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Weighted exchange distance of basis pairs

— Diploma Thesis —

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A **szakdolgozat** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

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Contents

Chapter 1

Introduction

The thesis presents the results appeared in [2] with the addition of Section 3.2 which builds upon results from discussions at the Emléktábla Workshop on Matroid Optimization, 2022.

Given a matroid M over a ground set S, the exchange axiom implies that for any pair of bases R and B there exists a sequence of exchanges that transforms R into B, and the shortest length of such a sequence is $|R - B|$. In the light of this, it is natural to ask whether analogous results hold for basis pairs instead of single bases. More precisely, let (R, B) be an ordered pair of disjoint bases of M, and let $e \in R$ and $f \in B$ be such that both $R' := R - e + f$ and $B' := B + e - f$ are bases. In such a case, we call the exchange **feasible** and say that the pair (R', B') is obtained from (R, B) by a **symmetric exchange**. Using this terminology, we define the **exchange distance** (or **distance** for short) of two pairs of disjoint bases $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ to be the minimum number of symmetric exchanges needed to transform the former into the latter if such a sequence exists and $+\infty$ otherwise. We call two pairs of disjoint bases **equivalent** if their exchange distance is finite. A sequence of symmetric exchanges starting from a pair $\mathbf{P}_1$ is called **strictly monotone with respect to another pair $\mathbf{P}_2$** (or **strictly monotone** for short when $\mathbf{P}_2$ is clear from the context) if each step decreases the difference between the first member of the current pair and that of $\mathbf{P}_2$. In other words, a strictly monotone exchange sequence uses elements only from $(R_1 \cap B_2) \cup (R_2 \cap B_1)$ and at most once.

At this point it is not clear (I) when the distance of two pairs will be finite, and (II) if their distance is finite, whether we can give an upper bound on it.

Regarding question (I), one can formulate an obvious necessary condition for the distance of $\mathbf{P}_1$ and $\mathbf{P}_2$ to be finite, namely $R_1 \cup B_1 = R_2 \cup B_2$ should certainly hold – two pairs with this property are called **compatible**. In [16], White conjectured that *two basis pairs $\mathbf{P}_1$ and $\mathbf{P}_2$ are equivalent if and only if they are compatible*. While the conjecture was verified for various matroid classes, it remains open in general. By relying on the constructive characterization of bispanning graphs, Farber, Richter, and

Shank [8] proved White's conjecture for graphic and cographic matroids, while Farber [7] settled the statement for transversal matroids. Bonin [5] verified the conjecture for sparse paving matroids. The case of strongly base orderable matroids was solved by Lasoń and Michałek [13]. McGuinness [14] extended the graphic case to frame matroids satisfying a certain linearity condition. Kotlar and Ziv [12] showed that any two elements of a basis have a sequential symmetric exchange with some two elements of any other basis. At the same time, Kotlar [11] proved that three consecutive symmetric exchanges exist for any two bases of a matroid, and that a full sequential symmetric exchange, of length at most 6, exists for any two bases of a matroid of rank 5.

The question can be formulated in a more general way. Let $\mathcal{X} = (X_1, \ldots, X_m)$ be a sequence of bases of a matroid M, and assume that there exist $e \in X_i - X_j$, $f \in X_j - X_i$ with $1 \leq i < j \leq m$ such that both $X_i - e + f$ and $X_j + e - f$ are bases. Then we say that $\mathcal{X}' = (X_1, \ldots, X_{i-1}, X_i - e + f, X_{i+1}, \ldots, X_{j-1}, X_j + e - f, X_{j+1}, \ldots, X_m)$ is obtained from $\mathcal{X}$ by a symmetric exchange. Two sequences $\mathcal{X}$ and $\mathcal{Y}$ are **equivalent** if $\mathcal{Y}$ can be obtained from $\mathcal{X}$ by a composition of symmetric exchanges. Furthermore, $\mathcal{X}$ and $\mathcal{Y}$ are called **compatible** if $|\{i : s \in X_i, 1 \leq i \leq m\}| = |\{i : s \in Y_i, 1 \leq i \leq m\}|$ for every $s \in S$. The more general version of White's conjecture is the following:

Conjecture 1.1 (White). *Two basis sequences $\mathcal{X}$ and $\mathcal{Y}$ of the same length are equivalent if and only if they are compatible.*

Much less is known about question (II), that is, the optimization version of the problem. Gabow [9] studied sequential symmetric exchanges and posed the following problem, which was later stated as a conjecture by Wiedemann [17] and by Cordovil and Moreira [6]:

Conjecture 1.2 (Gabow). *Let A and B bases of the same matroid of rank r. Then there are orderings $A = (a_1, \ldots, a_r)$ and $B = (b_1, \ldots, b_r)$ such that $\{a_1, \ldots, a_i, b_{i+1}, \ldots, b_r\}$ and $\{b_1, \ldots, b_i, a_{i+1}, \ldots, a_r\}$ are bases for $i = 0, \ldots, r$.*

In other words: *for any two disjoint bases R and B of a matroid M, there is a sequence of r symmetric exchanges that transforms the pair $\mathbf{P}_1 = (R, B)$ into $\mathbf{P}_2 = (B, R)$.* The rank of the matroid is a trivial lower bound on the minimum number of exchanges needed to transform a pair (R, B) into (B, R), and the essence of Gabow's conjecture is that that many steps might always suffice. The relation between the conjectures of White and Gabow is immediate: the latter would imply the former for sequences of the form (R, B) and (B, R). Gabow's conjecture was verified for partition matroids, matching and transversal matroids, and matroids of rank less than 4 in [9], and an easy reasoning shows that it also holds for strongly base orderable matroids as well. The graphic case was settled by Wiedemann [17], Kajitani, Ueno, and Miyano [10], and Cordovil and Moreira [6].

In general, if M has rank r, then $r - |R_1 \cap R_2|$ is an obvious lower bound on the exchange distance of $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$. However, it might happen that more symmetric exchanges are needed even if M is a graphic matroid; see [7] or Figure 5.2 for a counterexample. As a common generalization of the conjectures of White and Gabow, Hamidoune [6] proposed a rather optimistic variant stating that

Conjecture 1.3 (Hamidoune). *Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be compatible basis pairs of a rank-r matroid M over a ground set S. Then the exchange distance of $\mathbf{P}_1$ and $\mathbf{P}_2$ is at most r.*

Recently, Bérczi and Schwarcz [3] showed that Hamidoune's conjecture hold for split matroids, a large class that contains paving matroids as well. While studying a specific maker-breaker game on bispanning graphs, Andres, Hochstättler and Merkel [1] showed that there is an exchange sequence between any two pairs of disjoint spanning trees of a wheel of rank at least four using only so-called left unique exchanges. They also asked whether the exchange distance of compatible basis pairs of a matroid can be bounded by a polynomial of the rank – this latter problem is a weakening of Hamidoune's conjecture.

Let $w\colon S \to \mathbb{R}_+$ be a weight function on the elements of the ground set S. Given a pair (R, B) of disjoint bases, we define the **weight of a symmetric exchange** $R' := R - e + f$ and $B' := B + e - f$ to be $w(e) + w(f)$, that is, the sum of the weights of the exchanged elements. Analogously to the unweighted setting, we define the **weighted exchange distance** (or **weighted distance** for short) of two pairs of disjoint bases $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ to be the minimum total weight of symmetric exchanges needed to transform the former into the latter if such a sequence exists and $+\infty$ otherwise. As a weighted extension of Hamidoune's conjecture, we propose the following.

Conjecture 1.4. *Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be compatible basis pairs of a matroid M over a ground set S, and let $w\colon S \to \mathbb{R}_+$ be a weight function. Then the weighted exchange distance of $\mathbf{P}_1$ and $\mathbf{P}_2$ is at most $w(R_1 \cup B_1) = w(R_2 \cup B_2)$.*

By setting the weight function to be identically one, we get back Hamidoune's conjecture. It is worth mentioning that a strictly monotone exchange sequence transforming $\mathbf{P}_1$ into $\mathbf{P}_2$ is optimal in every sense, i.e., it has both minimum length and minimum weight.

Outline of the thesis. In **Chapter 2** we describe basic notations and explain the fundamental concepts used throughout the thesis. We also also look at some elementary observations on wheels.

In **Chapter 3** first we verify Conjecture 1.4 for strongly base orderable matroids, a class with distinctive structural properties. Then we describe in detail an algorithm for White's conjecture for graphic matroids and observe that the number of steps is $O(n^2)$. This latter result will follow from observing that every edge will be used at most n times.

In **Chapter 4**, we work with a further strengthening of the conjecture where both the length and the weight of the exchange sequence are ought to be bounded. The main result in the thesis is a proof of the conjecture for graphic matroids of wheels. Though wheels are structurally rather simple, the proof for this graph class is already non-trivial and requires a thorough understanding of feasible exchanges. As a byproduct, we show that the minimum number of steps required to transform $\mathbf{P}_1$ into $\mathbf{P}_2$ can be arbitrarily large compared to the lower bound $r - |R_1 \cap R_2|$.

Finally in **Chapter 5** we prove the conjecture for spikes, an important class of 3-connected matroids. Spikes are interesting because, as we show, one can define an arbitrarily large number of basis pairs without a strictly monotone exchange sequence between any two of them. This is in sharp contrast to the case of wheels, where for any three pairs of bases, there exist two with a strictly monotone exchange sequence between them.

Chapter 2

Preliminaries

General notation. The set of nonnegative real numbers is denoted by $\mathbb{R}_+$. For subsets $X, Y \subseteq S$, their **symmetric difference** is defined as $X \triangle Y := (X \setminus Y) \cup (Y \setminus X)$. When Y consist of a single element y, then $X \setminus \{y\}$ and $X \cup \{y\}$ are abbreviated as $X - y$ and $X + y$, respectively. Given a weight function $w \colon S \to \mathbb{R}_+$ and a subset $X \subseteq S$, we use the notation $w(X) = \sum_{s \in X} w(s)$.

Matroids. We present two ways one can define a matroid.

A **matroid** is a pair $(S, \mathcal{I})$ where S is a finite set and $\mathcal{I}$ is a family of subsets fulfilling the following properties:

(I1) $\emptyset \in \mathcal{I}$,

(I2) $X \subseteq Y \in \mathcal{I} \Rightarrow X \in \mathcal{I}$,

(I3) $X, Y \in \mathcal{I}, |X| < |Y| \Rightarrow \exists e \in Y \setminus X$ such that $X + e \in \mathcal{I}$.

We call the members of $\mathcal{I}$ **independent sets** and maximal independent sets are called **bases**. We can also define a matroid with its bases. Let $(S, \mathcal{B})$ be a pair where S is a finite set and $\mathcal{B}$ (these are the bases of the matroid) is a family of subsets with the following properties:

(B1) $\mathcal{B}$ is nonempty,

(B2) If A and B are distinct elements of $\mathcal{B}$ and $a \in A \setminus B$ then there exists an element $b \in B - A$ such that $A - a + b \in \mathcal{B}$.

Then the pair $(S, \mathcal{B})$ forms a matroid. Property (B2) is called the **basis exchange property**. It can be shown that (B2) is equivalent to the following:

(B2') If A and B are distinct elements of $\mathcal{B}$ and $a \in A \setminus B$ then there exists an element $b \in B - A$ such that $A - a + b \in \mathcal{B}$ and $B - b + a \in \mathcal{B}$.

Property (B2') is called the **symmetric exchange property**.

If M is a rank-r matroid over a ground set S of size $2r$ such that S decomposes into two disjoint bases R and B of M, then such a decomposition is called a **coloring** of M. Throughout the thesis, we will refer to the elements of R and B as 'red' and 'blue', respectively. A **feasible exchange** of elements $r \in R$ and $b \in B$ means that $R - r + b$ and $B - b + r$ are both bases of M; such an exchange is denoted by (r, b). We extend this notation to a **sequence of symmetric exchanges** as well by writing $(r_1, b_1), \ldots, (r_k, b_k)$, meaning that $(R \setminus \{r_1, \ldots, r_i\}) \cup \{b_1, \ldots, b_i\}$ and $(B \cup \{r_1, \ldots, r_i\}) \setminus \{b_1, \ldots, b_i\}$ are bases for $i = 1, \ldots, k$.

A matroid M is **strongly base orderable** if for any two bases B_1, B_2, there exists a bijection $\phi \colon B_1 \to B_2$ such that $(B_1 \setminus X) \cup \phi(X)$ is also a basis for any $X \subseteq B_1$, where we denote $\phi(X) := \{\phi(e) \mid e \in X\}$. In this case one can easily see that $(B_2 \setminus \phi(X)) \cup X$ is also a basis.

For a graph $G = (V, E)$ on n vertices, the **graphic matroid** $M = (E, \mathcal{I})$ of G is defined on E by considering a subset $F \subseteq E$ to be independent if it is a forest, that is, $\mathcal{I} = \{F \subseteq E \mid F \text{ does not contain a cycle}\}$. If the graph is connected, then the bases of the graphic matroid are exactly the spanning trees of G and the rank of the matroid is $n - 1$. A graph is called a **bispanning graph** if it is a union of two edge-disjoint spanning trees. In such a case the graphic matroid defined on this graph can be decomposed into the union of two disjoint bases.

Let $S = \{t, x_1, y_1, \ldots, x_r, y_r\}$ be a ground set of size $2r + 1$ for some $r \geq 3$, and let $\mathcal{C}_1 = \{\{t, x_i, y_i\} \mid 1 \leq i \leq r\}$ and $\mathcal{C}_2 = \{\{x_i, y_i, x_j, y_j\} \mid 1 \leq i < j \leq r\}$. Furthermore, let $\mathcal{C}_3 \subseteq \{Z \subseteq S \mid |Z| = r, \ |Z \cap \{x_i, y_i\}| = 1 \text{ for } 1 \leq i \leq r\}$ be such that the intersection of any two members of $\mathcal{C}_3$ has size at most $r - 2$. Note that $\mathcal{C}_3$ might be empty. Finally, define $\mathcal{C}_4 = \{C \subseteq S \mid |C| = r + 1, \ C' \not\subseteq C \text{ for } C' \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3\}$. Then the family $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$ satisfies the circuit axioms, hence $M = (S, \mathcal{C})$ is a rank-r matroid with circuit family $\mathcal{C}$. Matroids arising this way are called **spikes**, where t and the pairs $\{x_i, y_i\}$ are called the **tip** and the **legs** of the spike, respectively. It is not difficult to check that restricting M to any $2r$ of its elements (or in other words, deleting any of its elements) results in a matroid whose ground set decomposes into two disjoint bases.

Wheels. A graph $G = (V, E)$ is called a **wheel graph** (or **wheel** for short) if it is obtained by connecting a vertex, called the **center of the wheel**, to all the vertices of a cycle of length at least three, called the **outer cycle** of the wheel. In particular, wheels have at least four vertices. Edges connecting the center vertex with the vertices of the outer cycle are called **spokes**, while the edges of the outer cycle are called **rim edges**. Wheels are clearly planar, and so the order of the vertices on the outer cycle implies a natural cyclic ordering of the spokes and the rim edges as well.

(a) The red and blue intervals are denoted by I_1, I_3 and I_2, I_4, respectively.

(b) The coloring obtained by symmetrically exchanging e and its pair $\varphi_{\mathbf{P},-}(e)$.

Figure 2.1: Colorings containing four intervals. Thick rim edges correspond to boundary edges. Note that both colorings have positive orientation.

It is not difficult to check that any wheel is the disjoint union of two spanning trees. Therefore, a coloring of the graphic matroid of a wheel is basically a partition of its edge set into two colors R and B such that both color classes form a spanning tree. A nice property of wheels is that we have a fairly good understanding of the structure of their colorings. Indeed, in order to decompose a wheel into two spanning trees, we first need to split the spokes into two nonempty sets. Then, if a rim edge goes between the endpoints of two spokes having the same color, then it automatically has to be colored with the other color to obtain a basis. Hence it only remains to decide the color of the rim edges going between the endpoints of spokes having different colors. However, once we fix the color of any of those edges, it determines the color of all the remaining ones.

We call a maximal set of consecutive spokes of the same color an **interval**. By the **length** and **color of the interval** we mean the number and color of the spokes in it, respectively. Rim edges going between two intervals are called **boundary edges**. By the above, for $X \in \{R, B\}$, either every interval of color X is followed by a boundary edge of color X in a counterclockwise direction, or every interval of color X is followed by a boundary edge of color X in a clockwise direction. This property is referred to as the **orientation** of the coloring, and the orientation is called **positive** in the former and **negative** in the latter case; see Figure 2.1a for an example. Orientations will play a crucial role in whether one can go from one coloring to another using a small number of exchanges or not.

Given a coloring $\mathbf{P} = (R, B)$ of the wheel graph, each spoke e is incident to two rim edges. The rim edge sharing a vertex with e in the direction opposite of the orientation of the coloring can always be exchanged with e. We denote this rim edge by $\varphi_{\mathbf{P},-}(e)$, while the other rim edge incident to e is denoted by $\varphi_{\mathbf{P},+}(e)$; see Figure 2.1b for an example. Both $\varphi_{\mathbf{P},-}$ and $\varphi_{\mathbf{P},+}$ provide bijections between spokes and rim edges. Note that these bijections are already determined by the orientation of $\mathbf{P}$. If there are at least four intervals in the coloring, then the feasible exchanges are exactly the ones exchanging

e and $\varphi_{\mathbf{P},-}(e)$ for some spoke e. When there are only two intervals in the coloring, then there are other pairs that can be symmetrically exchanged.

Chapter 3

Strongly base orderable and graphic matroids

3.1 Strongly base orderable matroids

As a warm-up, we first consider the basic case of strongly base orderable matroids. For strongly base orderable matroids, the proof of Conjecture 1.4 can be read out directly from the proof of White's Conjecture 1.1 in [13] for strongly base orderable matroids. However, it might help the reader to get familiar with the notion of basis exchanges, and also sheds light to the difficulties caused by the presence of a weight function.

Theorem 3.1. *Let* $\mathbf{P}_1 = (R_1, B_1)$ *and* $\mathbf{P}_2 = (R_2, B_2)$ *be compatible pairs of disjoint bases of a rank-r strongly base orderable matroid* M *over a ground set* S, *and let* $w \colon S \to \mathbb{R}_+$ *be a weight function. Then there exists a sequence of exchanges of total weight at most* $w(R_1 \cup B_1) = w(R_2 \cup B_2)$ *that transforms* $\mathbf{P}_1$ *into* $\mathbf{P}_2$ *and uses each element at most twice.*

Proof. Let $\phi_1 \colon R_1 \to B_1$ and $\phi_2 \colon R_2 \to B_2$ be bijections such that $(R_i \setminus X) \cup \phi_i(X)$ is a basis for each $X \subseteq R_i$ and $i = 1, 2$. Consider the bipartite graph G with vertex set $R_1 \cup B_1 = R_2 \cup B_2$, and edges of the form $e\phi_1(e)$ for $e \in R_1$ and $f\phi_2(f)$ for $f \in R_2$. We denote the color classes of G by S and T. Note that

$$S = (S \cap R_1) \cup (S \cap B_1) = (R_1 \setminus (R_1 \setminus S)) \cup \phi_1(R_1 \setminus S)$$

and

$$T = (T \cap R_1) \cup (T \cap B_1) = (R_1 \setminus (R_1 \setminus T)) \cup \phi_1(R_1 \setminus T),$$

hence both S and T are bases of M. Let us define the basis pairs $\mathbf{P} = (S, T)$ and $\mathbf{P}' = (T, S)$.

By exchanging the elements between $R_1 \setminus S$ and $S \setminus R_1$ according to ϕ_1, we get a sequence of weight $w(R_1 \triangle S)$ that transforms $\mathbf{P}_1$ into $\mathbf{P}$. By exchanging the elements between $S \setminus R_2$ and $R_2 \setminus S$ according to ϕ_2, we get a sequence of weight $w(R_2 \triangle S)$ that transforms $\mathbf{P}$ into $\mathbf{P}_2$. The concatenation of these two sequences transforms $\mathbf{P}_1$ into $\mathbf{P}_2$, has total weight $w(R_1 \triangle S) + w(R_2 \triangle S)$, and uses each element at most twice.

By exchanging the elements between $R_1 \setminus T$ and $T \setminus R_1$ according to ϕ_1, we get a sequence of weight $w(R_1 \triangle T)$ that transforms $\mathbf{P}_1$ into $\mathbf{P}'$. By exchanging the elements between $T \setminus R_2$ and $R_2 \setminus T$ according to ϕ_2, we get a sequence of weight $w(R_2 \triangle T)$ that transforms $\mathbf{P}$ into $\mathbf{P}_2$. The concatenation of these two sequences transforms $\mathbf{P}_1$ into $\mathbf{P}_2$, has total weight $w(R_1 \triangle T) + w(R_2 \triangle T)$, and uses each element at most twice.

Since

$$w(R_1 \triangle S) + w(R_2 \triangle S) + w(R_1 \triangle T) + w(R_2 \triangle T)$$
$$= (w(R_1 \triangle S) + w(R_1 \triangle T)) + (w(R_2 \triangle S) + w(R_2 \triangle T))$$
$$= (w(R_1) + w(B_1)) + (w(R_2) + w(B_2)),$$

at least one of the above defined sequences has total weight at most $w(R_1 \cup B_1) = w(R_2 \cup B_2)$ and uses each element at most twice. This concludes the proof of the theorem. $\qquad\square$

3.2   Graphic matroids

The rank of the graphic matroid of a connected graph on n vertices is $n - 1$. Though it is not stated explicitly, the algorithms of [8] and [4] that prove White's conjecture for graphic matroids give a sequence of exchanges of length at most $O(n^2)$, as was observed by Bérczi, Kobayashi, Király and Szabó at the Emléktábla Workshop on Matroid Optimization, 2022. It remains an intriguing open problem to improve the bound to $O(n)$, matching the order of the bound in the conjecture of Hamidoune. We present here the algorithm in detail and study the number of steps it requires.

In what follows, we prove White's conjecture for graphic matroids, together with an upper bound on the number of exchanges.

Theorem 3.2. *Let $G = (V, E)$ be a bispanning graph and let M be its graphic matroid. Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be two pairs of disjoint bases of M. Then there exists a sequence of $O(n^2)$ exchanges that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$.*

The proof will rely on a lemma made on bispanning graphs

Lemma 3.3. *Let $G = (V, E)$ be a bispanning graph. Then there exists a vertex with degree at most 3.*

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Proof. Let $|V| = n$, then $|E| = 2n - 2$. Suppose to the contrary that $d(v) \geq 4$ for every $v \in V$. Then we have

$$2n - 2 = |E| = \sum_{v \in V} d(v)/2 \geq 2n, \tag{3.1}$$

a contradiction. $\qquad\square$

Now we prove Theorem 3.2.

Proof of Theorem 3.2. We will use induction on $|V| = n$. The case when $n \leq 3$ is not difficult to check.

Suppose that the statement holds for graphs with at most n vertices. We prove that it also holds for graphs having $n + 1$ vertices. Using Lemma 3.3, we get that there exists a vertex v, whose degree is at most 3. We discuss two cases based on the degree of v. Recall that we refer to the elements of R_i and B_i as **red** and **blue** elements.

Case 1. v has degree 2.

Let e_1 and e_2 denote the two edges adjacent to v. e_1 and e_2 have different colors. Since after every exchange both the red and blue subgraphs must stay spanning trees, these two edges can only be exchanged with one another.

Delete v from the graph together with the two edges incident to it. Let $G' = G - v$ denote the remaining graph, and let $R_i' = R_i \setminus \{e_1, e_2\}$ and $B_i' = B_i \setminus \{e_1, e_2\}$ for $i = 1, 2$. Applying the inductive hypothesis to the pairs $\mathbf{P}_1' = (R_1', B_1')$ and $\mathbf{P}_2' = (R_2', B_2')$ of disjoint spanning trees of the graph G' we get an exchange sequence from $\mathbf{P}_1'$ to $\mathbf{P}_2'$. Now apply the exact same sequence to G. At the end of the sequence the basis pair we acquire is either (R_2, B_2) or $(R_2 - e_1 + e_2, B_2 - e_2 + e_1)$. In the former case we have a sequence that fulfills the required conditions. In the latter case the symmetric exchange of e_1 and e_2 is feasible and exchanging them yields an exchange sequence from $\mathbf{P}_1$ to $\mathbf{P}_2$.

Case 2. v has degree 3.

Let the edges incident to v be a, b and c. We may assume that $a, b \in R_1$ and $c \in B_1$, that is, a and b are red and c is blue at the start. At first we deal with the case when the coloring of a, b and c remains the same, or all of them change in the second basis pair.

Case 2.1 Either $a, b \in R_2$ and $c \in B_2$, or $a, b \in B_2$ and $c \in R_2$.

By deleting v and the edges a, b, c the remaining graph will have n vertices, but it will not be a bispanning graph. In order to overcome this we are going to insert a new edge e into the graph. The new edge shall connect the end points of a and b that are not v and let $\mathbf{P}_1' = (R_1 - a - b + e, B_1 - c)$. If $a, b \in R_2$ and $c \in B_2$ then let $\mathbf{P}_2' = (R_2 - a - b + e, B_2 - c)$ and let $\mathbf{P}_2' = (R_2 - c, B_2 - a - b + e)$ otherwise. Let this new graph be G'. $\mathbf{P}_1'$ and $\mathbf{P}_2'$ will form two disjoint spanning trees of G'. Now apply the inductive hypothesis to G'.

If e does not change color, then we can just use the same sequence in G.

If e changes color we have to simulate this in the original graph as well. Let f be the edge which would be symmetrically exchanged with e. Now consider G. By the symmetric exchange property c can be symmetrically exchanged with some edge. In order to maintain both color classes being a spanning tree this edge can only be a or b. We may assume it is exchangeable with b and exchange them. After this step a can be symmetrically exchanged with f because exchanging e and f was feasible.

This way we could simulate an exchange of e with the edges a, b, c. Continuing the process we will acquire a desired exchange sequence concluding this case.

Case 2.2 The coloring of a, b, c in $\mathbf{P}_2$ is not like in the previous case.

As we have observed before the edge c can be symmetrically exchanged with at least one of a and b. We may assume that c and b can be exchanged.

Exchanging these two edges will result in a and c being red and b being blue. Observe that in this case if $\mathbf{P}_2$ is colored the same way, that is $a, c \in R_2$ and $b \in B_2$ or they are colored the opposite way, that is $a, c \in B_2$ and $b \in R_2$, then we are in exactly the same situation as in Case 2.1. So we can use the same algorithm with these two extra steps.

We are left with only two possibilities for the coloring of a, b, c in $\mathbf{P}_2$. If $a \in B_2$ and $b, c \in R_2$ then we can symmetrically exchange a with b or c. In either case we are left with a case already solved. Similarly if $a \in R_2$ and $b, c \in B_2$ then again we can symmetrically exchange a with either b or c, and both colorings will give us already solved cases. This way we have found an exchange sequence for all possible colorings for an $n + 1$ vertex graph.

To bound the length of the sequence thus obtained, we analyse the steps of the algorithm. In particular, we claim that every edge is used in at most n exchanges (where n is the number of vertices).

The proof is by induction on $|V|$. Indeed, suppose we know the statement for some n. Consider a graph G with $n + 1$ vertices. If we are in the case when G has a degree 2 vertex then we only have to use the edges incident to it at most once. If there is a vertex with degree 3, then the steps we used to simulate the exchange made in $G - v$, used each edge incident to v once. By the induction hypothesis, this results in at most n exchanges for these edges, but we may need an extra exchange for each of them to make the colorings to look like as in Case 2.1. The edges not incident to v are used at most n times by the induction hypothesis. This results in at most $n + 1$ uses for each edge.

Since G is a bispanning graph, it has $2n - 2$ edges. As we used each edge at most n times, the total number of exchanges is bounded by $(2n - 2)n/2$. $\qquad\square$

Remark 3.4. The same analysis shows that if a weight function $w \colon E \to \mathbb{R}_+$ is given, then the algorithm results in a sequence of exchanges of total weight at most $n \cdot w(E)$.

Chapter 4

Wheels

Our first main result is a proof of Conjecture 1.4 for the graphic matroid of wheels. In fact, we prove a much stronger statement: we verify that for any pair $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ of colorings of a wheel $G = (V, E)$, there exists a sequence of exchanges of length at most r and total weight at most $w(E)$ that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ and uses each edge at most twice.

Throughout the chapter, we assume that $\mathbf{P}_1$ has positive orientation. For ease of notation, we introduce $\varphi_\oplus := \varphi_{\mathbf{P}_1,+}$ and $\varphi_\ominus := \varphi_{\mathbf{P}_1,-}$. Recall that both $\varphi_\ominus$ and $\varphi_\oplus$ provide a bijection between spokes and rim edges.

First we settle the case when the two colorings have the same orientation.

Lemma 4.1. *Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be colorings of a wheel $G = (V, E)$ with the same orientation. Then there exists a strictly monotone sequence of exchanges that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$.*

Proof. Exchange each spoke e that has different color in $\mathbf{P}_1$ than in $\mathbf{P}_2$ with its pair $\varphi_\ominus(e)$ in an arbitrary order, only paying attention to always have at least one spoke in both color classes. Once the spokes have the right colors, that is, they are colored as in $\mathbf{P}_2$, the rim edges are also colored as required. Indeed, the orientation was not changed during the procedure and the coloring of the spokes together with the orientation of the coloring uniquely determines the colors of the rim edges. $\square$

Remark 4.2. Exchanging a spoke e with its pair $\varphi_\ominus(e)$ does not reverse the orientation of the coloring. Therefore, if the initial coloring $\mathbf{P}_1 = (R_1, B_1)$ and the target coloring $\mathbf{P}_2 = (R_2, B_2)$ have different orientations, then any sequence of feasible exchanges must go through a coloring of the graph with only two intervals. This results in reversing the orientation of a coloring to be costly, as one needs to make exchange steps that, at least seemingly, bring the colorings further away from each other in the sense that they increase the symmetric difference of R_1 and R_2. The class of wheels is particularly interesting as
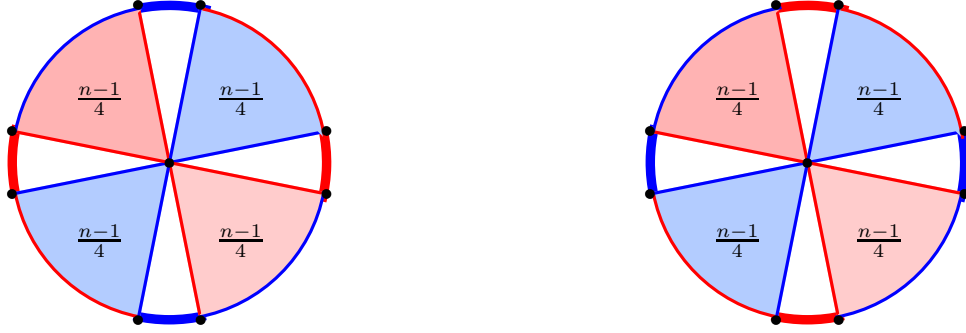
(a) A coloring $\mathbf{P}_1 = (R_1, B_1)$ with positive orientation. Thick edges denote the boundary edges.

(b) A coloring $\mathbf{P}_2 = (R_2, B_2)$ with negative orientation. Thick edges denote the boundary edges.

Figure 4.1: A pair of $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ with $(n-1) - |R_1 \cap R_2| = 2$. The colorings have opposite orientations, hence any sequence of exchanges that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ goes through a coloring having two intervals, implying that their exchange distance is at least $\frac{n-1}{4}$.

they admit colorings $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ for which the minimum number of steps required to transform $\mathbf{P}_1$ into $\mathbf{P}_2$ is arbitrarily large compared to the trivial lower bound $(n-1) - |R_1 \cap R_2|$; see Figure 4.1 for an example.

Next we consider colorings with different orientations and a bounded number of intervals in one of the color classes.

Lemma 4.3. *Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be colorings of a wheel $G = (V, E)$ with different orientations where $\mathbf{P}_1$ has at most four intervals, and let $w \colon E \to \mathbb{R}_+$ be a weight function. Then there exists a sequence of exchanges of length at most $n - 1$ and total weight at most $w(E)$ that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ and uses each edge at most twice.*

Proof. We will distinguish two main cases based on the number of intervals in $\mathbf{P}_1$. Recall that $\mathbf{P}_1$ is assumed to have positive orientation throughout.

Case 1. $\mathbf{P}_1$ has two intervals.

Case 1.1. $\mathbf{P}_1$ has an interval of length one.

We may assume that there exists a red interval of length one, and let c denote the unique spoke in it. We denote by a the spoke following c in negative direction, and further define $b := \varphi_\oplus(a) = \varphi_\ominus(c)$ and $d := \varphi_\oplus(c)$. Note that a and b are blue, while d is a red edge; see Figure 4.2a. As $\mathbf{P}_2$ has negative orientation, a and $\varphi_\oplus(a) = b$ have different colors in $\mathbf{P}_2$, and the same holds for c and $\varphi_\oplus(c) = d$. Hence the set of edges among a, b, c and d that have different colors in $\mathbf{P}_1$ and $\mathbf{P}_2$ is either $\{a, c\}, \{a, d\}, \{b, d\}$ or $\{b, c\}$. In the first three cases, changing the color of the two edges is a feasible exchange which reverses the orientation. Once the orientation of the coloring is reversed, there exists

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a strictly monotone exchange sequence to $\mathbf{P}_2$ by Lemma 4.1, altogether resulting in a strictly monotone exchange sequence from $\mathbf{P}_1$ to $\mathbf{P}_2$.

The only remaining case is when the set of edges among a, b, c and d that need to change color is $\{b, c\}$. In this case, the difficulty comes from the fact that these edges do not define a feasible exchange between the two color classes. In order to overcome this, extra steps are needed to reverse the orientation. Let s be an arbitrary red spoke of $\mathbf{P}_2$. Note that $s \notin \{a, c\}$ as we are in the case when a and c are blue in $\mathbf{P}_2$. As $\mathbf{P}_1$ has a unique red spoke, namely c, we get that s is blue in $\mathbf{P}_1$, and $\varphi_\oplus(s)$ is red in $\mathbf{P}_1$ and blue in $\mathbf{P}_2$ since $\mathbf{P}_2$ has negative orientation. Consider the two exchange sequences of length three (b, d), $(s, \varphi_\oplus(s))$, (c, d) and (a, c), $(s, \varphi_\oplus(s))$, (a, b). Both of these sequences reverse the orientation of the coloring and fix the colors of the edges a, b, c and d. Therefore, after applying any of them, there exists a strictly monotone exchange sequence to $\mathbf{P}_2$ by Lemma 4.1 that uses all the remaining edges in $E - \{a, b, c, d\}$ at most once. Thus in overall, we get an exchange sequence that uses each edge in $E - \{a, d\}$ at most once, does not use one of a and d and uses the other twice. Hence the length of the sequence is at most half of the number of edges, that is, $n - 1$. If $w(a) \geq w(d)$, then starting the sequence with (b, d), $(s, \varphi_\oplus(s))$, (c, d), while if $w(a) < w(d)$, then starting the sequence with (a, c), $(s, \varphi_\oplus(s))$, (a, b) ensures that total weight of the exchange sequence is at most $w(E)$, concluding the proof of the case.

Case 1.2. Both intervals of $\mathbf{P}_1$ have length at least two.

Let c denote the last spoke of the red interval in positive direction and let $d := \varphi_\oplus(c)$. Furthermore, let a be the last spoke of the blue interval in positive direction and let $b := \varphi_\oplus(a)$, see Figure 4.2b. Similarly to Case 1.1, the set of edges among a, b, c and d that have different colors in $\mathbf{P}_1$ and $\mathbf{P}_2$ is either $\{a, c\}$, $\{a, d\}$, $\{b, d\}$ or $\{b, c\}$. However, now fixing the orientation is even simpler than before as any of the exchanges (a, c), (a, d), (b, c) and (b, d) is feasible. After reversing the orientation using one of these exchanges, there exists a strictly monotone exchange sequence to $\mathbf{P}_2$ by Lemma 4.1, altogether resulting in a strictly monotone exchange sequence from $\mathbf{P}_1$ to $\mathbf{P}_2$.

We denote the number of spokes in R_1, R_2, B_1 and B_2 by r_1, r_2, b_1 and b_2, respectively. Let $2q$ denote the number of intervals in $\mathbf{P}_1$, and let $I_1, \ldots, I_{2q}$ denote the intervals in a positive direction, where intervals with odd indices have color red and intervals with even indices have color blue. Furthermore, for $1 \leq i \leq 2q$, we define

$$x_i := \sum[w(e) \mid e \in I_i \cup \varphi_\ominus(I_i), \ e \text{ has the same color in } \mathbf{P}_1 \text{ and } \mathbf{P}_2],$$
$$y_i := \sum[w(e) \mid e \in I_i \cup \varphi_\ominus(I_i), \ e \text{ has different colors in } \mathbf{P}_1 \text{ and } \mathbf{P}_2].$$

By the above definitions, we have $w(I_i \cup \varphi_\ominus(I_i)) = x_i + y_i$ for $1 \leq i \leq 2q$, and $\sum_{i=1}^{2q}(x_i + y_i) = w(E)$.

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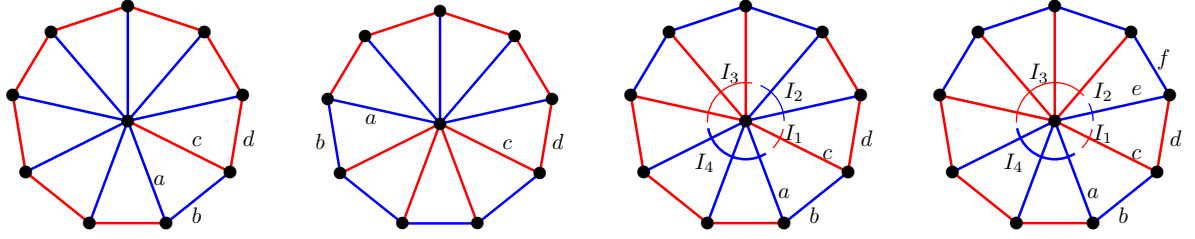
(a) $\mathbf{P}_1$ contains two intervals, one of them having length one.

(b) $\mathbf{P}_1$ contains two intervals, both having length at least two.

(c) $\mathbf{P}_1$ contains four intervals, one of them having length one.

(d) $\mathbf{P}_1$ contains both a red and a blue interval of length one.

Figure 4.2: Illustration of the cases in the proof of Lemma 4.3.

Case 2. $\mathbf{P}_1$ has four intervals.

Since $r_1 + r_2 + b_1 + b_2 = 2(n-1)$, we have $\min\{r_1 + r_2, b_1 + b_2\} \leq n - 1$. We may assume that $r_1 + r_2 \leq n - 1$. We distinguish two cases based on the structure of the red intervals in $\mathbf{P}_1$.

Case 2.1. $\mathbf{P}_1$ has no red interval of length one.

Take one of the intervals $I_j \in \{I_1, I_3\}$ and exchange each spoke p in I_j with its pair $\varphi_\ominus(p)$. The coloring thus obtained has only two intervals from which the red has length more than one, hence there exists a strictly monotone exchange sequence to $\mathbf{P}_2$ as in Case 1.2. Note that each edge is used at most twice during this process, and the edges used twice are exactly the edges of I_j having the same color in $\mathbf{P}_1$ and $\mathbf{P}_2$. The length of the exchange sequence thus obtained is

$$|I_j| + |(I_j \cup I_2 \cup I_4) \cap R_2| + |I_{4-j} \cap B_2| \leq |I_j| + |(I_1 \cup I_2 \cup I_3 \cup I_4) \cap R_2| + |I_{4-j}|$$
$$= r_1 + r_2$$
$$\leq n - 1.$$

In order to bound the total weight of the exchanges, note that we used edges twice only from $I_j \cup \varphi_\ominus(I_j)$, and all the other edges were used only if their color had to change. Thus the total weight of the exchange sequence is $2x_j + y_1 + y_2 + y_3 + y_4$. As $2\min\{x_1, x_3\} \leq x_1 + \cdots + x_4$, we get $2\min\{x_1, x_3\} + y_1 + y_2 + y_3 + y_4 \leq w(E)$. Hence choosing $j \in \{1, 3\}$ such that $x_j = \min\{x_1, x_3\}$ leads to an exchange sequence with the required properties.

Case 2.2. $\mathbf{P}_1$ has a red interval of length one.

Our approach is similar to that of Case 2.1 with slight modifications. By applying the same algorithm, the analysis goes through if the the red interval whose edges are not used before reversing the orientation has length at least two. Indeed, the bounds on the length and weight of the exchange sequence remain valid since we did not use the length of the intervals in the proof. However, a problem occurs if at the orientation reversal step

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we have an interval of length one, that is, when the algorithm starts with a red interval, say I_3 due to $x_3 \le x_1$, and the other red interval I_1 has length one.

Let $\mathbf{P}_1'$ denote the coloring obtained by the exchanges so far, that is, by exchanging each spoke p in I_3 with its pair $\varphi_\ominus(p)$. At this point, our goal is to reverse the orientation and then find a strictly monotone sequence to $\mathbf{P}_2$. Note that this is exactly the same scenario we had in Case 1.1, hence we use an analogous notation that we had there. That is, let c denote the unique red spoke of $\mathbf{P}_1'$ and a be the spoke following c in a negative direction. Furthermore, define $b := \varphi_\oplus(a) = \varphi_\ominus(c)$ and $d := \varphi_\oplus(c)$, see Figure 4.2c. Similarly to Case 1.1, the set of edges among a, b, c and d that have different colors in $\mathbf{P}_1'$ and $\mathbf{P}_2$ is either $\{a, c\}$, $\{a, d\}$, $\{b, d\}$ or $\{b, c\}$. In the first three cases, changing the color of the two edges is a feasible exchange which reverses the orientation and fixes the color of all four edges a, b, c and d. Once the coloring is reversed, there exists a strictly monotone exchange sequence to $\mathbf{P}_2$ by Lemma 4.1. The length and weight of the exchange sequence can be bounded analogously to Case 2.1.

The only remaining case is when the set of edges among a, b, c and d that need to change color is $\{b, c\}$. Let s be an arbitrary red spoke of $\mathbf{P}_2$. Note that $s \notin \{a, c\}$ as we are in the case when a and c are blue in $\mathbf{P}_2$. As $\mathbf{P}_1'$ has a unique red spoke, namely c, we get that s is blue in $\mathbf{P}_1'$, and $\varphi_\oplus(s)$ is red in $\mathbf{P}_1'$ and blue in $\mathbf{P}_2$ since $\mathbf{P}_2$ has negative orientation. Consider the two exchange sequences of length three (b, d), $(s, \varphi_\oplus(s))$, (c, d) and (a, c), $(s, \varphi_\oplus(s))$, (a, b). Both of these sequences change the orientation of the coloring and fix the colors of the edges a, b, c and d. Therefore, after applying any of them, there exists a strictly monotone exchange sequence to $\mathbf{P}_2$ by Lemma 4.1. If $w(a) \ge w(d)$, then use the sequence (b, d), $(s, \varphi_\oplus(s))$, (c, d) for reversing the orientation, and use (a, c), $(s, \varphi_\oplus(s))$, (a, b) otherwise. In what follows, we bound the length and weight of the sequence obtained.

Suppose first that $r_1 + r_2 < n - 1$. The length of the exchange sequence is

$$|I_3| + 3 + |(I_2 \cup I_3 \cup I_4 - s) \cap R_2| = (r_1 - 1) + 3 + (r_2 - 1) = r_1 + r_2 + 1 \le n - 1.$$

We used edges twice only from $I_3 \cup \varphi_\ominus(I_3) \cup \{d\}$ if $w(a) \ge w(d)$, and only from $I_3 \cup \varphi_\ominus(I_3) \cup \{a\}$ otherwise. Thus the total weight of the exchange sequence is at most

$$2x_3 + 2\min\{w(a), w(d)\} + y_1 + y_2 + y_3 + y_4 \le 2x_3 + x_2 + x_4 + y_1 + y_2 + y_3 + y_4 \le w(E),$$

where we used that $2\min\{w(a), w(d)\} \le 2\min\{x_4, x_2\} \le x_2 + x_4$ and $x_3 \le x_1$.

We are left with the case when $r_1 + r_2 = n - 1$, there exists a red interval consisting of a single spoke c, and the edges c and $\varphi_\ominus(c)$ have different colors in $\mathbf{P}_1'$ and $\mathbf{P}_2$. Then, by $r_1 + r_2 + b_1 + b_2 = 2n - 2$, we get that $b_1 + b_2 = n - 1$ also holds. Up to this point the two colors played different roles in the proof of Case 2 as we used the fact that $r_1 + r_2 \le n - 1$.

However, now the same inequality holds for the number of blue spokes as well, hence we can switch the roles of the two colors. In particular, if none of the blue intervals in $\mathbf{P}_1$ have length one, then we are done. Thus suppose that the red interval I_1 consisting of the single spoke c is followed in positive direction by a blue interval I_2 consisting of the single spoke e; the case when a blue interval of length one is followed in positive direction by a red interval of length one can be proved analogously. We denote by a the spoke before c in negative direction, and further define $b := \varphi_\oplus(a) = \varphi_\ominus(c)$, $d := \varphi_\oplus(c) = \varphi_\ominus(e)$ and $f := \varphi_\oplus(e)$, see Figure 4.2d. Recall that we are in the case when the set of edges among a, b, c and d that need to change color is $\{b, c\}$. However, this means that among the edges c, d, e and f, the set of edges that need to change color cannot be $\{d, e\}$. That is, if we start by exchanging each spoke p in I_4 with its pair $\varphi_\ominus(p)$, then the resulting coloring $\mathbf{P}_1''$ can be transformed into $\mathbf{P}_2$ by a strictly monotone sequence of exchanges, and the analysis of the second paragraph of Case 2.2 applies. $\qquad\square$

Our last technical lemma shows that when one of the colorings has at least six intervals, then there exists a sequence of exchanges that has low weight with respect to two arbitrary weight functions simultaneously.

Lemma 4.4. *Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be colorings of a wheel $G = (V, E)$ with different orientations such that $\mathbf{P}_1$ has at least six intervals, and let $w_1, w_2 : E \to \mathbb{R}_+$ be weight functions. Then there exists a sequence of exchanges of total w_i-weight at most $w_i(E)$ for $i = 1, 2$ that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ and uses each edge at most twice.*

Proof. Let $w \in \{w_1, w_2\}$, and let us define I_i, x_i and y_i for $1 \le i \le 2q$ as in the proof of Lemma 4.3, where the x_i and y_i values are computed with respect to w. We distinguish two cases based on the remainder of the number of intervals modulo four.

Case 1. $q = 2k + 1$ for some integer $k \ge 1$.

For an index $1 \le j \le 4k+2$, exchange each spoke $e \in \bigcup_{i=1}^{k} I_{j+2i-1}$ with its pair $\varphi_\ominus(e)$, and do the same for each spoke $e \in \bigcup_{i=k+1}^{2k+1} I_{j+2i}$ (all indices are meant in a cyclic order). After these exchanges, the resulting coloring $\mathbf{P}_1'$ has two intervals: $I_j \cup I_{j+1} \cup \cdots \cup I_{j+2k}$ has the same color in $\mathbf{P}_1'$ as I_j in $\mathbf{P}_1$, and $I_{j+2k+1} \cup I_{j+2k+2} \cup \cdots \cup I_{j+4k+1}$ has the other color. Note that none of these two intervals has length one as $k \ge 1$. Therefore, there exists a strictly monotone exchange sequence from $\mathbf{P}_1'$ to $\mathbf{P}_2$ by Case 1.2 of Lemma 4.3. Our goal is to bound the w-weight of the above defined sequence of exchanges.

Exchanging each spoke e in $\bigcup_{i=1}^{k} I_{j+2i-1} \cup \bigcup_{i=k+1}^{2k} I_{j+2i}$ with its pair $\varphi_\ominus(e)$ has weight

$$\sum_{i=1}^{k}(x_{j+2i-1} + y_{j+2i-1}) + \sum_{i=k+1}^{2k}(x_{j+2i} + y_{j+2i}).$$

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Then the strictly monotone sequence to $\mathbf{P}_2$ has weight

$$\sum_{i=0}^{k} y_{j+2i} + \sum_{i=1}^{k} x_{j+2i-1} + \sum_{i=k}^{2k} y_{j+2i+1} + \sum_{i=k+1}^{2k} x_{j+2i}.$$

The total weight is then

$$2 \cdot \left(\sum_{i=1}^{k} x_{j+2i-1} + \sum_{i=k+1}^{2k} x_{j+2i} \right) + \sum_{i=1}^{4k+2} y_i.$$

Therefore the total w-weight of the exchange sequence is at most $w(E) = \sum_{i=1}^{4k+2}(x_i + y_i)$ if and only if

$$\sum_{i=1}^{k} x_{j+2i-1} + \sum_{i=k+1}^{2k} x_{j+2i} \le \sum_{i=0}^{k} x_{j+2i} + \sum_{i=k}^{2k} x_{j+2i+1}. \qquad (\mathrm{A}_w(j))$$

Consider inequalities $\mathrm{A}_w(j)$ and $\mathrm{A}_w(j+1)$. The sum of these two inequalities gives

$$\left(\sum_{i=1}^{4k+2} x_i \right) - (x_j + x_{j+2k+1}) \le \left(\sum_{i=1}^{4k+2} x_i \right) + (x_j + x_{j+2k+1}).$$

As this inequality clearly holds, at least one of $\mathrm{A}_w(j)$ and $\mathrm{A}_w(j+1)$ must hold as well. Furthermore, $\mathrm{A}_w(j)$ is identical to $\mathrm{A}_w(j+2k+1)$. These together imply that $\mathrm{A}_w(j)$ holds for at least $k+1$ choices of j from $\{1,\dots,2k+1\}$ for $w \in \{w_1, w_2\}$. Therefore, there exists an index j for which both $\mathrm{A}_{w_1}(j)$ and $\mathrm{A}_{w_2}(j)$ are satisfied. As each edge is used at most twice, the statement follows.

Case 2. $q = 2k$ for some integer $k \ge 2$.

Our approach is similar to that of Case 1. For an index $1 \le j \le 4k$, exchange each spoke $e \in \bigcup_{i=1}^{k-1} I_{j+2i-1}$ with its pair $\varphi_{\ominus}(e)$, and do the same for each spoke $e \in \bigcup_{i=k}^{2k-1} I_{j+2i}$. After these exchanges, the resulting coloring $\mathbf{P}'_1$ has two intervals: $I_j \cup I_{j+1} \cup \dots \cup I_{j+2k-1}$ has the same color in $\mathbf{P}'_1$ as I_j in $\mathbf{P}_1$, and $I_{j+2k} \cup I_{j+2k+1} \cup \dots \cup I_{j+4k-1}$ has the other color. Note that none of these two intervals has length one as $k \ge 2$. Therefore, there exists a strictly monotone exchange sequence from $\mathbf{P}'_1$ to $\mathbf{P}_2$ by Case 1.2 of Lemma 4.3. Our goal is to bound the w-weight of the above defined sequence of exchanges.

Exchanging each spoke e in $\bigcup_{i=1}^{k-1} I_{j+2i-1} \cup \bigcup_{i=k}^{2k-1} I_{j+2i}$ with its pair $\varphi_{\ominus}(e)$ has weight

$$\sum_{i=1}^{k-1}(x_{j+2i-1} + y_{j+2i-1}) + \sum_{i=k}^{2k-1}(x_{j+2i} + y_{j+2i}).$$

Then the strictly monotone sequence to $\mathbf{P}_2$ has weight

$$\sum_{i=0}^{k-1} y_{j+2i} + \sum_{i=1}^{k-1} x_{j+2i-1} + \sum_{i=k}^{2k} y_{j+2i-1} + \sum_{i=k}^{2k-1} x_{j+2i}.$$

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The total weight is then

$$2 \cdot \left(\sum_{i=1}^{k-1} x_{j+2i-1} + \sum_{i=k}^{2k-1} x_{j+2i} \right) + \sum_{i=1}^{4k} y_i.$$

Therefore the total w-weight of the exchange sequence is at most $w(E) = \sum_{i=1}^{4k}(x_i + y_i)$ if and only if

$$\sum_{i=1}^{k-1} x_{j+2i-1} + \sum_{i=k}^{2k-1} x_{j+2i} \leq \sum_{i=0}^{k-1} x_{j+2i} + \sum_{i=k-1}^{2k-1} x_{j+2i+1}. \qquad (\mathrm{B}_w(j))$$

Consider inequalities $\mathrm{B}_w(j)$ and $\mathrm{B}_w(j+1)$. The sum of these two inequalities gives

$$\left(\sum_{i=1}^{4k} x_i \right) - (x_j + x_{j+2k-1}) \leq \left(\sum_{i=1}^{4k} x_i \right) + (x_j + x_{j+2k-1}).$$

As this inequality clearly holds, at least one of $\mathrm{B}_w(j)$ and $\mathrm{B}_w(j+1)$ must hold as well. This implies that $\mathrm{B}_w(j)$ holds for at least $2k$ choices of j from $\{1, \ldots, 4k\}$. Note that if the number of such choices is exactly $2k$, then $\mathrm{B}_w(j)$ holds either for all odd or for all even indices.

Now consider inequalities $\mathrm{B}_w(j)$ and $\mathrm{B}_w(j+2k)$. The sum of these two inequalities gives

$$\left(\sum_{i=1}^{4k} x_i \right) - (x_{j+2k-1} + x_{j+4k-1}) \leq \left(\sum_{i=1}^{4k} x_i \right) + (x_{j+2k-1} + x_{j+4k-1}).$$

As this inequality clearly holds, at least one of $\mathrm{B}_w(j)$ and $\mathrm{B}_w(j+2k)$ must hold as well. As the parities of j and $j+2k$ are the same, this, together with the above observation, implies that $\mathrm{B}_w(j)$ holds for at least $2k+1$ choices of j from $\{1, \ldots, 4k\}$ for $w \in \{w_1, w_2\}$. Therefore, there exists an index j for which both $\mathrm{B}_{w_1}(j)$ and $\mathrm{B}_{w_2}(j)$ are satisfied. As each edge is used at most twice, the statement follows. $\qquad \square$

With the help of Lemmas 4.1, 4.3 and 4.4, we are ready to prove the main result of the thesis.

Theorem 4.5. *Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be colorings of a wheel $G = (V, E)$, and let $w \colon E \to \mathbb{R}_+$ be a weight function. Then there exists a sequence of exchanges of length at most $n-1$ and total weight at most $w(E)$ that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ and uses each edge at most twice.*

Proof. If the colorings have identical orientation, then the theorem follows by Lemma 4.1. Indeed, the length and the weight of any strictly monotone sequence of exchanges that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ achieves the natural lower bounds $(n-1) - |R_1 \cap R_2| \leq n-1$ and $w(R_1 \triangle R_2) \leq w(E)$, respectively, and uses each edge at most once.

Hence assume that the colorings have different orientations. If $\mathbf{P}_1$ has at most four intervals, then the theorem immediately follows by Lemma 4.3. Otherwise, Lemma 4.4 with the choice $w_1 := w$ and $w_2 \equiv 1$ ensures the existence of a sequence of exchanges of total weight at most $w_1(E) = w(E)$ and length at most $w_2(E)/2 = |E|/2 = n - 1$ that uses each edge at most twice, concluding the proof. $\qquad\square$

Chapter 5

Spikes

In this chapter, we prove that a strengthening of Conjecture 1.4 analogous to Theorem 4.5 holds for spikes as well. That is, consider a rank-r spike M over ground set S, and let $w\colon S \to \mathbb{R}_+$ be a weight function. We show that for any two compatible basis pairs $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$, there exists a sequence of exchanges of length at most r and total weight at most $w(S)$ that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ and uses each element at most twice.

Recall that $S = \{t, x_1, y_1, \ldots, x_r, y_r\}$, where t is the tip and $\{x_i, y_i\}$ for $1 \le i \le r$ are the legs of the spike. Hence there is exactly one element $s \in S$ that is not contained in $R_1 \cup B_1 = R_2 \cup B_2$; for short, we say that the pairs $\mathbf{P}_1$ and $\mathbf{P}_2$ **miss** the element s. We distinguish two cases depending on whether this element is the tip of M or not.

Lemma 5.1. *Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be compatible pairs of disjoint bases of a rank-r spike M over a ground set S missing the tip t, and let $w\colon S \to \mathbb{R}_+$ be a weight function. Then there exists a sequence of exchanges of length at most r and total weight at most $w(S - t)$ that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ and uses each element at most twice.*

Proof. Let $Z \subseteq S - t$ be a subset of size r. Then Z is a basis of M if and only if one of the followings hold.

1. There exist indices k and ℓ such that $\{x_k, y_k\} \subseteq Z$, $\{x_\ell, y_\ell\} \cap Z = \emptyset$ and $|\{x_i, y_i\} \cap Z| = 1$ for all $1 \le i \le r$, $i \notin \{k, \ell\}$.

2. $|Z \cap \{x_i, y_i\}| = 1$ for all $1 \le i \le r$ and Z is not a circuit of M.

Bases of the latter type are called **transversal bases** as they intersect every leg of M, while bases of the former type are called **non-transversal bases**.

Case 1. Both R_1 and R_2 are non-transversal.

We may assume that both x_1 and y_1 are red and both x_r and y_r are blue in $\mathbf{P}_1$. Note that in this case exchanging the elements x_i and y_i is feasible for $2 \le i \le r - 1$. As R_2 is

$x_1 = x_k$ R_1 x_ℓ x_r

y_1 R_2 y_ℓ y_r

(a) Case $k = 1$.

x_1 x_k R_1 x_ℓ x_r

y_1 R_2 y_ℓ y_r

(b) Case $2 \le k \le r - 1$.

x_1 R_1 x_ℓ $x_r = x_k$

y_1 R_2 y_ℓ y_r

(c) Case $k = r$ and $2 \le \ell \le r - 1$.

$x_1 = x_\ell$ R_1 $x_r = x_k$

y_1 R_2 y_ℓ y_r
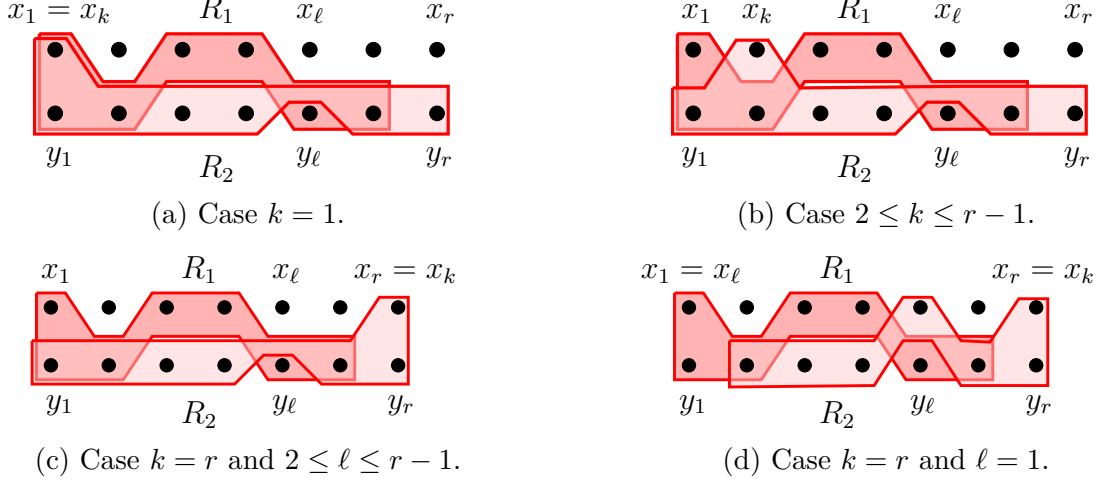
(d) Case $k = r$ and $\ell = 1$.

Figure 5.1: Illustration of Case 1 of Lemma 5.1.

non-transversal, there exist indices $1 \le k, \ell \le r$ such that x_k and y_k are both red and x_ℓ and y_ℓ are both blue in $\mathbf{P}_2$. We distinguish cases based on the values of k and ℓ.

Case 1.1. $k = 1$.

Exchange x_i and y_i for all $2 \le i \le r - 1$ for which $i \ne \ell$ and the color of x_i differs in $\mathbf{P}_1$ and $\mathbf{P}_2$. If $\ell \ne r$, then exchange the unique element of $\{x_\ell, y_\ell\} \cap R_1$ with the unique element of $\{x_r, y_r\} \cap R_2$.

Case 1.2. $2 \le k \le r - 1$.

We may assume that x_1 and x_r are blue in $\mathbf{P}_2$ and x_k is red in $\mathbf{P}_1$. Exchange x_1 with y_k, then exchange x_i and y_i for all $2 \le i \le r - 1$ for which $i \notin \{k, \ell\}$ and the color of x_i differs in $\mathbf{P}_1$ and $\mathbf{P}_2$. Moreover, if $\ell \ne r$ then exchange the unique element of $\{x_\ell, y_\ell\} \cap R_1$ with y_r.

Case 1.3. $k = r$ and $2 \le \ell \le r - 1$.

Exchange the unique element of $\{x_\ell, y_\ell\} \cap R_1$ with x_r. Then exchange x_i and y_i for all $2 \le i \le r - 1$ for which $i \ne \ell$ and the color of x_i differs in $\mathbf{P}_1$ and $\mathbf{P}_2$. Finally, exchange the unique element of $\{x_1, y_1\} \cap B_2$ with y_r.

Case 1.4. $k = r$ and $\ell = 1$.

Suppose first that there exists an index $2 \le j \le r - 1$ such that x_j has different colors in $\mathbf{P}_1$ and $\mathbf{P}_2$. We may assume that x_j is red in $\mathbf{P}_1$. Exchange x_1 with y_j, then exchange x_i and y_i for all $2 \le i \le r - 1$ for which $i \ne j$ and x_i has different colors in $\mathbf{P}_1$ and $\mathbf{P}_2$. Finally, exchange y_1 and x_r, and exchange x_j with y_r. This results in a strictly monotone exchange sequence that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$.

The only remaining case is when x_i has the same color in $\mathbf{P}_1$ and $\mathbf{P}_2$ for all $2 \le i \le r - 1$, we may assume that they are all red. We may also assume that $w(y_2) \le w(x_2)$. Consider the exchange sequence (x_1, y_2), (y_1, x_r), (y_2, y_r) of length $3 \le r$. The only

element that is used twice is y_2, while x_2 was not used at all, hence the sequence has total weight at most $w(E)$ by $w(y_2) \leq w(x_2)$.

Case 2. Both R_1 and R_2 are transversal bases.

If R_1 and R_2 differ in only one leg, say $R_1 \cap \{x_i, y_i\} \neq R_2 \cap \{x_i, y_i\}$, then exchanging x_i and y_i is feasible. Otherwise we may assume that $x_1, x_2 \in R_1$ and $y_1, y_2 \in R_2$. Exchange x_1 and y_2, then exchange x_i and y_i for all $3 \leq i \leq r$ for which the color of x_i differs in $\mathbf{P}_1$ and $\mathbf{P}_2$. Finally, exchange x_2 and y_1.

Case 3. Exactly one of R_1 and R_2 is a transversal basis.

We may assume that R_1 is non-transversal such that both x_1 and y_1 are red and both x_r and y_r are blue in $\mathbf{P}_1$. Exchange x_i and y_i for all $2 \leq i \leq r-1$ for which the color of x_i differs in $\mathbf{P}_1$ and $\mathbf{P}_2$. Then exchange the unique element of $\{x_1, y_1\} \cap B_2$ with the unique element of $\{x_r, y_r\} \cap R_2$.

Concluding the above, in all cases except Case 1.4, the resulting sequence of exchanges is strictly monotone and uses each element at most twice, hence the bounds on the length and the total weight follow. In Case 1.4, we showed that the bounds hold, concluding the proof of the lemma. $\qquad\square$

Now we consider the case when $\mathbf{P}_1$ and $\mathbf{P}_2$ miss a non-tip element, say, x_1.

Lemma 5.2. *Let* $\mathbf{P}_1 = (R_1, B_1)$ *and* $\mathbf{P}_2 = (R_2, B_2)$ *be compatible pairs of disjoint bases of a rank-r spike M over a ground set S missing the non-tip element x_1, and let $w \colon S \to \mathbb{R}_+$ be a weight function. Then there exists a sequence of exchanges of length at most r and total weight at most $w(S - x_1)$ that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ and uses each element at most twice.*

Proof. Let $Z \subseteq S - x_1$ be a subset of size r. Then Z is a basis of M if and only if one of the followings hold.

1. $t \in Z$, $y_1 \in Z$, $\{x_\ell, y_\ell\} \cap Z = \emptyset$ for an index $2 \leq \ell \leq r$, $|\{x_i, y_i\} \cap Z| = 1$ for all $i \in \{2, \ldots, r\}$, $i \neq \ell$.

2. $t \in Z$, $y_1 \notin Z$, $|\{x_i, y_i\} \cap Z| = 1$ for all $2 \leq i \leq r$.

3. $t \notin Z$, $y_1 \in Z$, $|\{x_i, y_i\} \cap Z| = 1$ for all $2 \leq i \leq r$, Z is not a circuit.

4. $t \notin Z$, $y_1 \in Z$ and there exist indices k and ℓ such that $\{x_k, y_k\} \subseteq Z$, $\{x_\ell, y_\ell\} \cap Z = \emptyset$ and $|\{x_i, y_i\} \cap Z| = 1$ for all $2 \leq i \leq r$, $i \notin \{k, l\}$.

5. $t \notin Z$, $y_1 \notin Z$, $\{x_\ell, y_\ell\} \subseteq Z$ for an index $2 \leq \ell \leq r$, $|Z \cap \{x_i, y_i\}| = 1$ for all $i \in \{2, \ldots, r\}$, $i \neq \ell$.

Note that a basis Z of type 4 does not appear as a member of a pair of disjoint bases since $(S - x_1) \setminus Z$ is not a basis. We will distinguish two cases based on the coloring of t and y_1.

Case 1. t and y_1 have the same color in at least one of $\mathbf{P}_1$ and $\mathbf{P}_2$.

We may assume that t and y_1 are both red, and x_r and y_r are both blue in $\mathbf{P}_1$. Exchange x_i and y_i for all $2 \le i \le r-1$ for which $|R_2 \cap \{x_i, y_i\}| = |B_2 \cap \{x_i, y_i\}| = 1$ and x_i have different colors in $\mathbf{P}_1$ and $\mathbf{P}_2$. Let $\mathbf{P}'_1 = (R'_1, B'_1)$ denote the coloring obtained this way. Note that all elements have the same color in $\mathbf{P}'_1$ and $\mathbf{P}_2$, except possibly t, y_1, x_r, y_r and at most one leg $\{x_\ell, y_\ell\}$. We distinguish further cases based on the colors of t and y_1 in $\mathbf{P}_2$.

Case 1.1. t and y_1 are both red in $\mathbf{P}_2$.

Let ℓ denote the unique index for which both x_ℓ and y_ℓ are blue in $\mathbf{P}_2$. If $\ell = r$, we need no further exchanges. If $\ell \ne r$, exchange the element of the leg $\{x_\ell, y_\ell\}$ which is red in $\mathbf{P}_1$ with the element of leg $\{x_r, y_r\}$ which is red in $\mathbf{P}_2$.

Case 1.2. Exactly one of t and y_1 is red in $\mathbf{P}_2$. In this case exactly one of t and y_1 and exactly one of x_r and y_r need to change colors and their exchange is feasible.

Case 1.3 Both t and y_1 are blue in $\mathbf{P}_2$.

Let ℓ denote the unique index for which x_ℓ and y_ℓ are both red in $\mathbf{P}_2$. If $\ell = r$, then consider the exchange sequences $(y_1, x_r), (t, y_r)$ and $(y_1, y_r), (t, x_r)$ of length two. At least one of them is feasible since at most one of $B'_1 - x_r + y_1$ and $B'_1 - y_r + y_1$ forms a circuit by $|(B'_1 - x_r + y_1) \cap (B'_1 - y_r + y_1)| = r - 1$. Adding this to the exchange sequence that transformed $\mathbf{P}_1$ into $\mathbf{P}'_1$ we get a strictly monotone exchange sequence from $\mathbf{P}_1$ to $\mathbf{P}_2$.

If $2 \le \ell \le r-1$, then we may assume that x_ℓ is red and y_ℓ is blue in $\mathbf{P}_1$, while x_r is red and y_r is blue in $\mathbf{P}_2$. If the exchange (y_1, x_r) is feasible, then adding $(y_1, x_r), (t, y_\ell)$ to the exchange sequence that transformed $\mathbf{P}_1$ into $\mathbf{P}'_1$ results in a strictly monotone exchange sequence. From now on we assume that (y_1, x_r) is not feasible which means that $B'_1 - x_r + y_1$ forms a circuit. As the intersection of two circuits cannot have size $r-1$, this implies that both $B'_1 - y_r + y_1$ and $B'_1 - \{x_r, y_\ell\} + \{x_\ell, y_1\}$ are bases.

If $w(x_\ell) \ge w(y_r)$, then extend the exchange sequence from $\mathbf{P}_1$ to $\mathbf{P}'_1$ by the exchanges $(y_1, y_r), (y_\ell, t), (x_r, y_r)$. The feasibility of these exchanges follows from $B'_1 - y_r + y_1$ being a basis. The total length of the exchange sequence is at most $(r-3) + 3 = r$. The sequence does not use x_ℓ, it uses y_r twice, and all the other elements at most once, so it has weight at most $w(E)$ by the assumption $w(x_\ell) \ge w(y_r)$.

If $w(x_\ell) < w(y_r)$, then extend the exchange sequence from $\mathbf{P}_1$ to $\mathbf{P}'_1$ by the exchanges $(x_\ell, x_r), (y_1, y_\ell), (x_\ell, t)$. The feasibility of these exchanges follows from $B'_1 - \{x_r, y_\ell\} + \{x_\ell, y_1\}$ being a basis. The total length of the exchange sequence is at most $(r-3)+3 = r$. The sequence does not use y_r, it uses x_ℓ twice, and all the other elements at most once,

so it has weight at most $w(E)$ by the assumption $w(x_\ell) < w(y_r)$.

Case 2. t and y_1 have different colors in both $\mathbf{P}_1$ and $\mathbf{P}_2$.

We may assume that y_1 is red and t is blue in $\mathbf{P}_1$. Observe that each leg contains one red and one blue element in any of the colorings $\mathbf{P}_1$ and $\mathbf{P}_2$. If every leg is colored the same way in $\mathbf{P}_1$ and $\mathbf{P}_2$, then the exchange distance of $\mathbf{P}_1$ and $\mathbf{P}_2$ is at most one. Hence we may assume that $x_2 \in B_1 \cap R_2$ and $y_2 \in B_2 \cap R_1$.

Case 2.1. y_1 is blue and t is red in $\mathbf{P}_2$.

Exchange x_2 with y_1. After this step, both y_1 and t is blue, hence we can exchange x_i and y_i for all $3 \le i \le r$ for which x_i has different colors in $\mathbf{P}_1$ and $\mathbf{P}_2$. Finally, exchange y_2 with t.

Case 2.2. y_1 is red and t is blue in $\mathbf{P}_2$.

If $w(y_1) \le w(t)$, exchange y_1 with x_2. After this step, exchange x_i and y_i for all $3 \le i \le r$ for which x_i has different colors in $\mathbf{P}_1$ and $\mathbf{P}_2$. Finally, exchange y_1 with y_2. This exchange sequence does not use t, uses y_1 twice and all the other elements at most once. These imply that its length is at most r and total weight is at most $w(E)$ by $w(y_1) \le w(t)$.

If $w(y_1) > w(t)$, exchange t with y_2. After this step, exchange x_i and y_i for all $3 \le i \le r$ for which x_i has different colors in $\mathbf{P}_1$ and $\mathbf{P}_2$. Finally, exchange t with x_2. This exchange sequence does not use y_1, uses t twice and all the other elements at most once. These imply that its length is at most r and total weight is at most $w(E)$ by $w(y_1) > w(t)$.

Concluding the above, in all cases except Cases 1.3 and 2.2, the resulting sequence of exchanges is strictly monotone and uses each element at most twice, hence the bounds on the length and the total weight follow. In Cases 1.3 and 2.2, we showed that the bounds hold, concluding the proof of the lemma. $\square$

Theorem 5.3. *Let $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ be compatible pairs of disjoint bases of a rank-r spike M over a ground set S, and let $w \colon S \to \mathbb{R}_+$ be a weight function. Then there exists a sequence of exchanges of length at most r and total weight at most $w(S)$ that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ and uses each element at most twice.*

Proof. The theorem follows by combining Lemmas 5.1 and 5.2. $\square$

Remark 5.4. The orientation of colorings of wheels played a crucial role in the existence of strictly monotone exchange sequences. In particular, among any three basis pairs there exists two having the same orientation, and for those there exists a strictly monotone sequence of exchanges. Spikes are interesting because one can define an arbitrarily large number of colorings without a strictly monotone exchange sequence between any two of them.

To see this, consider the spike[1] M defined by

$$\mathcal{C}_3 := \{C \subseteq S : |C| = r, \ |C \cap \{x_i, y_i\}| = 1 \text{ for } 1 \leq i \leq r, \ |C \cap \{x_1, \ldots, x_r\}| \text{ is odd}\}.$$

Consider all pairs (R, B) of disjoint sets of size r for which $y_1 \in R$, $t \in B$, $|R \cap \{x_i, y_i\}| = |B \cap \{x_i, y_i\}| = 1$ for all $2 \leq i \leq r$, and $|R \cap \{x_2, \ldots, x_r\}|$ is even. There are 2^{r-2} such pairs. Note that all such pairs (R, B) are colorings of the matroid obtained from M by deleting x_1. Further observe that any feasible exchange uses at least one of y_1 and t. Indeed, exchanging x_i with y_j for some $2 \leq i, j \leq r$ results in an odd number of red elements in $\{x_2, \ldots, x_r\}$, meaning that the red elements form a circuit, showing that the exchange is non-feasible. Exchanging x_i with x_j or y_i with y_j for some $2 \leq i, j \leq r$, $i \neq j$ results in either $\{t, x_i, y_i\}$ or $\{t, x_j, y_j\}$ forming a blue circuit, hence the exchange is non-feasible. This implies that there exists no strictly monotone exchange sequence between any two distinct colorings of this form.

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[1]This matroid is interesting on its own as it is the unique rank-r binary spike, see [15] for details.

Conclusions

We proposed a weighted generalization of Hamidoune's conjecture on the exchange distance of compatible basis pairs. We verified the conjecture for strongly base orderable matroids, and its strengthening in which the exchange sequence had small length and weight simultaneously for split matroids, graphic matroids of wheels, and spikes. For the latter three classes, our proofs also imply polynomial-time[2] algorithms that determine the required exchange sequences. For strongly base orderable matroids, we get a similar result if, for any pair of bases, a bijection ϕ between them ensured by strongly base orderability can be computed in polynomial time.

Motivated by Lemma 4.4, it would be tempting to formulate a conjecture stating that there always exists an exchange sequence that has small weight with respect to two weight functions w_1 and w_2 simultaneously. However, this is not true in general. Let $G = (V, E)$ be a complete graph on four vertices with edge set $E = \{a, b, c, d, e, f\}$, and let M be the graphic matroid of G, see Figure 5.2.

(a) Coloring $\mathbf{P}_1 = (R_1, B_1)$ of K_4. (b) Coloring $\mathbf{P}_2 = (R_2, B_2)$ of K_4.

Figure 5.2: Example showing that the statement of Lemma 4.4 is not true in general. Any exchange sequence that transforms (R_1, B_1) into (R_2, B_2) uses one of b and e twice. If $w_1(b) = 1$ and 0 otherwise, and $w_2(e) = 1$ and 0 otherwise, then any sequence violates the weight constraint for at least one the weight functions.

Interestingly, G is a wheel, but M can be also obtained by deleting the tip of a spike. Consider two colorings $\mathbf{P}_1 = (R_1, B_1)$ and $\mathbf{P}_2 = (R_2, B_2)$ of G where $R_1 = \{a, b, c\}$, $B_1 = \{d, e, f\}$, $R_2 = \{b, d, f\}$ and $B_2 = \{a, c, e\}$. It is not difficult to check that any sequence of exchanges that transforms $\mathbf{P}_1$ into $\mathbf{P}_2$ uses at least one of the edges b and e twice. This

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[2]In matroid algorithms, it is usually assumed that the matroids are given by independence oracles, and the complexity of the algorithm is measured by the number of oracle calls and other conventional elementary steps.

implies that if we set $w_1(b) := 1$ and 0 otherwise, and we set $w_2(e) := 1$ and 0 otherwise, then any exchange sequence has w_1-weight or w_2-weight at least $2 = 2w_1(E) = 2w_2(E)$.

On the other hand, no counterexample is known for the case when $w_2 \equiv 1$, that is, when one would like to find an exchange sequence that has small length and weight simultaneously. Even more, both Theorems 4.5 and 5.3 ensured the existence of an exchange sequence that uses each element at most twice. A rather optimistic conjecture would state the existence of a sequence of exchanges that, besides having small length and weight, also has this property.

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