal{S}$. Then μ_b is witness measure for every $S \in \mathcal{S}$, therefore $\mu_b(S) = 0$. Using [17, Theorem 17.41] for μ_b, we get that the system of sets with μ_b measure zero is isomorphic with the system of sets with Lebesgue measure zero. Thus by $|\mathcal{S}| < \mathrm{cov}(\mathcal{N}(\mathbb{R}))$ we get that $\bigcup \mathcal{S} \neq G$, and $\mathrm{cov}(\mathcal{GHN}(\mathbb{Z}^\omega)) \geq \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N}(\mathbb{R}))\}$ follows.

To prove $\mathrm{non}(\mathcal{GHN}(\mathbb{Z}^\omega)) \leq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N}(\mathbb{R}))\}$ take a dominating set $D \subseteq \mathbb{N}^\omega$ of size $\mathfrak{d}$. Then using [17, Theorem 17.41] again, we can fix for every $d \in D$ a set $N_d \subseteq G$, such that $|N_d| = \mathrm{non}(\mathcal{N}(\mathbb{R}))$ and $\mu_d(N_d) \neq 0$. Take $N := \bigcup_{d \in D} N_d$, then $|N| \leq \mathfrak{d} \cdot \mathrm{non}(\mathcal{N}(\mathbb{R})) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N}(\mathbb{R}))\}$. On the other hand, suppose that N is generalized Haar null, then using Remark 8.22 there is $c \in \mathbb{N}^\omega$ such that $\mu_c(N) = 0$, and μ_d is witness measure for N for every $d \geq^* c$. By the definition of D, there is $d \in D$ such that $c \leq^* d$, consequently $\mu_d(N) = 0$, which is a contradiction. □

Now we turn to the additivity of $\mathcal{GHN}(G)$. T. Banakh proved Theorem 8.24 in [1] for countable products of locally compact abelian groups, but for the sake of simplicity, we will work only in $\mathbb{Z}^\omega$.

Theorem 8.24. $\mathrm{add}(\mathcal{GHN}(\mathbb{Z}^\omega)) = \mathrm{add}(\mathcal{N}(\mathbb{R}))$.

Proof. First, we will need a strengthening of Lemma 8.6, the proof of which can be found in [1, Theorem 1].

Lemma 8.25. *Let G and H be Polish groups, furthermore assume that H is locally compact. Let $\varphi : G \to H$ be a continuous surjective homomorphism. Then $A \subseteq H$ is generalized Haar null if and only if $\varphi^{-1}(A) \subseteq G$ is generalized Haar null.*

First we prove that $\mathrm{add}(\mathcal{GHN}(\mathbb{Z}^\omega)) \leq \mathrm{add}(\mathcal{N}(\mathbb{R}))$. From Theorem 8.1 we know that $\mathrm{add}(\mathcal{N}(\mathbb{R})) = \mathrm{add}(\mathcal{GHN}(2^\omega))$, thus it is enough to prove that $\mathrm{add}(\mathcal{GHN}(\mathbb{Z}^\omega)) \leq \mathrm{add}(\mathcal{GHN}(2^\omega))$. Let $\mathrm{add}(\mathcal{N}(2^\omega)) = \kappa$, then there is a system $\{X_\alpha\}_{\alpha<\kappa} \subseteq \mathcal{GHN}(2^\omega)$ such that $\bigcup_{\alpha<\kappa} X_\alpha \notin \mathcal{GHN}(2^\omega)$. There is a continuous surjective homomorphism $\varphi : \mathbb{Z}^\omega \to$

2^ω, namely $\varphi(x)(n) := x(n)(\text{mod } 2)$. Thus by Lemma 8.25 we know that $\varphi^{-1}(X_\alpha) \in \mathcal{GHN}(\mathbb{Z}^\omega)$, but $\bigcup_{\alpha<\kappa} \varphi^{-1}(X_\alpha) = \varphi^{-1}(\bigcup_{\alpha<\kappa} X_\alpha) \notin \mathcal{GHN}(\mathbb{Z}^\omega)$, which completes the proof of the first inequality.

To show the other direction, take any family of universally measurable sets $\mathcal{S} \subseteq \mathcal{GHN}(\mathbb{Z}^\omega)$ such that $|\mathcal{S}| < \text{add}(\mathcal{N}(\mathbb{R}))$. Then from Theorem 8.21 and Remark 8.22 we know that for every $S \in \mathcal{S}$ there exists $a_S \in \mathbb{N}^\omega$ such that μ_{a_S} is witness measure for S, moreover μ_b is witness measure for S for every $b \geq^* a_S$. Using that $|\mathcal{S}| < \text{add}(\mathcal{N}(\mathbb{R})) \leq \mathfrak{b}$ (by Theorem 7.8), there is $b \in \mathbb{N}^\omega$ such that $a_S \leq^* b$ for every $S \in \mathcal{S}$. Then μ_b is witness measure for every $S \in \mathcal{S}$, therefore $\mu_b(S) = 0$. Using [17, Theorem 17.41] for μ_b, we get that the system of sets with μ_b measure zero is isomorphic with the system of sets with Lebesgue measure zero. Thus using that $|\mathcal{S}| < \text{add}(\mathcal{N}(\mathbb{R}))$ and that $\mu_b(x + S) = 0$ for every $x \in \mathbb{Z}^\omega$, we get that $\mu_b(x + \bigcup_{S\in\mathcal{S}} S) = 0$ for any $x \in \mathbb{Z}^\omega$. Furthermore [1, Lemma 3] we know that $\bigcup \mathcal{S}$ is universally measurable. So $\bigcup \mathcal{S} \in \mathcal{GHN}(\mathbb{Z}^\omega)$ and $\text{add}(\mathcal{GHN}(\mathbb{Z}^\omega)) \geq \text{add}(\mathcal{N}(\mathbb{R}))$ follows. □

Finally, in the case of cofinality we will prove only a consistent inequality (we will assume the Continuum Hypothesis). Even though this result may seem weaker than the other cases, it is very interesting since cardinal invariants typically never exceed the continuum. We remark that in [1] T. Banakh proved a stronger theorem in ZFC, so we state it here.

Theorem 8.26. *Let G be a Polish group that is not locally compact and admits a two-sided invariant metric. Then* $\text{cof}(\mathcal{GHN}(G)) > \min\{\mathfrak{d}, \text{non}(\mathcal{N}(\mathbb{R}))\}$.

Notice that using Theorem 7.8 we know that $\min\{\mathfrak{d}, \text{non}(\mathcal{N}(\mathbb{R}))\} = \mathfrak{c}$ is consistent with ZFC, thus $\text{cof}(\mathcal{GHN}(G)) > \mathfrak{c}$ is consistent with ZFC. Now we will prove the followig weaker theorem.

Theorem 8.27. *Let G be a non-locally compact Polish group that admits a two-sided invariant metric and let us assume the Continuum Hypothesis. Then* $\text{cof}(\mathcal{GHN}(G)) > \mathfrak{c}$.

Proof. Suppose to the contrary that $\text{cof}(\mathcal{GHN}(G)) \leq \mathfrak{c}$, and let $\{M_\alpha\}_{\alpha<\omega_1}$ be a cofinal system of generalized Haar null sets. As we assumed the Continuum Hypothesis, there are only ω_1 many σ-compact sets, so let $\{C_\alpha\}_{\alpha<\omega_1}$ be an enumeration of them. Now let us define a set X with transfinite recursion such that for any cardinal α we pick any point x_α from $G \setminus (\bigcup_{\beta<\alpha} C_\beta \cup M_\alpha)$. Since G has a two-sided invariant metric, using Lemma 8.9 and the fact that compact sets are o-bounded, we get that $C_\alpha \in \mathcal{GHN}(G)$. Thus $\bigcup_{\beta<\alpha} C_\beta \cup M_\alpha \in \mathcal{GHN}(G)$, consequently $G \setminus (\bigcup_{\beta<\alpha} C_\beta \cup M_\alpha)$ is nonempty. Now let $X := \{x_\alpha : \alpha < \omega_1\}$. We claim that $X \in \mathcal{GHN}(G)$, but it cannot be covered by any set M_α from the cofinal system (which is a contradiction). The second statement is easy to see, since $x_\alpha \in X \setminus M_\alpha$. To see that $X \in \mathcal{GHN}(G)$ we will prove that it is a universally null set. To see this, let μ be any continuous Borel probability measure on G, then there is a σ-compact set C_α such that $\mu(C_\alpha) = 1$ (see [17, Theorem 17.11]). Therefore $\mu(X \setminus C_\alpha) = 0$. On the other hand $X \cap C_\alpha$ is countable, thus $\mu(X) \leq \mu(X \setminus C_\alpha) + \mu(X \cap C_\alpha) = 0$. We know from Remark 4.20 that every universally null set is universally measurable, so it is enough to find a witness measure μ, such that $\mu(gXh) = 0$ for every $g, h \in G$. In fact, every continuous Borel probability measure μ on G is a witness measure.

This is so, since $\mu'(B) := \mu(gBh)$ is also a continuous Borel probability measure, thus $\mu(gXh) = \mu'(X) = 0$, which completes the proof. □

8.5 Cardinal invariants of generalized Haar meager sets

The results of this section are recent results of M. Elekes, M. Pálfy, and the author of the thesis.

We start this section by computing the covering number and the uniformity.

Theorem 8.28. *Let G be a non-locally compact Polish group that admits a continuous surjective homomorphism to a non-discrete locally compact Polish group H. Then* $\mathrm{cov}(\mathcal{GHM}(G)) = \mathrm{cov}(\mathcal{M}(\mathbb{R}))$ *and* $\mathrm{non}(\mathcal{GHM}(G)) = \mathrm{non}(\mathcal{M}(\mathbb{R}))$.

Proof. From Theorem 6.14 we know that $\mathcal{HM}(G) \subseteq \mathcal{GHM}(G) \subseteq \mathcal{M}(G)$. Thus from Lemma 7.7 we get that $\mathrm{cov}(\mathcal{HM}(G)) \geq \mathrm{cov}(\mathcal{GHM}(G)) \geq \mathrm{cov}(\mathcal{M}(G))$ and $\mathrm{non}(\mathcal{HM}(G)) \leq \mathrm{non}(\mathcal{GHM}(G)) \leq \mathrm{non}(\mathcal{M}(G))$ hold. From Theorem 8.17 and Theorem 8.4 we know that $\mathrm{cov}(\mathcal{HM}(G)) = \mathrm{cov}(\mathcal{M}(G)) = \mathrm{cov}(\mathcal{M}(\mathbb{R}))$ and $\mathrm{non}(\mathcal{HM}(G)) = \mathrm{non}(\mathcal{M}(G)) = \mathrm{non}(\mathcal{M}(\mathbb{R}))$, from which the statement immediately follows. □

Now let us turn to the cofinality of $\mathcal{GHM}(G)$ in non-locally compact Polish groups. Here we only have a consistent result, that is, assuming the Continuum Hypothesis we know that $\mathrm{cof}(\mathcal{GHM}(G)) > \mathfrak{c}$. This result is really peculiar on one hand since all the usual cardinal invariants are at most the continuum. On the other hand, in the generalized Haar null case the same inequality holds, thus Theorem 8.29 strengthens the connection of generalized Haar meager and generalized Haar null sets.

Theorem 8.29. *Let G be a non-locally compact group that admits a two-sided invariant metric, furthermore let us assume the Continuum Hypothesis. Then* $\mathrm{cof}(\mathcal{GHM}(G)) > \mathfrak{c}$.

To prove the theorem, we will need the following lemmas, that are interesting in their own right.

Lemma 8.30. *Every universally meager set $M \subseteq G$ is generalized Haar meager.*

Proof. From Remark 4.30 we know that M is universally Baire, thus it is enough to find a function f such that $f^{-1}(gMh)$ is meager in 2^ω for any $g, h \in G$. We claim that any continuous and nowhere locally constant function $f : 2^\omega \to G$ is witness function. To see this, take any $g, h \in G$, then it is easy to check that the function $f'(x) := g^{-1}f(x)h^{-1}$ is nowhere locally constant. Thus $f^{-1}(gMh) = f'^{-1}(M)$ is meager in 2^ω, which concludes the proof of the lemma. □

Let us introduce the following notation.

Notation 8.31. Let $\alpha < \omega_1$ be an arbitrary countable ordinal. Let us define $X_\alpha := \{f : f \text{ is a function from } \alpha \text{ to } 2^\omega\}$ and $X := \bigcup_{\alpha<\omega_1} X_\alpha$. There is a natural partial ordering on X: let us say that $f \geq g$ for any $f, g \in X$ if f extends g.

Lemma 8.32. *There exists a function $\Phi : X \to \mathcal{P}(2^\omega)$, such that*

1) $\Phi(f) \in \mathbf{\Pi}^1_1$ *for every* $f \in X$*,*

2) $\Phi(f) \subsetneq \Phi(g)$ *if* $f > g$*,*

3) $\Phi(f) \cap \Phi(g) = \emptyset$*, if* $f \ngeq g$ *and* $g \ngeq f$

Proof. It is clear that it is enough to find a function $\Phi : X \to \mathcal{P}\left((2^\omega)^\omega \times 2^{\omega\times\omega}\right)$ with properties 1), 2), and 3). Let us denote with WO $\subseteq 2^{\omega\times\omega}$ the set of countable well-orderings, that is, WO $= \{\prec \in 2^{\omega\times\omega} : \prec$ defines a well-ordering on $\omega\}$.

For any $\prec \in$ WO and $n \in \omega$ we denote with $\prec|_n$ the set $\{k \in \omega : k \prec n\}$ with the well-ordering inherited from $\prec$. Let us fix α and a function $f \in X_\alpha$. Then let us define:

$$\Phi(f) := \{(x, \prec) \in (2^\omega)^\omega \times \text{WO} : \alpha < \operatorname{otp}(\prec) \wedge (\operatorname{otp}(\prec|_n) < \alpha \Rightarrow x(n) = f(\operatorname{otp}(\prec|_n)))\}.$$

To see that 2) holds, take any $\alpha > \beta$ and $f \in X_\alpha, g \in X_\beta$ such that $f > g$. We will show that $(x, \prec) \in \Phi(f) \Rightarrow (x, \prec) \in \Phi(g)$. By definition, $\beta < \alpha < \operatorname{otp}(\prec)$ and if $\operatorname{otp}(\prec|_n) < \beta$, then $\operatorname{otp}(\prec|_n) < \alpha$, therefore $x(n) = f(\operatorname{otp}(\prec|_n)) = g(\operatorname{otp}(\prec|_n))$, since f extends g. Consequently, $(x, \prec) \in \Phi(g)$. To see that $\Phi(f) \setminus \Phi(g)$ is nonempty, take any $\prec \in$ WO such that $\beta \leq \operatorname{otp}(\prec) < \alpha$, and pick $x \in (2^\omega)^\omega$ for which $x(n) = f(\operatorname{otp}(\prec|_n))$. Then clearly $(x, \prec) \in \Phi(f) \setminus \Phi(g)$.

For 3), let us suppose to the contrary that $f \ngeq g$ and $g \ngeq f$ for some $\alpha, \beta < \omega_1, f \in X_\alpha, g \in X_\beta$ and $(x, \prec) \in \Phi(f) \cap \Phi(g)$. Since $f \ngeq g$ and $g \ngeq f$, there exists $\gamma < \alpha, \beta$ such that $f(\gamma) \neq g(\gamma)$. Using that $\prec$ defines a well-ordering, and $\alpha, \beta < \operatorname{otp}(\prec)$, there exists n for which $\operatorname{otp}(\prec|_n) = \gamma$. But then $g(\gamma) = g(\operatorname{otp}(\prec|_n)) = x(n) = f(\operatorname{otp}(\prec|_n)) = f(\gamma)$, which is a contradiction.

Now we have to show that $\Phi(f)$ is $\mathbf{\Pi}^1_1$ for fixed α and $f \in X_\alpha$. By [17, Theorem 32.B] WO is coanalytic in $2^{\omega\times\omega}$. From this and the fact that $\{\prec \in \text{WO} : \operatorname{otp}(\prec) \leq \alpha\}$ is Borel in $2^{\omega\times\omega}$ (see [17, Exercise 34.17]), we get that $\{(x, \prec) \in (2^\omega)^\omega \times \text{WO} : \alpha < \operatorname{otp}(\prec)\}$ is coanalytic. Let us fix n, then it is enough to show that

$$\{(x, \prec) \in (2^\omega)^\omega \times \text{WO} : \operatorname{otp}(\prec|_n) \geq \alpha\} \cup \{(x, \prec) \in (2^\omega)^\omega \times \text{WO} : x(n) = f(\operatorname{otp}(\prec|_n))\}$$

$$= \{(x, \prec) \in (2^\omega)^\omega \times \text{WO} : \operatorname{otp}(\prec|_n) \geq \alpha\} \cup$$

$$\cup \bigcup_{\beta<\alpha} \left(\{(x, \prec) \in (2^\omega)^\omega \times \text{WO} : \operatorname{otp}(\prec|_n) = \beta\} \cap \{(x, \prec) \in (2^\omega)^\omega \times \text{WO} : x(n) = f(\beta)\}\right)$$

is coanalytic. Clearly, it is enough to show the following claim.

Claim 8.33. *Fix n and $\alpha < \omega_1$. Then $\{\prec \in WO : \operatorname{otp}(\prec|_n) < \alpha\}$ is Borel in WO.*

Proof. We will proceed by transfinite induction on α. For $\alpha = 1$ this means that n is the smallest element of $\prec$, that is:

$$\{\prec \in \text{WO} : \operatorname{otp}(\prec|_n) < 1\} = \bigcap_{k\neq n} \{\prec \in 2^{\omega\times\omega} : n \prec k\} \cap \text{WO},$$

which is clearly Borel in WO (in fact, it is closed). Assume that we have proved the claim for all $\beta < \alpha$ and $n \in \omega$. If α is a limit ordinal, then

$$\{\prec \in \mathrm{WO} : \mathrm{otp}(\prec|_n) < \alpha\} = \bigcup_{\beta<\alpha} \{\prec \in \mathrm{WO} : \mathrm{otp}(\prec|_n) < \beta\},$$

thus by the induction hypothesis $\{\prec \in \mathrm{WO} : \mathrm{otp}(\prec|_n) < \alpha\} \in \mathcal{B}(\mathrm{WO})$. If $\alpha = \beta + 1$ then

$$\{\prec \in \mathrm{WO} : \mathrm{otp}(\prec|_n) < \alpha\} = \bigcap_{l \neq n} \{\prec \in \mathrm{WO} : l \prec n \Rightarrow \mathrm{otp}(\prec|_l) < \beta\} =$$

$$\bigcap_{l \neq n} \Big(\{\prec \in \mathrm{WO} : n \prec l\} \cup \{\prec \in \mathrm{WO} : \mathrm{otp}(\prec|_l) < \beta\} \Big).$$

The former set is closed in WO, while the latter is Borel in WO by the induction hypothesis, which completes the proof. □

□

Proof of Theorem 8.29. Let us suppose to the contrary that $\mathrm{cof}(\mathcal{GHM}(G)) \leq \omega_1$ (we assumed $\omega_1 = \mathfrak{c}$), so there is a cofinal system $\{M_\alpha\}_{\alpha<\omega_1}$. Let $\{f_\alpha\}_{\alpha<\omega_1}$ be an enumeration of every continuous nowhere locally constant function from $\mathbb{N}^\omega$ to G such that for any limit ordinal α there exists $\beta < \alpha$ for which $f_\alpha = f_\beta$. Let us recall Theorem 6.16, and let $\varphi : F \to 2^\omega$ be the function from the theorem. Finally, let us recall the notations X_α and Φ from Notation 8.31. We will define by transfinite recursion $g_\alpha \in X_\alpha$, $H_\alpha \subseteq G$ and $h_\alpha \in G$ for every $\alpha < \omega_1$ such that the following hold:

1) $g_\beta \leq g_\alpha$ for every $\beta < \alpha$ (where $\leq$ denotes extension),

2) $H_\alpha = \varphi^{-1}(\Phi(g_\alpha))$,

3) $f_\alpha^{-1}(H_\alpha)$ is meager in $\mathbb{N}^\omega$,

4) $h_\alpha \in H_\alpha \setminus \bigcup_{\beta\leq\alpha} M_\beta$.

First, let us assume that we have constructed g_α, H_α, and h_α this way. Then let us define $H := \{h_\alpha : \alpha < \omega_1\}$. We claim that $H \in \mathcal{GHM}(G)$, but $H \nsubseteq M_\alpha$ for every $\alpha < \omega_1$ which is a contradiction since M_α is a cofinal system. From Lemma 8.30 we know that every universally meager set is generalized Haar meager, thus for $H \in \mathcal{GHM}(G)$ it is enough to prove the following.

Claim 8.34. *H is universally meager.*

Proof. Using Remark 4.29, it suffices to prove that for every continuous nowhere locally constant function $f_\alpha : \mathbb{N}^\omega \to G$ the set $f_\alpha^{-1}(H) = f_\alpha^{-1}(H \cap H_\alpha) \cup f_\alpha^{-1}(H \setminus H_\alpha)$ is meager. Notice that $H \setminus H_\alpha$ is countable (since $H_\alpha \subseteq H_\beta$ if $\alpha \geq \beta$), thus $f_\alpha^{-1}(H \setminus H_\alpha)$ is meager (the preimage of any singleton by f_α is nowhere dense closed set in $\mathbb{N}^\omega$). On the other hand by 3) $f_\alpha^{-1}(H_\alpha \cap H) \subseteq f_\alpha^{-1}(H_\alpha)$ is meager, thus $f_\alpha^{-1}(H)$ is meager. □

It is easy to see that $H \nsubseteq M_\alpha$ for every $\alpha < \omega_1$, since $h_\alpha \notin M_\alpha$. Thus the proof will be finished if we show that we can define g_α, H_α, and h_α with the properties listed above.

So let $\alpha < \omega_1$ be any countable ordinal, and assume that we have constructed g_β, H_β and h_β for every $\beta < \alpha$. If α is a limit ordinal, then there exists exactly one function in X_α that extends g_β for every $\beta < \alpha$, let this function be g_α. Let $H_\alpha := \varphi^{-1}(\Phi(g_\alpha))$. We assumed that $f_\alpha = f_\beta$ for some $\beta < \alpha$. We know that g_α extends g_β, thus $H_\alpha \subseteq H_\beta$, consequently we get that $f_\alpha^{-1}(H_\alpha) \subseteq f_\beta^{-1}(H_\beta) \in \mathcal{M}(\mathbb{N}^\omega)$. To define h_α, notice that $\Phi(g_\alpha)$ is nonempty, therefore by Theorem 6.16 we get that H_α is a compact catcher set. On the other hand, $\bigcup_{\beta \leq \alpha} M_\alpha$ is countable union of generalized Haar meager sets, consequently, it is generalized Haar meager itself. Using Lemma 6.17 we get that $H_\alpha \setminus \bigcup_{\beta \leq \alpha} M_\alpha \neq \emptyset$, so let h_α be an arbitrary element of the difference.

For a successor ordinal $\alpha = \beta + 1$ there are continuum many functions in X_α that extend g_β. Let us call them g_x ($x \in 2^\omega$) according to the value they assign to α, that is, $g_x(\alpha) = x$. Clearly, all of the functions $\{g_x\}_{x \in 2^\omega}$ are pairwise incompatible (that is, $g_x \ngeq g_y$ and $g_y \ngeq g_x$ for any $x \neq y$), thus $\Phi(g_x) \cap \Phi(g_y) = \emptyset$ for any $x \neq y$. Consequently $f_\alpha^{-1}(\varphi^{-1}(\Phi(g_x))) \cap f_\alpha^{-1}(\varphi^{-1}(\Phi(g_y))) = \emptyset$, furthermore $f_\alpha^{-1}(\varphi^{-1}(\Phi(g_x)))$ is $\mathbf{\Pi}_1^1$ for every $x \in 2^\omega$ (continuous preimage of a $\mathbf{\Pi}_1^1$ set is $\mathbf{\Pi}_1^1$). Recall that every $\mathbf{\Pi}_1^1$ set possesses the Baire property (see Remark 4.23), thus we have continuum many pairwise disjoint sets in $\mathbb{N}^\omega$ with the Baire property. From this follows that there exists $x \in 2^\omega$ such that $f_\alpha^{-1}(\varphi^{-1}(\Phi(g_x)))$ is meager. Accordingly, let us define $g_\alpha := g_x$ and $H_\alpha := \varphi^{-1}(\Phi(g_x))$. Then 1) and 3) hold due to our choice on g_α, and using Theorem 6.16 we know that H_α is compact catcher, thus the difference $H_\alpha \setminus \bigcup_{\beta \leq \alpha} M_\beta$ is nonempty by Lemma 6.17. Let h_α be any element in the difference. So the conditions of the transfinite recursion hold, and this completes the proof. □

9 Isomorphism theorems for Haar null and Haar meager sets

In this section we will prove that consistently (assuming the Continuum Hypothesis) the σ-ideals $\mathcal{HM}(G), \mathcal{HN}(G), \mathcal{M}(\mathbb{R})$, and $\mathcal{N}(\mathbb{R})$ are isomorphic for an uncountable Polish group G with two-sided invariant metric . Moreover, there is an Erdős-Sierpiński duality between the σ-ideals $\mathcal{HN}(\mathbb{Z}^\omega)$ and $\mathcal{HM}(\mathbb{Z}^\omega)$. On the contrary, we prove that for Polish groups with two-sided invariant metric, there is neither a Borel nor an additive isomorphism. The results of this section are recent results of M. Elekes, M. Pálfy, and the author of the thesis.

In Section 7 we have already defined isomorphism between two σ-ideals. Let us now define the following special type of isomorphism:

Definition 9.1 (Erdős-Sierpiński duality)**.** Let X be a given set with two σ-ideals $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(X)$. We call a bijection $f : X \to X$ an *Erdős-Sierpiński duality* between the σ-ideals $\mathcal{I}$ and $\mathcal{J}$, if f is an isomorphism, moreover, f is an involution, that is, $f(f(x)) = x$ for every $x \in X$.

The name refers to the well-known paper of P. Erdős [15], in which he showed that there is an Erdős-Sierpiński duality between the σ-ideals $\mathcal{N}(\mathbb{R})$ and $\mathcal{M}(\mathbb{R})$. With this, he extended the result of W. Sierpiński: the existence of an isomorphism between $\mathcal{N}(\mathbb{R})$ and $\mathcal{M}(\mathbb{R})$.

Let us recall well-known sufficient conditions for the existence of an isomorphism and an Erdős-Sierpiński duality. For proof, see [20, Theorem 19.5, Theorem 19.6].

Theorem 9.2. *Let X be a set with cardinality ω_1 and let $\mathcal{I} \subseteq \mathcal{P}(X)$ be a σ-ideal on X with the following properties:*

1) $\bigcup \mathcal{I} = X$,

2) $\mathrm{cof}(\mathcal{I}) \leq \omega_1$,

3) for every $I \in \mathcal{I}$ there exists $I' \in \mathcal{I}$ such that $|I'| = \omega_1$ and $I' \subseteq X \setminus I$.

Then X can be decomposed into ω_1 disjoint sets X_α such that $|X_\alpha| = \omega_1$ and $I \in \mathcal{I}$ if and only if I is contained in a countable union of them.

Remark 9.3. Let X and Y be sets with cardinality ω_1 and let $\mathcal{I} \subseteq \mathcal{P}(X), \mathcal{J} \subseteq \mathcal{P}(Y)$ be σ-ideals on X and Y satisfying the properties 1), 2), and 3) from Theorem 9.2. Then we get a decomposition of X into disjoint sets X_α and respectively, a decomposition of Y into Y_α. Let $f : X \to Y$ be a bijection defined by the union of partial bijections $f_\alpha : X_\alpha \to Y_\alpha$. It is easy to see that f is an isomorphism of $\mathcal{I}$ and $\mathcal{J}$.

Theorem 9.4. *Let X be a set with cardinality ω_1 and let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(X)$ be two σ-ideals on X with properties 1), 2) and 3) from Theorem 9.2. Suppose furthermore that there exists $I \in \mathcal{I}$ and $J \in \mathcal{J}$ such that $X = I \cup J$. Then there exists an Erdős-Sierpiński duality f between $\mathcal{I}$ and $\mathcal{J}$.*

Now let us prove that assuming the Continuum Hypothesis the σ-ideals $\mathcal{N}(\mathbb{R}), \mathcal{M}(\mathbb{R})$, $\mathcal{HN}(G)$, and $\mathcal{HM}(G)$ are pairwise isomorphic.

Theorem 9.5. *Let G be an uncountable Polish group that admits a two-sided invariant metric. Assuming the Continuum Hypothesis, there exists an isomorphism between any two of the following σ-ideals: $\mathcal{HN}(G), \mathcal{HM}(G), \mathcal{N}(\mathbb{R}), \mathcal{M}(\mathbb{R})$. Moreover, if $G = \mathbb{Z}^\omega$, then there is an Erdős-Sierpiński duality between $\mathcal{HN}(\mathbb{Z}^\omega)$ and $\mathcal{HM}(\mathbb{Z}^\omega)$.*

Proof. We would like to apply Theorem 9.2 and Remark 9.3 after it for the σ-ideals $\mathcal{HN}(G), \mathcal{HM}(G), \mathcal{N}(\mathbb{R}), \mathcal{M}(\mathbb{R})$. It is easy to see that the σ-ideals $\mathcal{N}(\mathbb{R})$ and $\mathcal{M}(\mathbb{R})$ satisfy the conditions of Theorem 9.2. Thus we have to show that the conditions hold for $\mathcal{HN}(G)$ and $\mathcal{HM}(G)$, too. 1) trivially holds for both σ-ideals. Since we assumed the Continuum Hypothesis, from Theorem 8.12 and Theorem 8.18 2) also follows. Thus for the first part of the theorem it is enough to show property 3) for both $\mathcal{HN}(G)$ and $\mathcal{HM}(G)$, which we manage in the following two lemmas.

Lemma 9.6. *Let G be an uncountable Polish group that admits a two-sided invariant metric. Then for every $H \in \mathcal{HN}(G)$ there exists an uncountable Haar null set $H' \subset G \setminus H$.*

Proof. Let μ be a witness measure for H. Without loss of generality, we may assume that H is Borel. In Polish groups with two-sided invariant metric every compact set is Haar null (consider Lemma 8.9 and the fact that every compact set is o-bounded), in particular, there exists an uncountable Haar null set. So the lemma is trivial if H is countable. So we can assume that $|H| \geq \omega_1$, and thus by [17, Theorem 13.6] H contains a set C which is homeomorphic to 2^ω. Let λ be any continuous Borel probability measure on C. Let $B = \{(x,y) \in G \times G : xy \in H\}$. Notice that B is Borel, as it is a preimage of the Borel set H by the multiplication function, which is continuous. Thus we may apply Fubini's theorem in the product space $G \times G$ to the product measure $\mu \times \lambda$.

$$(\mu \times \lambda)(B) = \int_G \int_G \mathbb{1}_{B_x} \, \mathrm{d}\lambda(y) \, \mathrm{d}\mu(x) = \int_G \lambda(x^{-1}H) \, \mathrm{d}\mu(x) =$$
$$= \int_G \int_G \mathbb{1}_{B^y} \, \mathrm{d}\mu(x) \, \mathrm{d}\lambda(y) = \int_G \mu(Hy^{-1}) \, \mathrm{d}\lambda(y) = 0,$$

where the last equality follows from H being Haar null. Thus there exists $x \in G$ such that $\lambda(x^{-1}H) = 0$, which means that $|C \setminus x^{-1}H|$ must be uncountable. Therefore $|xH \setminus H|$ is uncountable. Notice that $xH \setminus H$ is Haar null (μ is a witness measure), which completes the proof. □

Lemma 9.7. *Let G be an uncountable Polish group that admits a two-sided invariant metric. Then for every $H \in \mathcal{HM}(G)$ there exists an uncountable Haar meager set $H' \subset G \setminus H$.*

Proof. Let C be a compact metric space and $f : C \to G$ a witness function of H. Without loss of generality, we may assume that H is Borel. In Polish groups with two-sided invariant metric every compact set is Haar meager (see [10, Corollary 22]), in particular, there exists an uncountable Haar meager set. So the lemma is trivial if H is countable. So we can assume that $|H| \geq \omega_1$ and thus by [17, Theorem 13.6] H contains a set K which is homeomorphic to 2^ω. Let $B = \{(x,y) \in K \times C : xf(y) \in H\}$. It is easy to see that B is Borel in the product space $K \times C$ since it is the preimage of H by the continuous function $(x,y) \mapsto xf(y)$. Thus we may apply the Kuratowski-Ulam theorem (Theorem 4.24) in the product space $K \times C$ for the set B. Notice that $B_x = f^{-1}(x^{-1}H)$ is meager in C for every $x \in K$ by the fact that f is witness function, thus B is meager in $K \times C$. Consequently, there exists $y \in C$ such that $B^y = Hf(y)^{-1}$ is meager in K. But then $|K \setminus Hf(y)^{-1}|$ must be uncountable, therefore $|Hf(y) \setminus H|$ is uncountable. Notice that $Hf(y) \setminus H$ is Haar meager (the same function f and compact space C witness this), which completes the proof. □

The second part of the theorem follows from Theorem 9.4 and [9, Example 24], which states that $\mathbb{Z}^\omega$ is the union of a Haar null and a Haar meager set. □

We show on the other hand that there is no Borel isomorphism between Haar null and Haar meager σ-ideals of Polish groups that admit a two-sided invariant metric.

Theorem 9.8. *Let G and H be uncountable Polish groups, moreover suppose that H admits a two-sided invariant metric. Then there is no Borel isomorphism between the σ-ideals $\mathcal{HM}(G)$ and $\mathcal{HN}(H)$, that is, there is no Borel isomorphism $f : G \to H$ such that $X \in \mathcal{HM}(G)$ if and only if $f(X) \in \mathcal{HN}(H)$.*

Proof. Suppose to the contrary that there is such an f. It is well-known that every Borel function between Polish spaces is continuous on a comeager set. Thus let U be a comeager set, such that $f\mid_U$ is continuous. We will construct a comeager set $V \subset U$ such that $f(V)$ is Haar null in H. We know from Theorem 6.14 that $\mathcal{HM}(G) \subseteq \mathcal{M}(G)$, thus the image of the non-Haar meager set V is in $\mathcal{HN}(H)$ which is a contradiction.

Let us fix metrics on G and H such that the metric fixed on H is two-sided invariant. Due to our assumption U is comeager, thus it contains a dense G_δ set. Using that f is a Borel isomorphism, we get that $f(U)$ is uncountable and Borel, thus (because of [17, Theorem 13.6]) it contains a set C that is homeomorphic with 2^ω. Let μ be an arbitrary continuous Borel probability measure with support C, and let us fix a constant $c > 0$. It is easy to check that since μ is a continuous Borel measure with compact support, for any fixed $\varepsilon > 0$ there exists $\delta > 0$ such that if $\operatorname{diam}(B) < \delta$ then $\mu(B) < \varepsilon$. Accordingly let us fix $\delta_n > 0$ for $\varepsilon = c2^{-n-1}$ for every $n \in \omega$. Let $(x_n)_{n\in\omega}$ be an enumeration of a dense set in U. From the continuity of f and the invariance of the metric on H there is r_n for every n such that $\operatorname{diam}(hf(B(x_n, r_n))g) < \delta_n$ for all $g, h \in H$. Thus $\mu\left(hf\left(\bigcup_{n\in\omega} B(x_n, r_n)\right) g\right) < c$ for every $g, h \in H$. Notice that $U \cap \bigcup_{n\in\omega} B(x_n, r_n) := U_c$ is a dense open set in U. Let us construct U_{c_n} for a zero-sequence c_n, and let $V := \bigcap_n U_{c_n}$. Then V is comeager in U (as it is dense G_δ). But $f(V)$ is Haar null, as it is Borel and $\mu(hf(V)g) = 0$ for all $g, h \in H$, which completes the proof. □

Finally, we finish this section with an interesting corollary of Theorem 9.8.

Corollary 9.9. *There is no isomorphism between the σ-ideals $\mathcal{HN}(\mathbb{Z}^\omega)$ and $\mathcal{HM}(\mathbb{Z}^\omega)$ that is a homomorphism of $\mathbb{Z}^\omega$.*

Proof. Suppose that there exists an isomorphism $f : \mathbb{Z}^\omega \to \mathbb{Z}^\omega$ between the σ-ideals $\mathcal{HM}(\mathbb{Z}^\omega)$ and $\mathcal{HN}(\mathbb{Z}^\omega)$ which is also a homomorphism. From [22, Theorem 3.2] we know that every homomorphism from $\mathbb{Z}^\omega$ to $\mathbb{Z}$ is continuous, consequently so is every homomorphism from $\mathbb{Z}^\omega$ to $\mathbb{Z}^\omega$. So f is a continuous bijection, then from [17, Theorem 15.1] we know that it is a Borel isomorphism. But this contradicts Theorem 9.8, which concludes the proof. □

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