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Intersection of matroids

MSc Thesis

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Budapest, 2023

Acknowledgement

First and foremost, I would like to express my gratitude to my supervisor, Kristóf Bérczi for introducing me to these fascinating research topics and providing a lot of support throughout the process of writing this thesis. He answered my questions extensively and enhanced the quality of this work with useful comments and suggestions. His enthusiasm has been an incredible source of motivation, and I look forward to continuing our work together.

I would also like to thank my girlfriend, family and friends for their unwavering belief in me and their constant support, especially during the most challenging times.

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1 Introduction

I met with matroids at the very beginning of my university career. In my first year, Gyula O.H. Katona gave an introduction on the topic, and then I started to read about matroids from Oxley's excellent book [36] in the next year. It fascinated me from the beginning that matroids are connected to various areas of mathematics such as combinatorics and algebra, but only years later, the problem-solving and research seminar on matroids organized by Kristóf Bérczi and Tamás Schwartz made me decide to dive into this topic deeper with Kristóf Bérczi as supervisor.

Whitney [45] introduced matroids as a combinatorial abstraction of linear independence in vector spaces. It turned out that matroids also appear naturally in combinatorics, the probably most notable example being the cycle-free subsets of edges of a graph. The theory of matroids became a powerful tool in combinatorial optimization. In particular, one of the most fundamental results of matroid theory is Edmonds' matroid intersection theorem [11] that provides a min-max formula for the maximum size of a common independent set of two matroids, which can be extended to weighted matroids as well [12, 16], providing another general theorem with several applications.

Unfortunately, the analogous problem for the intersection of three or more matroids results in hard problems. Indeed, we will see that the NP-complete Hamiltonian $s-t$ path problem in a directed graph reduces to finding a maximum sized common independent set in the intersection of three matroids, hence there is no hope for giving a good characterization in general. Still, there is a long list of open problems that can be formulated as special cases, hence identifying tractable instances is of interest. In this thesis, we gather five famous, widely open conjectures that can be formulated as a matroid intersection problem for three matroids. The ultimate goal is to examine the structure of these instances, identify some general pattern fulfilled by every triple of matroids in these conjectures, and prove a general theorem for the intersection of three matroids having this special property.

As the intersection of two matroids defines a simplicial complex, the intersection of three matroids can be looked at as the intersection of a simplicial complex and a matroid. Simplicial complexes received significant interest due to their nice combinatorial properties in various areas, see for example [5, 25, 32]. By using a topological approach, Aharoni and Berger [1] formulated several results that aim at bounding the maximum size a set can have in the intersection of a complex and a matroid. The homotopical connectivity of the simplicial complex plays a key role in these results, hence understanding and computing the value of this parameter is of fundamental interest.

Though we hardly know anything about the intersection of three matroids, there are various interesting questions on the intersection of even more matroids. Clearly, the intersection of matroids is a simplicial complex, and it is well-known that the converse is also true: every simplicial complex can be written as the intersection of finite number of matroids. We examine the problem of recognizing whether a simplicial complex is the intersection of k matroids, and we give an exponential lower bound for the minimal number $\kappa(n)$ such that every simplicial complex on n vertices can be written as the intersection of $\kappa(n)$ matroids. Combining this result with the trivial upper bound, we find the order of magnitude of $\kappa(n)$.

1.1 Structure of the thesis

In the next subsection, we introduce the main notation used throughout the thesis.

In Section 2, we give several equivalent definitions of matroids and overview the fundamental concepts of matroid theory. We show several examples of matroids appearing naturally in combinatorics and present some essential methods for generating matroids from other matroids. Finally, we give a list of fundamental results of matroid theory that will be needed later on.

In Section 3, we state the main problem of the thesis on the intersection of matroids and explain that it is solvable for two matroids but hard for more than two matroids.

In Section 4, we survey the main results of Aharoni and Berger. We begin with the necessary topological preliminaries followed by the presentation of the main theorem of Aharoni and Berger. Then we consider the limitations of this theorem and its applicability for proving conjectures related to the intersection of three or more matroids.

In Section 5, we list five famous conjectures and provide a reformulation of each of them as a matroid intersection problem involving three matroids.

In Section 6, we consider the structure of the intersection of more matroids. Specifically, we talk about the problem of recognizing whether a simplicial complex can be written as the intersection of k matroids. Additionally, we consider the extremal problem of determining the minimal number $\kappa(n)$ such that all simplicial complexes on n vertices can be written as the intersection of $\kappa(n)$ matroids.

1.2 Notation

We use calligraphic letters for denoting families of sets, particularly simplicial complexes or families arising from matroids. We use capital letters for sets and matroids, and we use lowercase letters for elements of a set. We use $A + x$ for $A \cup \{x\}$ and $A - x$ for $A \setminus \{x\}$. We denote by 2^X the family of subsets of X. For a function $f : S \to T$ and a set $X \subseteq S$, let

$$f(X) = \{y \in T \ : \ \text{there exists } x \in X \text{ with } f(x) = y\}.$$

We use $\mathbb{N}$ for the set of nonnegative integers. We denote the symmetric difference of two sets X and Y by $X \Delta Y$, and we denote the disjoint union by $X \dot{\cup} Y$. We always denote the set of independent sets, bases and circuits of matroids by the calligraphic letters $\mathcal{F}$, $\mathcal{B}$ and $\mathcal{C}$, respectively, possibly with some index or prime symbol. Similarly, we always use r to denote the rank function.

2 Matroid prerequisites

In this section, we introduce the basic definitions, examples and theorems about matroids to keep this thesis self-contained. We do not prove these results as they are well-known and can be found for example in the lecture notes of Frank [17] or the book of Oxley [36].

2.1 Definitions of matroids

A matroid can be characterized in several ways, we overview the definitions through independent sets, bases, rank function and circuits in this order.

Let $\mathcal{F} \subseteq 2^S$ be a family over the finite ground set S. We call the pair $(S, \mathcal{F})$ a matroid if the following three conditions hold.

(F1) $\emptyset \in \mathcal{F}$,

(F2) If $X \subseteq Y \in \mathcal{F}$ then $X \in \mathcal{F}$,

(F3) If $X, Y \in \mathcal{F}$ with $|X| < |Y|$ then there exists an $y \in Y \setminus X$ with $X + y \in \mathcal{F}$.

We call the members of $\mathcal{F}$ *independent* and call the rest of the subsets of S *dependent*.

The *bases* of a matroid $M = (S, \mathcal{F})$ are the maximal independent sets with respect to set inclusion, we denote the family of bases by $\mathcal{B}$. It is easy to see that all bases have the same size $r(M)$, called the *rank* of M. The bases of a matroid have the following properties.

(B1) $\mathcal{B}$ is nonempty,

(B2) For any $B_1, B_2 \in \mathcal{B}$ and $x_1 \in B_1 \setminus B_2$ there exists an $x_2 \in B_2 \setminus B_1$ such that $B_1 - x_1 + x_2 \in \mathcal{B}$.

Note that from the set of bases $\mathcal{B}$, we can determine the set of independent sets,

$$\mathcal{F} = \{F \subseteq S \ : \ \text{there exists } B \in \mathcal{B} \text{ with } F \subseteq B\}.$$

It can be proved that if a family $\mathcal{B}$ satisfies (B1) and (B2) then $(S, \mathcal{F})$ is a matroid with family independent sets $\mathcal{F}$, i.e. the family $\mathcal{F}$ satisfies (F1), (F2) and (F3). Hence, we can define a matroid by giving the family of its bases instead of the family of independent sets. We will use this, we write the pair $(S, \mathcal{B})$ instead of $(S, \mathcal{F})$ when we define a matroid through its bases.

The bases exchange property (B2) can be strengthened to obtain the so-called symmetric bases exchange property (B2'), and it is also equivalent to the so-called co-exchange axiom (B2") in which the role of B_1 and B_2 is reversed.

(B2') For any $B_1, B_2 \in \mathcal{B}$ and $x_1 \in B_1 \setminus B_2$ there exists an $x_2 \in B_2 \setminus B_1$ such that $B_1 - x_1 + x_2 \in \mathcal{B}$ and $B_2 - x_2 + x_1 \in \mathcal{B}$.

(B2") For any $B_1, B_2 \in \mathcal{B}$ and $x_1 \in B_1 \setminus B_2$ there exists an $x_2 \in B_2 \setminus B_1$ such that $B_2 + x_1 - x_2 \in \mathcal{B}$.

Note that if a matroid $M = (S, \mathcal{F})$ is restricted to a subset $Z \subseteq S$, i.e. we consider the family $\{F \in \mathcal{F} : F \subseteq Z\}$ over Z, then we obtain a matroid. Hence, by the earlier observation that every maximal independent set is of equal size, this property also holds for all subsets of S. Let $r(Z)$ be the size of a maximal independent set with respect to inclusion contained in Z. By the above discussion, this value is well-defined. The function $r : 2^S \to \mathbb{N}$ is called the rank function of the matroid M. The following propositions always hold for the rank function of a matroid.

(R1) $r(\emptyset) = 0$,

(R2) $r(X) \leq r(Y)$ if $X \subseteq Y$,

(R3) $r(X) \leq |X|$,

(R4) $r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y)$.

The first three properties are obvious, while (R4) is called the submodularity property of r. Interestingly, we can define a matroid by its rank function. If a function $r : 2^S \to \mathbb{N}$ satisfies (R1), (R2), (R3) and (R4) then $M = (S, \mathcal{F})$ is a matroid where

$$\mathcal{F} = \{X \subseteq S \ : \ r(X) = |X|\},$$

and the rank function of M is exactly r.

Finally, we define a matroid with its circuits. In a matroid $M = (S, \mathcal{F})$ a circuit is a minimal dependent set (with respect to inclusion). The family of circuits of M, denoted by $\mathcal{C}$, meet the following conditions.

(C1) $\emptyset \notin \mathcal{C}$,

(C2) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$ then $C_1 = C_2$, (C3) If $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$ and $e \in C_1 \cap C_2$ then there exists $C \in \mathcal{C}$ with $C \subseteq C_1 \cup C_2 - e$.

Condition (C3) is called the weak circuit axiom, and it is the only one which is not obvious. These properties characterize a matroid in the sense that if a family $\mathcal{C}$ fulfills (C1), (C2) and (C3), then $\mathcal{C}$ is the set of circuits of the matroid $(S, \mathcal{F})$ where

$$\mathcal{F} = \{F \subseteq S \ : \ \text{there is no } C \in \mathcal{C} \text{ with } C \subseteq F\}.$$

2.2 Examples for matroids

In this subsection, we consider some important examples of matroids that will be useful in later sections of the thesis.

Example 2.1 (Uniform matroid)**.** Let S be an n-element set and let $0 \leq k \leq n$. Define $\mathcal{F}$ to be the family that contains all subsets of S with cardinality at most k. Then $(S, \mathcal{F})$ is called a *uniform matroid*.

Example 2.2 (Partition matroid)**.** Let k be a positive integer, $P_1, P_2, \dots, P_k$ be a partition of S and $a_1, a_2, \dots, a_k$ be positive integers with $a_i \leq |P_i|$ for all $1 \leq i \leq k$. Define

$$\mathcal{F} = \{A \subseteq S \ : \ |A \cap P_i| \leq a_i \text{ for all } 1 \leq i \leq k\}.$$

The pair $(S, \mathcal{F})$ is a matroid called a *partition matroid*.

Example 2.3 (Graphic matroid)**.** Let $G = (V, E)$ be a graph. Let $\mathcal{F}$ contain the subsets of E that contain no cycles. Then $(E, \mathcal{F})$ is a matroid, called the *graphic matroid* associated with the graph G.

In the previous examples, we saw that there are various structures in mathematics that form a matroid. In the following examples, we define new matroids from existing ones.

Example 2.4 (Restriction of a matroid)**.** Given a matroid $M = (S, \mathcal{F})$ and a subset $X \subseteq S$, define $M|X = (X, \mathcal{F}|X)$ with

$$\mathcal{F}|X = \{F \in \mathcal{F} \ : \ F \subseteq X\}.$$

It is easy to see that $M|X$ is a matroid called the *restriction* of M to X.

Example 2.5 (Contraction of a matroid)**.** Let $M = (S, \mathcal{F})$ be a matroid, let $X \subseteq S$ and let B be a maximal independent set contained in X. The *contraction* of M by X is the matroid $M/X = (S \setminus X, \mathcal{F}/X)$ where

$$\mathcal{F}/X = \{F \subseteq S \setminus X \ : \ F \cup B \in \mathcal{F}\}.$$

It can be proved that M/X is a matroid

Example 2.6 (Direct sum of matroids)**.** Let $S_1, S_2, \dots, S_n$ be pairwise disjoint sets, and let $M_1 = (S_1, \mathcal{F}_1), M_2 = (S_2, \mathcal{F}_2), \dots, M_n = (S_n, \mathcal{F}_n)$ be matroids. Then $M = (S, \mathcal{F})$ is the *direct sum* of the above matroids, denoted by $M = M_1 \oplus M_2 \oplus \dots \oplus M_n$ if $S = \dot{\bigcup}_{i=1}^n S_i$, and

$$\mathcal{F} = \{F \subseteq S \ : \ F \cap S_i \in \mathcal{F}_i \text{ for all } 1 \leq i \leq n\}.$$

It is easy to see that M is indeed a matroid.

Example 2.7 (Dual matroid)**.** Given a matroid $M = (S, \mathcal{B})$, define

$$\mathcal{B}^* = \{B \subseteq S \ : \ S \setminus B \in \mathcal{B}\}.$$

It can be proved that $\mathcal{B}^*$ is the family of bases of a matroid on S. We call $M^* = (S, \mathcal{B}^*)$ the *dual matroid* of M. Note that the dual of the dual matroid M^* is M.

2.3 Further definitions

Matroids $(S_1, \mathcal{F}_1)$ and $(S_2, \mathcal{F}_2)$ are *isomorphic* if there exists a bijection $\varphi : S_1 \to S_2$ with the property that $\varphi(X) \in \mathcal{F}_2$ if and only if $X \in \mathcal{F}_1$ for all $X \subseteq S_1$. It is easy to see that if G is a planar graph and G^* is its dual, then the dual of the graphic matroid associated with G is isomorphic to the graphic matroid of G^*.

Given a matroid $M = (S, \mathcal{F})$ with rank function r, we call a subset $X \subseteq S$ a *flat* if $r(X + x) > r(X)$ for all $x \in S \setminus X$. The flats of rank $r(S) - 1$ are called *hyperplanes*. It can be proved that the intersection of flats is also flat.

In a matroid $M = (S, \mathcal{F})$ we call a set $X \subseteq S$ a *cocircuit* if it is a circuit in the dual matroid M^*. Cocircuits are also called *cuts* of the matroid M. It is easy to see that cocircuits are exactly the complements of hyperplanes. Therefore, the sets that intersect all cocircuits are exactly the sets that contain a basis.

3 The fundamental problem

The most general form of the problem we are interested in is the following.

Problem 3.1. Given k matroids on the same ground set S, what is the maximum cardinality of a set that is independent in all k matroids?

The problem is motivated by the fact that numerous problems can be formulated in this way, as we will see later on. Note that Problem 3.1 includes the problem of determining whether there exists a common basis of k matroids with the same rank.

3.1 Case of two matroids

It turns out that in the case of $k = 2$, there exists a beautiful min-max characterization for Problem 3.1 given by Edmonds [11].

Theorem 3.2 (Edmond's theroem)**.** *Let $M_1 = (S, \mathcal{F}_1)$ and $M_2 = (S, \mathcal{F}_2)$ be two matroids over the same ground set S with rank functions r_1 and r_2, respectively. There exists a set $A \subseteq S$ with $|A| = k$ and $A \in \mathcal{F}_1 \cap \mathcal{F}_2$ if and only if*

$$r_1(X) + r_2(S \setminus X) \geq k$$

for all $X \subseteq S$.

Note that the only if direction is easy, as $|A \cap X| \leq r_1(X)$ and $|(S \setminus A) \cap X| \leq r_2(X)$ for all $A \in \mathcal{F}_1 \cap \mathcal{F}_2$. The hard part is to prove that there exists a common independent set having

$$\max_{X \subseteq S} \{r_1(X) + r_2(S \setminus X)\}$$

elements. Though we will not go into the details of the proof, we would like to highlight that there exist algorithmic proofs showing that it is possible to find a maximal common independent set in polynomial time. For matroids with rank r, Cunningham [7] presented an algorithm with $O(nr^{1.5})$ independence oracle calls, where n denotes the size of the ground set S.

The matroid intersection theorem is a cornerstone result in combinatorial optimization, and many combinatorial quantities can be written in this form. For example, Edmond's theorem and the corresponding algorithms can be used to find maximum-sized matchings in a bipartite graph, see the following example.

Example 3.3. Let $G = (S, T, E)$ be a bipartite graph, and let E_v denote the set of edges incident to v for all $v \in S \cup T$. Define partition matroids

- $M_1 = (E, \mathcal{F}_1)$ with $\mathcal{F}_1 = \{X \subseteq E \ : \ |E_v \cap X| \leq 1 \text{ for all } v \in S\}$, and
- $M_2 = (E, \mathcal{F}_2)$ with $\mathcal{F}_2 = \{X \subseteq E \ : \ |E_v \cap X| \leq 1 \text{ for all } v \in T\}$.

Note that $\{E_v\}_{v \in S}$ and $\{E_v\}_{v \in T}$ are partitions of E, hence M_1 and M_2 are indeed partition matroids. Clearly, a set $X \subseteq E$ is independent in both matroids if and only if X is a matching.

We mention that there exist generalizations of Edmond's theorem for weighted matroids, see [12, 16] for details.

3.2 Case of more than two matroids

Unfortunately, calculating the maximum size of a common independent set of more than two matroids is hard. For instance, the NP-complete Hamiltonian path problem can be formulated this way.

Proposition 3.4. *Let $D = (V, A)$ be a directed graph, and let $s, t \in V$ be two given vertices. The problem of determining whether there exists a Hamiltonian path from s to t can be written as a case of Problem 3.1 using three matroids.*

Proof. Consider the following three matroids on the ground set A.

- Let $M_1 = (A, \mathcal{F}_1)$ be the partition matroid with

  $$\mathcal{F}_1 = \{X \subseteq A \ : \ |X \cap E_v| \leq 1 \text{ for all } v \in V - s \text{ and } |X \cap E_s| = 0\},$$

  where $E_v \subseteq A$ are the edges ending in v for all $v \in V$.

- In a similar manner, let $M_2 = (A, \mathcal{F}_2)$ be a partition matroid, where

  $$\mathcal{F}_2 = \{X \subseteq A \ : \ |X \cap E'_v| \leq 1 \text{ for all } v \in V - t \text{ and } |X \cap E'_t| = 0\},$$

  where E'_v is the set of edges starting in v for all $v \in V$.

- Let $M_3 = (A, \mathcal{F}_3)$ be the graphic matroid of the underlying undirected graph of G.

M_1 guarantees that all indegrees are at most one, and there is no edge ending in s. In the same way, M_2 contains subsets of edges with all outdegrees being at most one, and the outdegree of vertex t is zero. Hence, it is easy to see that the existence of a Hamiltonian path from s to t in D is equivalent to the size of the largest common independent set of M_1, M_2 and M_3 being $|V(G)| - 1$. ☐

Though the problem is hard in general, it would be still intriguing to characterize Problem 3.1 for special matroid classes. The 3-dimensional matching problem shows that in the case of one of the most basic classes of matroids, partition matroids, the problem is still NP-hard even for three matroids.

Definition 3.5. Let X, Y and Z be disjoint, finite sets, and let $\mathcal{T}$ be a 3-uniform hypergraph over $X \cup Y \cup Z$ with $|T \cap X| = |T \cap Y| = |T \cap Z| = 1$ for all $T \in \mathcal{T}$. A *3-dimensional matching* is a subhypergraph $\mathcal{T}' \subseteq \mathcal{T}$ consisting of pairwise disjoint sets.

Proposition 3.6. *The maximum size of a 3-dimensional matching can be formulated as a special case of Problem 3.1 with three partition matroids.*

Proof. We use the notation introduced in Definition 3.5. For any $a \in X \cup Y \cup Z$, let $\mathcal{T}_a \subseteq \mathcal{T}$ be the subhypergraph consisting of the sets containing a. Consider the following three partition matroids over $\mathcal{T}$.

- $M_1 = (\mathcal{T}, \mathcal{F}_1)$ with $\mathcal{F}_1 = \{\mathcal{S} \in \mathcal{T} \ : \ |\mathcal{S} \cap \mathcal{T}_x| \leq 1 \text{ for all } x \in X\}$.
- $M_2 = (\mathcal{T}, \mathcal{F}_2)$ with $\mathcal{F}_2 = \{\mathcal{S} \in \mathcal{T} \ : \ |\mathcal{S} \cap \mathcal{T}_y| \leq 1 \text{ for all } y \in Y\}$.
- $M_3 = (\mathcal{T}, \mathcal{F}_3)$ with $\mathcal{F}_3 = \{\mathcal{S} \in \mathcal{T} \ : \ |\mathcal{S} \cap \mathcal{T}_z| \leq 1 \text{ for all } z \in Z\}$.

Clearly, a subhypergraph $\mathcal{T}' \subseteq \mathcal{T}$ is in $\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3$ precisely if it is a 3-dimensional matching. $\square$

Although we usually cannot solve Problem 3.1 for $k = 3$, it is still meaningful to try to find manageable cases and applicable methods, as several unsolved problems can be formulated this way, see Section 5. In the next section, we examine a topological approach for attacking the problem based on the results of Aharoni and Berger [1].

4 Intersection of a matroid and a simplicial complex

I learned the basic concepts of simplicial complexes and combinatorial homotopy theory from the Combinatorial Algebraic Topology book of D. Kozlov [32] and the lecture notes of L. Lovász on Topological Methods in Combinatorics [34], hence we will mainly follow the notation introduced in these works.

4.1 Topological preliminaries

An *abstract simplicial complex* $\mathcal{K}$ is a down-closed family over a finite set S, i.e. $\mathcal{K} \subseteq 2^S$ with the property that $X \in \mathcal{K}$ if $X \subseteq Y \in \mathcal{K}$. In combinatorial optimization, an abstract simplicial complex is usually called an *independence system*. We stick to the former, as we are also interested in the link between abstract simplicial complexes and topology. The *vertices* of $\mathcal{K}$ are the elements contained in at least one set $K \in \mathcal{K}$, i.e. the elements $x \in S$ with $\{x\} \in \mathcal{K}$. The sets in $\mathcal{K}$ are called the *simplices* of $\mathcal{K}$.

A *(geometric) simplex* is the convex hull of finitely many affine independent points (the *vertices* of the simplex) in $\mathbb{R}^k$ for some positive integer k. A *geometric simplicial complex* is a set of simplices $\mathcal{K}$ in $\mathbb{R}^k$ with the properties that if $K \in \mathcal{K}$ then every face of K is in $\mathcal{K}$, and if $K_1, K_2 \in \mathcal{K}$ then $K_1 \cap K_2$ is either empty or a face of both of them. Denote the topological space given by the union of all simplices in $\mathcal{K}$ by $|\mathcal{K}|$. Note that an abstract (geometric) simplex can also be considered as an abstract (geometric) simplicial complex by taking itself and all of its subsets (subsimplices).

Let $\mathcal{K}$ be an abstract simplicial complex with vertex set S where $|S| = n$. Consider the space $\mathbb{R}^n$ and denote by e_i the i-th standard basis vector with all coordinates being 0 except for the i-th one, which is 1. Label the elements of S with the set $\{1, 2, \dots, n\}$, and to each element $i \in S$ assign the standard basis vector e_i. This creates a bijection between the sets $X \subseteq S$ and the simplices of $\mathbb{R}^n$ with vertices $\{e_i\}_{i \in X}$. The *standard geometric realization* of $\mathcal{K}$ is the topological space assigned to the geometric simplicial complex consisting of the simplices in $\mathbb{R}^n$ corresponding to the simplices of $\mathcal{K}$. This space is also denoted by $|\mathcal{K}|$. Note that if we embed the points of S as affine independent points into $\mathbb{R}^k$ for some k, and consider the union of the corresponding simplices of $\mathcal{K}$, we obtain a topological space homeomorphic to the standard geometric realization of $\mathcal{K}$. Consequently, we assigned a topological space to each abstract simplicial complex. When we write simplicial complex, or complex for short, we always mean abstract simplicial complex, and when we are referring to the corresponding topological space or the corresponding geometric simplex of an abstract simplex, we use the $|.|$ notation as above.

Similar to matroids, we denote by

$$\mathcal{K}|X = \{K \in \mathcal{K} \ : \ K \subseteq X\}$$

the restriction of the simplicial complex $\mathcal{K}$ to X. Clearly, $\mathcal{K}|X$ is also a simplicial complex.

Let $\mathcal{K}$ be a simplicial complex and $x \in |\mathcal{K}|$. The smallest simplex $K \in \mathcal{K}$ with respect to inclusion, such that $x \in |K|$ is called the support of x and is denoted by $\mathrm{supp}_{\mathcal{K}}(x)$.

A *simplicial map* between two abstract simplicial complexes $\mathcal{K}$ and $\mathcal{L}$ with vertex sets $V(\mathcal{K})$ and $V(\mathcal{L})$, respectively, is a map $f : V(\mathcal{K}) \to V(\mathcal{L})$ with the property that for every simplex $K \in \mathcal{K}$ the image of K under f is a simplex of $\mathcal{L}$. Note that f is not necessarily injective. A simplicial map can be extended to a continuous map $F : |\mathcal{K}| \to |\mathcal{L}|$ as follows. Any point x in the standard geometric realization of $\mathcal{K}$ can be written as the convex combination of vertices in $\mathrm{supp}_{\mathcal{K}}(x)$, let $x = \sum_{i=1}^{k} \alpha_i v_i$ where $\alpha_i \in [0, 1]$ with $\sum_{i=1}^{k} \alpha_i = 1$ and $\mathrm{supp}_{\mathcal{K}}(x) = \{v_1, v_2, \dots, v_k\}$. Let $F(x) = \sum_{i=1}^{k} \alpha_i f(v_i)$. As f is a simplicial map, the image of $\{v_1, v_2, \dots, v_k\}$ under f is in $\mathcal{L}$, hence $F(x) \in |\mathcal{L}|$. It is easy to see that F is a continuous function, and F is called the *affine extension* of f. Note that $\mathrm{supp}_{\mathcal{L}}(F(x))$ is the image of $\mathrm{supp}_{\mathcal{K}}(x)$ under f.

Let Δ be a simplex. The boundary of the geometric simplex $|\Delta| \in \mathbb{R}^k$ is the boundary in the usual topological sense, and is denoted by $\partial|\Delta|$. A point $x \in \partial|\Delta|$ precisely if $\mathrm{supp}_{\Delta}(x) \subsetneq \Delta$.

For a simplicial complex $\mathcal{K}$, let $\beta(\mathcal{K})$ denote its *barycentric subdivision*, i.e. the complex with vertex set equal to the simplices of $\mathcal{K}$, and a subset $\mathcal{X} \subseteq \mathcal{K}$ is a simplex of $\beta(\mathcal{K})$ if the simplices in $\mathcal{X}$ can be ordered $\mathcal{X} = \{X_1, X_2, \dots, X_k\}$ in such a way that $X_1 \subseteq X_2 \subseteq \dots \subseteq X_k$. The following theorem about barycentric subdivision is well-known, the proof is omitted.

Theorem 4.1. *Let $\mathcal{K}$ be a simplicial complex with barycentric subdivision $\beta(\mathcal{K})$. Then there exists a homeomorphism $f : |\mathcal{K}| \to |\beta(\mathcal{K})|$ with the property that for every $x \in |\mathcal{K}|$ and every simplex $K \in \mathcal{K}$ with $K \in \operatorname{supp}_{\beta(\mathcal{K})}(f(x))$, we have $K \subseteq \operatorname{supp}_{\mathcal{K}}(x)$.*

Let S^k denote the k-dimensional sphere and B^k denote the k-dimensional ball. It can be proved that $\partial|\Delta|$ is homeomorphic to S^{n-2} where Δ is a simplex with n vertices. A topological space T is called *(homotopically) k-connected* if for every $0 \le r \le k$, any continuous map $f : S^r \to T$ has a continuous extension $B^{r+1} \to T$. Let $\eta(T)$ be the largest k for which T is k-connected, plus 2. If T is k-connected for every nonnegative integer k, then we define $\eta(T) = \infty$.

We will need the famous Knaster-Kuratowski-Mazurkiewicz (KKM) Theorem [28].

Theorem 4.2 (Knaster-Kuratowski-Mazurkiewicz Theorem)**.** *Let Δ be a simplex with vertex set $V(\Delta)$ and assign an open subset $A_v \subseteq |\Delta|$ to every vertex $v \in V(\Delta)$. Assume that for all $x \in |\Delta|$ there exists a vertex $v \in \operatorname{supp}_{\Delta}(x)$ with $x \in A_v$. Then $\bigcap_{v\in V(\Delta)} A_v$ is nonempty.*

4.2 The main theorem of Aharoni and Berger

Aharoni and Berger [1] considered the general question of finding large sets in the intersection of a matroid $M = (S, \mathcal{F})$ and a simplicial complex $\mathcal{K}$ on the vertex set S. Note that the intersection of any number of matroids is a simplicial complex. Therefore, we cannot expect to have an efficient algorithm that calculates the maximum size of a set in $\mathcal{F} \cap \mathcal{K}$. However, their main result is a sufficient condition for $\mathcal{K}$ containing a basis of M.

We state and prove the main theorem of Aharoni and Berger [1].

Theorem 4.3. *Let $M = (S, \mathcal{B})$ be a matroid with rank function r and let $\mathcal{K}$ be a simplicial complex with vertex set S. Assume that $\eta(\mathcal{K}|X) \ge r(M/(S \setminus X))$ for all $X \subseteq S$. Then there exists a set $B \in \mathcal{B} \cap \mathcal{K}$.*

Proof. Let $\mathcal{Z}(M)$ be the complex with vertices being the flats of M except S, and $\{Z_1, Z_2, \ldots, Z_\ell\} \in \mathcal{Z}(M)$ if they form a chain, i.e. $Z_{\varphi(1)} \subseteq Z_{\varphi(2)} \subseteq \ldots \subseteq Z_{\varphi(\ell)}$ for some bijection φ of $\{1, 2, \ldots, \ell\}$. The main lemma of the proof is the following.

Lemma 4.4. *There exists a continuous map $g : |\mathcal{Z}(M)| \to |\mathcal{K}|$ such that for every $x \in |\mathcal{Z}(M)|$ there exists a flat $Z \in \operatorname{supp}_{\mathcal{Z}(M)}(x)$ with $Z \cap \operatorname{supp}_{\mathcal{K}}(g(x)) = \emptyset$.*

Proof. We define g by induction on the size of the simplices of $\mathcal{Z}(M)$, i.e. in the k-th step, under the assumption that g is defined on all simplices with size at most $k-1$, we define it in the interior of the simplices of $\mathcal{Z}(M)$ with k vertices. For the base case, let $g(\{Z\})$ be any vertex of $S \setminus Z$ for all singleton subsets $\{Z\} \in \mathcal{Z}(M)$.

Let $\Delta = \{Z_1, Z_2, \dots, Z_k\} \in \mathcal{Z}(M)$ with $Z_1 \subseteq Z_2 \subseteq \dots \subseteq Z_k$ and assume that we already defined g on each point of $|\mathcal{Z}(M)|$ contained in a simplex with at most $k-1$ vertices. In particular, the function g is defined on $\partial|\Delta|$. By the induction hypothesis, for each $x \in \partial|\Delta|$ there exists a flat $Z \in \operatorname{supp}_{\mathcal{Z}(M)}(x)$ with $Z \cap \operatorname{supp}_{\mathcal{K}}(g(x)) = \emptyset$. As $x \in \partial|\Delta|$, we have $Z_1 \subseteq Z$, hence $Z_1 \cap \operatorname{supp}_{\mathcal{K}}(g(x)) = \emptyset$. It follows that the image of $\partial|\Delta|$ under g is contained in $\mathcal{K}|(S \setminus Z_1)$. Notice that

$$r(M) > r(Z_k) > r(Z_{k-1}) > \dots > r(Z_1),$$

hence $r(Z_1) \leq r(M) - k$. Consequently,

$$\eta(\mathcal{K}|(S \setminus Z_1)) \geq r(M/Z_1) \geq k.$$

Therefore, the function g defined on the boundary of $|\Delta|$ isomorphic to S^{k-2} can be extended continuously to the whole simplex $|\Delta|$ such that $g(x) \in |\mathcal{K}|(S \setminus Z_1)|$ for all $x \in |\Delta|$. This means that for any $x \in |\mathcal{Z}(M)|$ with support Δ, the flat Z_1 satisfies $Z_1 \cap \operatorname{supp}_{\mathcal{K}}(g(x)) = \emptyset$, as required. $\square$

Let Δ be the simplex with vertex set $V(\Delta)$ equal to the set of all cocircuits of M. Recall that $\beta(\Delta)$ denotes the barycentric subdivision of Δ. For every vertex $\mathcal{D}$ of $\beta(\Delta)$ corresponding to a set of cocircuits of M, define

$$\pi(\mathcal{D}) = S \setminus \left(\bigcup_{D \in \mathcal{D}} D \right) = \bigcap_{D \in \mathcal{D}} (S \setminus D).$$

The complements of cocircuits are flats, and the intersection of flats is flat, hence $\pi(\mathcal{D})$ is a flat of M for all vertex $\mathcal{D}$ of $\beta(\Delta)$. Therefore, it can be considered as a vertex of $\mathcal{Z}(M)$, hence we can regard π as a function from $\beta(\Delta)$ to $\mathcal{Z}(M)$. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are vertices of $\beta(\Delta)$ with $\mathcal{D}_1 \subseteq \mathcal{D}_2$, then $\pi(\mathcal{D}_1) \supseteq \pi(\mathcal{D}_2)$. Hence, for any simplex $\{\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_k\}$ of $\beta(\Delta)$ with $\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \dots \subseteq \mathcal{D}_k$, the corresponding flats after applying π also form a chain, $\pi(\mathcal{D}_1) \supseteq \pi(\mathcal{D}_2) \supseteq \dots \supseteq \pi(\mathcal{D}_k)$. Consequently, the image of $\{\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_k\}$ under π is a simplex of $\mathcal{Z}(M)$ which means that π is a simplicial map between the complexes $\beta(\Delta)$ and $\mathcal{Z}(M)$. This implies that π can be extended affinely to a continuous map $\pi' : |\beta(\Delta)| \to |\mathcal{Z}(M)|$.

Using Theorem 4.1, there exists a homeomorphism $f : |\Delta| \to |\beta(\Delta)|$ such that for every $x \in |\Delta|$ and every subset of cocircuits $\mathcal{D} \in \operatorname{supp}_{\beta(\Delta)}(f(x))$ we have $\mathcal{D} \subseteq \operatorname{supp}_{\Delta}(x)$.

For each cocircuit D of M let

$$A_D = \{x \in |\Delta| \;:\; D \cap \mathrm{supp}_{\mathcal{K}}(g(\pi'(f(x)))) \neq \emptyset\}.$$

The idea is to prove that the sets A_D satisfy the conditions of the KKM Theorem (Theorem 4.2). If this is true, then there exists an $x \in \bigcap_{D \in V(\Delta)} A_D$. It follows that $\mathrm{supp}_{\mathcal{K}}(g(\pi'(f(x))))$ intersects every cocircuit of M, hence $\mathrm{supp}_{\mathcal{K}}(g(\pi'(f(x))))$ is a simplex of $\mathcal{K}$ containing a bases of M, which is exactly what we want. It remained to show that the sets A_D satisfy the conditions of the KKM Theorem.

First, we show that A_D is open for all cocircuit D of M. As g, π' and f are continuous functions, their composition is also continuous, hence it is enough to show that

$$A'_D = \{x \in |\mathcal{K}| \;:\; \mathrm{supp}_{\mathcal{K}}(x) \cap D \neq \emptyset\}$$

is open, as A_D is the preimage of A'_D under the map $g \circ \pi' \circ f$. The complement of A'_D in $|\mathcal{K}|$ is exactly $|\mathcal{K}|(S \setminus D)|$, which is clearly closed, as it is a subcomplex of $|\mathcal{K}|$, thus A'_D is open.

We want to show that for every $x \in |\Delta|$ there exists a cocircuit $D \in \mathrm{supp}_{\Delta}(x)$ with $D \cap \mathrm{supp}_{\mathcal{K}}(g(\pi'(f(x)))) \neq \emptyset$. By the definition of the map g, there exists a flat $Z \in \mathrm{supp}_{\mathcal{Z}(M)}(\pi'(f(x)))$ with $Z \cap \mathrm{supp}_{\mathcal{K}}(g(\pi'(f(x)))) = \emptyset$. By the properties of the affine extension, there exists a set of cocircuits $\mathcal{D} \in \mathrm{supp}_{\beta(\Delta)} f(x)$ such that

$$Z = \pi'(\mathcal{D}) = \pi(\mathcal{D}) = S \setminus \left(\bigcup_{D \in \mathcal{D}} D \right).$$

Consequently, $\mathrm{supp}_{\mathcal{K}}(g(\pi'(f(x)))) \cap D \neq \emptyset$ for some cocircuit $D \in \mathcal{D}$, as

$$\mathrm{supp}_{\mathcal{K}}(g(\pi'(f(x)))) \subseteq S \setminus Z = \left(\bigcup_{D \in \mathcal{D}} D \right).$$

From Theorem 4.1, $\mathcal{D} \subseteq \mathrm{supp}_{\Delta}(x)$, hence $D \in \mathrm{supp}_{\Delta}(x)$, concluding the proof of the theorem. □

Remark 4.5. The theorem also holds if we only assume the $\eta(\mathcal{K}|X) \geq r(M/(S \setminus X))$ condition for sets $X \subseteq S$ such that $S \setminus X$ is a flat of M.

4.3 Applications and limitations

In order to apply Theorem 4.3 for matroid intersection problems, we need lower bounds for the connectivity of simplicial complexes that arise as the intersection of matroids. There are few general results about this, Aharoni and Berger [1] proved the following two theorems.

Theorem 4.6. *Given matroids* $M_1 = (S, \mathcal{F}_1), M_2 = (S, \mathcal{F}_2), \ldots, M_k = (S, \mathcal{F}_k)$*, let* ν *be the size of the largest set independent in all of them. Then*

$$\eta\Big(\bigcap_{i=1}^{k} \mathcal{F}_i\Big) \geq \frac{\nu}{k}.$$

Theorem 4.7. *For any pair of matroids* $M_1 = (S, \mathcal{F}_1)$ *and* $M_2 = (S, \mathcal{F}_2)$*,*

$$\eta(\mathcal{F}_1 \cap \mathcal{F}_2) \geq |S| - r(M_1^*) - r(M_2^*) = r(M_1) + r(M_2) - |S|.$$

There are theorems that were solved using these bounds and Theorem 4.3. Most notably, these results imply the best known bound for the covering number of the intersection of two matroids. For a matroid $M = (S, \mathcal{F})$, let $\beta(M)$ be the minimum number such that S can be covered with $\beta(M)$ sets from $\mathcal{F}$. A theorem of Edmonds [10] implies that $\beta(M) = \max\{\lceil |X|/r(X) \rceil : \emptyset \neq X \subseteq S\}$. Similarly, for a pair of matroids M_1, M_2 on S, let $\beta(M_1, M_2)$ be the minimal number such that S can be covered with $\beta(M_1, M_2)$ common independent sets of M_1 and M_2.

Theorem 4.8. *For every pair of matroids* M_1 *and* M_2*,*

$$\beta(M_1, M_2) \leq 2 \cdot \max\{\beta(M_1), \beta(M_2)\}.$$

However, the results of Aharoni and Berger are usually weak to prove the strongest bound. In this case, the following much stronger conjecture is still open.

Conjecture 4.9. If $\beta(M_1) \neq \beta(M_2)$, then $\beta(M_1, M_2) \leq \max\{\beta(M_1), \beta(M_2)\}$, and $\beta(M_1, M_2) \leq \max\{\beta(M_1), \beta(M_2)\} + 1$ otherwise.

The main limitation of applying Theorem 4.3 is the fact that we usually cannot calculate, or even give sufficiently strong bounds on the connectivity of a combinatorially defined simplicial complex. However, there are some methods for determining connectivity, and possibly we will have even stronger tools in the future. In addition, there are several simplicial complexes arising as the intersection of two matroids which were possibly not investigated thoroughly yet. These topological methods go beyond the scope of this thesis.

In the next section, we consider five famous open problems that can be formulated as the intersection of three matroids. In every case, we have three possibilities to choose a pair of matroids out of the three and consider the intersection of them as a simplicial complex. It would be enough to prove an appropriate lower bound for any of these three simplicial complexes to prove an unsolved conjecture, hence this line of research is of particular interest.

5 Conjectures as the intersection of three matroids

In the first subsection we show that another famous, hard problem, the common basis packing problem can be formulated in the form of Problem 3.1. In the following subsections, we talk about five open problems. In each subsection, we present the conjecture and summarize the associated results. Then we show how to formulate the problem as a case of Problem 3.1 with three matroids. At the end of each subsection, we reflect on considering the intersection of two matroids as a simplicial complex and explore potential applications of Theorem 4.3.

5.1 Packing common bases

Given two matroids, $M_1 = (S, \mathcal{F}_1)$ and $M_2 = (S, \mathcal{F}_2)$, the following problem arises naturally as several problems and conjectures in combinatorics can be formulated this way, see Subsections 5.5 and 5.6.

Problem 5.1. Does S contain k pairwise disjoint sets that are bases in both M_1 and M_2?

Bérczi and Schwartz [4] showed that Problem 5.1 is difficult, i.e. there is no algorithm that decides whether there exist k disjoint common bases of two matroids by using a polynomial number of independence queries. For strongly base orderable matroids Davies and McDiarmid [8] found a nice characterization which also implies that the problem is decidable in polynomial time.

Problem 5.1 is strongly connected with the problem of covering the set S with common independent sets. As we mentioned in Subsection 4.3, the strongest known result is due to Aharoni and Berger [1] who proved that $\beta(M_1, M_2) \leq 2 \cdot \max\{\beta(M_1), \beta(M_2)\}$ for any two matroids M_1 and M_2, see Theorem 4.8.

We prove that Problem 5.1 can be formulated as a case of Problem 3.1 with three matroids. Let $M_1 = (S, \mathcal{F}_1)$ and $M_2 = (S, \mathcal{F}_2)$ be rank-r matroids, and let k be a positive integer. Let $S_1, S_2, \ldots, S_k$ be pairwise disjoint copies of the ground set S, i.e. for all $s \in S$ there exists a corresponding $s_i \in S_i$ for all $1 \leq i \leq k$. Let A_s denote the set of these copies for all $s \in S$. Let $T = \dot{\bigcup}_{i=1}^{k} S_i$ and let M_1^i and M_2^i be the copies of the matroids M_1 and M_2 on the ground set S_i for all $1 \leq i \leq k$. Define the following three matroids on the ground set T.

- Let $N_1 = \bigoplus_{i=1}^{k} M_1^i$.
- Let $N_2 = \bigoplus_{i=1}^{k} M_2^i$.
- Let $N_3 = (T, \mathcal{F}_3)$ be the partition matroid with

$$\mathcal{F}_3 = \{X \subseteq T \;:\; |A_s \cap X| \leq 1 \text{ for all } s \in S\}.$$

Proposition 5.2. *The matroids M_1 and M_2 have k disjoint common bases if and only if the matroids N_1, N_2 and N_3 have a common independent set of size kr.*

Proof. If $B_1, B_2, \dots, B_k \subseteq S$ are disjoint sets that are bases both in M_1 and M_2, then taking the union of the corresponding sets of $B_1, B_2, \dots, B_k$ in $S_1, S_2, \dots, S_k$, respectively, gives a set with kr elements which is clearly independent in N_1, N_2 and N_3.

Conversely, if F is a common independent set of N_1, N_2 and N_3 with $|F| = kr$, then $F \cap S_i$ is independent in M_1^i and M_2^i, hence $|F \cap S_i| \leq r$. Consequently, $|F \cap S_i| = r$ for all $1 \leq i \leq k$, so $F \cap S_i$ is a basis in both M_1^i and M_2^i. Let $B_i \subseteq S$ be the set corresponding to $F \cap S_i$. The sets $B_1, B_2, \dots, B_k$ are bases in M_1 and M_2 and are pairwise disjoint as F is independent in N_3. $\square$

5.2 Barnette's conjecture

Tait [42] conjectured that every cubic polyhedral graph is Hamiltonian, which would imply the famous Four Colour Theorem. However, this conjecture turned out to be false, as shown by Tutte's counterexample [43]. Barnette [3] proposed the following weaker variant.

Conjecture 5.3 (Barnette)**.** Every bipartite, 3-regular, planar, 3-vertex-connected graph has a Hamiltonian cycle.

An extensive survey about the conjecture was written by Hertel [22]. We call a graph that satisfies the above conditions, that is, bipartite, 3-regular, planar and 3-vertex-connected, a *Barnette graph*. Let G be a Barnette graph. For our purposes, we need the graphic matroid of the dual graph G^*, which is isomorphic to the dual of the graphic matroid of G. It is not unprecedented that Barnette's conjecture was examined using the dual of G. For example, the following lemma considering the existence of a Hamiltonian cycle in a dual graph already appeared implicitly in the work of Stein [41], and used by several authors since. It is explicitly stated and proved by Alt, Payne, Schmidt and Wood [2].

Lemma 5.4. *Let $G = (V, E)$ be a triangulated planar graph and denote its dual by G^*. The following statements are equivalent.*

- *G^* is Hamiltonian.*
- *G contains an induced subtree that meets every face of G.*
- *G contains two disjoint induced subtrees that meet every face of G such that their vertex sets partition V.*

However, considering Barnette's conjecture as the intersection of matroids is an approach that we have not come across yet. Given a Barnette graph $G = (S, T, E)$ on $2n$ vertices, let E_v denote the set of edges incident to v for all $v \in V$. Consider the following three matroids.

- Let $M_1 = (E, \mathcal{F}_1)$ be the partition matroid with

$$\mathcal{F}_1 = \{X \subseteq E \ : \ |E_v \cap X| \leq 1 \text{ for all } v \in S\}.$$

- Let $M_2 = (E, \mathcal{F}_2)$ be the partition matroid with

$$\mathcal{F}_2 = \{X \subseteq E \ : \ |E_v \cap X| \leq 1 \text{ for all } v \in T\}.$$

- Let $M_3 = (E, \mathcal{F}_3)$ be the dual of the graphic matroid corresponding to G.

Proposition 5.5. *The Barnette graph G is Hamiltonian if and only if there exists a set in $\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3$ with n elements.*

Proof. We prove that $X \subseteq \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3$ with $|X| = n$ is equivalent to $E \setminus X$ being a Hamiltonian cycle of G. Note that G is 3-regular, hence the complement of a Hamiltonian cycle is a perfect matching. The intersection of the first two matroids, M_1 and M_2, are exactly the matchings in G, see Example 3.3.

We show that $X \in \mathcal{F}_3$ if and only if the spanning subgraph of G with edges $E \setminus X$ is a connected graph. As the edges of G and its dual G^* are in a natural one-to-one relation with each other, we denote the edge set of both of them by E. Also, recall that M_3 is isomorphic to the graphic matroid of G^*. If $X \notin \mathcal{F}_3$, then it contains a cycle in G^* determining a closed Jordan curve in the plane cutting it into two parts. This means that the regions, which correspond to the vertices of G are not connected with edges in $E \setminus X$. Conversely, if $E \setminus X$ is not connected in G, then the corresponding regions are not connected in G^*, hence they must be separated by edges in X, implying that X contains a cycle, which means $X \notin \mathcal{F}_3$.

Consequently, the n-element common independent sets of M_1, M_2 and M_3 are exactly the perfect matchings with connected complements. The complement of a matching is a 2-regular subgraph which is connected precisely if it is a Hamiltonian cycle, which finishes the proof. □

As mentioned in the proof, the intersection of M_1 and M_2 are exactly the set of matchings. The matchings of a graph as a simplicial complex is called the matching complex, and it has been investigated in depth earlier, see e.g. Jonsson [25] who dedicated a chapter to them in his book. Unfortunately, the connectivity properties of the matching complex seem to be way too weak to apply Theorem 4.3. For this reason, we considered the intersection of M_1 and M_3. At first sight, it looked promising in small examples, however, we believe that it also does not fulfill the conditions of Theorem 4.3.

5.3 Equitability of matroids

A matroid $M = (S, \mathcal{B})$ is called *equitable* if for any set $X \subseteq S$ there exists a basis $B \in \mathcal{B}$ such that its complement is also a basis and $\lfloor |X|/2 \rfloor \leq |B \cap X| \leq \lceil |X|/2 \rceil$. Note that we obtain the same definition if we only require the above condition for $X \subseteq S$ with even elements, because if $Y \subseteq S$ is of odd cardinality then any appropriate $B \in \mathcal{B}$ for a set X with $|X \Delta Y| = 1$ is also suitable for Y.

The fact that the following simple conjecture remains unsolved is somewhat unexpected [27].

Conjecture 5.6 (Equitability)**.** If the ground set of a matroid M can be partitioned into two bases, then M is equitable.

A natural generalization of the problem is when S is the union of k disjoint bases and we want to find $B_1, \dots, B_k \in \mathcal{B}$ with $\lfloor |X|/k \rfloor \leq |B_i \cap X| \leq \lceil |X|/k \rceil$. However, it is easy to prove that the two versions are actually equivalent, hence it suffices to concentrate on the first one.

The equitability conjecture is closely linked to several other unsolved problems which confirms its importance. We highlight two such relevant open problems. Gabow [19] formulated the following beautiful conjecture which would be a strengthening of the strong basis exchange property (B2').

Conjecture 5.7 (Gabow)**.** Let $M = (S, \mathcal{B})$ be a matroid and $A, B \in \mathcal{B}$. Then there exists an ordering of the elements $A = \{a_1, a_2, \dots, a_r\}$ and $B = \{b_1, b_2, \dots, b_r\}$ such that $\{a_1, \dots, a_i, b_{i+1}, \dots, b_r\}$ and $\{b_1, \dots, b_i, a_{i+1}, \dots, a_r\}$ are both bases for all $0 \leq i \leq r$.

The following proposition shows the connection between the above conjectures.

Proposition 5.8. *Gabow's conjecture implies the equitability conjecture*

Proof. Let $S = A \dot\cup B$ with $A, B \in \mathcal{B}$ in a matroid $M = (S, \mathcal{B})$, and $X \subseteq S$ is an arbitrary subset with even elements for the sake of easier notation. Assume that Gabow's conjecture is true and let $A_i = \{a_1, \dots, a_i, b_{i+1}, \dots, b_r\}$ and $B_i = \{b_1, \dots, b_i, a_{i+1}, \dots, a_r\}$ with $A_i, B_i \in \mathcal{B}$ for all $0 \leq i \leq r$ with an appropriate ordering of the elements of A and B. Let $f(i) = |X \cap A_i|$. Note that $A_0 = B$ and $A_r = A$, hence $f(0) + f(r) = |X|$, implying that either $f(0) \leq |X|/2$ and $f(r) \geq |X|/2$ or the other way around. The inequality $|f(i + 1) - f(i)| \leq 1$ holds for all $0 \leq i \leq r - 1$ as $|A_{i+1} \Delta A_i| = 1$. It follows that the sequence $f(0), f(1), \dots, f(r)$ must take the value $|X|/2$. Assume that $|X \cap A_k| = |X|/2$. As $A_k \dot\cup B_k = S$, the matroid M is indeed equitable. $\square$

A different extension of the bases exchange axiom was examined by White [44]. Let k be an arbitrary positive integer, and let $\mathcal{X} = (X_1, X_2, \dots, X_k)$ be a sequence of bases of a matroid $M = (S, \mathcal{B})$. If there exist $x_i \in X_i$ and $x_j \in X_j$ for some $i \neq j$ with $X_i - x_i + x_j \in \mathcal{B}$ and $X_j - x_j + x_i \in \mathcal{B}$ then the sequence of bases $\mathcal{X}'$ where we change X_i to $X_i - x_i + x_j$ and X_j to $X_j - x_j + x_i$ in $\mathcal{X}$ is said to be obtainable from $\mathcal{X}$ by a symmetric exchange. Let $\mathcal{X} = (X_1, X_2, \dots, X_k)$ and $\mathcal{Y} = (Y_1, Y_2, \dots, Y_k)$ be basis sequences. Then $\mathcal{X}$ and $\mathcal{Y}$ are called *equivalent* if one can be obtained by the other with a sequence of symmetric exchanges, and they are called *compatible* if

$$|\{i \in \{1, 2, \dots, k\} \ : \ s \in X_i\}| = |\{i \in \{1, 2, \dots, k\} \ : \ s \in Y_i\}|$$

for every $s \in S$. White conjectured the following.

Conjecture 5.9 (White)**.** Two basis sequences $\mathcal{X}$ and $\mathcal{Y}$ of the same length are equivalent if and only if they are compatible.

Equivalent basis sequences are clearly compatible, as a symmetric exchange does not change the cardinality of the sets in the definition of compatibility. White's conjecture also implies the equitability conjecture. To see this briefly, if $A \dot\cup B = S$ in a matroid $M = (S, \mathcal{B})$ with $A, B \in \mathcal{B}$, then (A, B) and (B, A) are compatible, and in a similar way as in the proof of Proposition 5.8, if we move from (A, B) to (B, A) by symmetric exchanges then there must be a pair (C, D) on the way with $|X \cap C| = |X \cap D| = |X|/2$ for an arbitrary $X \subseteq S$ with even elements.

The equitability conjecture was settled in several special cases. Fekete and Szabó [14] showed that graphic matroids and weakly base orderable matroids are equitable. Szabó also checked that all matroids on at most eight elements are equitable. Király [27] proved that hypergraphic matroids are also equitable using the equitability of graphic matroids.

We illustrate how the equitability conjecture can be written as a special case of Problem 3.1 with the intersection of three matroids. Assume that $X \subseteq S$ with $|X|$ being even and a matroid $M = (S, \mathcal{B})$ is given, with $S = A \dot\cup B$ for some $A, B \in \mathcal{B}$. Consider the following three matroids.

- The matroid M.
- The dual matroid $M^* = (S, \mathcal{B}^*)$ of M.
- The partition matroid $M_X = (S, \mathcal{B}_X)$ with

$$\mathcal{B}_X = \{B \subseteq S \ : \ |B \cap X| = |X|/2 \text{ and } |B \cap (S \setminus X)| = |S \setminus X|/2\}.$$

Proposition 5.10. *The matroid M is equitable if and only if the matroids M, M^* and M_X have a common basis for all $X \subseteq S$ with even elements.*

Proof. By definition, a set B is in $\mathcal{B} \cap \mathcal{B}^* \cap \mathcal{B}_X$ if and only if it is a basis of M, its complement is also a basis and it intersects X in $|X|/2$ elements, which is exactly the condition of equitability for the set X. $\square$

It would be surprising if Theorem 4.3 could be applied to the intersection of M and M_X, as M can be any matroid and the condition of M_X being partition matroid of the given type is also not too strict, for example, we mentioned at the end of Subsection 5.2 that the matching complex, the intersection of two partition matroids, have weak topological properties for our purposes. On the other hand, the matroid and its dual are strongly connected, it is more likely that their intersection always satisfies some general property. Actually, we think that the study of understanding the structure of the intersection of a matroid and its dual is interesting on its own.

Problem 5.11. What can we say about the structure of the intersection of a matroid and its dual from topological or combinatorial perspective?

5.4 Rainbow arborescences

The rainbow arborescence problem is a recent, unpublished conjecture.

Conjecture 5.12. Given $n-1$ arborescences on an n element vertex set. Prove that there exists a rainbow arborescence, that is, an arborescence that intersects every given arborescence in exactly one edge.

The undirected case of the conjecture, when $n-1$ spanning tree is given and the problem is to find a rainbow spanning tree can be solved greedily. In the directed case, however, we obtain Conjecture 5.12 which seems to be extremely hard. The conjecture has been settled only in special cases. For instance, if the root nodes of the arborescences coincide, then the problem can be solved similarly as in the undirected case, starting from the joint root node and greedily building up a rainbow arborescence.

Let $D = (V, A)$ be a directed graph such that $A = A_1 \dot\cup A_2 \dot\cup \ldots \dot\cup A_{n-1}$ where the directed graphs $T_1 = (V, A_1), T_2 = (V, A_2), \ldots, T_{n-1} = (V, A_{n-1})$ are arborescences. Let $E_v \subseteq A$ denote the set of edges ending in V. Consider the following three matroids on the ground set A.

- The graphic matroid $M_1 = (A, \mathcal{F}_1)$ associated to underlying undirected graph of G.
- The partition matroid $M_2 = (A, \mathcal{F}_2)$ with

$$\mathcal{F}_2 = \{X \subseteq A \ : \ |E_v \cap X| \leq 1 \text{ for all } v \in V\}.$$

- The partition matroid $M_3 = (A, \mathcal{F}_3)$ with

$$\mathcal{F}_3 = \{X \subseteq A \ : \ |A_i \cap X| \leq 1 \text{ for all } 1 \leq i \leq n-1\}.$$

Proposition 5.13. *There exists a rainbow arborescence in D if and only if the matroids M_1, M_2 and M_3 have a common independent set of size $n-1$.*

Proof. The set of edges of an arborescence is clearly a common independent set of M_1, M_2 and M_3 with $n-1$ elements. Conversely, if $F \in \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3$ with $|F| = n-1$, then F is a spanning tree in the undirected sense because of $\mathcal{F}_1$, hence it is an arborescence as it is independent in $\mathcal{F}_2$, and finally, $\mathcal{F}_3$ guarantees that it is a rainbow arborescence. □

As M_2 and M_3 are partition matroids with bound 1 on each partition class, their intersection is a matching complex of a bipartite graph. If we intersect M_1 with one of the other matroids then we intersect a graphic matroid and a partition matroid. It seems too general to be able to have strong results, however, it is definitely interesting to examine these complexes. The complex $\mathcal{F}_1 \cap \mathcal{F}_2$ is the complex with maximal simplices being the arborescences. These form a greedoid if the root vertices of the arborescences are the same. Topological properties of greedoids have been considered earlier, see [5]. However, if not all the root vertices coincide, we obtain complexes that were not investigated yet, to the best of our knowledge.

5.5 Woodall's conjecture

In a directed graph $D = (V, A)$, we call a set of edges $A' \subseteq A$ a *dicut*, if there exists a partition $V = V_1 \dot{\cup} V_2$ of the vertices with no edges going from V_2 to V_1, and the set of edges pointing from V_1 to V_2 is exactly A'. A *dijoin* is a subset of edges that intersects every dicut. Equivalently, a set of edges $A' \subseteq A$ is a dijoin, if we obtain a strongly connected graph after contracting all the edges of A'. The famous theorem of Lucchesi and Younger [35] is a min-max theorem, stating that the minimum size of a dijoin is equal to the maximum number of pairwise disjoint directed cuts. Woodall's conjecture is a dual of this in some sense.

Conjecture 5.14 (Woodall)**.** In a directed graph D, the minimum size of a dicut is equal to the maximum number of pairwise disjoint dijoins.

The statement was proved only in special cases. Frank characterized the directed graphs containing two disjoint dijoins (see for example in [39]). The conjecture was proved for source-sink connected graphs by Schrijver [38] and independently Feofiloff [15], and for series-parallel graphs by Lee and Wakabayashi [33].

As mentioned in Subsection 5.1, Woodall's conjecture can be formulated as a bases packing problem, in the form of Problem 5.1. Frank and Tardos [18] noticed this reformulation which is definitely not trivial. We describe two matroids for a directed graph D such that disjoint common bases can be corresponded to disjoint dijoins in D.

Let $D = (V, A)$ be a directed graph. We denote the head of an edge $e \in A$ by $h(e)$, and its tail by $t(e)$. For each edge e, assign an element h_e for the head of e and k elements $t_e^1, t_e^2, \dots, t_e^k$ for the tail of e. Assume that all these assigned elements are distinct, let S_e be the set of all elements assigned to the edge e and define $S = \bigcup_{e \in A} S_e$. Let

$$\mathcal{Z} = \{Z \subseteq V \ : \ Z \neq \emptyset, \ Z \neq V \text{ and there is no edges leaving } Z\},$$

and let

$$S(Z) = \{h_e \in S \ : \ e \in A, \ h(e) \in Z\} \cup \{t_e^i \in S \ : \ e \in A, \ t(e) \in Z, \ 1 \leq i \leq k\}$$

for all $Z \in \mathcal{Z}$. Denote by $i(Z)$ the number of edges spanned by Z for a subset $Z \subseteq V$. Define the following two matroids on S.

- Let $M_1 = (S, \mathcal{B}_1)$ with

$$\mathcal{B}_1 = \{B \subseteq S \ : \ |B| = |A| \text{ and } |B \cap S(Z)| \geq i(Z) + 1 \text{ for all } Z \in \mathcal{Z}\}.$$

- Let $M_2 = (S, \mathcal{B}_2)$ be a partition matroid with

$$\mathcal{B}_2 = \{B \subseteq S \ : \ |B| = |A| \text{ and } |B \cap S_e| \leq 1 \text{ for all } e \in A\}.$$

It is not trivial that M_1 is a matroid, we prove it in Proposition 5.16, but first, we prove that this is indeed a base packing reformulation of Woodall's conjecture.

Proposition 5.15. *The matroids M_1 and M_2 contain k disjoint common bases precisely if D contains k disjoint dijoins.*

Proof. We show that the common bases of M_1 and M_2 can be corresponded to the dijoins of D. Let B be a common basis of M_1 and M_2. The partition matroid M_2 ensures that for all $e \in A$, B contains exactly one element corresponding to e. Let $H \subseteq A$ be the edges with $h_e \in B$. For all dicut C of D, there exists a $Z \in \mathcal{Z}$ such that C are exactly the set of edges between $V \setminus Z$ and Z. As $B \in \mathcal{B}_1$, $|B \cap Z| \geq i(Z) + 1$ meaning that there is an edge $e \in C$ such that $h_e \in B$, hence $H \cap C \neq \emptyset$. This is true for all dicut C, so H is a dijoin.

Conversely, assume that H is a dijoin, and let $B \subseteq S$ be a set that contains one element from each edge, and $h_e \in B$ precisely if $e \in H$. Clearly, $B \in \mathcal{B}_2$, we show that

$B \in \mathcal{B}_1$ also holds. For any $Z \in \mathcal{Z}$, one element is in B for all edges spanned by Z, and H intersects the dicut C_Z corresponding to the partition $\{Z, V \setminus Z\}$ which means that there is an edge $e \in C_Z$ with $h_e \in B$. Therefore, $|B \cap S(Z)| \geq i(Z) + 1$ indeed holds for all $Z \in \mathcal{Z}$, proving that $B \in \mathcal{B}_1$.

Consequently, if there exist k disjoint common bases of M_1 and M_2, then for each $e \in A$ the corresponding h_e is contained in at most one of the common bases, hence the corresponding dijoins are disjoint. Conversely, if there are k disjoint dijoins, then for each dijoin we can assign a common basis of M_1 and M_2 such that these bases are disjoint because for all $e \in A$ there are k elements corresponding to the tail of e, hence we can choose a distinct one for all common bases. $\square$

Therefore, finding k disjoint dijoins can be formulated as a case of Problem 5.1, hence it is also a case of Problem 3.1 with $k = 3$ from Proposition 5.2.

Finally, we prove that M_1 is a matroid.

Proposition 5.16. *The family $\mathcal{B}_1$ forms the bases of a matroid on S.*

Proof. We prove (B1) and (B2"). It is easy to see that $\bigcup_{e \in A}\{h_e\} \in \mathcal{B}_1$, proving (B1). For (B2") let $B_1, B_2 \in \mathcal{B}_1$ and let $x \in B_1 \setminus B_2$. We need to prove that there exists a $y \in B_2 \setminus B_1$ such that $B_2 + x - y \in \mathcal{B}_1$.

For easier notation, let $m(Z) = |B_2 \cap S(Z)|$ and $p(Z) = i(Z) + 1$ for all $Z \in \mathcal{Z}$. Note that by definition, $m(Z) \geq p(Z)$ for all $Z \in \mathcal{Z}$. Call a set $Z \in \mathcal{Z}$ *tight*, if $m(Z) = p(Z)$.

Let $y \in B_2 \setminus B_1$, and assume that $B_2 + x - y \notin \mathcal{B}_1$. This means that there is a $Z \in \mathcal{Z}$ with $|(B_2 + x - y) \cap S(Z)| < p(Z)$. Note that $B_2 \in \mathcal{B}_1$, hence

$$p(Z) \leq m(Z) \leq |(B_2 + x - y) \cap S(Z)| + 1 < p(Z) + 1.$$

This can only happen if $p(Z) = m(Z) = |(B_2 + x - y) \cap S(Z)| + 1$, meaning that Z is tight, and $y \in S(Z)$, $x \notin S(Z)$.

We say that $Z, Z' \in \mathcal{Z}$ are *crossing*, if all the sets $Z \setminus Z', Z' \setminus Z, Z \cap Z', Z \cup Z'$ are nonempty. If $Z, Z' \in \mathcal{Z}$ then clearly, $Z \cap Z', Z \cup Z' \in \mathcal{Z} \cup \{\emptyset, V\}$. Furthermore, if $Z, Z' \in \mathcal{Z}$ are crossing sets, then the supermodular inequality

$$p(Z) + p(Z') \leq p(Z \cap Z') + p(Z \cup Z')$$

holds. This is true as $p(Z) = i(Z) + 1$, hence it is enough to prove the supermodular inequality for $i(Z)$. On the left-hand side, we count every edge once which is spanned by Z or Z', and we count the edges that are spanned by both twice. On the right-hand side, we also count every edge which is spanned by Z or Z', and we also count the edges that are spanned by both twice. Maybe even more, as we also count the edges with one endpoint in $Z \setminus Z'$ and the other in $Z' \setminus Z$. Hence the right-hand side is indeed cannot be smaller than the left-hand side.

Notice that if $Z, Z' \in \mathcal{Z}$ are crossing, then

$$m(Z) + m(Z') = m(Z \cap Z') + m(Z \cup Z').$$

Combining these, and using that $p(Z) \leq m(Z)$ for all $Z \in \mathcal{Z}$, we get that if $Z, Z' \in \mathcal{Z}$ are crossing and tight, then

$$m(Z \cap Z') + m(Z \cup Z') = m(Z) + m(Z') = p(Z) + p(Z') \leq$$

$$\leq p(Z \cap Z') + p(Z \cup Z') \leq m(Z \cap Z') + m(Z \cup Z'),$$

hence all inequalities are actually equalities, implying that $Z \cap Z'$ and $Z \cup Z'$ are tight.

Assume by contradiction, that for all $y \in B_2 \setminus B_1$, we have $B_2 + x - y \notin \mathcal{B}_1$. Let $Z_1, Z_2, \ldots, Z_\ell \in \mathcal{Z}$ be the maximal tight sets with respect to inclusion, such that $x \notin S(Z_i)$ for all $1 \leq i \leq \ell$. These sets are pairwise disjoint, as the union of tight crossing sets is also tight. Also, for all $y \in B_2 \setminus B_1$ there exists an index i with $y \in S(Z_i)$. We have

$$|B_2 \setminus B_1| + \left|(B_1 \cap B_2) \cap \bigcup_{i=1}^{\ell} S(Z_i)\right| = \sum_{i=1}^{\ell} |S(Z_i) \cap B_2| = \sum_{i=1}^{\ell} m(Z_i) =$$

$$= \sum_{i=1}^{\ell} p(Z_i) \leq \sum_{i=1}^{\ell} |S(Z_i) \cap B_1| = \left|(B_1 \setminus B_2) \cap \bigcup_{i=1}^{\ell} S(Z_i)\right| + \left|(B_1 \cap B_2) \cap \bigcup_{i=1}^{\ell} S(Z_i)\right|,$$

hence

$$|B_2 \setminus B_1| \leq \left|(B_1 \setminus B_2) \cap \bigcup_{i=1}^{\ell} S(Z_i)\right|,$$

which is a contradiction, as $|B_2 \setminus B_1| = |B_1 \setminus B_2|$ and $x \notin (B_1 \setminus B_2) \cap \bigcup_{i=1}^{\ell} S(Z_i)$. □

Remark 5.17. We only used about $p(Z) = i(Z) + 1$ that

$$p(Z) + p(Z') \leq p(Z \cap Z') + p(Z \cup Z')$$

for any crossing sets $Z, Z' \in \mathcal{Z}$. Therefore, the above proposition can be stated in a much more general form with any family $\mathcal{Z}$ provided that $Z \cap Z', Z \cup Z' \in \mathcal{Z}$ for crossing sets $Z, Z' \in \mathcal{Z}$, and with any function $p : \mathcal{Z} \to \mathbb{N}$ satisfying the supermodular inequality for all crossing sets $Z, Z' \in \mathcal{Z}$.

Examining Woodall's conjecture as the intersection of a simplicial complex and a matroid does not seem to lead to results, as among the three matroids there are two partition matroids, with bound 1 on each partition class, hence their intersection is a matching complex, and the third matroid is defined in a complicated manner, so it looks difficult to work with it.

5.6 Rota's conjecture

Rota [23] formulated a conjecture motivated by linear algebra. Let $M = (S, \mathcal{B})$ be a matroid with rank r such that its ground set can be partitioned into r bases, i.e. $S = B_1 \dot\cup B_2 \dot\cup \dots \dot\cup B_r$ with $B_i \in \mathcal{B}$ for all $1 \le i \le n$. We call a basis $B \in \mathcal{B}$ *transversal* with respect to this partition if $|B \cap B_i| = 1$ for all $1 \le i \le r$. It is a folklore result that M contains a transversal basis. Rota conjectured the following much stronger statement.

Conjecture 5.18 (Rota)**.** If $M = (S, \mathcal{B})$ is a matroid with rank r and $B_1, B_2, \dots, B_r$ partition S, where $B_i \in \mathcal{B}$ for all $1 \le i \le r$, then there exists r pairwise disjoint transversal bases with respect to $B_1, B_2, \dots, B_r$.

Equivalently, the conjecture states that the elements of S can be arranged in an $r \times r$ table such that the rows are exactly $B_1, B_2, \dots, B_r$ and the columns also form bases of M. The conjecture was proved by Davies and McDiarmid [8] for strongly base orderable matroids as the problem is a special case of packing common bases (Subsection 5.1). Rota's conjecture was also proved in the case of paving matroids by Geelen and Humphries [20]. A natural approach is to prove weaker results about the number of disjoint traversal bases, which has a long research history. Geelen and Webb [21] proved that there always exist $\Omega(\sqrt{r})$ disjoint traversal bases. Dong and Geelen [9] improved this to $\Omega(r/\log r)$. Most recently, Bucić, Kwan, Pokrovskiy and Sudakov [6] verified the bound $(1/2 - o(1))r$. In another recent paper, Pokrovskiy [37] considered the problem of finding large independent sets intersecting each B_i in at most one element, and proved that $n - o(n)$ such disjoint sets can be found, all having $n - o(n)$ elements.

To see that Rota's conjecture is indeed a special case of the common basis packing problem (Problem 5.1), consider the following two matroids.

- The matroid M.
- The partition matroid $M' = (S, \mathcal{B}')$ with

$$\mathcal{B}' = \{B \subseteq S \ : \ |B \cap B_i| \le 1 \text{ for all } 1 \le i \le r\}.$$

Proposition 5.19. *Rota's conjecture is equivalent to M and M' having r pairwise disjoint common bases.*

Proof. Trivial from definitions. □

Applying Proposition 5.2, we obtain that Rota's conjecture is a special case of the matroid intersection problem (Problem 3.1) with $k = 3$ matroids. Examining Rota's conjecture as the intersection of a simplicial complex and a matroid seems unproductive, similar to Woodall's conjecture. This is because two of the three matroids involved are partition matroids with bound 1 on each partition class, hence their intersection is a matching complex, while the third matroid has no significant constraints.

6 Simplicial complexes as the intersection of matroids

As we saw in the earlier sections, we understand the structure of the intersection of k matroids only if $k \leq 2$. There are plenty of open problems in connection with the intersection of three matroids. Still, it would be interesting to examine the intersection of even more matroids. Clearly, the intersection of any number of matroids is a simplicial complex. For a simplicial complex $\mathcal{K}$, let $\mu(\mathcal{K})$ denote the smallest number such that there exist $\mu(\mathcal{K})$ matroids whose intersection is $\mathcal{K}$.

First, we prove that $\mu(\mathcal{K})$ is a finite number for all simplicial complexes $\mathcal{K}$. This was proved several times before, for instance [13, 26, 30, 31].

Lemma 6.1. *For every simplicial complex $\mathcal{K}$, there exist finitely many matroids, such that their intersection is $\mathcal{K}$.*

Proof. Let S be the set of vertices of $\mathcal{K}$ and let $\mathcal{C} = \{C_1, C_2, \ldots, C_\ell\}$ be the minimal sets, with respect to inclusion, not contained in $\mathcal{K}$. Define $\mathcal{C}_i = \{C_i\}$ for $1 \leq i \leq \ell$. Clearly, $\mathcal{C}_i$ forms the set of circuits of a matroid $M_i = (S, \mathcal{F}_i)$. A set $X \subseteq S$ is in $\mathcal{K}$ precisely if $C_i \nsubseteq X$ for each $1 \leq i \leq \ell$, and this is also the condition that X is independent in all of the matroids M_i. Consequently, $\mathcal{K} = \bigcap_{i=1}^{\ell} \mathcal{F}_i$. □

We prove a well-known theorem observed by Jenkyns [24] and Korte and Hausmann [30], which says that the greedy algorithm approximates the size of the largest set in the intersection of k matroids by a multiplicative factor of k. This means that for simplicial complexes $\mu(\mathcal{K})$ is a measure of complexity in some sense.

Theorem 6.2. *Let $\mathcal{K}$ be a simplicial complex over S that can be written as the intersection of k matroids. Build up a set by picking arbitrary elements $s_1, s_2, \ldots \in S$ one by one in such a way that after the ℓ-th step, for every ℓ, $\{s_1, s_2, \ldots, s_\ell\} \in \mathcal{K}$. We stop if we cannot add any more elements in this way, i.e. if we arrive at a maximal element $F = \{s_1, s_2, \ldots, s_k\}$ of $\mathcal{K}$ with respect to inclusion. Then*

$$k \cdot |F| \geq |G|,$$

where $G \in \mathcal{K}$ is a set with maximum size. Equality can hold precisely if $k \leq |S| - 1$.

Proof. Let $\mathcal{K} = \bigcap_{i=1}^{k} \mathcal{F}_i$ with matroids $M_1 = (S, \mathcal{F}_1), M_2 = (S, \mathcal{F}_2), \ldots, M_k = (S, \mathcal{F}_k)$. The set $G \cup F$ has rank at least $|G|$ in each matroid M_i, as $G \in \mathcal{F}_i$. Hence F, which is also independent in each matroid by definition, can be extended to a set $F\dot{\cup}G_i \in \mathcal{F}_i$ with $G_i \subseteq G \setminus F$ and $|F \cup G_i| \geq |G|$. It follows that $|G \setminus G_i| \leq |F|$. As F is maximal in $\mathcal{K}$, for all $x \in G$ there exists a matroid M_i with $F + x \notin \mathcal{F}_i$. Consequently,

$$|G| \leq \sum_{i=1}^{k} |G \setminus G_i| \leq k \cdot |F|,$$

as we wanted.

For the case of equality, first, assume that $k \geq |S|$. The greedily obtained F contains at least one element, hence equality can only occur if $|F| = 1$ and $|G| = |S|$, but this is not possible, as the latter equality means that $\mathcal{K}$ is the whole 2^S, and in this case, the greedy algorithm also finds the set S.

If $k \leq |S| - 1$, then let $x \in S$, $X \subseteq S$ with $|X| = k$ and $x \notin X$, and let

$$\mathcal{K} = \{K \subseteq S \;:\; K \subseteq X \text{ or } K = \{x\}\}.$$

This is clearly a simplicial complex. The greedy algorithm can end in the set $\{x\}$, as it is maximal in $\mathcal{K}$. However, the set with maximum size in $\mathcal{K}$ is X, and $|X| = k$. In this case, the greedy algorithm that starts with element x approximates $|X|$ within a factor of k.

Because of the first part of the theorem, $\mathcal{K}$ cannot be the intersection of less than k matroids. We show that $\mathcal{K}$ can be written as the intersection of k matroids as follows. Let $X = \{x_1, x_2, \ldots, x_k\}$. Define

$$\mathcal{B}_i = \{X, X + x - x_i\}$$

for $1 \leq i \leq k$. It is easy to check, that $\mathcal{B}_i$ forms the bases of a matroid $M_i = (S, \mathcal{F}_i)$. Clearly, the sets X and $\{x\}$ are in $\bigcap_{i=1}^{k} \mathcal{F}_i$, and for all $1 \leq i \leq k$ we have $\{x, x_i\} \notin \mathcal{F}_i$. Also, $\{z\} \notin \mathcal{F}_1$ for any $z \in S \setminus (X + x)$, hence $\mathcal{K} = \bigcap_{i=1}^{k} \mathcal{F}_i$ indeed holds. □

We considered two problems after these observations. Recognizing whether a simplicial complex is the intersection of k matroids, and finding the minimal number $\kappa(n)$, such that any matroid on n vertices can be written as the intersection of $\kappa(n)$ matroids.

6.1 Recognizing matroid intersection

We examined the following problem.

Problem 6.3. Let k be a positive integer. Characterize simplicial complexes $\mathcal{K}$ satisfying $\mu(\mathcal{K}) \leq k$.

The problem is especially important for $k = 2$, as we understand the structure of the intersection of two matroids. Surprisingly, it is wide open to finding a characterization for a simplicial complex to be the intersection of two matroids. There are results only in special cases. Fekete, Firla and Spille [13] examined this problem for the matching complex, i.e. the matchings of a graph. They characterized the graphs for which the matching complex is the intersection of two matroids and gave an integer programming formulation for the more general, $\mu(\mathcal{K}) \leq k$ problem. Kashiwabara, Okamoto and Uno [26] investigated the case of clique complexes, the complexes coming from complete subgraphs of a graph, and gave a characterization for $\mu(\mathcal{K}) \leq k$ using stable-set partitions. Their result implies that determining whether $\mu(\mathcal{K}) \leq k$ for a clique complex $\mathcal{K}$ belongs to NP.

There is a double exponential number of matroids on n vertices. Clearly, there cannot be more, as there are 2^{2^n} subsets of 2^S where $|S| = n$. Knuth [29] gave the first double exponential lower bound, he showed that there are at least $2^{2^{n-3/2\log(n)-O(1)}}$ matroids by constructing a large family of so-called sparse paving matroids. Hence, even in exponential time, it is not trivial the recognize whether a complex is the intersection of two matroids, as there is no chance to check all the possibilities. We cannot solve this problem, even in expontential time.

Problem 6.4. Given a simplicial complex $\mathcal{K}$ on n vertices by a list of all independent sets. Is it possible to decide in exponential time in the function of n whether $\mathcal{K}$ is the intersection of two matroids?

6.2 An extremal question of matroid intersection

We considered the following extremal question.

Problem 6.5. What is the minimal number $\kappa(n)$ such that for every simplicial complex $\mathcal{K}$ over an n element set, $\mu(\mathcal{K}) \leq \kappa(n)$?

Korte and Hausmann [30] proved that $\kappa(n) \geq n-1$ for all n with the construction appearing in Theorem 6.2, and conjectured that there are families of complexes with $\mu(\mathcal{K}_n)$ being super-linear, or even exponential in n, where $\mathcal{K}_n$ has n vertices. Surprisingly, other than this, we found nothing about Problem 6.5 in the literature other than special cases. Fekete, Firla and Spille [13] proved that $\Omega(\log\log n) \leq \kappa_{\text{matching}}(n) \leq O(\log n/\log\log n)$, where $\kappa_{\text{matching}}(n)$ is the minimal number such that every matching complex on n elements can be written as the intersection of $\kappa_{\text{matching}}(n)$ matroids. Kashiwabara, Okamoto and Uno [26] proved that $\kappa_{\text{clique}}(n) = n-1$ for all $n > 1$, where analogously, $\kappa_{\text{clique}}(n)$ is the minimal number such that every clique complex on n vertices can be written as the intersection of $\kappa_{\text{clique}}(n)$ matroids.

We prove that the conjecture of Korte and Hausmann is correct by showing a family of simplicial complexes $\mathcal{K}_n$ for which $\mu(\mathcal{K}_n)$ is exponential in n. We need a celebrated theorem from extremal combinatorics.

Theorem 6.6 (Sperner's Theorem, [40])**.** *Let $\mathcal{C} \subseteq 2^S$ be a family of sets over the n element set S, and assume that there are no distinct $C_1, C_2 \in \mathcal{C}$ with $C_1 \subseteq C_2$. Then*

$$|\mathcal{C}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

Our theorem is the following.

Theorem 6.7.

$$\binom{n-1}{\lfloor (n-1)/2 \rfloor} \leq \kappa(n) \leq \binom{n}{\lfloor n/2 \rfloor}.$$

Proof. Recall that in the proof of Lemma 6.1, we proved that for a simplicial complex $\mathcal{K}$, the number of minimal sets of $\mathcal{K}$ is an upper bound of $\mu(\mathcal{K})$. These sets satisfy the condition of Sperner's Theorem, which proves the upper bound.

For the lower bound, consider

$$\mathcal{K} = \{K \subseteq S \;:\; x \notin K \text{ or } |K| \leq \lfloor (n-1)/2 \rfloor\}$$

over the n-element vertex set S where $x \in S$ is arbitrary. Notice that the minimal sets not in $\mathcal{K}$ are exactly the sets $C \subseteq S$ with $x \in C$ and $|C| = \lfloor (n+1)/2 \rfloor$. Call the family of these sets $\mathcal{C}$. Clearly, $|\mathcal{C}| = \binom{n-1}{\lfloor (n-1)/2 \rfloor}$. Let $M_1 = (S, \mathcal{F}_1), M_2 = (S, \mathcal{F}_2), \ldots, M_\ell = (S, \mathcal{F}_\ell)$ be matroids such that their intersection is $\mathcal{K}$ where $\ell = \mu(\mathcal{K})$, and let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\ell$ be the family of their circuits, respectively. For every $C \in \mathcal{C}$, there is a matroid M_i with $C \notin \mathcal{F}_i$. As C is a minimal set among the sets not in $\mathcal{K}$, and $\mathcal{K} \subseteq \mathcal{F}_i$, C is also minimal that is not in $\mathcal{F}_i$, so $C \in \mathcal{C}_i$. If for some distinct $C_1, C_2 \in \mathcal{C}$ we have $C_1, C_2 \in \mathcal{C}_i$ for some i, then by $x \in C_1 \cap C_2$ and the axiom (C3), there exists a set $C \in \mathcal{C}_i$ with $C \subseteq C_1 \cup C_2 - x \subseteq S - x$. However, this means that $S - x$ is dependent in M_i, which is a contradiction as $S - x \in \mathcal{K}$. It follows that $|\mathcal{C} \cap \mathcal{C}_i| \leq 1$ for every $1 \leq i \leq \ell$, and $\mathcal{C} \subseteq \bigcup_{i=1}^{\ell} \mathcal{C}_i$, hence

$$\kappa(n) \geq \mu(\mathcal{K}) = \ell \geq |\mathcal{C}| = \binom{n-1}{\lfloor (n-1)/2 \rfloor}.$$

□

Remark 6.8. From Theorem 6.7, using Stirling's formula, we obtain that

$$\kappa(n) = \Theta\left(\frac{2^n}{\sqrt{n}}\right),$$

hence $f(n)$ is indeed exponential.

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