Eötvös Loránd University

Faculty of Science

Institute of Mathematics

Rational homology disk smoothings of isolated surface singularities

Supervisor:
András Stipsicz
Alfréd Rényi Institute of Mathematics

Author:
Márton Beke
Mathematician Msc

Budapest, 2023

Acknowledgements

I would like to thank András Stipsicz for the many enlightneing conversations, and his patient, helpful supervision during the writing of this thesis. I'm also grateful to my family for their continued care and support, and further wish to express gratitude towards Marco Golla, László Koltai and Alexander Kubasch for their help.

Introduction

In this thesis we wish to give an exposition on the known literature regarding the titular rational homology disk smoothings of isolated surface singularities. As discussed in the introduction of [30], this construction is helpful for constructing interesting smooth 4-manifolds, because a generalisation of the blowdown process can be carried out for a configuration of spheres in a manifold representing a resolution of such a singularity. Besides the applications, this class of singularities is interesting in its own right, being at the intersection of surface singularity theory, and various branches of 4-manifold theory.

We begin with deriving a necessary condition for a graph to be eligible for this generalised blowdown (codified in 1.1.4), purely combinatorial in nature but motivated by geometric principles. The first chapter presents the main methods and arguments used in [30], where a complete classification of graphs satisfying the requirements derived is given.

This condition is only necessary, but for "star-shaped" graphs the classification is complete, the precise statement, and the proof of a subcase comprises chapter 2. We tried minimizing the use of symplectic geometric notions as possible, the idea of the proof should be accessible to readers with little to no background in symplectic topology as well.

The third chapter is devoted to the construction of the objects used for the generalised blowdown procedure, which are smoothings of the given singularities. Much of the star-shaped case is given in detail, and a different method is sketched as well. Afterwards we present some obstructions, which may be used to finish the classification and hopefully prove that the list conjectured by Wahl is complete. A possible approach to this is [25], the continuation of which is planned for the future.

Contents

Acknowledgements **i**

Introduction **ii**

1 Symplectic plumbing trees **1**

1.1 Motivation 1

1.2 The list 3

1.2.1 Some generalities 5

1.3 Case A: cyclic quotient singularities 11

2 Stars **14**

2.1 Weighted homogeneous singularities 15

3 Constructions and obstructions **22**

3.1 Constructions 22

3.1.1 Smoothings of negative weights 22

3.1.2 Quotients 27

3.2 Obstructions 28

3.2.1 The geometric genus 28

3.2.2 The mubar invariant 29

3.2.3 Dimension count 31

3.2.4 McDuff's theorem and lattice embeddings 33

A Algebraic geometry **35**

A.1 The Milnor fibration 35

A.2 Taut singularities 37

B Graph algorithms in SageMath **41**

Bibliography **46**

1 Symplectic plumbing trees

1.1 Motivation

1.1.1. Consider a normal surface singularity (X, o), with minimal good resolution $\tilde{X}$, and suppose further, that there exists a smoothing $\mathfrak{X}$, that is a contractible Stein space and a (proper flat analytic) map $f : (\mathfrak{X}, o) \to (D, o) \subset (\mathbb{C}, o)$ over a small disk, which has a rational homology disk fibre $M = f^{-1}(\epsilon)$. By [16, Theorem 4.10. i)] (see also [23, Example 6.9.11]) we see that X is a rational singularity (see Definition 3.2.1), so the resolution dual graph is a tree, and every exceptional divisor is a $\mathbb{C}P^1$ ([22, 3.9.]).

1.1.2. The complex structure on $\tilde{X}$ and M induces an almost contact structure on the link of the singularity $Y := \partial\tilde{X} = \partial M$, the planes are given by the complex tangent lines $TY \cap iTY$ to the boundary. The homotopy type of such a 2-plane field cannot change under a small deformation. From the Hirzebruch signature theorem it follows, that $c_1^2 - 3\sigma - 2\chi$ is an invariant of almost contact structures. Since the plumbing is negative definite ([23, Proposition 2.1.12]), we have $\sigma(\tilde{X}) = -n$ ($n := dim H^2(\tilde{X}; \mathbb{Q})$), and $\chi(\tilde{X}) = 1 + n$, and similarly for the smoothing $\sigma(M) = 0$, since it has no second rational homology (in particular $c_1^2(M) = 0$), and $\chi(M) = 1$. Equating the two sides gives us

$$c_1^2(\tilde{X}) + 3n - 2(1 + n) = c_1^2(M) - 0 - 2 \Rightarrow c_1^2(\tilde{X}) + n = 0$$

By the long exact sequence of the pair (M, Y), the boundary of a rational homology disk is a rational homology sphere, hence rationally $c_1(\tilde{X})|_Y = 0$, so it has compact support in $\tilde{X}$. If we take $Z = \tilde{X} \cup -M$ this first Chern class extends rationally to it, and is zero outside a compact subset of $\tilde{X}$, so its previously calculated integral $-n$ doesn't change.

1.1.3. Considering the CW homology chain complex of M we see that $H_2(M; \mathbb{Z}) = 0$ (free and rank 0). The adjunction identity in $\tilde{X}$ shows, that the first Chern class is a characteristic element of the lattice $H^2(\tilde{X}; \mathbb{Z})$ equipped with the intersection form, since

$$c_1(\tilde{X})[\Sigma] = 2 + [\Sigma] \cdot [\Sigma] \equiv [\Sigma] \cdot [\Sigma] \mod 2$$

The manifold Z is smooth, closed and negative definite (since M is a $\mathbb{Q}HD^4$) so by Donaldson's theorem ([7, Theorem 1.3.1]) we get, that its intersection form is diagonalizable. In this diagonal basis $E_1, \dots, E_n$ the first Chern class has to have odd multiplicities with each basis element, and square $-n$, as previously calculated. This implies, that it is of the form

$$\pm E_i \pm \cdots \pm E_n$$

by another change of basis we can presume, that all coefficients are $+1$. Collecting the above we have

Definition 1.1.4. A negative definite dual plumbing graph Γ on n vertices is called a *symplectic plumbing tree*, if the following are satisfied

- Γ is a tree
- the associated intersection form $(\mathbb{Z}\langle v_1, \dots, v_n\rangle, Q_\Gamma)$ admits a lattice embedding φ into the Euclidean lattice of the same rank $(\mathbb{Z}\langle E_1, \dots, E_n\rangle, -I_n)$
- defining the canonical element $K = \sum E_i \in (\mathbb{Z}\langle E_1, \dots, E_n\rangle, -I_n)$ the adjunction identity is satisfied for all vertices $-I_n(v_i, K) + -I_n(v_i, v_i) = -2$.

1.1.5. The main purpose of investigating these objects is the construction of interesting 4-manifolds. The general strategy is to find spheres embedded according to a symplectic plumbing tree, remove their neighborhood, and if it actually exists, glue back the rational homology disk smoothing instead. This decreases the Euler characteristic, while leaving other interesting invariants unchanged. The first motivating example was the rational blowdown procedure of [9], and more generally [26], where it is shown, that path graphs can be blown down in this way if and only if the decorations correspond to the Hirzebruch-Jung continued fraction expansion of $\frac{p^2}{pq-1}$, where $p > q > 0$ are relatively prime integers.

We will be working mainly in the symplectic category, the previous construction still holds if the spheres are symplectic submanifolds, instead of holomorphic ones, the adjunction identity stays true, and we have a contractible choice of compatible almost complex structures (which we will start exploiting shortly). The symplectic form extends over the glued back smoothing by [10, Theorem 1.1], thus we can "blow down" these more complicated arrangements, like -1 spheres.

1.2 The list

Firstly we will give a complete classification of symplectic plumbing trees. The laborious proof will not be fully provided (it was carried out in [30, 3-6.]), but we will indicate the main methods, which will be mostly combinatorial. Note, that Γ being a symplectic plumbing tree is a purely combinatorial condition, which does not guarantee the existence of a $\mathbb{Q}HD^4$ smoothing for the corresponding plumbed 4-manifold. The question as to which of them do is still open, and will be discussed later.

To state the theorem we make the following

Definition 1.2.1. Let $\mathcal{S}$ denote the set of minimal* symplectic plumbing trees. Define the graph classes $\mathcal{G}, \mathcal{W}, \mathcal{N}, \mathcal{M}, \mathcal{A}, \mathcal{B}, \mathcal{C}$ as follows:

- The class $\mathcal{G}$ consists of path graphs, with framings given by the negatives of the Hirzebruch-Jung continued fraction expansion of $\frac{p^2}{pq-1}$ for some relatively prime $0 < q < p$.
- $\mathcal{W}$ is depicted by Figure 1.1 (a), the parameters are nonnegative.
- $\mathcal{N}$ consists of Figure 1.1 (b) with $p, q, r \geq 0$, and a further degenerate case 1.1 (c).
- $\mathcal{M}$ is shown mainly by Figure 1.1 (g) with $p, q, r \geq 0$, and certain degenerations as previously (with p or r " $= -1$") defined by (d), (e).
- Finally the $\mathcal{A}, \mathcal{B}, \mathcal{C}$ classes are defined inductively as follows. For $\mathcal{A}$ consider the graph depicted on figure 1.2 (a), and blow up the central -1 vertex, or one of its intersection points with its neighbors. This produces a new graph, which again has a unique -1 vertex. Blow it up, or one of the edges eminating from it, and so on. Finally replace the -1 framing with -4. The graphs arising by this algorithm constitute the class $\mathcal{A}$.

*no vertex has decoration -1

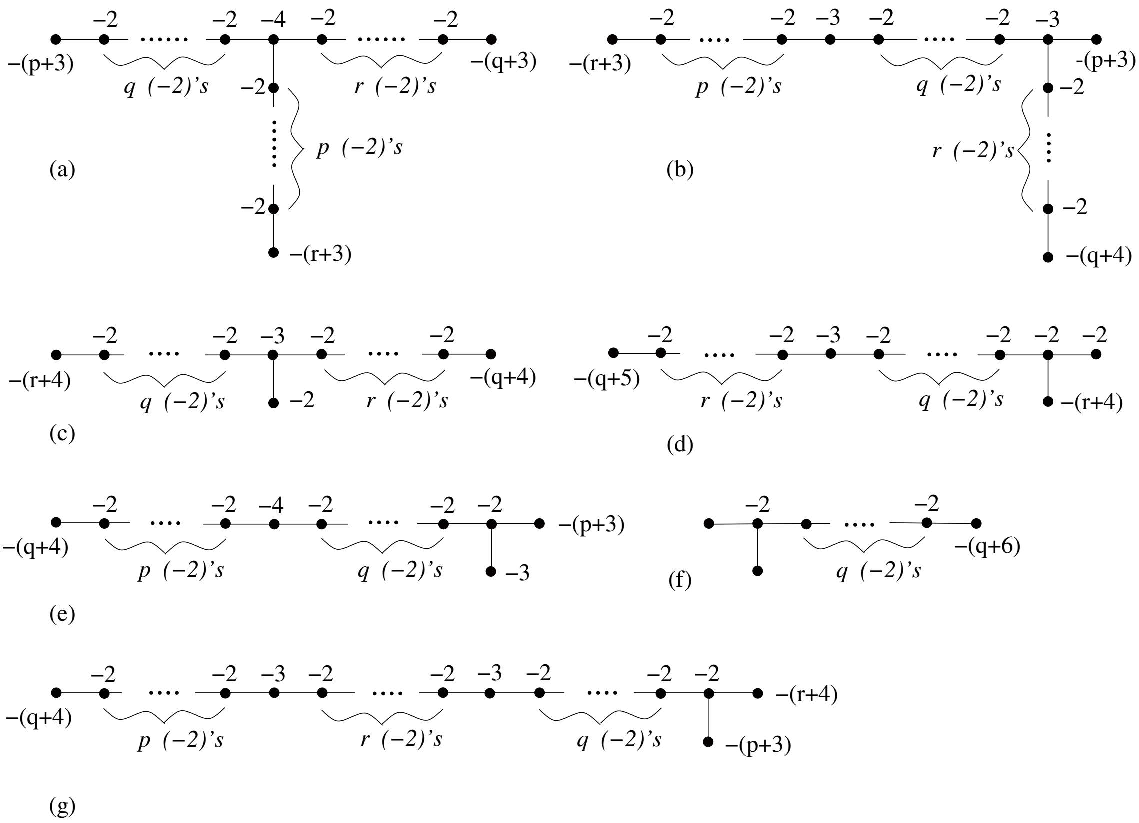


Figure 1.1: Definition of the classes $\mathcal{W}, \mathcal{N}, \mathcal{M}$

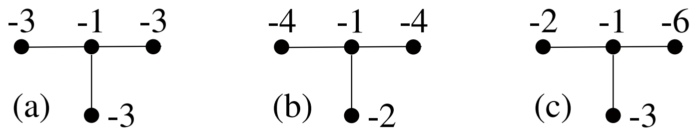


Figure 1.2: The graphs used for the definition of the $\mathcal{A}, \mathcal{B}, \mathcal{C}$ classes

- For $\mathcal{B}$ start the same process with figure 1.2 (b), and at the end change the -1 to -3.

- $\mathcal{C}$ is done in much the same way beginning from Figure 1.2 (c), and after the blowups changing the -1 to a -2.

Remark 1.2.2. It is worth noting, that the degenerations in the definition of the $\mathcal{N}, \mathcal{M}$ classes disappear if one considers their dual graphs* instead, see 3.1.2.

Now we can state

Theorem 1.2.3 ([30, Theorem 1.8]).

$$\mathcal{S} = \mathcal{G} \cup \mathcal{W} \cup \mathcal{N} \cup \mathcal{M} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$$

*defined in 2.1.3, 2.1.4

1.2.1 Some generalities

We follow [30, Section 3.]. In the following we identify the generator v_i with its image $\varphi(v_i) = \sum_j \alpha_i^j E_j$. To get a better understanding of the relations between the α_i^j consider the adjunction identity

$$\sum E_j \cdot \sum \alpha_i^j E_j + \sum \alpha_i^j E_j \cdot \sum \alpha_i^j E_j = -2 \Rightarrow -\sum_i \alpha_i^j(\alpha_i^j + 1) = -2$$

Since $\alpha_i^j \in \mathbb{Z}$, either there is one α_i^j which equals 1, and the others are 0 or -1, or one coefficient is -2, and all others are again 0, -1. This is true because if $\alpha_i^j \in \mathbb{Z} \setminus \{-2, -1, 0, 1\}$, then $\alpha_i^j(\alpha_i^j + 1) \geq 6$. This proves

Lemma 1.2.4. *The image of a vertex $v \in \Gamma$ has one of the following forms in the diagonal basis E_i*

i. $v = E_{i_v} - \sum_{J_v} E_j$

ii. $v = -2E_{i_v} - \sum_{J_v} E_j$

for some index set J_v depending on the vertex with $i_v \notin J_v$.

Remark 1.2.5. Notice, that if v_i and v_j are adjacent in Γ, i.e. $v_i \cdot v_j = 1$, then $K \cdot (v_i + v_j) + (v_i + v_j)^2 = K \cdot v_i + K \cdot v_j + v_i^2 + v_j^2 + 2 = -2 - 2 + 2$, so the adjunction identity still holds. Repeating the same argument, and using induction we get, that the sum of vertices over a connected subgraph $\Gamma' \subset \Gamma$ still has one of the two above forms in the E_i basis.

Next we claim, that

Theorem 1.2.6. *If there exists a vertex of type ii, then it is unique.*

Proof. Aiming for a contradiction, suppose that there are two such vertices v, w. Consider the vertices of the path connecting them in Γ, $v = v_1, \ldots, v_k = w$. By shortening the path if necessary, we can assume, that $v_2, \ldots, v_{k-1}$ are all of the form i in Lemma 1.2.4. Using Remark 1.2.5 and the previous assumption on $v_2, \ldots, v_{k-1}$, the sums $v_1 + v_2, \ldots, v_1 + \cdots + v_{k-1}$ all have a coordinate -2. This is seen by induction. For the sum $-2E_{i_{v_1}} - \sum_{J_{v_1}} E_j + E_{i_{v_2}} - \sum_{J_{v_2}} E_j$ to be of form i we would need $i_{v_1} = i_{v_2}$, but then since all other coefficients are negative, a $+E_*$ cannot appear.

Continuing the path to the very end we need to look at $\langle v_1 + \cdots + v_{k-1}, v_k\rangle = 1$ (since the path $v_1 \ldots, v_{k-1}$ and v_k are connected by an edge, and having more than one edge would create a cycle in the graph). In the diagonal basis this is

$$1 = \langle -2E_\alpha - \sum_{j\in J_\alpha} E_j, -2E_{i_{v_k}} - \sum_{j'\in J_{v_j}} E_j\rangle$$

Using the Kronecker we can expand as follows

$$= -4\delta_{\alpha,i_{v_k}} - 2\sum_{j'\in J_{v_k}} \delta_{\alpha,j'} - 2\sum_{j\in J_\alpha}\delta_{j,i_{v_k}} - \sum_{j\in J_\alpha}\sum_{j'\in J_{v_k}}\delta_{j,j'} \leq 0$$

a contradiction. □

Using the same method we get the following

Corollary 1.2.7. *If $\Gamma_1, \Gamma_2 \subset \Gamma$ are disjoint connected subtrees, then at least one of the vectors $\sum_{\Gamma_i} v$ has form i.*

Moreover, the '+' indices for the type i vertices are unique.

Lemma 1.2.8. *If $v, w \in \Gamma$, $v \neq w$ are two vertices of type i, then the distinguished indices i_v, i_w are not equal.*

Proof. Suppose on the contrary, that the indices are equal. The two vertices pair as either 0 or 1 depending on their adjacency, from this we get

$$0 \leq \langle E_{i_v} - \sum_{j\in J_v} E_j, E_{i_w} - \sum_{j'\in J_w} E_{j'}\rangle = -1 - \sum_{j\in J_v}\sum_{j'\in J_w}\delta_{j,j'}$$

since $i_v = i_w \notin J_v \cup J_w$. This gives $0 \leq -1$, a contradiction. □

Proposition 1.2.9. *For a minimal symplectic plumbing tree fix a basis vector E_i in the diagonal lattice. This basis element can appear in the image of 1,2,3 or 4 vectors with multiplicities as shown by the following table, or none at all.*

$\#\backslash^{\Sigma}$	-2	-1	0	1
1	(-2)	(-1)		(1)
2	(-1,-1)	(1,-2)	(1,-1)	
3	(1,-1,-2)	(1,-1,-1)		
4	(1,-1,-1,-1)			

Proof. Applying Remark 1.2.5 to the whole Γ (a connected subgraph of itself), we see that the sum of the coefficients of any basis element E_i is either $-2,-1,0$ or 1. Combining this with Theorem 1.2.6 and Lemma 1.2.8 we only have to find the number of ways there are to make a sum of $-2,-1,0,1$ while using at most one -2, at most one 1, and possibly many -1's. This is done in the above table. $\square$

Remark 1.2.10. This observation only uses some of the information we have at our disposal. To refine the statement, notice that E_i cannot be of type $(1,-1,-2)$, since if $v=-2E_i-\sum_{j\in J_v}E_j$ and $\langle w,E_i\rangle=-1$, the product $\langle v,w\rangle$ is at most -1, a contradiction. If an E_i does not appear in any vertex, then if we change the graph and the embedding by adding $-E_i$ to an arbitrary vertex v we still get a minimal symplectic plumbing tree. It is still embedded into $(\mathbb{Z}^n,-I_n)$. The intersections between vertices, i.e. the structure of the undecorated graph doesn't change, since E_i is in no other vertex. The self-intersection of v decreases by 1 and the adjunction identity stays intact*, since

$$\langle\sum E_j,v-E_i\rangle+(v-E_i)^2=\langle\sum E_j,v\rangle+v^2-\langle\sum E_j,E_i\rangle+(-E_i)^2=-2+1-1=-2.$$

So from the last possibility we can produce a graph with a basis element of type (-1). Note, that the reverse construction does not always (in fact it will turn out never to) produce a minimal plumbing tree.

Now we continue the restrictions on the possible embeddings with

Theorem 1.2.11. *If Γ is a minimal symplectic plumbing tree consisting of more than just a vertex, then there cannot be a basis element of type* (-2).

Proof. We proceed by induction. Consider the graph consisting of two vertices with an edge connecting them. By assumption one of the two vertices has the form $-2E_i$ or $-2E_i-E_j$ for $\{i,j\}=\{1,2\}$. Now by 1.2.6 the other vertex is type i, and has the form $+E_j$ since E_i is type (-2), in contradiction with the minimality assumption.

Suppose that the embedding of Γ has a basis element of type (-2), and no graph on fewer vertices has such an embedding. Denote the basis vector of type (-2) by E_1. Since it is not contained in any other vertex, it has to be the distinguished basis

*every other vertex stays unchanged, so we only need to check v

element in $x = \sum_\Gamma v$ as well (in particular it is of type ii). This rules out basis elements of types $(-1,-1),(1,-1,-1,-1),(1)$.

By assumption, the graph has more than one vertex. Therefore we know that there is a leaf (valency 1 vertex) of type i, call it v. Since it contains the vertex of type ii, the sum $\sum_{\Gamma\setminus\{v\}} w$ is of type ii. From this we see, that E_{i_v} cannot be of type $(1,-2),(1,-1,-1)$, since this would mean, that the sum has both $-2E_1$ and $-2E_{i_v}$, which is a contradiction.

This means, that E_{i_v} is of type $(1,-1)$; we argue that this cannot happen either. Consider the unique vertex $v' \in \Gamma$, neighboring v, and first suppose, that the $-E_{i_v}$ is in v'. Replace the edge $\overline{vv'}$ by a new vertex, the framing of which is given by $v+v'$. Notice, that $(v+v')^2 = v^2+v'^2+2 < -1$. This eliminates E_{i_v} from the embedding, and we get a new minimal symplectic plumbing tree on $n-1$ vertices, with an embedding containing a (-2) basis element, a contradiction.

Similarly, if the vertex containing $-E_{i_v}$ is not v', then delete v, and add E_{i_v} to this vertex, again eliminating this basis element from the embedding. We have to check, that the new graph stays minimal after this modification. If this modified vertex v'' becomes a -1 vertex, then originally it had the form $E_{i_{v''}} - E_{i_v}$. Consider $0 = \langle v, v''\rangle$. Substituting the expression for v'' we get $\langle E_{i_v}, -E_{i_v}\rangle + \langle -\sum_{J_v} E_j, E_{i_{v''}}\rangle$, the other terms pair as zero, since $i_{v''} \neq i_v$. This expression is at least one, which is a contradiction. The new graph is again a minimal symplectic plumbing tree on fewer vertices than Γ, still containing a (-2) element in its embedding, in contradiction with the inductive assumption. □

Remark 1.2.12. The graph containing a single vertex with decoration -4 with embedding $-2E$ shows, that our assumption on the vertex set is necessary, having a basis element of type (-2) is equivalent to $|\Gamma| = 1$.

Repeating the same argument* we also get

Corollary 1.2.13. *Basis elements of type* (1) *cannot exist for a minimal symplectic plumbing tree.*

*or noticing, as in Remark 1.2.10, that a type (1) can be changed to a type (-2) while staying in $\mathcal{S}$

Proof. The induction starts easier, since a single vertex graph with basis vector of type (1) is not minimal. Choose a graph Γ with a type (1) basis element and minimal vertex set, denote the basis element of type (1) by E_1. $\sum_\Gamma v$ will be of type i, since E_1 cannot disappear, so no basis elements of types $(-1,-1),(1,-1,-1,-1)$. This means, that there are no vertices of type ii, if there would be one, then the whole sum would by type ii. This forbids basis elements of type $(1,-2)$.

Consider a leaf v not containing E_1 (this exists, since there are at least two vertices in Γ). By the previous argument, the sum $\sum_{\Gamma\setminus\{v\}} w$ is also of type i, so the distinguished coordinate of v cannot be of type $(1,-1,-1)$, so it is of type $(1,-1)$.

If the vertex containing $-E_{i_v}$ is the vertex adjacent to v, we can replace the edge by a new vertex and get a contradiction from the inductive assumption. Otherwise delete v, and add E_{i_v} to the vertex containing $-E_{i_v}$, the same proof shows that the new graph is again minimal, and the proof is complete by induction. □

1.2.14. This means, that $\sum_\Gamma v$ is always of type ii. For a more systematic treatment of the actual list of indices, which can occur consider the sum $\sum_{i<j}\langle v_i, v_j\rangle$, which counts the number of edges in Γ, which is $n-1$, since it is assumed to be a connected tree. Expand this sum in terms of the diagonal basis E_i. Since these all pair as $-\delta_{ij}$, the contribution to the sum only depends on the type of the basis element. By a simple calculation we see that the contributions of the basis elements are as follows:

- $(1,-2)$ contributes 2
- $(1,-1)$ and $(1,-1,-1)$ both contribute 1
- (-1) and $(1,-1,-1,-1)$ both contribute 0
- $(-1,-1)$ contributes -1

This helps us organize our investigation. Since $\sum_\Gamma v$ is of type ii, there has to be a single basis element of type $(-1,-1)$ or $(1,-1,-1,-1)$, and we collect the different cases that can occur in

Proposition 1.2.15. *The following three collections of basis element types can occur in a minimal symplectic plumbing tree:*

(A) one $(-1,-1)$, one $(1,-2)$, and $n-2$ are a combination of $(1,-1)$ and $(1,-1,-1)$

(B) one $(1,-1,-1,-1)$, one $(1,-2)$, one (-1), and the remaining $n-3$ can be $(1,-1)$ or $(1,-1,-1)$

(C) one $(1,-1,-1,-1)$ and $n-1$ can be $(1,-1)$ or $(1,-1,-1)$

Proof. We saw that $x = \sum_\Gamma v$ is of type ii, so there has to be a basis element of type $(-1,-1)$ or $(1,-1,-1,-1)$, since these are the only ones which provide a $-2E_*$ to x. Consider the sum $\sum_{i<j}\langle v_i, v_j\rangle = n-1$, expanding this in terms of the E_i basis and collecting the terms corresponding to a single basis vector we see an n term sum, each term of which is $\in \{-1,0,1,2\}$ as above.

First suppose, that there is a basis element of type $(-1,-1)$, which contributes -1 to the sum, we have to produce a contribution of n out of the remaining $n-1$ terms. For this there has to be an $(1,-2)$ basis element contributing 2, and there cannot by more by 1.2.6. This means, that the others have to contribute 1, i.e. a mix of basis elements of types $(1,-1)$ and $(1,-1,-1)$, this is case (A).

If the distinguished -2 in the sum of the vertices is provided by a basis element of type $(1,-1,-1,-1)$, then our sum stays 0, and we have to produce $n-1$ out of $n-1$ terms. There are two ways to do this, either all terms contribute 1, or a term contributes 2 (there can be at most one as before), another 0, and all others contribute 1, which are cases (C) and (B) respectively. □

In the proofs of 1.2.11 and 1.2.13 we already saw that basis elements of type $(1,-1)$ require more care to deal with, than the others. We solidify this distinction in the following

Definition 1.2.16. A vertex $v \in \Gamma$ is called *full*, if it is of type i, and its distinguished basis element is of type $(1,-2), (1,-1,-1)$ or $(1,-1,-1,-1)$. Further, Γ is called *full*, if it has an embedding without a basis element of type $(1,-1)$.

Lemma 1.2.17. *Γ is full if and only if $\sum_\Gamma v = -E_1 - \cdots - E_{n-1} - 2E_n$ (after possibly reordering the basis).*

Proof. We ruled out either generally, or by definition the types, which have multiplicity 1 or 0 in the sum, i.e. every basis element has multiplicity -1 or -2, and there can only be one of the latter. □

The next statement motivates the nomenclature.

Corollary 1.2.18. *Γ is full if and only if $\sum_\Gamma v^2 = -3n - 1$.*[*]

Proof. If it is full, then $w = \sum_\Gamma v = -E_1 - \cdots - 2E_n$, and using the fact that Γ is a connected tree $w^2 = \sum_\Gamma v^2 + 2(n-1) = -(n-1) - 4$, the other direction follows similarly: if w does not contain an E_i, then w^2 grows. □

We are ready to state the main

Theorem 1.2.19. *If $\Gamma \in \mathcal{S}$ is in case (A), then $\Gamma \in \mathcal{G}$, if it is in case (B), then $\Gamma \in \mathcal{C}$, and if it is in case (C), then $\Gamma \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{M} \cup \mathcal{N} \cup \mathcal{W}$.*

Along with constructions showing that each class is in $\mathcal{S}$, this will prove 1.2.3. In the following we present the proof for case (A).

1.3 Case A: cyclic quotient singularities

Firstly we need a characterisation of the continued fraction coefficients for the rational numbers defining the graphs in $\mathcal{G}$.

Proposition 1.3.1 ([30, Proposition 4.1])**.** *$(-a_1, \ldots, -a_k)$ is the vector of decorations for a graph $\Gamma \in \mathcal{G}$ if and only if it is in the minimal set of vectors containing (-4), and for which if $(-a_1, \ldots, -a_k)$ is contained, then so are $(-2, -a_1, \ldots, -a_k - 1)$ and $(-a_1 - 1, \ldots, -a_k, -2)$.*

The proof consists of simple induction reliant on the crucial fact, that

Proposition 1.3.2 ([28, Corollary 5.7])**.** *If the Hirzebruch-Jung continued fraction $[a_1, a_2, \ldots, a_n] = \frac{p}{q}$, then $[a_n, a_{n-1}, \ldots, a_1] = \frac{p}{q'}$, where $qq' \equiv 1 \mod p$ and $0 < q' < p$.*

This allows for an inductive proof for case (A), but first

[*]it is the least possible

Proposition 1.3.3. *The graphs in $\mathcal{G}$ are full.*

Proof. The first graph on a single vertex is clearly full by 1.2.18, since $-4 = -3\cdot 1 - 1$. One step of the inductive definition raises the vertex count by 1, and lowers the square sum of the vertices by 3. Now since $-3(n-1) - 1 - 3 = -3n - 1$ we get the statement. □

Theorem 1.3.4. *If Γ contains a basis element E_t of type $(-1,-1)$ (i.e. it is of case (A)), then $\Gamma \in \mathcal{G}$, and the two $-E_t$'s are in the endpoints of the chain.*

Proof. To start our induction we check the $n = 2$ case. Say E_1 is the type $(-1,-1)$, the distinguished basis element has to be E_2 in both vertices, v and w so by Lemma 1.2.8 and Theorem 1.2.6 up to reordering of the vertices the only possibility is $v = E_2 - E_1,\ w = -2E_2 - E_1$, which clearly satisfies the requirements of the statement.

Now proceed by induction. Suppose the statement is known for $k < n$, where $n \geq 3$, and denote the $(-1,-1)$ basis element E_t. First consider the case, when the $-E_t$'s are in the endpoints v_1, v_n. Add E_t to both, and delete the one which has smaller decoration in absolute value. This procedure gives us a graph on $n-1$ vertices, removes E_t from the embedding, so it is now embedded into $\mathbb{Z}^{n-1}$. Since $v_1 \cdot v_n = 0$, the adjunction identity stays intact (cf. Remark 1.2.10). Finally the graph will be minimal, since if it were non-minimal, then both $v_1,\ v_n$ would be of the form $E_{v_i} - E_t$ in the original Γ, which implies that they pair to -1, a contradiction.

Γ' is also of case (A), since we only removed a -1 from the types of the remaining basis elements, or left them unchanged, and this cannot produce a type $(1,-1,-1,-1)$. Now by induction we know that $\Gamma' \in \mathcal{G}$. In particular Γ' is full. For Γ the sum of decorations is at least $-3n-1$, we raise the sum by 1 when adding E_t to the vertex which will remain, thus the deleted vertex has to have decoration -2, i.e. it looks like $E_* - E_t$ (the decoration cannot be -1). Following the removal process backwards we see, that it coincides with the inductive characterisation of the graphs in $\mathcal{G}$. We subtract 1 from one end, and concatenate a new vertex of decoration -2.

The other case to consider is if there is a non-leaf vertex containing $-E_t$, say it is v_i. Since $n \geq 3$ there is at least one leaf v with $v \cdot E_t = 0$. v cannot be full, since full vertices contribute $-2E_{i_v}$ to the sum $\sum_{\Gamma\setminus\{v\}} u$, but this also contains $-2E_t$, a contradiction. v is a leaf, so $\Gamma \setminus \{v\}$ is also connected, and the sum of its vertices is of type ii (it has E_t as its distinguished basis element), v is of type i. Comparing with the possible types in case (A), we get that E_{i_v} is type $(1,-1)$.

We split into two further cases based on whether the unique neighbor of v is the vector containing $-E_{i_v}$, or not, and begin with the latter. So let $E_{i_v} \cdot w = 1$, with $v \cdot w = 0$. Now delete v, and replace w with $w + E_{i_v}$ to get a graph on $n-1$ vertices, with an embedding into $\mathbb{Z}^{n-1}$. The adjunction identity stays intact at w, just like before (Remark 1.2.10). It will also be minimal, the only problematic vertex is w, if it squares to -1, then it is of the form $E_{i_w} - E_{i_v}$, which means that $v \cdot w \geq 1$, contradiction. In conclusion this new $\Gamma' \in \mathcal{S}$, still without a type $(1,-1,-1,-1)$ basis element, i.e. case (A). By induction $\Gamma' \in \mathcal{G}$, in particular it's full, repeating the same argument as in the previous case provides that $v = E_{i_v} - E_\iota$, implying that its neighbor v' has the form $E_\iota - \sum_{J_{v'}} E_j$. This contradicts the fullness of Γ', E_ι could have type $(1,-1,-1)$ or $(1,-1)$, in any case after removing v, Γ' will not be full in the former case, or $|\Gamma'| = 1$ in the latter, in contradiction with Proposition 1.3.3 or the inductive hypothesis.

Lastly if v' and w coincide, modify v' to $v + v'$ and delete v. By Remark 1.2.5 the adjunction identity stays intact, $(v+v')^2 = v^2 + (v')^2 + 2 \leq -2$ so it stays minimal and E_{i_v} gets eliminated from the embedding, so we produce $\Gamma'' \in \mathcal{G}$ by induction. By the additional inductive hypothesis we get that v_i is one of the endpoints of Γ'', but it was not a leaf in Γ, so v_i coincides with the modified v'. Now we can justify deleting v from the original Γ, and adding E_{i_v} to v', since it contains one of the $-E_t$'s, the graph will stay minimal, and the previous case's argument finishes the proof. □

1.3.5. Finally we must show, that $\mathcal{G} \subset \mathcal{S}$. The proof above already indicates how this can be done inductively. The single -4 vertex graph is readily embedded by $v \mapsto -2E$. $(-5,-2)$ is embedded by $(-2E_1 - E_2, E_1 - E_2)$, as seen at the beginning of the proof. Here E_2 is the unique $(-1,-1)$ basis vector of our graph. Now the inductive step can be completed by mapping the new (-2) vertex to $E_t - E_n$, and modifying the other end of the chain to $v_1 - E_n$ (E_n is the new basis element, and E_t is still the unique type $(-1,-1)$).

2 Stars

Now we begin the discussion about which resolution graphs actually admit $\mathbb{Q}HD^4$ smoothings. The case, when the graph admits a unique vertex of valency > 2 (also called star-shaped) is completely classified as follows:

Theorem 2.0.1 ([2, Theorem 1.4, 1.6])**.** *Suppose Γ is a minimal star-shaped plumbing tree with at least three branches and the weight of the node* [*] *in the dual* [†] *Γ' is at least -1, then the following are equivalent:*

1. *There is a singularity with minimal good resolution graph Γ admitting a $\mathbb{Q}HD^4$ smoothing*
2. *The Milnor fillable contact structure* [‡] *of the link of the plumbed 3-manifold corresponding to Γ admits a weak symplectic $\mathbb{Q}HD^4$ filling*
3. *$\Gamma \in \mathcal{W} \cup \mathcal{N} \cup \mathcal{M} \cup$ the subfamilies of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ depicted on Figure 2.1*

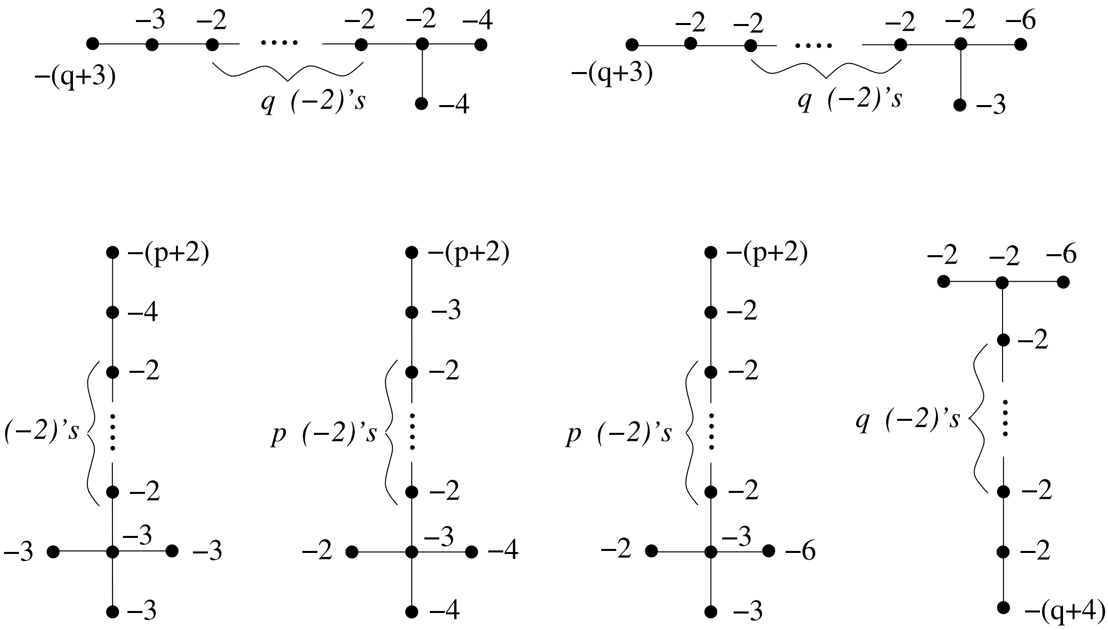


Figure 2.1: The star-shaped subfamilies of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ admitting a $\mathbb{Q}HD$ smoothing

[*] the vertex with valency > 2
[†] see Remark 2.1.4
[‡] see section 1.1 and section A.1 or [5]

Remark 2.0.2. We see in Example A.2.7 that the graphs where the node has valency 3 are taut (Definition A.2.1). The framing assumption is automatically satisfied in this case. It is also worth noting, that this assumption on the valency 4 case places no additional restrictions on the holomorphic result*, since our singularities are rational, the framing of any vertex cannot be less than the valency of the vertex minus one in absolute value (cf. Remark 3.2.3). In [32] it is conjectured, that this list is complete, i.e.:

Conjecture 2.0.3. *The only normal surface singularities admitting a rational homology disk smoothing are the known weighted homogeneous examples.*

2.1 Weighted homogeneous singularities

Now following [2, Section 2-3] we expose the main arguments and methods used in the proof. The first important construction is the Hirzebruch surface (see [12, 3.4.7], [17, II.1-2]), used for the definition of the dual graph later.

Definition 2.1.1. Consider the complex plane bundle $\mathcal{O}(0) \oplus \mathcal{O}(n) \to \mathbb{C}P^1$, where $\mathcal{O}(n)$ is the complex line bundle over $\mathbb{C}P^1$ with Euler number n. We call its projectivization $\mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(n))$ a *Hirzebruch surface*, and denote it by Σ_n.

Example 2.1.2. $\Sigma_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$ and $\Sigma_1 = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Up to diffeomorphism these are the only two cases, dependent only on the parity of n.

2.1.3. These surfaces have two obvious (smooth) sections, corresponding to the origin, and the "point at infinity" on each fiber, we call these the zero and infinity sections respectively (c.f. the coordinate description in [17]). These are two disjointly embedded $\mathbb{C}P^1$'s, with self intersections n and $-n$ respectively. Now let there be given a star-shaped negative definite graph Γ with the central vertex having valency m and decoration b. Pick m different fibers in Σ_b. By blowing up the intersection points of these fibers with the infinity section and then the new divisors as depicted on Figure 2.2, recreate Γ as the neighborhood of the zero section and some curves on the strict transforms of the fibers. Disregarding the -1 curves "in the middle", the complement of this configuration is called the dual graph of Γ, denoted Γ'.

Remark 2.1.4. This construction is needed only to obtain the compactifying divisor

*but the question of symplectic fillability is still an open question if the central framing is $= -2$

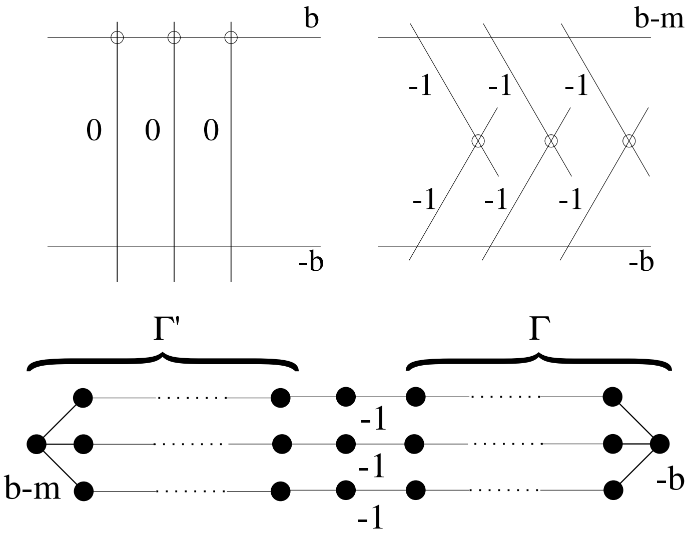


Figure 2.2: $m = 3$ example for the construction of the dual graph and the intersections of the spheres at the end of the construction

$X_{\Gamma'}$ (an open neighborhood of $\Gamma' \subset \Sigma_b \# n\overline{\mathbb{C}P^2}$)), one does not need to run this algorithm to determine the dual graph Γ'. Consider instead the Hirzebruch-Jung continued fraction given by reading the negatives of the decorations of an arm of the graph from the central vertex outwards, this encodes each arm as a rational number $\frac{p_i}{q_i}$.* If Γ had m arms, and the central vertex had decoration $-b$, then the dual graph Γ' will have the same number of arms, central decoration $b-m$, and each arm will have decorations given by the negatives of the continued fraction coefficients of $\frac{p_i}{p_i - q_i}$ (again written from the center outwards). A particularly simple algorithmic way of obtaining these numbers is the *Riemenschneider point rule* ([28, Proposition 2.8], see also [23, 2.3.5]). Consider $\frac{p}{q} = [a_1, a_2, \ldots, a_k]$, and write $a_1 - 1$ dots in a row, next begin a new row of $a_2 - 1$ dots, the first one of which is under the last one of the previous row, and so on. Reading the diagram by columns, instead of rows gives us back on less, than the coefficients of $\frac{p}{p-q}$.

Example 2.1.5. The following diagram represents $\frac{64}{23} = [3, 5, 3, 2]$, reading by columns we see $1, 2, 1, 1, 2, 2$, thus $\frac{64}{41} = [2, 3, 2, 2, 3, 3]$.

	1	2	1	1	2	2
2	•	•				
4		•	•	•	•	
2					•	•
1						•

*In our case of interest each decoration is at most -2, so this number is greater than 1

2.1.6. By construction, the corresponding $X_{\Gamma'}$ has boundary $-\partial X_\Gamma =: -Y_\Gamma$. Now suppose that there is a $\mathbb{Q}HD$ smoothing K of the singularity of X_Γ. This smoothing provides a strong convex filling of the Milnor fillable contact structure of Y_Γ as well, since the deformation induces a homotopy of ξ_M through contact structures on the boundary. Now use [8, Proposition 4] to cut out a sufficiently small neighborhood of X_Γ from the just constructed $\Sigma_b \# n\overline{\mathbb{C}P^2}$, and replace it with K to obtain a new closed symplectic manifold X containing $X_{\Gamma'}$. This can be done, since by [8, Corollary 6.] any sufficiently small neighborhood of the spheres encoded by Γ will be ω-convex, allowing us to preform the surgery symplectically, see [loc. cit.].

2.1.7. The case which is of interest to us are the star-shaped graphs $\Gamma \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, since the other graphs already have $\mathbb{Q}HD$ smoothings constructed, see Theorem 3.1.2. We split the classes according to the arm, where the blowup process defining the class begins, and according to whether the central vertex has valency 3 or 4. Denote them by $\mathcal{A}_3, \mathcal{B}_2, \mathcal{B}_4, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_6, \mathcal{A}^4, \mathcal{B}^4, \mathcal{C}^4$ respectively*. Each of these classes have simple enough shapes for us to be able to describe the corresponding dual graphs with the use of Remark 2.1.4. The strategy of the proof relies on the following theorem:

Theorem 2.1.8 (McDuff, [18, Theorem 1.4])**.** *If a closed symplectic 4-manifold (M^4, ω) contains a symplectically embedded sphere L with self-intersection 1, then (M, L) is symplectomorphic to a blowup of $(\mathbb{C}P^2, \mathbb{C}P^1)$ away from L (i.e. the blown up -1 spheres are disjoint from L, which descends to a projective line).*

2.1.9. We wish to apply this theorem to the above constructed symplectic manifold $X = K \cup_{Y_\Gamma} X_{\Gamma'}$. By blowing up and down the configuration in the $X_{\Gamma'}$ component we create a suitable L with $[L]^2 = 1$. A simple application of the Mayer-Vietoris sequence shows that $\operatorname{rk} H^2(X) = \operatorname{rk} H^2(X_{\Gamma'})$. Now Theorem 2.1.8 guarantees us the existence of suitably many disjoint (-1)-spheres to blow down, but we wish to follow what happens to the dual graph during this process, so we choose a tame almost complex structure for which every curve of Γ' is pseudoholomorphic. A consequence of the above theorem ([2, Lemma 2.2]) tells us, that the remaining curves are all -1 spheres, used to initiate the blowdown process. Now the strategy is clear, since the configuration Γ' does not contain -1 curves, these new curves must intersect it to

subscripts indicate the decoration of the arm where we first blow up, superscripts indicate that the inductive process started by blowing up the central vertex start the blowdown. We must find the possible places they meet the already present curves and see, that these are precisely the cases stated in the theorem, thus ruling out every other member of the families from having $\mathbb{Q}HD$ fillings.

Remark 2.1.10. Some helpful facts aid us to this end. Since $b_2^+(X) = 1$, there are no symplectic spheres of nonnegative self-intersection in the complement of L. There also cannot exist irreducible singular rational curves, or higher genus curves in the complement ([2, 2.5, 2.3]). This also means, that there cannot exist a cycle of rational curves in the complement of L, since we could glue them together to produce a higher genus embedded curve in contradiction with the previous statement. Now we begin discussing the simplest case, that of $\mathcal{C}_6$.

Theorem 2.1.11. *If the link Y_Γ of a singularity associated to a singularity with minimal good resolution graph $\Gamma \in \mathcal{C}_6$ admits a weak $\mathbb{Q}HD$ filling, then the blown up arm has decorations $(-2, \dots, -2, -n-5)$, where $n \geq 2$ is the length of the arm.*

Remark 2.1.12. The proof is modified slightly from the presentation in [2], since the symplectic geometrical arguments made there are difficult to pin down precisely. For some background see [20, Remark 3.2.3].

Proof. The dual graph Γ' can be described simply, the -2 arm dualizes to another -2 arm, for -3, we see $\frac{3}{3-1} = [2, 2]$. The third arm we can only describe abstractly of course, but there are a few pieces of information we note. First, the arm has length at least 5, since the last vertex has decoration at most -6.† By the definition of the $\mathcal{C}$ class the central vertex has framing at most -2, so the dual graph has central framing at least -1. If the framing is not equal to -1 we blow up the intersection of the central sphere and the '6' arm to make it -1.

Now blow down this central -1 sphere, and the two -2 spheres constituting the dual of the -3 arm of Γ to obtain a $+1$ sphere L, a curve C triply intersecting it, and the rest of the dual of the 6 arm unchanged as on Figure 2.3. If the original graph had a rational homology disk smoothing, then we can obtain a closed symplectic manifold as discussed above, in which this configuration is embedded (note, that blowing

*since all symplectic curves are nonzero in homology every curve not intersecting L must disappear before we arrive at $\mathbb{C}P^2$

†at the end of the Hirzebruch surface construction we need to produce -6 or less, which brings at least 5 -2's to the end of the dual chain

up and down is possible symplectically, and it does not change the boundary, see [24, Appendix 3.3.3]). Applying McDuff's theorem, we need to be able to blow this configuration down, and L has to descend to a line. This implies, that C (which we cannot blow down either since it intersects L) will become a cubic curve, since $L \cdot C = 3$.

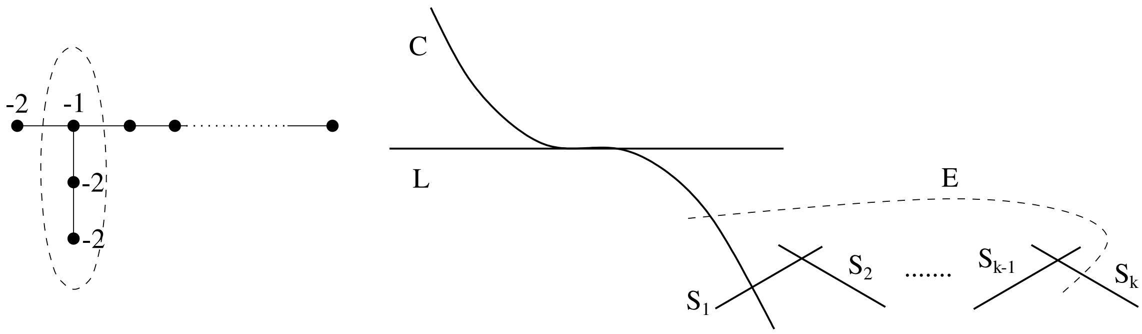


Figure 2.3: The dual configuration with the three blowdowns and the placement of the extra -1 curve indicated

Every blowdown removes a $\overline{\mathbb{C}P^2}$ connected summand from our manifold, thus $rkH^2(X)$ drops by one at every step until it reaches one, when we stop. This rank is coming from $X_{\Gamma'}$, since we closed it with a rational homology ball. After the setup we see that this rank is $k+2$, so $X \cong \mathbb{C}P^2\#(k+1)\overline{\mathbb{C}P^2}$ up to diffeomorphism. We see the generator of $\mathbb{C}P^2$, and k curves which need to be blown down, since they don't intersect L. This means, that by choosing a tame almost complex structure for which the current configuration is pseudoholomorphic, there will be one more -1 sphere E, which has to intersect the chain at some point for us to be able to blow down the configuration. This can only happen at a single point, otherwise it would create a cycle of spheres contradicting Remark 2.1.10. Since the image of the sphere C represents $3L$ after we blow everything down it has to be singular *, so E has to intersect it as well.

If the central framing is not -1, then after the initial constructions the curve C will have framing 2 since we had to blow up the central vertex, meaning that its framing must increase by precisely 7 after we blow everything down. Notice however, that every step increases its framing, and at least once it gets increased by† ≥ 4, when the loop created by E gets closed, i.e. $7 \geq k+4$, so $k \leq 3$ which is a contradiction,

*A smooth curve representing $3L$ has genus 1

†If $[S] \cdot [S'] = k$, with $[S]^2 = -1$, then blowing S down increases the self-intersection of S' by k^2 [12, Figure 5.17-18] since the dual arm has length at least 5 (by the general formula of Equation 2.1 one step in the construction cannot decrease the length of the dual arm, and in the beginning it is of length 5).

If the central framing is already -1, then this means, that during the construction of the 6 arm we don't blow up next to the central vertex (node) since the node of the dual graph will have framing $a-3$ if $-a$ was the framing of the original graph. By the Riemenschneider point rule (Remark 2.1.4) we see that if the original arm of the graph had framings $-c_1, \ldots, -c_n$, then the dual arm has length

$$k = -(n-1) + \sum_{1}^{n}(c_i - 1). \tag{2.1}$$

The sum gives the points in the diagram. The number of rows is this number, but we lose one for every new row we start beginning from the second. Using this formula simple induction shows, that in our case the length of the dual arm $= k+1 = n+4$. For the length 1 graph this is clear, since (-6) dualizes to $(-2,-2,-2,-2,-2)$. One step in the construction of the $\mathcal{C}$ class is a modification of the form: (the underline indicates the inductive -1 vertex, that changes to -2 when we finish the blowups)

$$(\ldots, c_{i-1}, \underline{-2}, c_{i+1}, \ldots) \to (\ldots, c_{i-1} - 1, \underline{-2}, -2, c_{i+1}, \ldots)$$

$$(\ldots, c_{i-1}, \underline{-2}, c_{i+1}, \ldots) \to (\ldots, c_{i-1}, -2, \underline{-2}, c_{i+1} - 1, \ldots)$$

(notice, that we are using here, that the blowup isn't happening next to the central vertex). This means, that Equation 2.1 changes by $2-1$, proving the statement.

We also see, that C has framing at least $3-(n+1)$ by the point rule, since in Γ the length of the arm is n, the corresponding diagram has n rows. The previous argument applies to this case as well. We blow down $n+4$ times, and at least one of these raises the framing of C by (at least) 4 instead of just 1, i.e. it gets raised to at least $3-(n+1)+n+3+4=9$, and it cannot be more since it will descend to a singular cubic.

Now if E does not intersect the chain at its far end, then we claim, that there will be more than one blowdown which increases the framing of C by more than 1. For if it intersects some S_i which is not the endpoint, then blowing down E, then S_i we see a configuration of three spheres intersecting at a point, one of which we have to blow down, say S_{i-1}. Doing so increases the intersection multiplicity of S_{i+1} with the (image of) C, thus when we blow S_{i+1} down we increase the framing by at least 4, and S_{i+2} will also intersect C with higher multiplicity. This contradicts the bound derived previously, since when the loop $C, E, S_i, S_{i-1}, \dots$ gets closed we also have an increase of (at least) 4. Thus the only possibility is for E to intersect the chain at its far end, and this forces the framings to be -2 for all S_i, and C has framing $2-n$. Notice, that to get back to the unmodified dual graph, we have to blow up the three spheres, which we blew down to produce L, so in the actual dual graph the first vertex of the 6 arm has framing $-n-1$.

Finally we can check, that this graph is indeed dual to the claimed configuration. A continued fraction containing n twos can be calculated by simple induction

$$[(2)^n] := [2, 2, \dots, 2] = \frac{n+1}{n} \tag{2.2}$$

thus the dual long arm is represented by $n+1-\frac{n+3}{n+4} = \frac{n^2+4n+1}{n+4}$, so $\frac{n^2+4n+1}{n^2+3n-3}$ encodes the long arm of the original Γ. Now since $n(n^2+3n-3) \equiv n(-n-4) \equiv 1 \mod n^2+4n+1$ we see that by Proposition 1.3.2 reversing the coefficients corresponds to $\frac{n^2+4n+1}{n} = n+5-\frac{n-1}{n} = [n+5, (2)^{n-1}]$, the claimed framings. □

3 Constructions and obstructions

3.1 Constructions

3.1.1 Smoothings of negative weights

In this section we introduce a method for constructing smoothings for all of the known cases (star-shaped graphs i.e. weighted homogeneous singularities). We use a corollary of a result of Pinkham

Theorem 3.1.1 ([30, Theorem 8.1], [27, Theorem 6.7])**.** *Let Γ be a star-shaped negative definite graph. If there exists a smooth projective rational surface Z and a collection of smooth rational curves $D \subset Z$ intersecting according to the dual graph of Γ, Γ' satisfying $\operatorname{rk} H_2(Z, \mathbb{Z}) = \operatorname{rk} H_2(D, \mathbb{Z})$, then Z is a smoothing of a singularity with resolution graph Γ, moreover the interior of the Milnor fiber is diffeomorphic to $Z \setminus D$.*

Using this theorem our aim is to find curve configurations in some blowup of $\mathbb{C}P^2$, a subcollection of which represents the dual graph of a member of the class under consideration. The complete* list of constructions can be found in [30, Section 8.1-2], [3, Section 3.] and [2, proof of theorem 1.4&1.6].

Theorem 3.1.2 ([30, Example 8.3-4])**.** *The graphs in $\mathcal{G}, \mathcal{M}, \mathcal{N}, \mathcal{W}$ all admit $\mathbb{Q}HD$ smoothings.*

Proof. For $\mathcal{G}$ consider a smooth quadric Q and a line L transversely intersecting it in $\mathbb{C}P^2$. Blow up the two intersection points of Q with L to obtain the square dual graph depicted on Figure 3.1. Given a graph $\Gamma \in \mathcal{G}$, we consider it a star-shaped graph with two legs (one possibly empty) by designating the image of the -4 vertex to be the node. We can build its dual graph from the configuration above by blowing up the intersection points of the image of L with its two neighbors.

*with regards to the star-shaped case

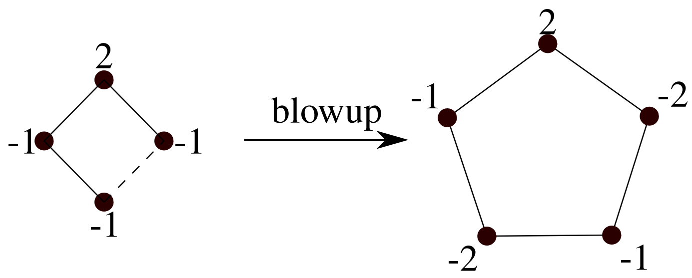


Figure 3.1: The basic configuration and the first blowup indicated by dashes

Identify this with the dual graph of Γ by considering the construction of the dual graph in the Σ_4 Hirzebruch surface. Blow up two fibres as usual, and then the construction is modified slightly as follows: remember that each step in the construction for $\mathcal{G}$ consists of lowering the framing on one end of the chain by one, and concatenating a new -2 onto the other. We can mimic this by doing blowups on both distinguished* -1 curves in the configuration as indicated by Figure 3.2. Building up Γ in this way we almost get back the original definition of the dual graph, but there is a trailing chain of a -1 curve followed possibly by some -2's on one of the arms of the claimed dual. This is a byproduct of our method, blowing them down corresponds precisely to the fact that the -4 curve has had its framing lowered. The length of this -1 subarm is precisely the number of times -4 had its framing lowered by construction (notice, that when -4 is no longer on one of the ends of the chain, the construction changes the last -2 of this portion of the dual arm to a -3, and thus the blowdown process stops at this point). The Γ portion of the

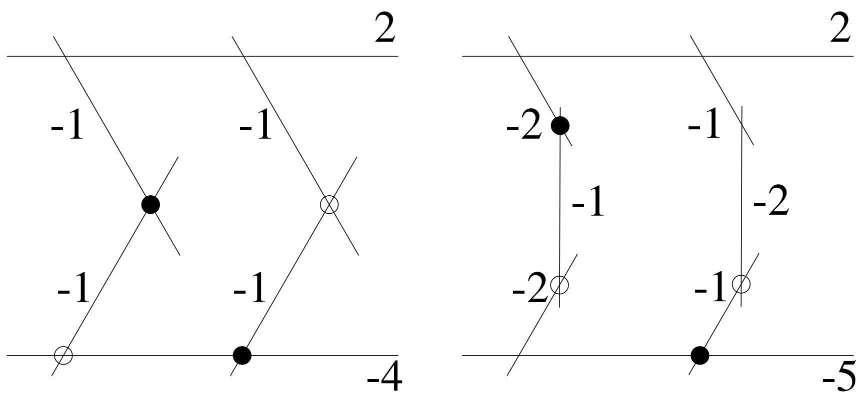


Figure 3.2: The first step of the modified construction with the two possible blowup-pairs indicated by empty and full circles.

configuration stays unchanged during this blowdown process, and we get the dual

*At the first step the -1 curves intersecting the -4 central curve, and afterwards always the "newest" -1 curve on both arms.

graph according to the original definition.

Now it is clear, that the previous construction provides the same dual graph. The starting configuration is identical, disregarding the "bottom" -1 curve on the left of Figure 3.1. A single blowup step produces the same result (namely one arm gets a new -2 vertex, and the other arm's last vertex's framing gets dropped by one) as the original construction of the dual graph. This is true by noticing the symmetry of the construction. If one arm of Γ gets a new -2 vertex, the blowup also lowers the framing of the last member of the corresponding dual arm, and vice versa on the other arm. The rank will also be equal to the rank of the ambient space by induction. It is so in the first step and after every blowup a single new generator is added to the configuration. Afterwards we blow down only vertices in the configuration, so the rank of the ambient space, and also of the subcomplex gets dropped by one.

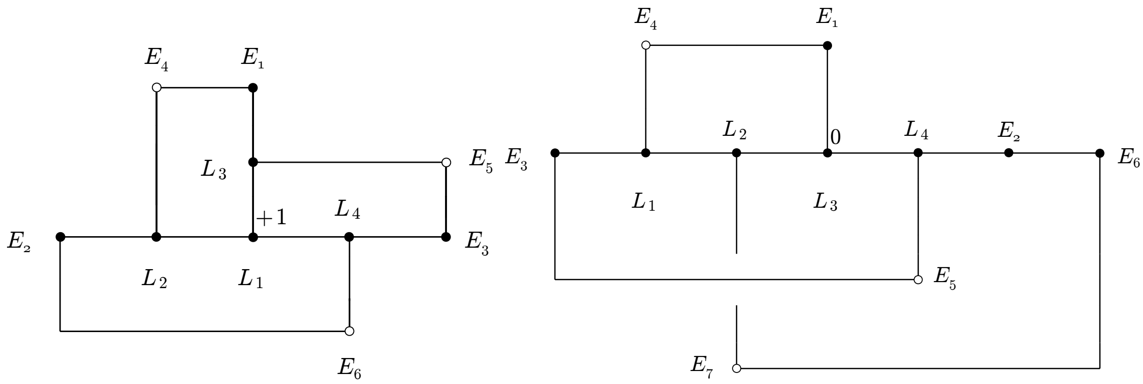


Figure 3.3: The blowup process used for $\mathcal{W}$ (left) and $\mathcal{N}$ (right)
Full circles are framed -2, empty circles are framed -1 unless otherwise indicated

For the $\mathcal{W}$ class consider four lines $L_1, \ldots, L_4 \subset \mathbb{C}P^2$ in general position. Blow up their intersection points 6 times as indicated by $E_1, \ldots, E_6$ on the left diagram of Figure 3.3, and a further p, q and r times at the intersections of L_2, L_3, L_4 with their respective neighboring -1 curves (Figure 3.4).

After the necessary blowups disregarding the three -1 curves gives an appropriate D for us, the central vertex will be L_1 with framing $+1$. From the construction we can read off that the arms will be represented by* $[p+2, (2)^{r+1}]$, $[q+2, (2)^{p+1}]$, $[r+2, (2)^{q+1}]$. The graph we get is the dual of $\Gamma_{p,q,r}$. This is readily checked by pairing up the arms by parameters and checking that the concatenated configuration blows

*see Equation 2.2 for this notation

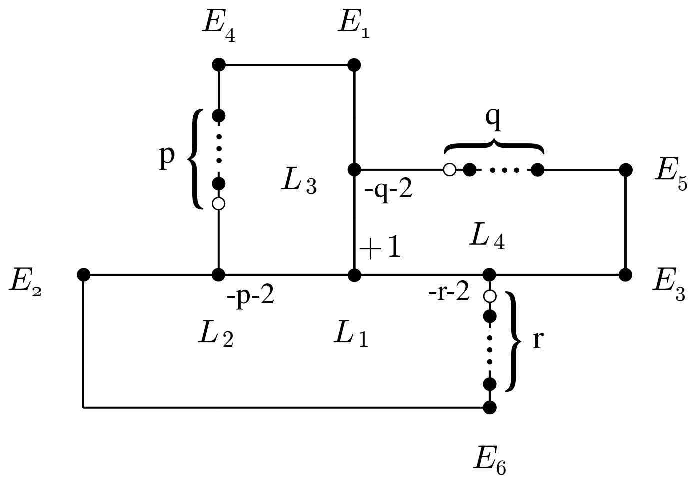


Figure 3.4: The configuration after the $p+q+r+6$-fold blowup

down to the appropriate Hirzebruch surface, by the symmetry of the construction it is enough to check one arm.

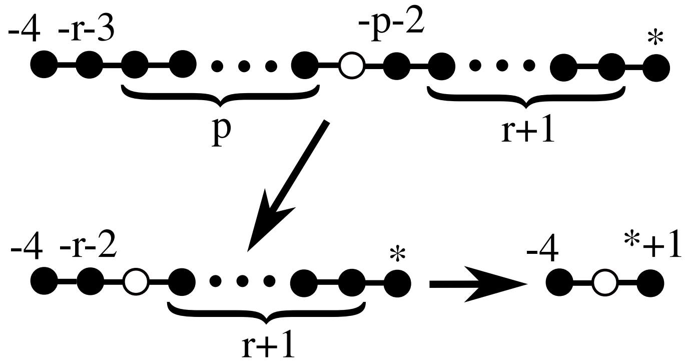


Figure 3.5: The blowdown process for the p, r arm

For $\mathcal{N}$ begin with the same configuration, and through a different sequence of blowups arrive at the configuration on the right of Figure 3.3. To obtain the dual of some specific $\Delta_{p,q,r}$ blow up the intersection between L_1, L_2, L_4 and its neighboring -1 curve p, q, r times respectively as before. Disregarding the three -1 curves we get the desired D with $Z = \mathbb{C}P^2\#(p+q+r+7)\overline{\mathbb{C}P^2}$. T
he homological assumption is clearly satisfied, and once again we can check arm by arm, that it is indeed dual to $\Delta_{p,q,r}$. The central vertex is L_3 of course, the arms pair up by parameters as previously. Concatenating as on Figure 3.5 the single $-p-2$ vertex arm of $\Delta_{p,q,r}$ gets blown down by the E_1 arm, the $[(2)^2, q+4]$ arm is dual to the L_4 arm of our configuration. The long arm is a bit more tricky, but is clear to follow.

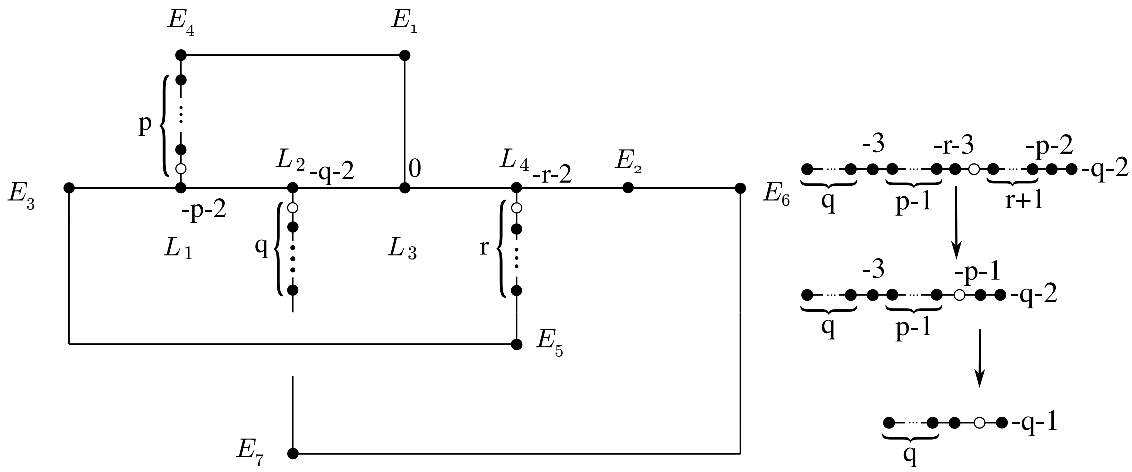


Figure 3.6: The configuration after the appropriate blowups, the full circles represent the dual of $\Delta_{p,q,r}$ and the blowdown of the long arm

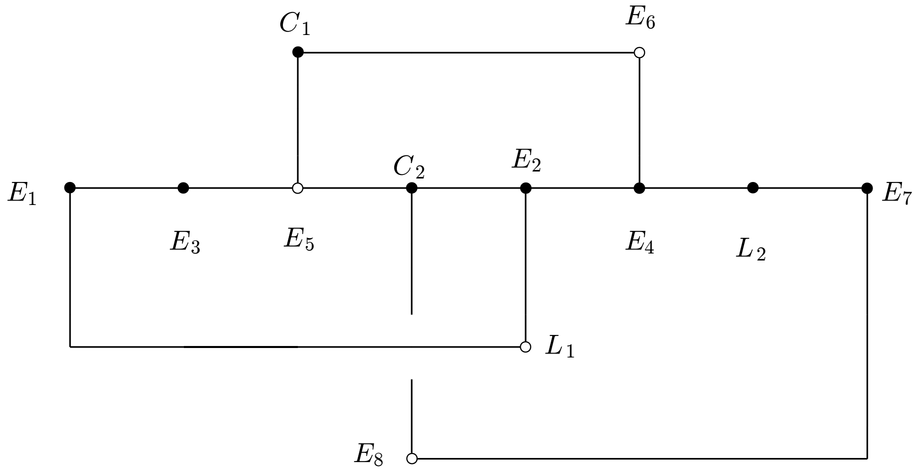


Figure 3.7: The starting blown up configuration with the indicies indicating the order of the blowups

For $\mathcal{M}$ a more elaborate configuration is required, to this end define the following curves in $\mathbb{C}P^2$:

- $C_1 = \{x^2 - yz = 0\}$
- $C_2 = \{x^2 - yz + xy = 0\}$
- $L_1 = \{x = 0\}$
- $L_2 = \{z = 0\}$

i.e. two smooth conics with a triple intersection, the line through the two intersection points ($[0 : 0 : 1]$, $[0 : 1 : 0]$) and the tangent of the first at the transverse intersection of the two conics. As before, blow up the intersections 8 times, to get the curve configuration of Figure 3.7. Further p, q, r-fold blowups of the intersection between E_4, C_2, E_2 and their -1 neighbors produces a configuration from which if we throw the -1's out represents the dual of $\Lambda_{p,q,r}$ (E_5 will be the root).

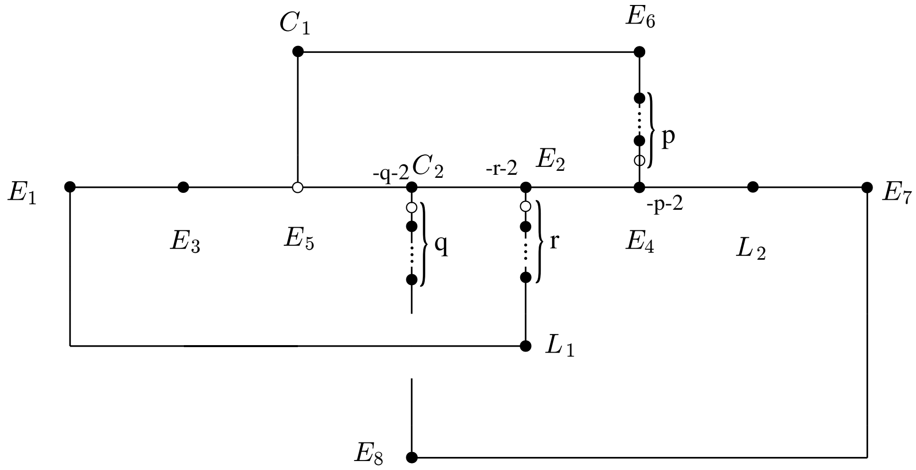


Figure 3.8: The auxiliary blowups producing the dual graph of $\Lambda_{p,q,r}$

By the conditional nature of the definition of the $\Lambda_{p,q,r}$ graphs this assertion requires more checking, but all cases are very similar. The dual arms are $[(2)^{p+1}]$, $[(2)^{r+2}]$, $[q+2, r+2, p+2, (2)^{q+2}]$. The $(2)^*$ arms pair up with the single vertex $p+2$ and $r+3$ arms of the original, and the long arm also blows down in each case, analogously to the long arm of the $\mathcal{N}$ graphs.

After these constructions appealing to Pinkham's theorem proves the stated result.

□

3.1.2 Quotients

3.1.3. Another method we shall mention briefly is the construction of smoothings via quotients ([30, 8.2], [31, 5.8]). Consider a normal threefold singularity germ $(\mathcal{Y}, o)$ on which a finite group G acts freely away from o. An invariant function $f \in \mathcal{O}^G_{\mathcal{Y},o}$, whose zero set (Y, o) is an isolated surface singularity actually defines a smoothing of Y/G, whose Milnor fiber is the factor of the free action of G on the Milnor fiber of Y. This implies that if one can find such an f and G where the group has order equal to the Euler characteristic of the Milnor fiber of Y (i.e. $1 + \mu(Y)$), then one has a $\mathbb{Q}HD$ smoothing of the factor X.

Example 3.1.4 ($\mathcal{G}$ revisited ([31, Example 5.9.1]))**.** Take $(\mathcal{Y}, o) = (\mathbb{C}^3, 0)$ and $0 < q < p$ relatively prime integers. Let ω be a primitive pth root of unity, and let $\mathbb{Z}/p\mathbb{Z}$ act on $\mathbb{C}^3$ via coordinatewise multiplication by $[x, y, z] \mapsto [\omega x, \omega^q y, \omega^{p-1} z]$. The function $f = xz - y^p$ is clearly invariant under this action, the factor is the $(p^2, pq-1)$ cyclic quotient singularity ([23, 2.3]). We can calculate the Milnor number* using [14, Korollar 3.10 a)], since f is clearly weighted homogeneous of degree p with weights $(1, 1, p-1)$, $\mu = (p-1)(p-1)(\frac{p}{p-1} - 1) = p - 1$.† This means, that the Milnor fiber of f has Euler characteristic p, so after factoring by G $p(1 + \mu(F/G)) = p$, and we get rational homology disk smoohtings for the $\mathcal{G}$ family.

Remark 3.1.5. These constructions make computing other invariants (i.e. π_1) of these spaces possible as well, for the valency 3 case see [30, 8.2], for valency 4, [32]. Before moving on we also mention, that in the $\mathcal{M}, \mathcal{N}, \mathcal{W}$ cases the above constructed spaces are the unique minimal fillings up to symplectic deformation, see [1].

3.2 Obstructions

There are a few obstructions for a singularity to admit a $\mathbb{Q}HD$ smoothing, with most depending on only the (smooth) topological type of the minimal good resolution X_Γ.

3.2.1 The geometric genus

As stated in §1.1.1, any singularity admitting a rational homology ball smoothing must be a rational singularity.

Definition 3.2.1 ([22, Definition 3.1-2])**.** The *geometric genus* of a normal surface singularity germ (X, o) is defined by $p_g := \dim H^1(\tilde{X}; \mathcal{O}_{\tilde{X}})$ for some good resolution $\tilde{X} \to X$. (X, o) is called a *rational singularity*, if $p_g = 0$.

Being rational is an analytic property of the singularity at first glance, but this turns out not to be the case. A variation of Laufer's algorithm makes it relatively simple to determine rationality from a good resolution graph for the singularity at hand. We follow [29], but also cf. [22, 2.10, 3.8-9] or [23, 7.1.2] for a more detailed exposition and proofs.

*The Milnor fiber has the homotopy type $\vee_\mu S^2$, we call μ the Milnor number, see [23, 3.2.13]

†The same formula has a typo in [23, 5.1.15 (a)] the terms should be multiplied, instead of summed

3.2.2. Denote the exceptional curves corresponding to the vertices of Γ by E_i. Consider $Z_0 := \sum[E_i] \in H_2(X_\Gamma, \mathbb{Z})$ and the products $(Z_0, E_i) \in \mathbb{Z}$. Now we split the cases according to if among these products there is an E_i with (Z_0, E_i)

- > 1 stop, the singularity is not rational
- $= 1$ replace Z_0 with $Z_0 + E_i$ and repeat
- < 1 then repeat with E_{i+1}

We stop when $(Z_0, E_i) \leq 0\ \forall i$, and this shows the singularity at hand to be rational.

Remark 3.2.3. Note that this algorithm uses only topological data, so rationality does not depend on the analytical type of the singularity, only the topological type of its resolution(s). It is also worth noting, that by this algorithm any graph in which every vertex v has valency* $|N(v)| \leq |v^2|$ *automatically* corresponds to a rational singularity, and if there is a vertex with $|v^2| \leq |N(v)| - 2$ it automatically fails.

Example 3.2.4. Any element of the $\mathcal{A}, \mathcal{B}, \mathcal{C}$ classes where one begins the construction by blowing up the central vertex and then the new vertex will automatically be non-rational and thus will not admit a $\mathbb{Q}HD$ smoothing by the previous remark.

3.2.2 The $\bar{\mu}$ invariant

Consider a plumbing graph with $2 \not| \det\Gamma$. From this it follows that $H_1(Y_\Gamma; \mathbb{Z}_2) = 0$, where $Y_\Gamma = \partial X_\Gamma$ is the boundary of the plumbed 4-manifold corresponding to Γ. This also means that Y_Γ has a unique spin structure, which will be important for us, since generally $\bar{\mu}$ only obstructs Y_Γ from bounding a spin rational homology 4-ball. However in the special case when $\det\Gamma$ is odd, the unique spin structure of Y_Γ always extends to a $\mathbb{Q}HD$ bounding it ([29, Proposition 4.2, Theorem 1.4]), and thus the to be defined $\bar{\mu}$ can be used to rule out certain plumbing graphs. This invariant can be calculated combinatorially as follows ([29, Section 2]).

3.2.5. Begin with the plumbing graph Γ. First we reduce the graph until it consists of isolated points, and then build up a subset of the vertices following this reduction process backwards. The reduction step consists of two possible moves we can make

*$N(v) := \{w \in \Gamma : (v, w) \text{ is an edge of } \Gamma\}$

on the graph. If Γ does not consist of isolated points, consider a valency 1 vertex v and its unique neighbor $w \in \Gamma$.

1. if $v^2 \equiv 0 \mod 2$, delete* v and w from Γ
2. if $v^2 \equiv 1 \mod 2$ delete v, and change the parity of w^2.

This produces a sequence of decorated graphs $\Gamma = \Gamma_1, \Gamma_2^{i_2}, \dots, \Gamma_k^{i_k}$ where we take note of the type of move used with $i_j \in \{1, 2\}$, and where the sets of vertices form a descending chain. Now if the resulting set of points do not all have odd framing, stop, the graph has even determinant.

Take the set of these points $S_1 = \Gamma_k^{i_k}$, and follow the previous reduction process in reverse. We omit the indices from the S_i for easier readability, and use primes instead to present a single step. Given $S \subset \Gamma_j^{i_j}$, if the next vertex to be re-added is $v \in \Gamma_{j-1}^{i_{j-1}}$ with neighbor w, then the new subset S' will be depending on the type of move used

1. $S = S'$ if $|N(w) \cap S'| \equiv w^2 \mod 2$ or $S = S' \cup \{v\}$ if not when $i_{j-1} = 1$.
2. $S = S'$ if $w \in S'$ or $S = S' \cup \{v\}$ if not for $i_{j-1} = 2$.

Running through all the steps we arrive at some subset of vertices $S \subset \Gamma$. Now using this set S we define $\bar{\mu}(Y_\Gamma) := -n - \sum_S v^2$.

Theorem 3.2.6 ([29, Corollary 1.2]). *Using the previous notations, if* $\det \Gamma$ *is odd, and* $\bar{\mu}(Y_\Gamma) \neq 0$*, then no normal surface singularity with resolution graph* Γ *admits a* $\mathbb{Q}HD$ *smoothing.*

Remark 3.2.7. By construction the set S represents a disjoint union of spheres. It is also simple to check, that $c_S := PD([\sqcup_S S_i]) \in H^2(X_\Gamma, \mathbb{Z})$ is a characteristic cohomology class, and since $\pi_1(X_\Gamma) = 0$ it corresponds to a unique spin^c structure on X_Γ, with first Chern class c_S. This means that this class has compact support in X_Γ, so the restriction to the boundary gives us not just a spin^c, but the spin structure on Y_Γ. S is called the *Wu set* corresponding to the unique spin structure of Y_Γ.

*along with any other edges w is an endpoint of

Example 3.2.8. The family depicted on Figure 3.9 has odd determinant, and $\bar{\mu}(Y_\Gamma) = 0$ for odd n, but no rational homology disk smoothing, as shown later.

Proposition 3.2.9. *Elements of $\mathcal{B}$ have even determinant.*

Proof. If we reduce the intersection matrix for the smallest element of $\mathcal{B}$ we see

$$\det \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -4 & 0 \\ 1 & 0 & 0 & -4 \end{bmatrix} \equiv \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \equiv 0 \mod 2$$

since the last three rows are the same. One step in the construction changes at most one of these rows, and further steps cannot touch these either. This means, that all intersection matrices have two rows, which are linearly dependent $\mod 2$, and thus have even determinant. □

Corollary 3.2.10. *By the same argument, if one begins with blowing up the vertex, or the -3 edge in $\mathcal{C}$, the resulting graphs will have even determinant.*

3.2.3 Dimension count

From the algebraic viewpoint we have the following

Theorem 3.2.11 ([32, Theorem 8.1])**.** *Let (X, o) be a rational surface singularity with minimal good resolution $(\tilde{X}, E_1 \cup \cdots \cup E_n)$. Let the sheaf $S_{\tilde{X}}$ be defined by the exact sequence $0 \to S_{\tilde{X}} \to TX \to \oplus_i \nu_{E_i \subset \tilde{X}} \to 0$ (see also [33, 2.2]). If $\dim H^1(\tilde{X}; S_{\tilde{X}}) - \sum_1^n (E_i^2 + 3) \leq 0$ then (X, o) does not admit a $\mathbb{Q}HD$ smoothing.*

Remark 3.2.12. The formula computes the dimension of any $\mathbb{Q}HD$ smoothing component in the base space of the semi-universal deformation space for (X, o), but this is outside the scope of this thesis. We only need an important corollary.

Corollary 3.2.13. *With the above notations, if (X, o) is taut, then $H^1(\tilde{X}; S_{\tilde{X}}) = 0$, in particular $\mathbb{Q}HD$ smoothings cannot exist if $\sum_1^n (-E_i^2 - 3) \leq 0$.*

Corollary 3.2.14 ([32, Theorem 8.6])**.** *The taut "H-shaped" (having exactly two vertices of valency 3, and the others of valency 1, 2) graphs in the three families $\mathcal{A}, \mathcal{B}, \mathcal{C}$ cannot have $\mathbb{Q}HD$ smoothings.*

Proof. To obtain an H-shaped graph in one of the families we have to first blow up one of the intersection points, and (possibly after a sequence of other edge blowups) blow up the vertex once, and then possibly the edges a few more times. It is easy to check, that the number $\sum_\Gamma(-v^2-3)$ does not change under edge blowups: this lowers the framing of a vertex by one, and creates a new vertex of framing -2, i.e. the sum changes by $1+2-3=0$. A similar check shows, that the sum decreases by 1 under a vertex blowup. Without any blowups, the graphs of Figure 1.2 (after changing the -1 to $-4,-3,-2$ respectively) have this sum equal 1, so if we have a taut H-shaped member of these families, the sum is zero, and thus the theorem rules them out from having a $\mathbb{Q}HD$ smoothing. □

Remark 3.2.15. Note, that tautness does not behave well under blowups, let alone the construction of the $\mathcal{A},\mathcal{B},\mathcal{C}$ classes. In particular, a taut graph has at most two vertices of valency >2 by Theorem A.2.2.

Example 3.2.16. Nonetheless, we get many members of $\mathcal{A},\mathcal{B},\mathcal{C}$ not admitting a QHD smoothing. In particular all H-shaped graphs, where the second node (obtained by blowing up the vertex during the construction) has framing at most -3 are taut of the form $(L_1)+(J_1)+(R_2)$ (see Table A.1).

Example 3.2.17 ([32, Corollary 8.7]*, cf. [11, Proposition 4.2])**.** For $n\geq 2$ the graph falls into the previous example. For $n=1$ the second node of the family on Figure 3.9 has framing -2, but is also taut of type $(L_1)+(J_2)+(R_4)-(C_3)$, and so it cannot admit a $\mathbb{Q}HD$ smoothing.

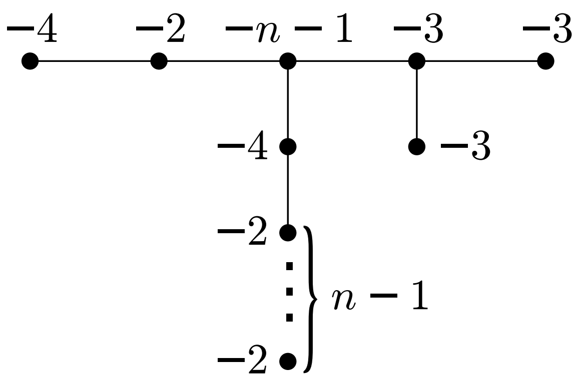


Figure 3.9: A subfamily of $\mathcal{A}$ admitting no $\mathbb{Q}HD$ smoothing

*Note that in [32, Remark 8.9] a graph is erroneously claimed to be non-taut, in fact it is $(L_1)+(J_1)+(R_8)-(C_4)$, see Example A.2.9. ([25, Remark 4.2])

Example 3.2.18. A simple search utilising Appendix B and manual checking of tautness shows, that all rational graphs with $\bar{\mu} = 0$ where we blow up at most 4 times are taut with one exception in $\mathcal{A}$ and two exceptions in $\mathcal{B}$ depicted on Figure 3.10. These graphs cannot be taut, since they contain a vertex of valency 4, but are rational and their $\bar{\mu}$ invariant vanishes for some spin structure (the graphs with the same shape in $\mathcal{C}$ all have nonzero $\bar{\mu}$). One of the smallest graphs which these

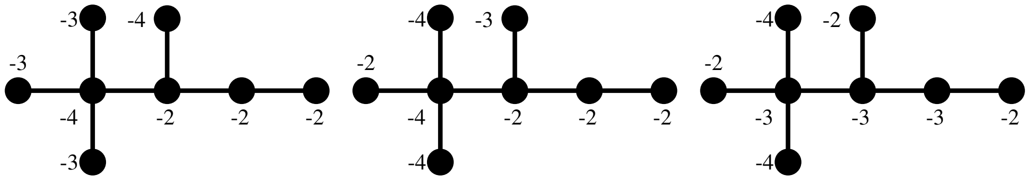


Figure 3.10: Three graphs not ruled out by our previous conditions

invariants cannot rule out in $\mathcal{C}$ is depicted on Figure 3.11 along with an H-shaped example from $\mathcal{A}$ and $\mathcal{B}$ as well. The latter two require 5 blowups, while the smallest example from $\mathcal{C}$ still needs 4. They cannot be taut, since a vertex of degree two cannot support two arms of length > 1.

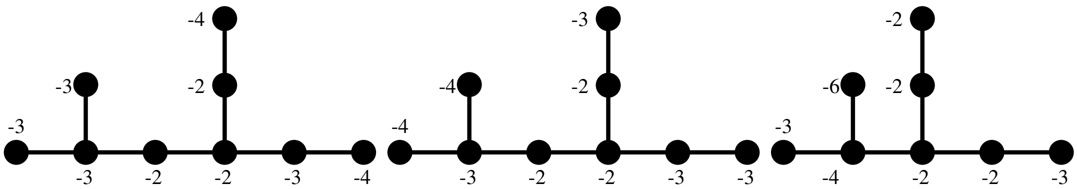


Figure 3.11: Smallest non-taut rational H-shaped graphs with $\bar{\mu} = 0$ in $\mathcal{A}, \mathcal{B}, \mathcal{C}$.

3.2.4 McDuff's theorem and lattice embeddings

A direct application of the strategy in chapter 2 can be used to rule out certain subfamilies of the remaining $\mathcal{A}, \mathcal{B}, \mathcal{C}$ classes. For suitable graphs one can construct a "dual" configuration in a Hirzebruch surface, manipulating it until one produces a $+1$ sphere allows for the application of Theorem 2.1.8. Ruling out every possible sequence of blowdowns shows that the given singularity link cannot admit a weak rational homology ball filling, thus no such smoothing either. An example of this strategy is found in [4]. Start with one of the graphs on Figure 3.12, and substitute the corresponding subgraph in place of the arm as indicated. The graphs produced this way admit no $\mathbb{Q}HD$ smoothing ([4]).*

*Note, that the family Γ_n^B of [4] is not in fact a member of $\mathcal{B}$, one of the short arms should be framed -4 instead of -2.

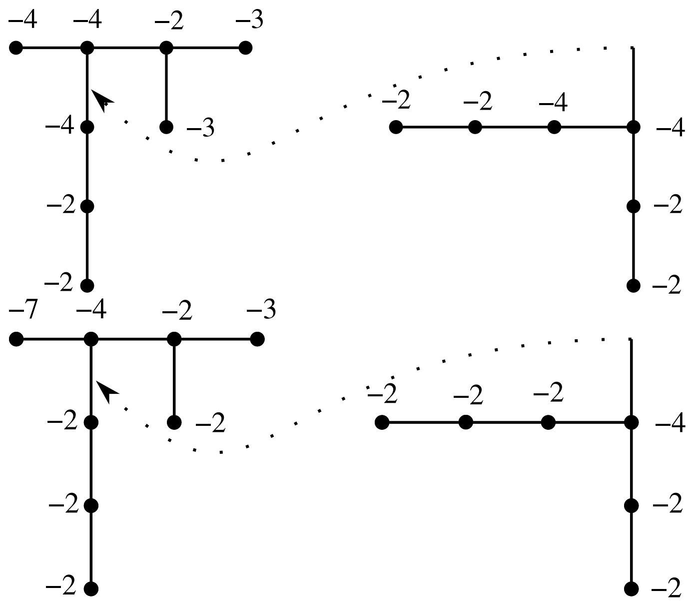


Figure 3.12: The starting graphs and the inductive subgraphs producing subfamilies of $\mathcal{A}, \mathcal{C}$ not admitting a rational homology disk smoothing.

One can also extend the strategy of chapter 2 more generally. In [11] an explicit construction is given for symplectic caps, i.e. concave fillings of plumbed 4-manifolds along a negative definite tree of spheres, with $-v_i^2 \geq |N(v)|$ (c.f. 3.2.3 and also with the dual graph construction in 2.1.6). This is done, so arguments in the spirit of 2.1.9 could be carried out in a more general setting. A Kirby diagram for the filling can be described as follows:

3.2.19. Choose a leaf $v \in \Gamma$. The diagram will be described as the closure of a braid, take $-w^2 - |N(w)|$ many strands for each vertex $v \neq w \in \Gamma$, and $-v^2 - 2$ many for v (note, that this number is nonnegative, since $-v^2 \geq 2$). Make a full -1 twist on this trivial braid, and for each edge e a full negative twist on the strands corresponding to connected components of $\Gamma \setminus \{e\}$, which don't contain v. The framing of the strings associated to v will be -2, and each string associated to a vertex $w \neq v$ will be framed $-d(v, w) - 3$, where $d(v, w)$ is the (vertex) distance between v, w in Γ.

Now using this construction we can suppose that the manifold ∂X_Γ has a rational homology sphere filling, and derive a contradiction by showing, that the intersection lattice (which can be read off from the Kirby diagram of the cap) cannot be embedded into the diagonal lattice of the same rank, contradicting Donaldson's theorem ([7, 1.3.1]). This is done for the family depicted on Figure 3.9 in [11, Proposition 4.4].

A Algebraic geometry

For the sake of completeness and easier reading we review briefly two unrelated, but equally important notions used multiple times in the main text.

A.1 The Milnor fibration

The first important construction is the Milnor fibration and the Milnor fiber associated to a holomorphic function. For a detailed exposition see [21].

Theorem A.1.1 ([23, Proposition 3.2.3]). *Let (X, o) be an isolated singularity germ, $\rho : X \to [0, \infty)$ a real analytic function with $\rho^{-1}(0) = \{o\}$ and $f : (X, o) \to (\mathbb{C}, o)$ a holomorphic function germ having an isolated singularity at o. Then for all sufficiently small* $0 < \delta < \epsilon$

1. *$\rho^{-1}([0, \epsilon]) \cap f^{-1}(\xi)$ is a smooth manifold, which is transverse to $\rho^{-1}(\epsilon)$ for all nonzero $\xi \in D^2_\delta \subset \mathbb{C}$ (i.e. $0 < |\xi| \leq \delta$).*
2. *$f : (\rho^{-1}([0, \epsilon]) \cap f^{-1}(S^1_\delta), \rho^{-1}(\epsilon) \cap f^{-1}(S^1_\delta)) \to \partial D^2_\delta = S^1_\delta$ is a locally trivial fibration of pairs*
3. *The "boundary" fibration $f : \rho^{-1}(\epsilon) \cap f^{-1}(S^1_\delta) \to S^1_\delta$ extends to the disc D^2_δ.*

Moreover these fibrations are independent of the choice of ρ.

Remark A.1.2. The above fibration is called the nearby fibration of the pair. The space $\rho^{-1}([0, \epsilon]) \cap f^{-1}(t)$ for some $t \in D^2_\delta \setminus \{0\}$ is called the *Milnor fiber*. By the third statement its boundary is oriented diffeomorphic to the link of the singularity of f.

The next ingredient we require is the notion of a deformation, but first an algebraic notion

Definition A.1.3 ([6, Definition 10.39.1]). If R is a unital commutative ring and M is an R-module, then we call M *flat*, if the functor $N \mapsto N \otimes_R M$ is exact, i.e. for all short exact sequences $0 \to N_1 \to N_2 \to N_3 \to 0$ the sequence $0 \to N_1 \otimes_R M \to$

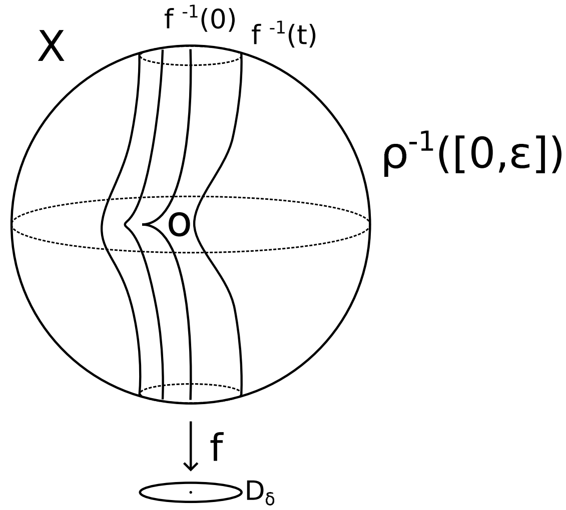


Figure A.1: Schematic picture of the nearby fibration with the singular fiber and the Milnor fiber depicted.

$N_2 \otimes_R M \to N_3 \otimes_R M \to 0$ is also exact. A map $\varphi : A \to B$ of commutative unital rings is called *flat*, if the induced A-module structure*on B makes it into a flat A-module. A map between germs $\phi : (X, o) \to (Y, o)$ is called *flat* if the mapping induced on the local ring $\phi^* : \mathcal{O}_{Y,o} \to \mathcal{O}_{X,o}$ is flat.

Remark A.1.4. The tensor functor is automatically right exact, only the first "$0 \to$ " of the sequence is needed. The geometric importance of this definition comes from a theorem of Frisch, which implies that the fibres of a flat map are equidimensional ([6, Lemma 10.112.9]) if the domain and the range have well defined dimensions ([13]).

Definition A.1.5. [13, Definition 1.1] Fix a normal surface singularity germ (X, o). A flat morphism $\phi : (\mathcal{Y}, o) \to (\mathbb{C}, o)$ together with an isomorphism $(X, o) \cong (\phi^{-1}(o), o)$ is called a *deformation* of the singularity (X, o).

The details of this theory are beyond our current scope, we only wish to apply the nearby fibration theorem to a given deformation. This shows us, as stated in §1.1.1, that the Milnor fiber of a deformation gives a filling of the link of the singularity, which we use repeatedly during the constructions.

*$a \cdot b = \phi(a)b$

A.2 Taut singularities

We also wish to reproduce here the list of taut singularities of Laufer ([15]).

Definition A.2.1. Let (X, o) be a normal surface singularity, with minimal resolution dual graph* Γ. We call (X, o) *taut* if for any other singularity (X', o) with the same minimal resolution dual graph $\Gamma' = \Gamma$ we have an analytic isomorphism $(X, o) = (X', o)$.

Theorem A.2.2 ([15, §2.2])**.** *The taut resolution graphs are precisely the type $I - V$ graphs of the following list.*

All genera are 0 (higher genus curves don't have unique complex structures). $\bullet_n$ will denote a vertex of framing at most n.

Remark A.2.3. Note, that if a graph is taut, the modified graph, where the framing of any vertex is lowered will be taut as well.

Remark A.2.4. Note, that some details are omitted in this exposition, the original paper goes through some trouble†, so that every graph fits into exactly one category, but for the sake of recognizing taut graphs this is superfluous.

For brevity let $\bullet := \bullet_{-2}$, and $\cdots$ a path of $\bullet$ vertices, which may consist of a single edge and no vertex at all.

Now the list begins as follows: a single vertex $\bullet$ (denoted type I), or a path $\bullet \cdots \bullet$ (type II) is always taut.

The possibilities with one star‡ (type III) are the graphs depicted on Figure A.2 together with E_6, E_7, E_8 (denoted here as $III.7, III.8, III.9$). Note the possibility of lowering the framing from -2 on the latter three as well.

For the graphs with two stars (type IV) there is a scheme for building taut graphs from two star-shaped components very similar to the ones of Figure A.2. We describe the building blocks and list the permitted combinations in Table A.1.

*here every vertex is decorated with not just the Euler number of the normal bundle, but the genus of the component as well

†some nodes are only allowed to be framed -2, otherwise they fall into some previous category

‡star-shaped subgraph

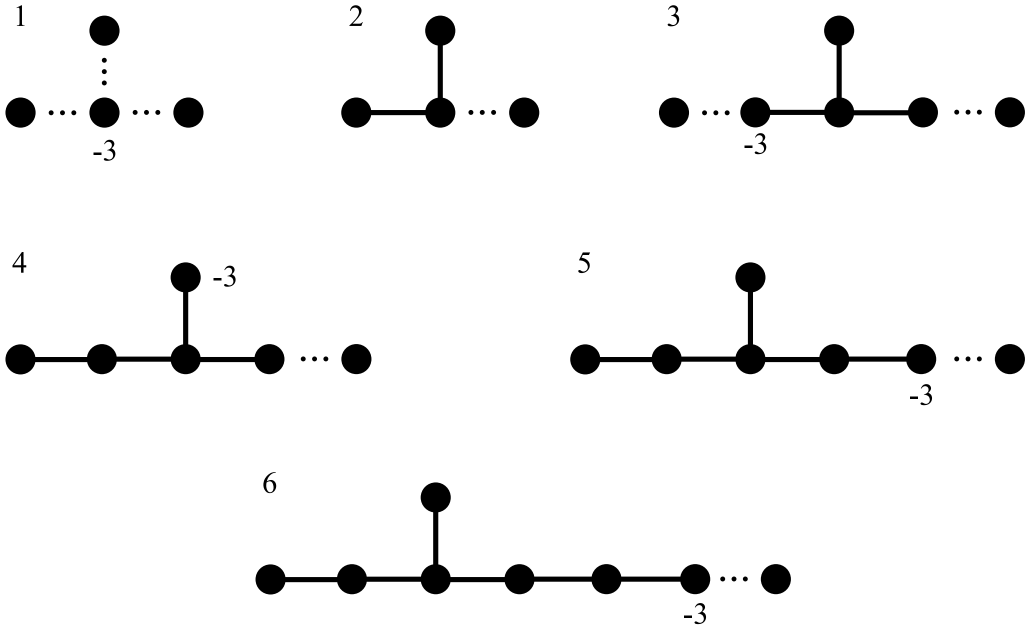


Figure A.2: The taut star-shaped graphs besides E_6, E_7, E_8.

The list of star-shaped subgraphs is on Figure A.3. We list only the left versions, the right graphs $(R_1), \ldots, (R_8)$ are simply the mirror images of the left ones.

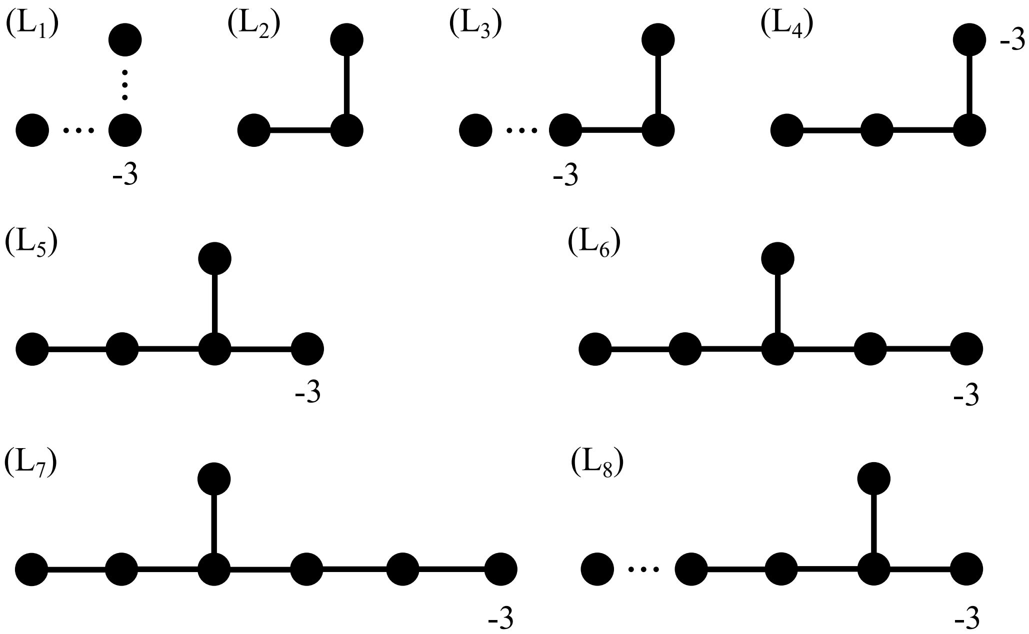


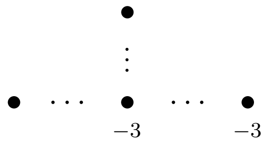
Figure A.3: The left star-shaped subgraphs.

These star-shaped subgraphs will be joined by $(J_1) = \cdots$, or $(J_2) = \cdots \bullet_{-3} \cdots$.

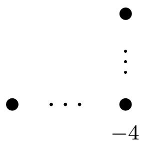
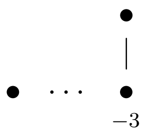
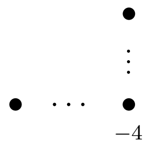
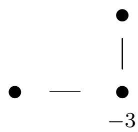
Some subgraphs *may* be contracted as follows: A $\bullet_{-3} \cdots \bullet_{-3}$ subgraph between the two nodes may be contracted to a -3 or -4 framed vertex:

$$(C_1) = \bullet_{-3} \qquad (C_2) = \bullet_{-4}$$

A subgraph of the form



may be replaced by the following subgraphs:

$(C_3) =$  or by  $(C_4) =$  or by 

Remark A.2.5. The two options for both subgraphs may seem redundant, but as mentioned previously not all combinations are permitted.

	R_1	R_2	$R_3\&R_4$	$R_5\&R_6$	$R_7\&R_8$
L_1	J_1	J_1	J_2, C_3	J_1, C_3	J_1, C_4
L_2		$J_1{}^*$	J_2	J_1	J_1
$L_3\&L_4$			J_2	J_1	J_2, C_2
$L_5\&L_6$				J_1, C_1	J_1, C_2
$L_7\&L_8$					J_1, C_2

Table A.1: The table of taut graphs with two stars.

Remark A.2.6. Some subgraphs share a column/row, since their allowed operations are the same, but note, that only such $(L_i), (R_j)$ combinations are allowed where $i \leq j$. All graphs produced by Table A.1 are taut, with one subtle exception. The $(L_2) + (J_1) + (R_2)$ (marked with *) entry is only negative semidefinite if all framings are -2, and thus cannot correspond to a singularity ([23, Proposition 2.1.12]). At least one vertex has to have its framing lowered to make it so.

For the sake of completeness (even though it does not come up in our investigations), the last (type V) taut graphs are depicted on Figure A.4.

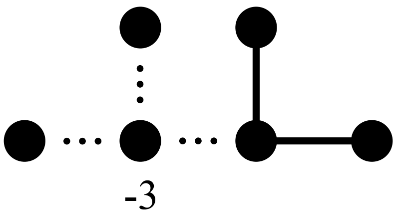
*At least one of the framings has to be chosen -3 or less.

Figure A.4: The cyclic taut graphs

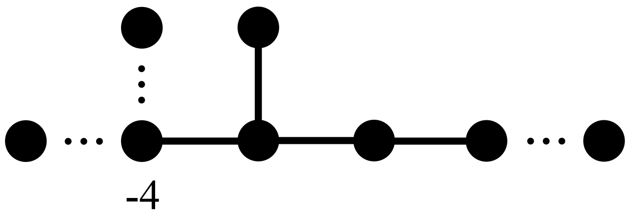
Example A.2.7 (The main families of Figure 1.1)**.** Elements of $\mathcal{G}$ are of type I and II, and so are taut. $\mathcal{W}, \mathcal{N}$ are type $III.1$, while $\mathcal{M}$ are of type $III.2$, and so are all taut as well. The star-shaped subfamilies of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of Figure 2.1 are also of type $III.2$.

Remark A.2.8. The valency 4 graphs of Figure 2.1 are non-taut, but are still special. Their analytical type is determined by the analytical type* of the exceptional curve configuration. Since spheres have a unique complex structure, this is in turn determined by the cross-ratio of the 4 intersection points of the central vertex with the arms, see [15, §4] and [32, proof of Corollary 8.2].

Example A.2.9. Two example entries from Table A.1.



$(L_1) + (J_1) + (R_2)$



$(L_1) + (J_1) + (R_8) - (C_4)$

*as opposed to the topological type, as is the definition of taut singularities

B Graph algorithms in SageMath

Given a graph G and some edge or vertex o, the following algorithm checks if o is a vertex of the graph, and blows it up, otherwise assumes that it is an edge, and blows that up. The vertices are encoded by numbers, and the edges by pairs as usual.

```
def blowup(G,o): #blow up the graph G at the vertex or edge o
    switch=False #blow up an edge by default
    for v in G.vertex_iterator(): #check if o is a vertex
        if v==o:
            switch=True
            break
    if switch: #blow up the vertex o
        new=G.add_vertex() #the new vertex
        G.add_edge(o,new) #connected to the one we blew up
        G.set_vertex(new,-1) #new vertex framed -1
        weight=G.get_vertex(o)-1 #lower the framing of o
        G.set_vertex(o,weight)
    else: #blow up the edge o
        new=G.add_vertex() #the new vertex
        G.delete_edge(o) #remove the blown up edge&connect the
        G.add_edges([(o[0],new),(o[1],new)]) #new vertex
        G.set_vertex(new,-1) #new vertex framed -1
        left=G.get_vertex(o[0])-1 #subtract one from the
        right=G.get_vertex(o[1])-1 #framings of both endpoints
        G.set_vertex(o[0],left)
        G.set_vertex(o[1],right)
    return G
```

The reduction steps of 3.2.5: The following algorithm recursively produces a list of ordered triples encoding the reduction process, consisting of the graph, which vertex was deleted and the type of step (either 1 or 2), and the input graph separately for easier usage.

```
def rank(graph,l=[]):
    G=graph.copy()
    if G.size()==0: #if there are no edges we are done
```

```
        return [G,l]
    for v in G.vertex_iterator():
        nbrs=G.neighbors(v) #get the neighbors of v
        if len(nbrs)==1: #find a leaf
            if G.get_vertex(v)%2==0: #type 1 reduction
                l.append([G.copy(),v,1]) #add to the list
                G.delete_vertices({v,nbrs[0]}) #delete them
                break
            else: #type 2 reduction
                l.append([G.copy(),v,2]) #add to the list
                G.set_vertex(nbrs[0],G.get_vertex(nbrs[0])-1)
                G.delete_vertex(v) #change parity and delete
                break
    return rank(G,l)
```

The `wuset` function is one step in the second part of 3.2.5. Requires a graph, the Wu set for the previous step, which vertex was added, and by which type of move.

```
def wuset(graph,Sprime,v,n):
    G=graph.copy() #the original G stays unmodified
    w=G.neighbors(v)[0] #the unique neighbor of the leaf v
    if n==1: #v was type 1
        number=(Sprime.intersection(Set(G.neighbors(w)))).
            cardinality() #number of neighbors of w in Sprime
        if (number-G.get_vertex(w))%2==0: #|N(w)|=w^2 (2)?
            S=Sprime
        else:
            S=Sprime.union(Set([v]))
    else: # v was type 2
        if w in Sprime:
            S=Sprime
        else:
            S=Sprime.union(Set([v]))
    return S
```

Putting together the previous two algorithms, the following function computes the $\bar{\mu}$ invariant for a given *negative definite* graph. Note also, that this implements the general version of the algorithm found in [29], and outputs the product of the $\bar{\mu}(Y,\mathfrak{s})$'s for each spin structure $\mathfrak{s}$ of Y.

```
def mubar(G):
```

```
    sigma=-G.order() #we assume G is negative definite
    var=rank(G.copy(),[]) #get the list of reductions
    l=var[1] #the list of reductive steps
    l.reverse() #we go through the reduction in reverse
    out=1 #initialise the output
    generators=Set([]) #initialise S
    R=var[0] #the graph consisting of disjoint vertices
    for v in R.vertex_iterator(): #go through all vertices
        dec=R.get_vertex(v) #get the framing
        if dec%2==0: #collect the even framed vertices
            generators=generators.union(Set([v]))
    ones=Set(var[0].vertices()).difference(generators)
    onesets=generators.subsets() #every subset of the even vertices
        gives a spin structure
    for S in onesets:
        wus=copy(S)
        wus=wus.union(ones)
        for H in l: #the backwards step of the wuset algorithm
            wus=wuset(H[0],wus,H[1],H[2])
        square=int(0) #initialise the square sum
        for v in wus: #add together the framings
            square+=int(G.get_vertex(v))
        out*=(sigma-square)
    return out
```

Given a list of vertices, the following function generates the graph in one of the three classes corresponding to it. Since the -1 vertex is always unique, the edge blowups are encoded by the neighboring vertex. The graph's vertices are indexed from -3, so in the beginning the -1 vertex has index 0, and in the further steps the -1 vertex will always be the one indexed by index of the vertex in the list l.

```
def generate_graph(l):
    G=Graph({0: [-1,-2,-3]}) #initialize the graph
    G.set_vertices({-3:-3,-2:-3,-1:-3,0:-1}) #replace the values
       with -2,-4,-4 or -2,-3,-6 for B and C
    for a in range(0,len(l)):
        if G.get_vertex(l[a])==-1:
            G=blowup(G,l[a]) #blow up at the -1 vertex
        else:
            G=blowup(G,(a,l[a],None)) #blow up the edge encoded by
```

```
                the neighbor a
    G.set_vertex(len(l),-4) #also replace with -3 or -2 for B or C
    return G
```

`intersection_form` produces the intersection form as a matrix from the given graph. Note, that this assumes, that the graph was created by `generate_graph`.

```
def intersection_form(G):
    M=G.adjacency_matrix()
    for v in G.vertex_iterator():
        M[v+3,v+3]=G.get_vertex(v) #the indexing is shifted by 3
    return M
```

`pg` implements Laufer's algorithm 3.2.2, outputting 1 if the graph is found to be rational and -1 if not.

```
def pg(G):
    Q=intersection_form(G)
    n=Q.dimensions()[0] #Q has dimension (n,n)
    v=[] # initialize Z_0
    for i in range(0,n): # Z_0 begins as the sum of the E_i
        v.append(1)
    v=vector(v) # convert v from list to vector
    vektor=v*Q #coordinates of this are the v*E_i
    while len([s for s in vektor if s>0])>0: #any product >0?
        vektor=v*Q #recalculate the products
        for i in range(0,n): #check them all
            prod=vektor[i]
            if prod>1: #product>1
                return -1 #not rational
            elif prod==1: #the multiplicity with E_i is 1
                v[i]+=1 #add E_i to v
                break
            elif prod<1: #multiplicity is negative
                continue #next vertex
    return 0 #if we came here, that means every multiplicity is <1
```

`search` is a simple breadth-first search. Beginning from a list of graphs additional blowups produce graphs from the ones on the list. It then checks the $\bar{\mu}$, $\det\Gamma$, p_g invariants.

```
def search(l):
 uj=[] #initialise the list for the current step
 while True:
  for lista in l: #go through every element of l
   G=generate_graph(lista)
   neighbors=G.neighbors(len(lista)) #neighbors of the -1
   neighbors.append(len(lista)) #search for vertex blowups as well
   for v in neighbors: #do each possible blowup
    w=copy(lista) #copy the list of blowups
    w.append(v) #take note of the new blowup
    H=generate_graph(w) #do the blowup
    mu=mubar(H) #calculate mubar
    genusz=pg(H) #calculate pg
    uj.append(w) # add w to the new list
    nodes=len([x for x in H.degree_sequence() if x>2]) # calculate the number of nodes in the graph
    if mu==0: #modify the conditions here for different searches
     print('constr=',w)
     print("mubar=",mu)
     print("pg=",genusz)
     print("det=",intersection_form(H).det()%2)
     print("# of nodes=",nodes)
     H.show(vertex_labels=H.get_vertices())
     #return(w)
  l=uj #update the list with the new blowup sequences
  uj=[] #zero out the temporary list
  print("# of blowups:",len(l[0])) #how many steps checked
```

After checking all possible graphs in $\mathcal{A},\mathcal{C}$ up to 11 blowups (this brought our attention to Proposition 3.2.9 as well, $\bar{\mu}$ cannot rule out any element of $\mathcal{B}$), we make the following conjecture:

Conjecture B.0.1. *For* $\Gamma \in \mathcal{A} \cup \mathcal{C}$ *if* $\det \Gamma$ *is odd, then* $\bar{\mu}(\Gamma) \in \{0, 8\}$.

This is quite surprising, as it would mean that even if the $\bar{\mu}$ invariant (and the associated Gauge theoretic d invariant) rule out a certain graph from having a rational homology disc filling, the value of the invariant is the least possible.

Bibliography

[1] M. Bhupal and K. Ono. Symplectic fillings of links of quotient surface singularities. *Nagoya Mathematical Journal*, 207:1–45, 2012.

[2] M. Bhupal and A. I. Stipsicz. Weighted homogeneous singularities and rational homology disk smoothings. *American journal of mathematics*, 133(5):1259–1297, 2011.

[3] M. Bhupal and A. I. Stipsicz. *Smoothings of Singularities and Symplectic Topology*, volume 23 of *Bolyai Society Mathematical Studies*, pages 57–97. Springer Berlin Heidelberg, Berlin, Heidelberg, 2013.

[4] M. Bhupal and A. I. Stipsicz. Nonexistence of rational homology disk weak fillings of certain singularity links. In *Proceedings of the Gökova Geometry-Topology Conference 2015*. International Press, 2016.

[5] C. Caubel, A. Némethi, and P. Popescu-Pampu. Milnor open books and milnor fillable contact 3-manifolds. *Topology*, 45(3):673–689, 2006.

[6] A. J. De Jong et al. The stacks project, 2013.

[7] S. K. Donaldson and P. B. Kronheimer. *The geometry of four-manifolds*. Oxford university press, 1997.

[8] J. B. Etnyre. Symplectic convexity in low-dimensional topology. *Topology and its Applications*, 88(1-2):3–25, 1998.

[9] R. Fintushel and R. J. Stern. Rational blowdowns of smooth 4-manifolds. *Journal of Differential Geometry*, 46(2):181–235, 1997.

[10] D. T. Gay and A. I. Stipsicz. Symplectic surgeries and normal surface singularities. *Algebraic & Geometric Topology*, 9(4):2203–2223, 2009.

[11] D. T. Gay and A. I. Stipsicz. *On Symplectic Caps*, pages 199–212. Birkhäuser Boston, Boston, MA, 2012.

[12] R. E. Gompf and A. I. Stipsicz. *4-manifolds and Kirby calculus*. Number 20 in Graduate Studies in Mathematics. American Mathematical Soc., 1999.

[13] G.-M. Greuel. Deformation and smoothing of singularities. *Handbook of Geometry and Topology of Singularities I*, pages 389–448, 2020.

[14] G.-M. Greuel and H. A. Hamm. Invarianten quasihomogener vollständiger durchschnitte. *Inventiones mathematicae*, 49:67–86, 1978.

[15] H. B. Laufer. Taut two-dimensional singularities. *Mathematische Annalen*, 205(2):131–164, 1973.

[16] E. Looijenga and J. Wahl. Quadratic functions and smoothing surface singularities. *Topology*, 25(3):261–291, 1986.

[17] M. Manetti. Lectures on deformations of complex manifolds. *arXiv preprint math/0507286*, 2005.

[18] D. McDuff. The structure of rational and ruled symplectic 4-manifolds. *Journal of the American Mathematical Society*, 3(3):679–712, 1990.

[19] D. McDuff. The local behaviour of holomorphic curves in almost complex 4-manifolds. *Journal of Differential Geometry*, 34(1):143–164, 1991.

[20] D. McDuff and D. Salamon. *J-holomorphic curves and symplectic topology*, volume 52. American Mathematical Soc., 2012.

[21] J. Milnor. Singular points of complex hyper sur faces. *Ann of Math. Stud*, 61, 1968.

[22] A. Némethi. *Five lectures on normal surface singularities*, volume 8 of *Bolyai Society Mathematical Studies*, pages 269–351. Bolyai Society Mathematical Studies, 1999.

[23] A. Némethi. *Normal surface singularities*, volume 74 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics*. Springer Cham, 2022.

[24] B. Ozbagci and A. I. Stipsicz. *Surgery on Contact 3-Manifolds and Stein Surfaces*, volume 13 of *Bolyai Society Mathematical Studies*. János Bolyai Mathematical Society, 2004.

[25] H. Park, D. Shin, and A. I. Stipsicz. Normal complex surface singularities with rational homology disk smoothings. *arXiv preprint*, 2013.

[26] J. Park. Seiberg-witten invariants of generalised rational blow-downs. *Bulletin of the Australian Mathematical Society*, 56(3):363–384, 1997.

[27] H. Pinkham. Deformations of normal surface singularities with c^* action. *Mathematische Annalen*, 232:65–84, 1978.

[28] P. Popescu-Pampu. The geometry of continued fractions and the topology of surface singularities. *Adv. Stud. Pure Math*, 46:119–195, 2007.

[29] A. Stipsicz. On the $\bar{\mu}$ invariant of rational surface singularities. *Proceedings of the American Mathematical Society*, 136(11):3815–3823, 2008.

[30] A. I. Stipsicz, Z. Szabó, and J. Wahl. Rational blowdowns and smoothings of surface singularities. *Journal of Topology*, 1(2):477–517, 2008.

[31] J. Wahl. Smoothings of normal surface singularities. *Topology*, 20(3):219–246, 1981.

[32] J. Wahl. On rational homology disk smoothings of valency 4 surface singularities. *Geometry & Topology*, 15(2):1125–1156, 2011.

[33] J. M. Wahl. Equisingular deformations of normal surface singularities, i. *Annals of Mathematics*, 104(2):325–356, 1976.