Borel equivalence relations and Ramsey theory

Master's thesis

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Contents

1 Introduction

The theory of Borel graphs has been a very active field of research in the last decades. Some of the most interesting results of the area include

- Circle squaring with Borel/measurable pieces [MU17, GMP17]
- The $\mathbb{G}_0$-dichotomy: an analytic digraph on a Polish space is Borel-colorable with countably many colors if and only if it does not contain a copy of a fixed graph $\mathbb{G}_0$ [KST99]
- The connection between chromatic numbers and Borel Determinacy [Mar16]

We refer the reader to [KM16] and [Pik20] for surveys of the present state of the field.

The study of definable objects is partially motivated by the phenomenon that in general, when extending finite structures into infinite ones, objects often perform counterintuitive behavior without the constraint of definability. For example, generalizing Hall's perfect matching theorem results in the well-known Banach-Tarski paradox, i.e., that the unit ball of $\mathbb{R}^3$ is decomposable into finitely many pieces which are rearrangeable into two unit balls, while this is impossible with measurable/Borel pieces.

The behavior of a Borel graph is strongly connected to the behavior of the corresponding connected component equivalence relation. Another motivation to study definable equivalence relations comes from the fact that a significant portion of mathematics is devoted to the investigation of objects modulo some notion of similarity (e.g., algebraic structures modulo isomorphism, topological spaces up to homeomorphism/homotopy equivalence etc.). The collection of objects often comes with a natural Borel structure, and similarity is a definable equivalence relation.

To compare the complexity of such relations, a partial ordering is defined:

Definition 1.0.1. Let X, Y be standard Borel spaces. An equivalence relation $E \subseteq X \times X$ is said to be Borel reducible to an equivalence relation $F \subseteq Y \times Y$, denoted $E \leq_B F$, if there exists a Borel map $\varphi : X \to Y$ such that $x_1 E x_2 \Leftrightarrow \varphi(x_1) F \varphi(x_2)$.

In this case we can think of E as being less complex than F. One can also see this hierarchy as the definable version of cardinalities: for example, when determining whether there are „more" countable groups or real numbers, the evidence for the former being less or equal than the other is a reduction from the trivial equivalence relation on the reals, $\Delta_{2^{\mathbb{N}}}$, to the isomorphism equivalence relation of countable groups. Thus, a Borel reduction between these equivalence relations can be interpreted as the definable cardinality of reals is at most the definable cardinality of countable groups. This gives a refinement of the standard notion of cardinalities, which is much more sensitive, as for example the definable cardinality of countable groups can be shown to be strictly larger than the definable cardinality of real numbers, while of course both cardinalities are equal to continuum in the regular sense.

One of the most of important goals of descriptive set theory is to gain a deeper understanding of this hierarchy, see e.g., [Kan08, KM04, JKL02, Kec19, Gao08].

Its behavior is clear at the beginning, with relations of $1, 2, \ldots, \mathbb{N}, 2^{\mathbb{N}}$ classes being the most simple ones. The following equivalence relation, which is bireducible with the Vitali equivalence relation on the real line, is the least complex of all the others, and has great significance:

Definition 1.0.2. Let $\mathbb{E}_0 \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ be such that $x \mathbb{E}_0 y \Leftrightarrow \exists n \forall k \geq n \; x(k) = y(k)$ for $x, y \in 2^{\mathbb{N}}$.

However, from this point, the hierarchy is no longer linear [AK00], and seems to be truly complicated [CM17], in fact, it is already not clear whether a given equivalence relation is Borel reducible to $\mathbb{E}_0$.

Here, one can restrict to the case of countable Borel equivalence relations (abbreviated CBER), i.e., Borel equivalence relations with countable classes[1]. These are exactly the equivalence relations generated by actions of countable groups by the theorem of Feldman and Moore.

For a CBER E, $E \leq_B \mathbb{E}_0$ is equivalent to the existence of a sequence of Borel equivalence relations $E_0 \subseteq E_1 \subseteq E_2 \dots$ whose classes are all finite such that $E = \bigcup_{n \in \mathbb{N}} E_n$. This property is called *hyperfiniteness*, which is an extremely important notion, having strong connections with group actions, graphs, and amenability.

In the investigation of Borel graphs, roughly speaking, there are four main tools:

- Baire-category [KST99]
- Measure theory [CK13]
- Infinite games, Borel determinacy [Mar16]
- Ramsey theory [TV21]

Ramsey theory in this context is about generalizing finite dimensional „pigeonhole-principle" results to infinite dimensional definable objects. A central example is the Galvin-Prikry theorem, which states that any finite Borel coloring of $[\mathbb{N}]^{\mathbb{N}}$ has a monochromatic set of form $[X]^{\mathbb{N}}$. This is again a situation where non-definability could ruin the nice behavior of an object, as this statement turns out to be false for arbitrary finite colorings.

Ramsey theoretic statements are often viewed as canonization results: an arbitrary object behaves nicely if we restrict it to a suitable large subset. For example, by the theorem of Prömel and Voigt [PV85], any Borel mapping $\Delta : [\omega]^{\omega} \to \mathbb{R}$ can be restricted in such a way, that it can be represented by a continuous mapping, which takes every set into a subset of his own. Many further examples can be found in [Tod10] and [KSZ13].

Ramsey theory is relatively underutilized in the context of Borel equivalence relations compared to the other three listed areas, but it is unfolding, and could be useful to tackle several problems that are shown to be unreachable by the standard tools of measure theory or category.

Our main goal in this thesis is to give a summary of some of the basic results from the rich theory of Borel equivalence relations and graphs, with an emphasis on Ramsey theory. Section 2 is devoted to hyperfiniteness, and its connection to amenability and the tools of measure and category theory. The goal of Section 3 is to present the core idea behind the machinery of infinite dimensional Ramsey theory, and to prove that all countable equivalence relations are hyperfinite on a Ramsey positive set. Finally, Section 4 contains our results and summarizes the possible further directions. We construct several new examples of graphs having close connections to Ramsey theory (Propositions 4.1.7., 4.2.1., 4.2.2., 4.2.5., 4.2.6., 4.2.7. and 4.2.8.), whose properties are yet to be better understood.

[1]This is a slightly unfortunate, nevertheless standard terminology

2 Hyperfiniteness

2.1 An overview of countable Borel equivalence relations

Firstly, we will take a look at how the initial segment of the partial ordering $\leq_B$ looks like. We write $E <_B F$ when $E \leq_B F$ and $F \not\leq_B E$, and $E \leq_c F$ when a continuous reduction exists (in such a situation the underlying space is assumed to be equipped with a topology generating the Borel structure).

Definition 2.1.1. For a Polish space X let Δ_X be the equality equivalence relation on X.

We equip $1, 2, \dots \mathbb{N}$ with the trivial Borel structure, while $2^{\mathbb{N}}$ is endowed with the product topology.

Proposition 2.1.2. $\Delta_1 <_B \Delta_2 <_B \dots <_B \Delta_{\mathbb{N}} <_B \Delta_{2^{\mathbb{N}}}$

Proof. Observe that for $i \in \{1, 2, \dots \mathbb{N}\}$ and any Borel equivalence relation E, we have that $\Delta_i \leq_B E$ if and only if E has at least i-many classes, and that $\Delta_{2^{\mathbb{N}}} \leq_B \Delta_{\mathbb{N}}$ is impossible for the same reason. □

Now we consider subclasses of Borel equivalence relations. One basic notion arising when classifying equivalence relations is the following:

Definition 2.1.3. A Borel equivalence relation E is *finite*, if every E-class is finite.

An only somewhat more complex yet important notion is *smoothness*.

Definition 2.1.4. A Borel equivalence relation E is *smooth*, if $E \leq_B \Delta_{2^{\mathbb{N}}}$.

The concept of smoothness naturally occurs if one intends to determine which equivalence relations can be inserted into the order involved in Proposition 2.1.2. An additional motivation comes from the interpretation of Borel reductions as classifications: the classification problems such that the corresponding equivalence relation is Borel reducible to $\Delta_{2^{\mathbb{N}}}$ are exactly those which can be charaterized by a real number valued invariant in a Borel way.

Definition 2.1.5. A *transversal* for an equivalence relation $E \subseteq X \times X$ is a set $T \subseteq X$ intersecting all E-classes in exactly one point.

We will show that for countable Borel equivalence relations, smoothness is equivalent to the existence of a Borel transversal. First recall the following central result of descriptive set theory (18.10. in [Kec12]):

Theorem 2.1.6 (Luzin-Novikov)**.** *Let X, Y be Polish spaces and suppose $P \subseteq X \times Y$ is a Borel set with all sections $(P_x)_{x \in X}$ countable. Then P has a Borel uniformization, i.e. a Borel set $B \subseteq P$ with $|B_x| \leq 1$ for all $x \in X$ and $\mathrm{proj}_X(B) = \mathrm{proj}_X(P)$.*

Moreover, P arises as $\bigcup_{n \in \mathbb{N}} \mathrm{gr}(f_n)$ where each $f_n : X \to Y$ is a Borel map.

Proposition 2.1.7. *A countable Borel equivalence relation $E \subseteq X \times X$ is smooth if and only if E has a Borel transversal.*

Proof. Let T be a Borel transversal for E. Then T is a standard Borel space with the inherited Borel structure, hence it is Borel isomorphic with $2^{\mathbb{N}}$ or otherwise it is countable (see 15.6 in [Kec12]), this implies $\Delta_T \leq_B \Delta_{2^{\mathbb{N}}}$ in both cases. Therefore, it is enough to see that $E \leq_B \Delta_T$. Let $\varphi(x) = y$ if y is the unique element of T which is E-equivalent to x, this is a Borel map as $\varphi(x) = y \Leftrightarrow y \in T \wedge (x, y) \in E$, and it is clear that φ reduces E to Δ_T.

Conversely, suppose that E is a smooth CBER, and φ reduces it to $\Delta_{2^{\mathbb{N}}}$. By the Luzin-Novikov theorem, if a Borel subset of a product space has countable sections, then it has a Borel uniformization. Therefore, the graph $\mathrm{gr}(\varphi) \subset X \times 2^{\mathbb{N}}$ has a Borel uniformization U, as all E-classes are countable, and hence all $2^{\mathbb{N}}$-sections of $\mathrm{gr}(\varphi)$ are countable. The set $\mathrm{proj}_X(U)$ is a Borel injective image of U, consequently it is Borel by the Luzin-Souslin theorem (15.1. in [Kec12]), and also it contains exactly one point of every E-class. $\square$

Corollary 2.1.8. *If a Borel equivalence relation is finite, then it is smooth.*

Proof. Suppose that $E \subseteq X \times X$ is a finite equivalence relation. Fix a Borel linear ordering $<$ of X, then the $<$-minimal elements of each E-class form a Borel transveral. $\square$

Definition 2.1.9. Suppose that Γ is a Polish group, X is a standard Borel space, and $(\gamma, x) \mapsto \gamma(x)$ is a Borel action of Γ on X (that is, the action is a Borel map of $\Gamma \times X$ into X). The *orbit equivalence relation* of the action of Γ on X is defined by

$$E_{\Gamma,X} = \{(x, y) \in X^2 : \exists \gamma \in \Gamma \; x = \gamma(y)\}.$$

The Luzin-Novikov theorem also implies the next characterization of countable Borel equivalence relations:

Theorem 2.1.10 (Feldman-Moore, [FM77])**.** *Every countable Borel equivalence relation E on a Borel set X in a standard Borel space arises as the orbit equivalence relation of a Borel action of a countable group Γ on X.*

Proof. The proof we present is based on 7.4.1. in [Kan08]. Without loss of generality, we can assume that $X = 2^{\mathbb{N}}$ (see 15.6 in [Kec12]). Take $(f_n)_{n\in\mathbb{N}}$ to be Borel maps such that $P = \bigcup_{n\in\mathbb{N}} \mathrm{gr}(f_n)$ by the Luzin-Novikov theorem. Define

$$H_n = \mathrm{gr}(f_n) \setminus \left(\bigcup_{k<n} \mathrm{gr}(f_k) \right).$$

For $H \subseteq X \times X$, let $H^T = \{(x, y) : (y, x) \in H\}$, and take $P_{nk} = H_n \cap H_k^T$, now the sets $\{P_{n,k} : n, k \in \mathbb{N}\}$ form a partition of E to countably many Borel injective sets, that is, Borel sets whose sections are all of cardinality at most 1. For $i < 2$ and $s \in 2^{<\mathbb{N}}$ with $|s| = k$ define

$$R_s^i = \{(x, y) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} : x|_{k+1} = s^\frown(i) \wedge y|_{k+1} = s^\frown(1 - i)\}.$$

Now, every set of form $H_{n,k} \cap R_s^i$ is a Borel partial function with disjoint domain and range. Let $(D_n)_{n\in\mathbb{N}}$ be an enumeration of these sets, and define Borel involutions $g_n : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by

$$g_n(x) = y \Leftrightarrow ((x, y) \in D_n \vee (y, x) \in D_n \vee x = y).$$

Let Γ be the group generated by $(g_n)_{n\in\mathbb{N}}$, then it is easy to see that Γ's natural action on $2^{\mathbb{N}}$ induces E. $\square$

Note that the converse is also true, as the equivalence relation induced by a countable group action is countable and Borel, however, the equivalence relation induced by an uncountable group action can be strictly analytic.

Observe that this proof yields the following useful claim as well:

Proposition 2.1.11. *Let E be a countable Borel equivalence relation on a Polish space X. Then there are $g_n : X \to X$ Borel involutions such that $E = \bigcup_{n\in\mathbb{N}} \mathrm{gr}(g_n)$.*

We arrive to a key notion of the area, hyperfiniteness, first introduced by Slaman and Steel [SS88]. Recall that a Borel equivalence relation E is hyperfinite, if there exists a sequence of finite equivalence relations $E_0 \subseteq E_1 \subseteq E_2 \subseteq \dots$ such that $E = \bigcup_{n\in\mathbb{N}} E_n$.

Note that it is relevant that the sequence of finite equivalence relations is increasing: in fact, any countable equivalence relation arises as the union of finite equivalence relations, or even equivalence relations with classes of at most 2 elements by Proposition 2.1.11. above.

Interestingly, Slaman and Steel introduced this notion in a different context: the investigation of certain recursion theoretic questions. In fact, they have shown that Turing equivalence of reals (x and y are *Turing-equivalent*, if x is Turing computable from y and vice-versa), which is also a CBER, is not hyperfinite.

Example 2.1.12. The equivalence relation $\mathbb{E}_0$ is hyperfinite, consider the finite equivalence relations

$$E_n = \{(x, y) \in (2^{\mathbb{N}})^2 : \ \forall k \geq n \ x(k) = y(k)\}.$$

Thus $\mathbb{E}_0$ is the most complex hyperfinite equivalence relation, in the sense of Borel reducibility.

Also all countable smooth equivalence relations are hyperfinite: Suppose that E is a countable smooth equivalence relation, so it can be written as $\bigcup_{n\in\mathbb{N}} \mathrm{gr}(f_n)$ with some Borel maps $(f_n)_{n\in\mathbb{N}}$, and has a Borel transversal T. Take $H_n = \{x | \, \exists k < n : \ f_k(x) \in T\}$ and define

$$E_n = \{(x, y) | \, xEy \wedge (x, y \in H_n \vee x = y)\}.$$

Clearly E_n is a finite Borel equivalence relation, and indeed $\bigcup_{i\in\mathbb{N}} E_n = E$, as for all x there exists some k such that $f_k(x) \in T$.

Therefore, the following inclusions hold: finite $\subset$ countable smooth $\subset$ hyperfinite $\subset$ countable, and also these inclusions are all strict: we will see in Section 2.2. that $\mathbb{E}_0$ is not smooth, and we will also prove the existence of a countable equivalence relation which is not hyperfinite in Section 2.4. Also there exists a most complex countable Borel equivalence relation E_∞ such that for all E CBER $E \leq_B E_\infty$, this is not hyperfinite either, otherwise all CBER would be hyperfinite. These inclusions are displayed on Figure 1. (Note that even though $\Delta_{2^{\mathbb{N}}}$ is finite, it is placed on the top of smooth relations to highlight that it is the most complex one in the sense of Borel reducibility, to get a smooth equivalence relation which is not finite one can take $E = \{((x_1, n_1), (x_2, n_2)) \in \left(2^{\mathbb{N}} \times \mathbb{N}\right)^2 : x_1 = x_2\}$.)

Definition 2.1.13. A Borel equivalence relation E is *hypersmooth* if there is an increasing sequence of smooth equivalence relations $E_0 \subseteq E_1 \subseteq \dots$ such that $E = \bigcup_{n\in\mathbb{N}} E_n$.

The benchmark equivalence relation related to hypersmoothness is the following:

Definition 2.1.14. Define $\mathbb{E}_1$ on $\left(2^{\mathbb{N}}\right)^{\mathbb{N}}$ by

$$(x, y) \in \mathbb{E}_1 \Leftrightarrow \exists n \forall k \geq n \ x(k) = y(k).$$

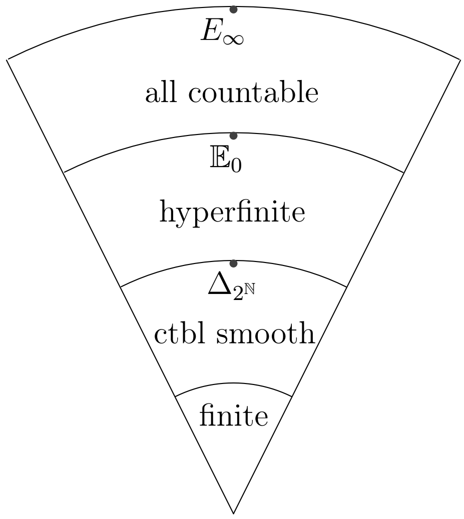

Figure 1: The structure of countable Borel equivalence relations

Proposition 2.1.15. *Suppose that E is a Borel equivalence relation on a Polish space X. E is hypersmooth if and only if $E \leq_B \mathbb{E}_1$.*

Proof. Assume first that E is hypersmooth, so there is an increasing sequence $(E_n)_{n\in\mathbb{N}}$ such that $E = \bigcup_{i\in\mathbb{N}} E_n$, and for each n there exists a Borel map $\varphi_n : X \to 2^\mathbb{N}$ with $xE_ny \Leftrightarrow \varphi_n(x) = \varphi_n(y)$. Then the Borel map $\varphi : X \to \left(2^\mathbb{N}\right)^\mathbb{N}$ defined by

$$(\varphi(x))(n) = \varphi_n(x)$$

reduces E to $\mathbb{E}_1$.

Conversely, if $\varphi : X \to \left(2^\mathbb{N}\right)^\mathbb{N}$ is a Borel reduction of E to $\mathbb{E}_1$, then the equivalence relations defined by

$$xE_ny \Leftrightarrow \forall k \geq n\ (\varphi(x))(k) = (\varphi(y))(k)$$

witness that E is hypersmooth: it is clear that $E_0 \subseteq E_1 \subseteq \ldots$. Define the map

$$\pi_{\geq n} : (x_0, x_1, \ldots) \mapsto (x_n, x_{n+1}, \ldots)$$

on $\left(2^\mathbb{N}\right)^\mathbb{N}$. Then E_n is smooth since $\pi_{\geq n} \circ \varphi$ reduces E_n to $\Delta_{(2^\mathbb{N})^\mathbb{N}} \cong \Delta_{2^\mathbb{N}}$. $\square$

The next theorem is the basic characterization result of hyperfiniteness.

Theorem 2.1.16. *The following are equivalent for a Borel equivalence relation E on a Polish space X:*

1. *$E \leq_B \mathbb{E}_0$ and E is countable*
2. *E is hyperfinite*
3. *E is hypersmooth and countable*
4. *there exists a Borel set $B \subseteq \left(2^\mathbb{N}\right)^\mathbb{N}$ such that the restricted equivalence relation $\mathbb{E}_1|_B$ is countable and isomorphic with E via a Borel bijection of X onto B*

5. *E is induced by a Borel action of $\mathbb{Z}$*

6. *there is a Borel partial order $\leq$ on the domain of E such that every E-class is $\leq$-ordered similarly to a subset of $\mathbb{Z}$*

Proof. The proof is based on 8.2. in [Kan08].

The implications 2. $\Rightarrow$ 3. and 1. $\Rightarrow$ 3. are quite straightforward.

We may assume without loss of generality that $X = 2^{\mathbb{N}}$.

3. $\Rightarrow$ 4. Let $E = \bigcup_{n\in\mathbb{N}} E_n$ be a countable hypersmooth equivalence relation on X with $E_0 \subseteq E_1 \subseteq \ldots$ smooth equivalence relations. We may assume that $E_0 = \Delta_{2^{\mathbb{N}}}$. Let $T_n \subseteq 2^{\mathbb{N}}$ be a Borel transveral for E_n (such set exists by Proposition 2.1.7.), and define $\vartheta_n(x)$ by

$$\vartheta_n(x) = y \Leftrightarrow xE_n y \wedge y \in T_n.$$

Let $\vartheta(x) = (\vartheta_n(x))_{n\in\mathbb{N}}$, then $\vartheta : 2^{\mathbb{N}} \to \left(2^{\mathbb{N}}\right)^{\mathbb{N}}$ is a Borel map such that $xEy \Leftrightarrow \vartheta(x)\mathbb{E}_1\vartheta(y)$, take B to be the image of ϑ. Observe that ϑ is injective, thus B is Borel by the Luzin-Souslin theorem (15.1. in [Kec12]).

4. $\Rightarrow$ 6. Let B be as indicated. For $x \in \left(2^{\mathbb{N}}\right)^{\mathbb{N}}$ define $x|_{>n} = x|_{(n,\infty)}$ and $B|_{>n} = \{x|_{>n} : x \in B\}$. The Luzin-Novikov theorem implies that there is a countable family of Borel functions $g_i^n : B|_{>n} \to B$ such that the set $B_\xi = \{x \in B : x|_{>n} = \xi\}$ is equal to $\{g_i^n(\xi) : \ i \in \mathbb{N}\}$ for all $\xi \in B|_{>n}$. In other words, $\{g_i^n(\xi)(n) : \ i \in \mathbb{N}\} = \{x(n) : \ x \in B_\xi\}$.

For every $x \in \left(2^{\mathbb{N}}\right)^{\mathbb{N}}$ define $\varphi(x) = (\varphi_n(x))_{n\in\mathbb{N}}$ by choosing $\varphi_n(x)$ to be the smallest $i \in \mathbb{N}$ such that $x(n) = g_i^n(x)(n)$. Let $\varphi_n'(x) = \max_{k\leq n}\varphi_k(x)$, and define the sequence

$$\mu(x) = (\varphi_0(x),\, \varphi_0'(x),\, \varphi_1(x)+1,\, \varphi_1'(x)+1,\, \varphi_2(x)+2,\, \varphi_2'(x)+2, \ldots)$$

For $x \neq y \in B$ with $x\mathbb{E}_1 y$, we have $x|_{>n} = y|_{>n}$ for some n, therefore, $\varphi(x)|_{>n} = \varphi(y)|_{>n}$, but $\varphi(x) \neq \varphi(y)$, $\mu(x) \neq \mu(y)$, and $\mu(x)|_{>m} = \mu(y)|_{>m}$ for some $m \geq n$.

Let $<_{\text{alex}}$ be the anti-lexicographical partial order on $\mathbb{N}^{\mathbb{N}}$, which is defined as

$$a <_{\text{alex}} b \Leftrightarrow (\exists n)(a|_{>n} = b|_{>n} \wedge a(n) < b(n)).$$

For $x, y \in B$ define $x <_0 y \Leftrightarrow \mu(x) <_{\text{alex}} \mu(y)$. Our observations so far yield that $<_0$ linearly orders every $\mathbb{E}_1$-class $[x]_{\mathbb{E}_1} \cap B$ for each $x \in B$. Moreover, it follows from the definition of $\mu(x)$ that every $<_{\text{alex}}$-interval between some $\mu(x) <_{\text{alex}} \mu(y)$ contains only finitely many elements of the form $\mu(z)$. Thus, for any $x \in B$ we have that $[x]_{\mathbb{E}_1} \cap B$ is ordered by $<_0$ similarly to a subset of $\mathbb{Z}$.

6. $\Rightarrow$ 5. Suppose that $\leq$ is an order as in 6. We convert $\leq$ into a Borel action of $\mathbb{Z}$ in the following way: If an E-class $[x]_E = \{\ldots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \ldots\}$ is ordered similarly to $\mathbb{Z}$ itself, then let $1 \cdot x_j = x_{j+1}$ for all $j \in \mathbb{Z}$.

If $[x]_E = \{x_0 < x_1 < \ldots\}$ is ordered similar to $\mathbb{N}$, then re-order it similarly to $\mathbb{Z}$ as $[x]_E = \{\ldots < x_3 < x_1 < x_0 < x_2 < x_4 < \ldots\}$, and apply the first case.

If $[x]_E = \{x_0 > x_1 > \ldots\}$ is ordered similar to $-\mathbb{N}$, then use the same method.

Finally, if $[x]_E = \{x_0 < x_1 < \ldots < x_n\}$ is finite, then apply the cyclic action $1 \cdot x_j = x_{j+1}$ for $j < n$ and $1 \cdot x_n = x_0$. It is not hard to check that the action of $\mathbb{Z}$ defined this way is Borel.

5. $\Rightarrow$ 2. We will define an increasing sequence of finite equivalence relations F_n separately on each E-class C. The resulting equivalence relations will be Borel on the whole space since the $\mathbb{Z}$-action allows us to replace quantifiers over an E-class by quantifiers over $\mathbb{Z}$.

Firstly, if we can choose a representative $x_C \in C$ from a class C in a Borel way, then we can define xF_ny if $\exists j,k \in \mathbb{Z}$, $|j|,|k| \leq n$ with $x = j \cdot x_C$ and $y = k \cdot x_C$. This applies in the cases when C is finite, thus, we can assume that it is infinite. Let $<_{\text{lex}}$ be the lexicographical ordering of $2^{\mathbb{N}}$, and $<_{\text{act}}$ be the partial order induced by the $\mathbb{Z}$-action, that is, $x <_{\text{act}} y$ if $y = j \cdot x$ with some $j > 0$. Then we can assume that neither $a = \inf_{<_{\text{lex}}} C$ nor $b = \sup_{<_{\text{lex}}} C$ is in C. Define $C_n = \{x \in C : \ x|_n \neq a|_n \wedge x|_n \neq b|_n\}$. Define xF_ny if $x = y$ or x, y belong to the same $<_{\text{act}}$-interval in C lying entirely within C_n.

5. $\Rightarrow$ 1. This implication is somewhat more complicated then the others. We show an outline for its proof, but omit some of the parts. Firstly, we show that $E \leq_B E(\mathbb{Z}, 2^{\mathbb{N}})$, where $E(\mathbb{Z}, 2^{\mathbb{N}})$ is the orbit equivalence induced by the shift action of $\mathbb{Z}$ on $\left(2^{\mathbb{N}}\right)^{\mathbb{Z}}$, that is, $(k \cdot x)(j) = x(j-k)$ for $k, j \in \mathbb{Z}$ and $x \in \left(2^{\mathbb{N}}\right)^{\mathbb{Z}}$. We can obtain a Borel reduction by $\vartheta(a) = (j \cdot a)_{j \in \mathbb{Z}}$, where $\cdot$ is the Borel action of $\mathbb{Z}$ which induces E. Thus it is enough to show that $E(\mathbb{Z}, 2^{\mathbb{N}}) \leq_B \mathbb{E}_0$.

Now let $W_n = 2^{n \times n}$, and fix an order $<_n$ on W_n so that $u <_{n+1} v$ implies $u|_{n \times n} <_n v|_{n \times n}$ for all $u, v \in W_{n+1}$. Put $W = \bigcup_{n \in \mathbb{N}} W_n$.

For $x \in \left(2^{\mathbb{N}}\right)^{\mathbb{Z}}$ and $w \in W_n$ let $A^x(w)$ be the set of all integers $a \in \mathbb{Z}$ satisfying $x(a+k)(i) = w(k,i)$ for all $k, i < n$. For $x \in \left(2^{\mathbb{N}}\right)^{\mathbb{Z}}$ and $n \in \mathbb{N}$, let w_n^x be the $<_n$-least element $w \in W_n$ such that $A^x(w) \neq \emptyset$. Observe that $w_n^x \subset w_{n+1}^x$, hence there is an element $\psi^x \in \left(2^{\mathbb{N}}\right)^{\mathbb{N}}$ such that $\psi^x(k)(i) = w_n^x(k,i)$ for all $k, i < n$. Finally let A^x denote the set of all integers $a \in \mathbb{Z}$ satisfying $x(a+k) = \psi^x(k)$ for all $k \in \mathbb{N}$.

Now we define the following partition of $\left(2^{\mathbb{N}}\right)^{\mathbb{Z}}$:

$X_0 = \{x \in \left(2^{\mathbb{N}}\right)^{\mathbb{Z}} : (\exists w \in W)(A^x(w) \neq \emptyset \text{ and bounded in } \mathbb{Z}\}$,

$X_1 = \{x \in \left(2^{\mathbb{N}}\right)^{\mathbb{Z}} \setminus X_0 : A^x \neq \emptyset \text{ and it is bounded in } \mathbb{Z} \text{ from below}\}$,

$X_2 = \{x \in \left(2^{\mathbb{N}}\right)^{\mathbb{Z}} \setminus X_0 : A^x \neq \emptyset \text{ and it is unbounded in } \mathbb{Z} \text{ from below}\}$, and

$Y = \{x \in \left(2^{\mathbb{N}}\right)^{\mathbb{Z}} \setminus X_0 : \ A^x = \emptyset\} = \left(2^{\mathbb{N}}\right)^{\mathbb{Z}} \setminus X_0 \cup X_1 \cup X_2$.

Now we claim that X_i and Y are all Borel, and that $E|_{X_i}$ is smooth for $i < 3$, these are not so hard to check. We also claim that $E|_Y \leq_B \mathbb{E}_0$, which requires a little more work. Once we checked this, all that is left is to prove that for an $X = \bigcup_{k<n} X_k$ partition and E equivalence relation with $E|_{X_k} \leq \mathbb{E}_0$ for all $k < n$, we have that $E \leq_B \mathbb{E}_0$. $\square$

2.2 $\mathbb{G}_0$, Silver's dichotomy and the $\mathbb{E}_0$-dichotomy

We define a graph on $2^{\mathbb{N}}$ which is a graph theoretic analogue of $\mathbb{E}_0$. We say that a sequence $(s_n)_{n \in \mathbb{N}}$ of elements of $2^{<\mathbb{N}}$ is *appropriate*, if $|s_n| = n$ for every n and for every $t \in 2^{<\mathbb{N}}$ with $|t| = k$ there exists an $n \in \mathbb{N}$ such that $t = s_n|_k$.

Definition 2.2.1 (Kechris-Solecki-Todorčević, [KST99]). Fix an appropriate sequence $(s_n)_{n \in \mathbb{N}}$. For $x, y \in 2^{\mathbb{N}}$ let $x \mathbb{G}_0 y$ if and only if

$$x = s_n{}^\frown(i)^\frown r \text{ and } y = s_n{}^\frown(1-i)^\frown r$$

for some $n \in \mathbb{N}$ and $r \in 2^{\mathbb{N}}$.

Proposition 2.2.2. *$\mathbb{G}_0$ is acyclic, and two points are in the same connected component of $\mathbb{G}_0$ if and only if they are $\mathbb{E}_0$-equivalent.*

Proof. For the sake of contradiction, suppose that $\mathbb{G}_0$ has a cycle, and choose one with minimal length. Let $n \in \mathbb{N}$ be the maximal such that the nth entry changes along the cycle. This means that at the edges where the nth entry changes, the n-prefix must be s_n, and also all the later entries are constant, so there would be a shorter cycle between two of these edges.

If x, y are $\mathbb{G}_0$-connected, then they differ in finitely many coordinates, so they are $\mathbb{E}_0$-equivalent. For the other direction, it is enough to show by induction, that the first n entries of an arbitrary element of $2^{\mathbb{N}}$ can be set to s_n along the edges of $\mathbb{G}_0$. For $n = 0$ this is trivial, and if this is true for n, then we can change the first n entries to s_n, change the $(n+1)$st to $s_{n+1}(n)$, and then use induction for $(s_{n+1})|_n$. $\square$

For a graph on a topological space X we define the *Baire measurable chromatic number* of G, $\chi_{BM}(G)$ to be the minimal element of $\{1, 2, \ldots \aleph_0\}$ such that G admits a Baire measurable coloring with that many colors (where the set of colors is endowed with the discrete topology). Note that a Baire measurable coloring is equivalent to a partition of X to Baire measurable G-independent sets, that is, sets not containing any edge of G. We define the *Borel chromatic number* of G, $\chi_B(G)$ similarly, by constraining to Borel-measurable colorings.

Proposition 2.2.3. *No set of second category having the Baire property is $\mathbb{G}_0$-independent, consequently, $\chi_{BM}(\mathbb{G}_0) \not\leq \aleph_0$.*

Proof. Suppose that H is a non-meager set with the Baire property, and choose a basic neighborhood $\mathcal{N}_t = \{s \in 2^{\mathbb{N}} : s|_k = t\}$ such that H is comeager in $\mathcal{N}_t$. There exists some n with $t = s_n|_k$, then H is comeager on $\mathcal{N}_{s_n{}^\frown 0}$ and also on $\mathcal{N}_{s_n{}^\frown 1}$, this means that there is an element $r \in 2^{\mathbb{N}}$ such that $s_n{}^\frown 0^\frown r \in H$ and $s_n{}^\frown 1^\frown r \in H$, as the map which changes the $(n+1)$st entry from 0 to 1 is category preserving. Therefore, H is not $\mathbb{G}_0$-independent. $\square$

Corollary 2.2.4. $\Delta_{2^{\mathbb{N}}} <_B \mathbb{E}_0$

Note that the next argument is essentially the same as the classical one for the non-existence of a Baire-measurable transversal for the Vitali-equivalence relation.

Proof. To show that $\Delta_{2^{\mathbb{N}}} \leq_B \mathbb{E}_0$, identify $2^{<\mathbb{N}}$ with $\mathbb{N}$, and to each $x \in 2^{\mathbb{N}}$ associate the set $\{x|_n : n \in \mathbb{N}\}$, this way the associated elements to distinct elements will differ in in co-finitely many entries.

Towards a contradiction, suppose that $\mathbb{E}_0 \leq_B \Delta_{2^{\mathbb{N}}}$. Then by Proposition 2.1.7., there exists a Borel transversal T for $\mathbb{E}_0$. Define $c : 2^{\mathbb{N}} \to [\mathbb{N}]^{<\mathbb{N}}$ by

$$c(x) = a \Leftrightarrow \exists y \in 2^{\mathbb{N}} : \ (y \in T) \wedge (a = \{n \in \mathbb{N} : \ x(n) \neq y(n)\}),$$

this is a Baire measurable countable coloring of $\mathbb{G}_0$, contradicting Proposition 2.2.3. $\square$

It turns out that not only $\mathbb{G}_0$ is not Baire measurably colorable, but also it is the canonical example for such a graph.

Definition 2.2.5. Let $G \subseteq X^2$ and $H \subseteq Y^2$ be graphs. A *homomorphism* from G to H is a map $f : X \to Y$ such that $(x, y) \in G \Rightarrow (f(x), f(y)) \in H$ for all $x, y \in X$. If X, Y are standard Borel spaces, we write $G \leq_c H$ resp. $G \leq_B H$ if there exists a continuous resp. Borel homomorphism from G to H.

Theorem 2.2.6 (Kechris-Solecki-Todorčević)**.** *Let G be an analytic graph on a Polish space. Then either $\chi_B(G) \leq \aleph_0$, or $\mathbb{G}_0 \leq_c G$.*

We do not give a proof for this theorem, but show that it yields Silver's dichotomy theorem and the $\mathbb{E}_0$-dichotomy for countable equivalence relations as a corollary.

Theorem 2.2.7 (Silver's dichotomy)**.** *Assume that E is a coanalytic equivalence relation on a Polish space X. Then either $E \leq_B \Delta_{\mathbb{N}}$, or $\Delta_{2^{\mathbb{N}}} \leq_c E$.*

Theorem 2.2.8 ($\mathbb{E}_0$-dichotomy; Glimm-Effros, Harrington-Kechris-Louveau)**.** *Let E be a Borel equivalence relation on a Polish space X. Then either E is smooth, or $\mathbb{E}_0 \leq_c E$.*

The fact that the G_0 dichotomy implies both results illustrates the power of the graph theoretic approach to descriptive set theory [Mil12] and the important role Borel graphs can play.

Now, assembling these theorems with Proposition 2.1.2. and Corollary 2.2.4. we have the following:

Corollary 2.2.9. $\Delta_1 <_B \Delta_2 <_B \ldots <_B \Delta_{\mathbb{N}} <_B \Delta_{2^{\mathbb{N}}} <_B \mathbb{E}_0$, *moreover, if $E \leq_B \mathbb{E}_0$ for a Borel equivalence relation E, then it is Borel bireducible with exactly one of the listed equivalence relations.*

Recall that the *hyperspace* $\mathcal{K}(X)$ of a metric space X is the set of its nonempty compact subsets with the *Hausdorff metric* defined as

$$d_H(K, L) = \inf\{\varepsilon > 0 : K \subseteq L_\varepsilon \wedge L \subseteq K_\varepsilon\}$$

where H_ε is the ε-neighborhood of a set H.

For a set A we use the notation $(A)^2 = \{(x, y) \in A^2 : x \neq y\}$.

We will use the following theorem (19.1. in [Kec12]):

Theorem 2.2.10 (Mycielski-Kuratowski)**.** *Suppose that X is a Polish space and $U \subseteq X^2$ is an open dense set. Then $\{K \in \mathcal{K}(X) : (K)^2 \subseteq U\}$ is co-meager.*

Consequently, if X is uncountable, and G is a co-meager graph on X, then it contains a perfect clique.

Now we prove Silver's dichotomy theorem. Note that the statement fails for certain analytic equivalence relations.

Proof of Theorem 2.2.7. Let G be the complement of E. If $\chi_B(G) \leq \aleph_0$, then E has only countably many equivalence classes, since a color class of a Borel coloring can intersect only one E-class.

Assume now that $\chi_B(G) > \aleph_0$, then, by the $\mathbb{G}_0$-dichotomy, and the fact that G is analytic, there is a continuous homomorphism $\varphi : 2^{\mathbb{N}} \to X$ of $\mathbb{G}_0$ to G. Let E' be the pullback of E by φ. We claim that E' is meager in $2^{\mathbb{N}}$: Since E' is coanalytic, it has the Baire property by the Luzin-Sierpiński theorem (21.6. in [Kec12]). The Kuratowski-Ulam theorem (8.41. in [Kec12]) implies that it is enough to see that every section of it is meager. If a section E'_x was non-meager, then it would contain a $\mathbb{G}_0$-edge by Proposition 2.2.3., which would contradict the fact that φ is a homomorphism to the complement of E.

Thus, there exists an injective continuous map $h : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that $h(2^{\mathbb{N}})$ is E'-independent by Theorem 2.2.10, in other words, $\varphi \circ h$ reduces $\Delta_{2^{\mathbb{N}}}$ to E. □

Proposition 2.2.11 (Miller, [Mil12]). *Let $E \subseteq F$ be equivalence relations on a Polish space X, such that E is dense in X^2 and can be expressed as the countable union of $X \to X$ homeomorphisms and F is meager. Then $\mathbb{E}_0 \leq_c F$.*

Proof. Fix a decreasing sequence of open dense sets $V_n \subseteq X^2$ with $(\bigcap_{n\in\mathbb{N}} V_n) \cap F = \emptyset$. We recursively define nonempty open sets U_n and homeomorphisms $\gamma_n : X \to X$ whose graphs are contained in E, and we associate to every $t \in \mathbb{N}^{<\mathbb{N}}$ with $|t| = n$ a homeomorphism $\gamma_t = \gamma_0^{t(0)} \circ \ldots \circ \gamma_{n-1}^{t(n-1)}$. We do all this with the following properties:

(1) $\forall n \in \mathbb{N}\ \forall s, t \in 2^n \gamma_{s^\frown(0)}(U_{n+1}) \times \gamma_{t^\frown(1)}(U_{n+1}) \subseteq V_n$,

(2) $\forall n \in \mathbb{N}\ \overline{U_{n+1}} \cup \overline{\gamma_n U_{n+1}} \subseteq U_n$,

(3) $\forall n \in \mathbb{N}\ \forall t \in 2^n\ \mathrm{diam}(\gamma_t U_n) \leq \frac{1}{n+1}$.

Assume that we have U_n, γ_n satisfying these properties. For $x \in 2^{\mathbb{N}}$ let $\varphi(x)$ be the unique element of $\bigcap_{n\in\mathbb{N}} \overline{\gamma_{x|_n} U_n}$. Note that φ is continuous by (3). We claim that it reduces $\mathbb{E}_0$ to F. Indeed, if $\neg(x\mathbb{E}_0 y)$, (1) guarantees that $(\varphi(x), \varphi(y)) \in \bigcap_{n\in\mathbb{N}} V_n$, thus $\neg(\varphi(x) F \varphi(y))$. To show that φ takes $\mathbb{E}_0$-equivalent elements to F-equivalent ones, we will check that if $x = (0^n)^\frown r$ and $y = t^\frown r$ with $|t| = n$, then $\varphi(x) F \varphi(y)$. By definition, and the fact that γ_t is a homeomorphism, $\gamma_t \circ \varphi(x) = \varphi(y)$ and the graph of γ_n is contained in $E \subseteq F$.

Now we construct a suitable U_n and γ_n. Set $U_0 = X$. Assume that we have already defined U_n and $(\gamma_k)_{k<n}$. Take a nonempty open set U'_n with $\overline{U'_n} \subseteq U_n$ and $\mathrm{diam}(\gamma_t(U'_n)) \leq \frac{1}{n+1}$. For $s, t \in 2^n$ let $V_{s,t} = (\gamma_s \times \gamma_t)^{-1}(V_n)$, this is dense and open, therefore, there are nonempty open sets $V, W \subseteq U'_{n+1}$ with $V \times W \subseteq \bigcap_{s,t\in 2^n} V_{s,t}$. As E is dense and covered by homeomorphisms, there exists a γ_n homeomorphism and $(x_0, x_1) \in E \cap (V \times W)$ with $\gamma_n(x_0) = x_1$. Then $U_{n+1} = V \cap \gamma_n{-}1(W)$ satisfies the needed requirements. $\square$

With the assistance of the results above, now we are ready to prove the $\mathbb{E}_0$-dichotomy theorem for countable Borel equivalence relations.

Proof of Theorem 2.2.8. for CBERs. Let $G = E \setminus \{(x, x) : x \in X\}$. If $\chi_B(G) \leq \aleph_0$, and $c : X \to \mathbb{N}$ is a Borel proper coloring of G, then we can define a Borel transversal for E by $T = \{x : yEx \Rightarrow y = x \vee c(y) > c(x)\}$, this implies that E is smooth.

If $\chi_B(G) \not\leq \aleph_0$, then there exists a continuous homomorphism $\varphi : \mathbb{G}_0 \to G$ by the $\mathbb{G}_0$ dichotomy theorem. Let E' be the pullback of E by φ.

We claim that E' is meager: suppose that it is of second category, then by the Kuratowski-Ulam theorem (8.41. in [Kec12]), there would be an x such that E'_x is of second category (as E' is Borel), consequently, as $\mathbb{E}_0 \subseteq E'$, there would be a comeager E'-class C. The map φ is a homomorphism, therefore, C is mapped into a single E-class, which is countable, hence there would be a point with non-meager preimage, this would yield a $\mathbb{G}_0$-independent non-meager Borel set, contradicting Proposition 2.2.3.

Thus E' is meager, $\mathbb{E}_0 \subseteq E'$, $\mathbb{E}_0$ is dense, and it can be expressed as the countable union of $2^{\mathbb{N}} \to 2^{\mathbb{N}}$ homeomorphisms (namely those involutions, which switch up two fixed prefixes with the same length), so we can use Proposition 2.2.11 to conclude $\mathbb{E}_0 \leq_c E'$, and by applying φ, $\mathbb{E}_0 \leq_c E$. $\square$

2.3 Hyperfiniteness on a comeager set

In this section, we show that Baire-category does not detect the complexity of a countable Borel equivalence relation, i.e. all CBER are hyperfinite on a comeager invariant Borel set.

Theorem 2.3.1 (Hjorth-Kechris [HK96], Sulliwan-Weiss-Wright [SWW86], Woodin)**.** *Let E be a countable Borel equivalence relation on a Polish space. Then there is a comeager invariant Borel set C such that $E|_C$ is hyperfinite.*

Proof. Throughout the proof, $\forall^*$ denotes "for comeager many".

By Proposition 2.1.11, there exists a sequence of Borel involutions $\{g_n : n \in \mathbb{N}\}$ whose graphs cover E. For every Borel set $S \subseteq X$ and $n \in \mathbb{N}$ we define an equivalence relation F_n^S on S by

$$F_n^S = \{(x, y) : x = y \vee g_n(x) = y\}.$$

Fix a Borel linear ordering $<$ on X. As each F_n^S-class has at most 2 elements, there is a smallest element in all of these classes, let $\phi_n(S)$ be the subset of S consisting of these smallest elements, this is Borel as $<$ is Borel. Note that if S contains an element from all E classes, then $\phi_n(S)$ does too. Let $f_n^S : S \to \phi_n(S)$ be the map defined by $f_n^S(x) = (y \Leftrightarrow xF_n^S y \wedge y \in \phi_n(S))$, so f_n^S is a Borel retraction.

To each $\alpha \in \mathbb{N}^{\mathbb{N}}$, associate $\{S_n^\alpha, f_n^\alpha : n \in \mathbb{N}\}$ defined recursively by $S_0^\alpha = X$, $S_{n+1}^\alpha = \phi_{\alpha(n)}(S_n^\alpha)$, and $f_n^\alpha = f_{\alpha(n)}^{S_n^\alpha}$. Consider the increasing finite equivalence relations

$$E_n^\alpha = \{(x, y) : f_n^\alpha f_{n-1}^\alpha \dots f_0^\alpha(x) = f_n^\alpha f_{n-1}^\alpha \dots f_0^\alpha(y)\},$$

and let $E_\infty^\alpha = \bigcup_{n\in\mathbb{N}} E_n^\alpha$, this is hyperfinite. We will show that there exists an α and a comeager Borel invariant set C such that $E_\infty^\alpha|_C = E|_C$. This follows from

$$\forall^*\alpha \in \mathbb{N}^{\mathbb{N}} \forall^* x \in X \left([x]_E = [x]_{E_\infty^\alpha}\right),$$

because if $C = \{x \in X : [x]_E = [x]_{E_\infty^\alpha}\}$ is comeager for some $\alpha \in \mathbb{N}^{\mathbb{N}}$, then this C is a comeager E-invariant Borel set with $E_\infty^\alpha|_C = E|_C$.

By the Kuratowski-Ulam theorem (8.41. in [Kec12]), a subset of $\mathbb{N}^{\mathbb{N}} \times X$ with the Baire property is comeager if and only if comeager many of its sections are comeager. As the set

$$\{(\alpha, x) \in \mathbb{N}^{\mathbb{N}} \times X : [x]_E = [x]_{E_\infty^\alpha}\}$$

is Borel, it has the Baire property, and it suffices to show that $\forall^*\alpha \in \mathbb{N}^{\mathbb{N}} \left([x]_E = [x]_{E_\infty^\alpha}\right)$ for all $x \in X$.

The intersection of countably many comeager sets is comeager, and $E_\infty^\alpha \subseteq E$, therefore, it is enough to show that for any fixed $y \in [x]_E$

$$\forall^*\alpha \in \mathbb{N}^{\mathbb{N}} (y \in [x]_{E_\infty^\alpha}).$$

The set $A = \{\alpha \in \mathbb{N}^{\mathbb{N}} : y \in [x]_{E_\infty^\alpha}\}$ is open as $[x]_{E_\infty^\alpha} = \bigcup_{n\in\mathbb{N}} [x]_{E_n^\alpha}$ and $[x]_{E_n^\alpha}$ only depends on the first $n + 1$ entries of α, so it is enough to show that it is dense. Fix a basic neighborhood $\mathcal{N}_s = \{\alpha \in \mathbb{N}^{\mathbb{N}} : \alpha|_{n+1} = s\}$ for some $s \in \mathbb{N}^{n+1}$. Consider $S_0, f_0, S_1, f_1, \dots, S_n, f_n, S_{n+1}$ associated to s. Let $x' = f_n f_{n-1} \dots f_0(x)$ and $y' = f_n f_{n-1} \dots f_0(y)$, then there exists a k with $g_k(x') = y'$ as $x'ExEyEy'$. Choose α_0 with $\alpha_0|_{n+1} = s$, $\alpha_0(n + 1) = k$, then $\alpha_0 \in A \cap \mathcal{N}_s$, because $(x, y) \in E_{n+1}^{\alpha_0}$. $\square$

2.4 An example for non-hyperfiniteness

In this section we give a non-standard argument for proving non-hyperfiniteness of combinatorial flavor.

Definition 2.4.1. Let $\mathcal{H}$ be a Borel graph equipped with a finite edge labeling $d : \mathcal{H} \to k$. Define the *edge labeled Borel chromatic number* of $\mathcal{H}$, $\chi_{elB}(\mathcal{H})$ to be the minimal n such that there is a Borel map $c : V(\mathcal{H}) \to n$ such that for all $i < n$ we have $|d(\mathcal{H} \cap c^{-1}(i) \times c^{-1}(i))| < k$.

Equivalently, $\chi_{elB}(\mathcal{H}) > n$ if and only if for any Borel map $c : V(\mathcal{H}) \to n$ there exists a color class spanning edges with all k labels.

For a Borel measure μ on X analogously define the *edge labeled μ measurable chromatic number* $\chi_{el\mu}$ and the *edge labeled Baire measurable chromatic number* χ_{elBM}.

Notation 2.4.2. Let G be a Borel graph on a Polish space X. Let E_G denote the Borel equivalence relation on X which is defined by

$$xE_Gy \Leftrightarrow \big(\exists \text{ a path } P \text{ in } G \text{ between } x \text{ and } y\big).$$

Theorem 2.4.3. *Let $\mathcal{H}$ be an acyclic Borel graph with a Borel edge 3-labeling d such that for every sequence of Borel functions $f_n : V(\mathcal{H}) \to [0,1]^3$ there exists a Borel set $B \subseteq V(\mathcal{H})$ satisfying the following properties:*

(1) $\chi_{elB}(\mathcal{H}|_B) > 3$

(2) there exists a sequence $g_n \in \mathrm{conv}(f_n, f_{n+1}, \ldots)$ such that $(g_n)_{n\in\mathbb{N}}$ converges pointwise on B. Then $E_{\mathcal{H}}$ is not hyperfinite.

Proof. For $x \in V(\mathcal{H})$ and $i < 3$ define

$$D_{i,x} = \{y \in [x]_{E_{\mathcal{H}}} : \text{ the (unique) injective walk from } x \text{ to } y \text{ in } \mathcal{H} \\ \text{starts with an edge labeled with } i\}.$$

Towards a contradiction, suppose that $E_{\mathcal{H}}$ is hyperfinite, and let the sequence $(F_n)_{n\in\mathbb{N}}$ be a witness for this fact. We can assume that $E_{\mathcal{H}}$ is aperiodic (i.e. all of its equivalence classes are infinite), as we can guarantee this by discarding a smooth set. For $n \in \mathbb{N}$ and $i < 3$ let

$$f_n(x)(i) = \frac{|D_{i,x} \cap [x]_{F_n}|}{|[x]_{F_n}|}.$$

Acyclicity and aperiodicity implies $\|f_n(x)\|_1 < 1$ and $\lim_{n\to\infty} \|f_n(x)\|_1 = 1$. Our assumption on $(f_n)_{n\in\mathbb{N}}$ provides a Borel set B and a sequence $(g_n)_{n\in\mathbb{N}}$. For $x \in B$, let $g(x) = \lim_{n\to\infty} g_n(x)$.

Define a Borel map $c : B \to 3$ by

$$c(x) = \min\{i \in 3 : g(x)(i) \text{ is minimal}\},$$

as $g_n \in \mathrm{conv}(f_n, f_{n+1}, \ldots)$, we have $\|g_n(x)\|_1 < 1$, and thus $g(x)(c(x)) \leq \frac{1}{3}$.

As $\chi_{elB}(\mathcal{H}) > 3$, there are $(x, x') \in (B \cap \mathcal{H})^2$ and $i < 3$ with $c(x) = c(x') = d(x, x') = i$. Note that $D_{i,x'} \supset \bigcup_{i' \neq i} D_{i',x}$. Fix some $0 < \varepsilon < \frac{1}{3}$, and choose N such that for all $n \geq N$ we have

- $\|f_n(x)\|_1 > \frac{2}{3} + \varepsilon$,
- $f_n(x')(i) \geq \sum_{i' \neq i} f_n(x)(i')$ for any n with xF_nx', and
- $|g(x)(i) - g_n(x)(i)| < \frac{\varepsilon}{2}$ and $|g(x')(i) - g_n(x')(i)| < \frac{\varepsilon}{2}$.

Then still for $n \geq N$

$$\frac{2}{3} + \varepsilon < \|f_n(x)\|_1 = \sum_{j<3} f_n(x)(i) \leq f_n(x)(i) + f_n(x')(i),$$

and by $g_n \in \operatorname{conv}(f_n, f_{n+1}, \ldots)$, $g(x)(i) \le \frac{1}{3}$ and $g(x')(i) \le \frac{1}{3}$, we have a contradiction with

$$\frac{2}{3} + \varepsilon < g_n(x)(i) + g_n(x')(i) \le \frac{2}{3} + \varepsilon.$$

□

This result yields some interesting corollaries, the first one we present is about a way to prove non-hyperfiniteness with the help of measures, and uses the following statement from functional analysis:

Lemma 2.4.4 (Mazur's lemma). *Let $(X, \|.\|)$ be a normed vector space and suppose that the sequence $(u_n)_{n\in\mathbb{N}}$ converges weakly to $u \in X$, then there exists $v_n \in \operatorname{conv}(u_n, u_{n+1}, \ldots)$ such that $\|v_n - u\| \to 0$.*

Corollary 2.4.5. *Let $\mathcal{H}$ be an acyclic Borel graph with a Borel 3-edge labeling and let μ be a Borel probability measure on $V(\mathcal{H})$ with $\chi_{el\mu}(\mathcal{H}) > 3$. Then $E_{\mathcal{H}}$ is not hyperfinite.*

Proof. We will use Theorem 2.4.3., to check (2), suppose that $f_n : V(\mathcal{H}) \to [0,1]^3$ are Borel functions. As μ is a probability measure, these are in the unit sphere of $L^2(V(\mathcal{H}), \mu)$, which is weakly compact, so there is a weakly convergent subsequence $(f_{n_k})_{k\in\mathbb{N}}$, hence by Lemma 2.4.4., there exists $g_n \in \operatorname{conv}(f_n, f_{n+1}, \ldots)$, with $\|g_n - f\|_2 \to 0$ for some $f \in L^2(V(\mathcal{H}), \mu)$. We can assume that $\|g_n - f\|_2 \le 2^{-n}$, then by the Borel-Cantelli lemma (g_n) converges pointwise on a Borel set B with $\mu(B) = 1$.

To verify (1) it is enough to show that $\chi_{el\mu}(\mathcal{H}|_B) > 3$. Towards a contradiction, suppose the contrary, then there exists a μ-measurable coloring $c : B \to 3$, with no colors spanning edges with all 3 labels. As $\mathcal{H}$ is acyclic, there is some coloring $c' : V(\mathcal{H}) \to 3$ with no monochromatic edges outside of B such that $c'|_B = c$. As $\mu(V(\mathcal{H}) \setminus B) = 0$, c' is μ-measurable as well, but this would imply $\chi_{el\mu}(\mathcal{H}) \le 3$. □

Example 2.4.6. Using the theory of local-global convergence of Lovász and Szegedy, the following example is constructed as a limit of edge labelled random finite regular graphs:

Theorem 2.4.7 (Grebík-Vidnyánszky, [GV22]). *Let $k \ge 1$ and $n \ge 3$. There exist disjoint Borel graphs $(\mathcal{G}_j)_{j<k}$ on a probability measure space (Y, μ) such that*

(1) $\bigcup_{j<k} \mathcal{G}_j$ is acyclic and has bounded degree, and

(2) for every integer $j < k$ and measurable sets $B, B' \subseteq Y$ with $\mu(B), \mu(B') \ge \frac{1}{n}$ there exists $z \in B$ and $z' \in B'$ that are adjacent in $\mathcal{G}_j$.

This contains everything we need: let $k = 3$ and $n = 3$, and consider $\mathcal{H} = \bigcup_{j<3} \mathcal{G}_j$, this is acyclic by (1). Consider the edge 3-labeling of $\mathcal{H}$ given by the partition to the edges of $(\mathcal{G}_j)_{j<3}$. To see that indeed $\chi_{el\mu}(\mathcal{H}) > 3$, take an arbitrary μ-measurable 3-coloring $c : Y \to 3$. Then for at least one of the color classes $\mu(c^{-1}(i)) \ge \frac{1}{3}$. Applying (2) for $B = B' = c^{-1}(i)$ proves that $c^{-1}(i)$ spans all three color classes, as required.

Therefore, the associated equivalence relation $E_{\mathcal{H}}$ is not hyperfinite.

Definition 2.4.8. A *mean* on a set X is a finitely additive probability measure which is invariant under finite modifications.

Finitely additive probability measures are naturally connected to positive linear functionals, if $\mathbf{m}$ is a left-invariant positive linear functional on a countable group $\mathbb{G}$, then $\varphi(X) = \mathbf{m}(1_X)$ (where 1_X is the characteristic function of X) is a left-invariant finitely additive probability measure, and conversely, if φ is a left-invariant mean on $\mathbb{G}$, then $\mathbf{m}(f) = \int f \, \mathrm{d}\varphi$ is a left-invariant positive linear functional. These notions are involved in one of the many definitions of *amenable groups*, a widely studied type of groups, which is also in connection with amenable equivalence relations (although not as directly as one would expect).

One can obtain a mean on $\mathcal{P}(\mathbb{N})$ using the axiom of choice, but with the constraint of definability in the category sense, this turns out to be impossible.

Corollary 2.4.9. *There is no Baire measurable mean on* $\mathcal{P}(\mathbb{N})$.

Proof sketch. Let $\mathcal{H}$ be the graph $\mathbb{G}_0$ and define the following labeling on its edges: label an edge (xy) with i if for the only n with $x(n) \neq y(n)$ we have

$$|\{k < n : s_k = s_n|_k\}| \equiv i \mod 3.$$

Now if we take an arbitrary $\aleph_0$-coloring of $2^{\mathbb{N}}$, one of the color classes will be non-meager, this implies that it spans edges with all labels, similarly as in the proof of Proposition 2.2.3. Thus $\chi_{elBM}(\mathbb{G}_0) > \aleph_0 > 3$. Similarly as in the previous proof, this implies that for any non-meager Borel subset B we have $\chi_{elBM}(\mathbb{G}_0|_B) > 3$, which implies $\chi_{elB}(\mathbb{G}_0|_B) > 3$.

One can check that the existence of a Baire measurable mean would imply the existence of a sequence of functions convergent on a non-meager Borel set, contradicting the fact that $E_{\mathbb{G}_0} = \mathbb{E}_0$ is hyperfinite. ☐

These observations suggest the following definition.

Definition 2.4.10. Let $\mathcal{I}$ be an ideal of Borel sets on some Polish space X. Say that $\mathcal{I}$ has the *average convergence property* if for any sequence $f_n : X \to [0,1]$ Borel functions there exists a Borel set $B \notin \mathcal{I}$ and a sequence $g_n \in conv(f_n, \dots)$ that is point-wise convergent on B.

We have seen that the measure ideal has this property, while the category ideal does not. Zapletal (personal communication) pointed out that the ideal of Ramsey-null sets (see Section 3.) also fails to have this property.

Question 2.4.11. Give examples of ideals with the average convergence property.

2.5 Amenability

In this section, which is mainly based on [Mar17], we discuss a more standard way of proving non-hyperfiniteness.

Definition 2.5.1. A Borel equivalence relation E on a Polish space X is *amenable*, if there exists an $\lambda^n : E \to [0,1]$ for $n \in \mathbb{N}$ such that $\lambda^n_x = \lambda^n(x, .)$ for all x satisfies the following conditions:

(1) $\|\lambda^n_x\|_1 = 1$ for every $x \in X$

(2) $(x,y) \in E \Rightarrow \lim_{n\to\infty} \|\lambda^n_x - \lambda^n_y\|_1 = 0$

This definition resembles the following equivalent formulation of amenability of groups: A countable group Γ is amenable if and only if there exists a sequence $f_n : \Gamma \to [0,1]$ such that $\|f_n\|_1 = 1$ for all n and $\|g \cdot f_n - f_n\|_1 \to 0$ for every $g \in \Gamma$, where $(g \cdot f_n)(h) = f_n(gh)$. Note that (one of the) original definition(s) of amenability requires the existence of a left-invariant finitely additive probability measure on Γ.

Example 2.5.2. Let E be a smooth CBER, then it has a Borel transversal T by Proposition 2.1.7. Define

$$\lambda^n(x,y) = \begin{cases} 0 & y \notin T \\ 1 & y \in T. \end{cases}$$

As we define λ^n_x on $[x]_E$, it is indeed a probability measure, i.e., has norm 1, and also λ^n does not depend on n, thus (2) is satisfied as well.

We can strengthen this in the following way:

Proposition 2.5.3. *Suppose E is a hyperfinite Borel equivalence relation. Then E is amenable.*

Proof. Let $(F_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite Borel equivalence relations with $E = \bigcup_{n\in\mathbb{N}} F_n$. For $(x,y) \in E$ define

$$\lambda^n(x,y) = \begin{cases} 0 & y \notin [x]_{F_n} \\ \frac{1}{|[x]_{F_n}|} & y \in [x]_{F_n}. \end{cases}$$

Then clearly $\|\lambda^n_x\|_1 = 1$ for every $x \in X$, and xF_ny implies $\lambda^n_x = \lambda^n_y$, hence if xEy and n is large enough then we have $\|\lambda^n_x - \lambda^n_y\|_1 = 0$. $\square$

Whether the converse is true is an open problem. However, from a measure theoretic point of view, we will see that amenability and hyperfiniteness are the same.

Definition 2.5.4. Suppose that μ is a Borel probability measure on X. A Borel equivalence relation E is *μ-amenable* if there exists a Borel set B with $\mu(B) = 1$ such that $E|_B$ is amenable.

Definition 2.5.5. Suppose that μ is a Borel probability measure on X. A Borel equivalence relation E is *μ-hyperfinite* if there exists a Borel set B with $\mu(B) = 1$ such that $E|_B$ is hyperfinite.

Definition 2.5.6. A Borel equivalence relation E on X is *measure-hyperfinite*, if it is μ-hyperfinite for all Borel probability measure μ on X.

Theorem 2.5.7 (Connes-Feldman-Weiss, [CFW81])**.** *Let μ be a Borel probability measure and E a Borel equivalence relation on X. Then E is μ-amenable if and only if it is μ-hyperfinite.*

We will give an outline of the proof, showing the key concepts and claims, and giving a proof for some of the parts.

Proposition 2.5.8. *Let E be the orbit equivalence relation of a countable group Γ acting on a Polish space X, and let μ be a Borel probability measure on X. Then E is μ-hyperfinite if and only if for every finite subset $S \subset \Gamma$ and every $\varepsilon > 0$ exists a finite Borel equivalence relation $F \subseteq E$ such that $\mu(\{x \in X : \forall s \in S\ (sx)Fx\}) > 1 - \varepsilon$.*

Proof. Suppose first that E is μ-hyperfinite, take a finite subset $S \subset \Gamma$ and let $\varepsilon > 0$. Choose B with $\mu(B) = 1$ and finite equivalence relations $F_0 \subseteq F_1 \subseteq \dots$ with $E|_B = \bigcup_{n\in\mathbb{N}} F_n$. Define

$$A_n = \{x \in B : \forall s \in S \ (sx) F_n x\},$$

then $A_0 \subseteq A_1 \subseteq \dots$ are Borel sets with $\bigcup_{n\in\mathbb{N}} A_n = B$, hence there exists $n \in \mathbb{N}$ such that $\mu(A_n) > 1 - \varepsilon$, which means that $F_n \subseteq E$ is a suitable finite equivalence relation.

For the opposite direction, enumerate the elements of Γ as $\{\gamma_0, \gamma_1 \dots\}$. Take $F_n \subseteq E$ finite equivalence relations for $S_n = \{\gamma_0, \gamma_0^{-1}, \dots, \gamma_n, \gamma_n^{-1}\}$ and $\varepsilon_n = 1 - \frac{1}{2^n}$, and define $F'_n = \bigcap_{k>n} F_n$. Observe that Borel sets

$$B_n = \bigcap_{k>n} \{x \in X : \forall s \in S_n \ (sx) F_n x\}$$

form an increasing sequence with $\mu(B_n) \geq 1 - \frac{1}{2^n}$ and $E|_{B_n} \subseteq F'_n$, consequently, $B = \bigcup_{n\in\mathbb{N}} B'_n$ is a Borel set such that $\mu(B) = 1$ and $E|_B$ is hyperfinite. □

Proposition 2.5.9. *Suppose $E_0 \subseteq E_1 \subseteq \dots$ are μ-hyperfinite CBERs, and $E = \bigcup_{n\in\mathbb{N}} E_n$. Then E is μ-hyperfinite as well.*

Proof. By Theorem 2.1.10, there are countable groups Γ_n generating E_n. Take the free product

$$\Gamma = \mathop{\ast}_{n\in\mathbb{N}} \Gamma_n,$$

clearly the orbit equivalence relation of Γ is E. Suppose that $S \subset \Gamma$ is finite, then there is some n with $S \subset \ast_{i<n} \Gamma_i$. Observe that the orbit equivalence relation of $\ast_{i<n} \Gamma_i$ is exactly E_n, so by Proposition 2.5.8. there exists a finite Borel equivalence relation $F \subseteq E_n$ such that $\mu(\{x \in X : \forall s \in S \ (sx) F x\}) > 1 - \varepsilon$. As $F \subseteq E$, we can apply Proposition 2.5.8. once again to complete the proof. □

Remark. This is a quite useful property of μ-hyperfiniteness, interestingly it is still an open problem whether the analogue holds for regular hyperfiniteness.

Definition 2.5.10. A Borel graph G is *hyperfinite*/*μ-hyperfinite* if E_G is hyperfinite/μ-hyperfinite, respectively.

Example 2.5.11. The graph $\mathbb{G}_0$ is hyperfinite, as $E_{\mathbb{G}_0} = \mathbb{E}_0$.

Definition 2.5.12. Suppose that G is a graph on a set X, and $A \subseteq X$. Define the *boundary of A* as $\partial_G(A) = \{x \in X : \exists (y, z) \in (A \times (X \setminus A)) \cap G$ such that $x \in \{y, z\}\}$.

Definition 2.5.13. Let X be a Polish space with a Borel measure μ, and let G be a locally finite Borel graph on X. Define the *isoperimetric constant* of G as

$$\mathrm{ic}(G) = \inf_{\substack{A \subseteq X,\, \mu(A)>0, \\ G|_A \text{ has finite} \\ \text{connected components}}} \frac{\mu(\partial_G(A))}{\mu(A)}.$$

Example 2.5.14. For $d \geq 2$ the isoperimetric constant of the d-regular tree T_d with μ being the counting measure is $d - 2$.

Remark. One can define the isoperimetric constant of a finitely generated countable group Γ by taking its Cayley-graph G and taking

$$\mathrm{ic}(\Gamma) = \inf_{A \subset \Gamma \text{finite}} \frac{|\partial_G(A)|}{|A|},$$

with this definition, $\mathrm{ic}(\Gamma)$ is 0 if and only if Γ is amenable [Ada93].

The following propositions show that μ-amenability and μ-hyperfiniteness are connected with the fact whether the isoperimetric constant of some Borel graph is 0.

Proposition 2.5.15. *Suppose that E is a μ-amenable equivalence relation, let $\mu(B) > 0$, and suppose that $G \subseteq E$ is a bounded degree Borel graph on B. Then $\mathrm{ic}(G) = 0$.*

Theorem 2.5.16 (Kaimanovich [Kai97], Elek [Ele12])**.** *Suppose G is a locally finite Borel-graph on a Polish space X equipped with a Borel measure μ. Then E_G is μ-hyperfinite if and only if for each Borel set $B \subseteq X$ with $\mu(B) > 0$ we have $\mathrm{ic}(G|_B) = 0$.*

Remark. This statement gives yet another way to contruct non-hyperfinite equivalence relations: suppose that G is a locally finite Borel-graph on a Polish space X equipped with a Borel measure μ, and $B \subseteq X$ is a Borel set with $\mu(B) > 0$ and $\mathrm{ic}(G|_B) > 0$. Then E_G cannot be hyperfinite, since this would imply that it is μ-hyperfinite as well.

Proof of Theorem 2.5.7. As hyperfiniteness implies amenability, it is clear that μ-hyperfiniteness implies μ-amenability. Now suppose that E is a μ-amenable CBER, and let Γ be a countable group generating E by Theorem 2.1.10. Take an increasing sequence of finite symmetric subsets $S_0 \subseteq S_1 \subseteq \ldots \subseteq \Gamma$ such that $\bigcup_{n \in \mathbb{N}} S_n = \Gamma$. Define

$$G(S_n) = \{(x, y) \in E : \exists s \in S_n \; sx = y\}.$$

Each $G(S_n)$ is a bounded degree subset of E, hence for any Borel set with $\mu(B) > 0$ we have that $\mathrm{ic}(G(S_n)|_B) = 0$ by Proposition 2.5.15. Applying Proposition 2.5.16. yields that for all n the equivalence relation $E_{G(S_n)}$ is μ-hyperfinite, consequently $E = \bigcup_{n \in \mathbb{N}} E_{G(S_n)}$ is hyperfinite by Proposition 2.5.9. □

Example 2.5.17. Consider the equivalence relation $\mathbf{Fr}(F_2, 2)$, that is, the free part of the shift action of F_2 on 2^{F_2}. (The shift action is defined as follows: for $x \in 2^{F_2}$ and $w \in F_2$ let $(w \cdot x)(u) = x(w^{-1}u)$ for all $u \in F_2$.) Observe that the 4-regular acyclic graph which naturally generates $\mathbf{Fr}(F_2, 2)$ has isoperimetric constant 2 with respect to the standard measure on $2^{F_2} \cong 2^{\mathbb{N}}$. Therefore, $\mathbf{Fr}(F_2, 2)$ is not hyperfinite.

2.6 Open problems related to hyperfiniteness

We list here a couple of the most important problems connected to hyperfiniteness, most of which are from [JKL02].

As we seen in Proposition 2.5.9. the increasing union of μ-hyperfinite equivalence relations remain μ-hyperfinite.

Problem 2.6.1. Let E be a countable Borel equivalence relation such that there are $E_0 \subseteq E_1 \subseteq \ldots$ hyperfinite equivalence relations with $E = \bigcup_{n \in \mathbb{N}} E_n$. Is it true that E is necessarily hyperfinite?

Problem 2.6.2. What is the complexity of the set $\{E : E \text{ is hyperfinite}\}$? Is it $\mathbf{\Sigma}^1_2$-complete?

Adams and Kechris established some of the earliest complexity results in the context of CBERs [AK00].

Such complexity results can yield very interesting corollaries. For example, the projective complexity of Borel graphs with finite Borel chromatic number can be used to prove that there is no similar dichotomy between Borel graphs with finite and with infinite chromatic number as the $\mathbb{G}_0$ dichotomy [TV21].

An affirmative answer would yield a negative one to the next problem.

Problem 2.6.3. Let E be measure-hyperfinite, i.e. μ-hyperfinite for all μ Borel probability measures. Does this imply the hyperfiniteness of E?

Note that the Continuum Hypothesis implies that measure-amenability is the same as regular amenability.

Problem 2.6.4. Suppose that E is the orbit equivalence relation of an amenable group Γ. Does this imply that E is hyperfinite?

In the past couple of years tremendous progress has been made towards proving that certain classes of amenable groups generate hyperfinite equivalence relations, e.g., Gao and Jackson for abelian groups [GJ15], Conley, Jackson, Seward, Marks and Tucker-Drob for policylcic groups [CJM+20].

3 Ramsey theory

Infinite dimensional Ramsey theory concerns itself with extending pigeonhole-principles of finite dimensional objects to their infinite dimensional counterparts, and thus finding monochromatic subsets for colorings being sufficiently nice with respect to a certain associated topology. There exists a general framework for problems of this type, see [Tod10], but first we present an important example of this phenomenon.

3.1 The Galvin-Prikry theorem

The infinite version of Ramsey's theorem states that any finite coloring of the edges of the complete graph on $\mathbb{N}$ admits an infinite monochromatic subgraph. In a more general version, we can also consider colorings of k-tuples instead, this can be obtained from the previous statement by induction.

Notation 3.1.1. For a set A and a cardinal κ, the notation $[A]^\kappa$ is used for the set of subsets of A with cardinality κ.

Theorem 3.1.2 (Ramsey)**.** *Suppose that $k, n \in \mathbb{N}$, for any coloring $c : [\mathbb{N}]^k \to n$ there exists a set $A \in [\mathbb{N}]^{\mathbb{N}}$ such that $c|_{[A]^k}$ is constant.*

The question arises whether this can be extended even further to colorings of infinite subsets. For the straightforward generalization the answer is negative, using the Axiom of Choice, a counterexample can be constructed: enumerate the infinite subsets of $\mathbb{N}$ transfinitely as $\{H_\xi : \xi < 2^{\mathbb{N}}\}$, and in each step, color two different uncolored subsets of H_ξ to two different colors.

We identify the set $[\mathbb{N}]^{\mathbb{N}}$ with a G_δ hence Polish subset of $2^{\mathbb{N}}$ by considering characteristic functions. We also equip $[\mathbb{N}]^{\mathbb{N}}$ with the measure inherited from the standard probality measure of the Cantor space $2^{\mathbb{N}}$

The following example shows that Lebesgue and Baire measurable counterexamples exist as well.

Example 3.1.3. Take a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and define $c : [\mathbb{N}]^{\mathbb{N}} \to \{-1, 1\}$ by

$$c(M) = \lim_{n \to \mathcal{U}} (-1)^{|M \cap n|}.$$

This coloring is not constant on any set of the form $[M]^\infty$, as $c(M) \neq c(M \setminus \min(M))$. Of course this coloring is not measurable yet, so define $b : [\mathbb{N}]^{\mathbb{N}} \to \{-1, 1\}$ by

$$b(M) = \min\{c(M), \min_{m,n \in M} (-1)^{m-n}\}.$$

Observe that the set $U = \{M \in [\mathbb{N}]^{\mathbb{N}} : \min_{m,n \in M} (-1)^{m-n} = -1\}$ is a dense open subset of full measure, so b is indeed Lebesgue and Baire measurable. Also for every $N \in [\mathbb{N}]^{\mathbb{N}}$, there is $M \in [N]^{\mathbb{N}}$ with $M \cap U = \emptyset$, hence $b|_{[M]^{\mathbb{N}}} = c|_{[M]^{\mathbb{N}}}$, so b is not constant on $N \in [N]^{\mathbb{N}}$.

Remark. With an appropriate modification to this construction, a universally measurable counterexample is also obtainable.

However, if we require the coloring to be Borel, then the generalization holds.

Theorem 3.1.4 (Galvin-Prikry)**.** *Let $n \in \mathbb{N}$. For any Borel coloring $c : [\mathbb{N}]^{\mathbb{N}} \to n$ there exists a set $A \in [\mathbb{N}]^{\mathbb{N}}$ such that $c|_{[A]^{\mathbb{N}}}$ is constant.*

Notation 3.1.5. For $a \in [\mathbb{N}]^{<\mathbb{N}}$ and $A \in [\mathbb{N}]^{\mathbb{N}}$ with $\max(a) < \min(A)$ let

$$[a, A] = \{S \in [\mathbb{N}]^{\mathbb{N}} : a \subset S \subseteq a \cup A\}.$$

Definition 3.1.6. A set $S \subseteq [\mathbb{N}]^{\mathbb{N}}$ is *Ramsey measurable* (or *completely Ramsey*) if for any $a \in [\mathbb{N}]^{<\mathbb{N}}$ and $A \in [\mathbb{N}]^{\mathbb{N}}$ with $\max(a) < \min(A)$ there exists $B \in [A]^{\mathbb{N}}$ such that $[a, B] \subseteq S$ or $[a, B] \subseteq [\mathbb{N}]^{\mathbb{N}} \setminus S$.

Note that for a Borel 2-coloring, Ramsey measurability of a color class would yield the desired statement with $a = \emptyset$ and $A = \mathbb{N}$, as $[\emptyset, B] = [B]^{\mathbb{N}}$. That is exactly the plan we will follow along.

Lemma 3.1.7. *Open sets are Ramsey-measurable. Furthermore, if $U \subseteq [\mathbb{N}]^{\mathbb{N}}$ is relatively open in some $[a, A]$, then there exists $B \in [A]^{\mathbb{N}}$ such that $[a, B] \subseteq U$ or $[a, B] \subseteq [\mathbb{N}]^{\mathbb{N}} \setminus U$.*

Proof. Fix $U \subseteq [\mathbb{N}]^{\mathbb{N}}$. We call a set $[a, A]$ *good* if there exists $B \in [A]^{\mathbb{N}}$ such that $[a, B] \subseteq U$, call it *bad* otherwise, and call it *very bad* if for every $n \in A$ the set $[a \cup \{n\}, A/n]$ is bad, where we use the notation $A/n = \{m \in A : m > n\}$.

Claim 3.1.8. *If $[a, A]$ is bad, then there exists $B \in [A]^{\mathbb{N}}$ such that $[a, B]$ is very bad.*

Proof. Otherwise, as $[a, A]$ is not very bad, there would be $n_0 \in A$ such that $[a \cup \{n_0\}, A/n_0]$ is good. Hence $[a \cup \{n_0\}, B_0] \subseteq U$ for some $B_0 \in [A/n_0]^{\mathbb{N}}$. Similarly, as $[a, B_0]$ is not very bad, there exists $n_1 \in B_0$ such that $[a \cup \{n_1\}, B_0/n_1]$ is good, so we can find $B_1 \in [B_0/n_1]^{\mathbb{N}}$ with $[a \cup \{n_1\}, B_1] \subseteq U$. Iterating this process, and defining $B = \{n_0, n_1, \ldots\}$, we would have a contradiction by $[a, B] \subseteq U$. ☐

Now suppose that U is relatively open in some set $[a, A]$. If $[a, A]$ is good, then we are done, otherwise we will find $B \in [A]^{\mathbb{N}}$ such that $[a, B] \subseteq [\mathbb{N}]^{\mathbb{N}} \setminus U$. By using Claim 3.1.8. iteratively, we can find a decreasing sequence $A \supseteq B_0 \supseteq B_1 \supseteq \ldots$ such that $n_i = \min(B_i)$ is strictly increasing, and for all $b \subseteq \{n_0, \ldots, n_{i-1}\}$ the set $[a \cup b, B_i]$ is very bad. Let $B = \{n_0, n_1, \ldots\}$. Towards a contradiction, suppose that $[a, B] \cap U \neq \emptyset$. As U is relatively open in $[a, A]$, there exists $a' \supseteq a$ with $[a', B/(\max a')] \subseteq U$, but then $a' = a \cup b \cup \{n_i\}$ for some $b \subseteq \{n_0, \ldots, n_{i-1}\}$, contradicting that $[a \cup b, B_i]$ is very bad. ☐

Lemma 3.1.9. *Ramsey measurable sets form a σ-algebra.*

Proof. Taking complements does not change Ramsey measurability, as the definition is symmetric for S and $[\mathbb{N}]^{\mathbb{N}} \setminus S$.

Now suppose that $(S_i)_{i \in \mathbb{N}}$ are Ramsey-measurable, take any $a \in [\mathbb{N}]^{<\mathbb{N}}$ and $A \in [\mathbb{N}]^{\mathbb{N}}$ with $\max(a) < \min(A)$. By using the Ramsey measurability of S_i iteratively 2^i times we can find a decreasing sequence $A \supseteq B_0 \supseteq B_1 \supseteq \ldots$ such that $n_i = \min(B_i)$ is strictly increasing, and for all $b \subseteq \{n_0, \ldots, n_{i-1}\}$ we have $[a \cup b, B_i] \subseteq S_i$ or $[a \cup b] \subseteq [\mathbb{N}]^{\mathbb{N}} \setminus S_i$. Set $B = \{n_0, n_1, \ldots\}$. Notice that whether some $H \in [a, B]$ is in S_i depends only on $H \cap \big(a \cup \{n_0, \ldots, n_{i-1}\}\big)$, so each S_i is relatively open in $[a, B]$, consequently $\bigcup_{i \in \mathbb{N}} S_i$ is relatively open in $[a, B]$ as well. Then by Lemma 3.1.7. there exists $B' \in [B]^{\mathbb{N}}$ such that $[a, B'] \subseteq \bigcup_{i \in \mathbb{N}} S_i$ or $[a, B'] \subseteq [\mathbb{N}]^{\mathbb{N}} \setminus \bigcup_{i \in \mathbb{N}} S_i$, thus $\bigcup_{i \in \mathbb{N}} S_i$ is Ramsey measurable. ☐

These two lemmas together imply the following:

Corollary 3.1.10. *Borel sets are Ramsey-measurable.*

Now we are ready to prove the Galvin-Prikry theorem.

Proof of Theorem 3.1.4. We use induction on n. For $n = 2$, applying Ramsey measurability with $a = \emptyset$ and $A = \mathbb{N}$ yields a suitable infinite subset. For $n > 2$, by merging two colors we get a Borel $(n-1)$-coloring c'. By the inductive hypothesis, there exists $A \in [\mathbb{N}]^{\mathbb{N}}$ where c' is constant, if this constant color happens to be the merged one, use the case $n = 2$ on $[A]^{\mathbb{N}}$. $\square$

Remark. The Galvin-Prikry theorem can also be proved by showing that the Ramsey measurable subsets of $\mathbb{N}$ are exactly those with the Baire property with respect to the so-called *Ellentuck topology*, which is the topology generated by basic open sets of the form

$$\{[a, A] : a \in [\mathbb{N}]^{<\mathbb{N}}, A \in [\mathbb{N}]^{\mathbb{N}}, \max(a) < \min(A)\}.$$

This can be shown with a similar level of difficulty by checking first that Ellentuck-open sets are Ramsey measurable (which is a stronger claim then Lemma 3.1.7., since the Ellentuck topology is finer than the usual topology of $[\mathbb{N}]^{\mathbb{N}}$), then checking that an Ellentuck meager set does not spoil this property.

Now we present a short graph-theoretic application of the Galvin-Prikry theorem, which will turn up again in Section 4.1.

Definition 3.1.11. Identify the elements of $[\mathbb{N}]^{\mathbb{N}}$ with their increasing enumeration, let S be the *shift map* defined by

$$S(\{n_0, n_1, n_2, \ldots\}) = \{n_1, n_2, n_3 \ldots\},$$

Denote by G_S the graph generated by the shift map, that is, $G_S = \{(x, y) \in \left([\mathbb{N}]^{\mathbb{N}}\right)^2 : S(x) = y \vee S(y) = x\}$. Note that $E_{G_S} = \mathbb{E}_0|_{[\mathbb{N}]^{\mathbb{N}}}$.

Proposition 3.1.12. $\chi_B(G_S) = \aleph_0$.

Proof. Observe that $c(x) = \min(x)$ is a countable Borel coloring of G_S.

To see $\chi_B(G_S) \geq \aleph_0$, suppose that we have a finite Borel coloring $c : [\mathbb{N}]^{\mathbb{N}} \to n$. Then by the Galvin-Prikry theorem, there exists $x \in [\mathbb{N}]^{\mathbb{N}}$ such that $c|_{[x]^{\mathbb{N}}}$ is constant, hence $c(x) = c(S(x))$. $\square$

3.2 The general framework

As we mentioned before, there is a general formalization of the type of problems which can be handled from the point of view of Ramsey colorings. One can consider a set of abstract axioms, which guarantee the possibility of finding a monochromatic subset of certain type for any sufficiently nice-behaving coloring.

Here we will only write about a somewhat more simple special case of the most general formalization, based on Chapter 5. of [Tod10]. This special case can be formulated as follows:

Consider a triple $(\mathcal{R}, \leq, r)$, where $\mathcal{R}$ is a nonempty set, $\leq$ is a quasi-ordering on $\mathcal{R}$, and $r : \mathcal{R} \times \mathbb{N} \to \mathcal{AR}$ is a mapping. The range of r can be thought of as the collection of finite approximations to elements of $\mathcal{R}$. We will use capitals $A, B, \ldots$ for the elements of $\mathcal{R}$, and $a, b, \ldots$ for their approximations. We denote $r_n(.) = r(., n)$. The first axiom is about r being an approximation mapping.

Axiom 1. (1) $r_0(A) = \emptyset$ for all $A \in \mathcal{R}$.

(2) $A \neq B$ implies $r_n(A) \neq r_n(B)$ for some n.

(3) $r_n(A) = r_m(B)$ implies $n = m$ and $r_k(A) = r_k(B)$ for $k < n$.

We write $a \sqsubseteq b$ when there exists $m \leq n$ and $A \in \mathcal{R}$ such that $a = r_m(A)$ and $b = r_n(B)$. The next axiom is about $\leq$ being compatible with the finite approximations in some sense.

Axiom 2. There is a quasi-ordering $\leq_{\text{fin}}$ on $\mathcal{AR}$ such that

(1) $\{a \in \mathcal{AR} : a \leq_{\text{fin}} b\}$ is finite for all $b \in \mathcal{AR}$,

(2) $A \leq B \Leftrightarrow \forall n \exists m\ r_n(A) \leq_{\text{fin}} r_m(B)$,

(3) $\forall a, b \in \mathcal{AR}\big((a \sqsubseteq b) \wedge (b \leq_{\text{fin}} c) \Rightarrow (\exists d \sqsubseteq c : a \leq_{\text{fin}} d)\big)$.

For the third axiom, we need the following notations: for $a \in \mathcal{AR}$ and $B \in \mathcal{R}$ let

$$\text{depth}_B(a) = \min\{n : a \leq_{\text{fin}} r_n(B)\}.$$

Also for $a \in \mathcal{AR}$ and $B \in \mathcal{R}$ define

$$[a, B] = \{A \in \mathcal{R} : A \leq B \wedge (\exists n)\ r_n(A) = a\}.$$

Axiom 3. (1) If $\text{depth}_B(a) = n < \infty$ then $[a, A] \neq \emptyset$ for all $A \in [r_n(B), B]$.

(2) $A \leq B$ and $[a, A] \neq \emptyset$ imply that for $n = \text{depth}_B(a)$ there is an $A' \in [r_n(B), B]$ such that $\emptyset \neq [a, A'] \subseteq [a, A]$.

The last axiom is about an abstract pigeonhole principle for the approximations. We use the notation $\mathcal{AR}_n$ for the range of r_n (so $\mathcal{AR} = \bigcup_{n \in \mathbb{N}} \mathcal{AR}_n$), and for $a \in \mathcal{AR}$, $|a|$ is the integer n such that $a = r_n(A)$ for some $A \in \mathcal{R}$.

Axiom 4. If $\text{depth}_B(a) = n < \infty$ and $\mathcal{O} \subseteq \mathcal{AR}_{|a|+1}$ then there is $A \in [r_n(B), B]$ such that $r_{|a|+1}[a, A] \subseteq \mathcal{O}$ or $r_{|a|+1}[a, A] \subseteq \mathcal{O}^c$.

Example 3.2.1. A prototype example of a triple $(\mathcal{R}, \leq, r)$ satisfying these axioms is what we encountered in the context of the Galvin-Prikry theorem: let $\mathcal{R} = [\mathbb{N}]^{\mathbb{N}}$, $\leq$ be the subset quasi-ordering, and let $r_n(A)$ be the first n elements of A. This way $\mathcal{AR} = [\mathbb{N}]^{<\mathbb{N}}$, and $\mathcal{AR}_n = [\mathbb{N}]^n$.

It is clear that these satisfy Axiom 1. To check Axiom 2., take

$$a \leq_{\text{fin}} b \Leftrightarrow (a \subseteq b \wedge \max(a) = \max(b)).$$

For Axiom 3., notice that the sets of form $[a, A]$ are nearly the same as those defined in Section 3.1., the only difference is that here we require A to contain a instead of assuming $\min(a) < \max(A)$ (for $[a, A]$ not to be empty), and that $\text{depth}_B(a) < \infty$ occurs exactly when $a \subset B$. Finally, to check Axiom 4., let

$$B_0 = \{k \in B : (k > \max(a)) \wedge (a \cup \{k\} \in \mathcal{O})\},$$

$$B_1 = \{k \in B : (k > \max(a)) \wedge (a \cup \{k\} \notin \mathcal{O})\}.$$

Then B_i will be infinite for at least one $i < 2$, and $A = r_n(B) \cup B_i$ will meet the conditions.

Definition 3.2.2. A subset $H \subseteq \mathcal{R}$ is *Ramsey measurable* if for every $\emptyset \neq [a, A]$ there is some $B \in [a, A]$ such that $[a, B] \subseteq H$ or $[a, B] \cap H = \emptyset$.

A subset $H \subseteq \mathcal{R}$ is *Ramsey null* if for every $\emptyset \neq [a, A]$ there is some $B \in [a, A]$ such that $[a, B] \cap H = \emptyset$, and *Ramsey co-null* if its complement is Ramsey null.

Equip $\mathcal{R}$ with the topology generated by the sets of form $[a, A]$ for some $a \in \mathcal{AR}$ and $A \in \mathcal{R}$ (this is again called the *Ellentuck topology*), and take $\mathcal{AR}$ with the discrete topology. We can identify $\mathcal{R}$ with a subspace of $\mathcal{AR}^{\mathbb{N}}$ by identifying $A \in \mathcal{R}$ with $(r_n(A))_{n\in\mathbb{N}}$. The topology on $\mathcal{R}$ inherited from $\mathcal{AR}$ is referred to as its metrizable topology, but without further specification we consider the Ellentuck topology.

Definition 3.2.3. A triple $(\mathcal{R}, \leq, r)$ is a *topological Ramsey space* if every subset of $\mathcal{R}$ with the Baire property is Ramsey measurable and every meager subset of $\mathcal{R}$ is Ramsey null.

Now we are ready to state the special case of the Abstract Ramsey Theorem about topological Ramsey spaces.

Theorem 3.2.4 (Abstract Ellentuck Theorem)**.** *Suppose that the triple $(\mathcal{R}, \leq, r)$ satisfies axioms 1., 2., 3. and 4., and that $\mathcal{R} \subseteq \mathcal{AR}$ is a closed subset. Then the triple $(\mathcal{R}, \leq, r)$ forms a topological Ramsey space.*

Using the classical result that in any topological space the property of Baire is preserved under the Souslin operation (see 29.14. in [Kec12]), we get the following conclusion:

Corollary 3.2.5. *Suppose that the triple $(\mathcal{R}, \leq, r)$ satisfies axioms 1., 2., 3. and 4., and that $\mathcal{R} \subseteq \mathcal{AR}$ is a closed subset. Then every Souslin measurable subset of $\mathcal{R}$ is Ramsey measurable.*

This yields the following result for the metrizable topology of $\mathcal{R}$:

Corollary 3.2.6 (Abstract Silver Theorem)**.** *Suppose that the triple $(\mathcal{R}, \leq, r)$ satisfies axioms 1., 2., 3. and 4., and that $\mathcal{R} \subseteq \mathcal{AR}$ is a closed subset. Then every metrically Souslin measurable subset of $\mathcal{R}$ is Ramsey measurable.*

Corollary 3.2.7 (Abstract Galvin-Prikry Theorem)**.** *Suppose that the triple $(\mathcal{R}, \leq, r)$ satisfies axioms 1., 2., 3. and 4., and that $\mathcal{R} \subseteq \mathcal{AR}$ is a closed subset. Then every metrically Borel subset of $\mathcal{R}$ is Ramsey measurable.*

We checked that $[\mathbb{N}]^{\mathbb{N}}$ satisfies axioms 1., 2., 3. and 4., and also $[\mathbb{N}]^{\mathbb{N}}$ is closed in $\left([\mathbb{N}]^{<\mathbb{N}}\right)^{\mathbb{N}}$. Therefore, this also proves the Galvin-Prikry theorem in the case of a 2-coloring, which easily yields the whole statement by induction.

It is worth pointing out that for every space $\mathcal{R}$ satisfying the conditions of Theorem 3.2.4. with no isolated points there are subsets which are not Ramsey measurable, so the constraint of the Baire property is not unnecessary.

For the rest of this section, we show another example for topological Ramsey spaces. In Chapter 5. of [Tod10], many more examples and beautiful applications can be found.

Example 3.2.8 (5.6. in [Tod10])**.** Let $\mathcal{E}_\infty = \mathcal{E}_\infty(\mathbb{N})$ be the collection of all equivalence relations E on $\mathbb{N}$ with infinitely many equivalence classes. Each E-class has a minimal element, let $p(E) = \{p_0(E), p_1(E), \dots\}$ be the set of these minimal representatives in an increasing enumeration. Observe that for any $E \in \mathcal{E}_\infty$ we have $0 \in p(E)$, consequently $p_0(E) = 0$.

For $E, F \in \mathcal{E}_\infty$, E *is coarser than* F, or $E \leq F$, if every equivalence class of E can be represented as the union of certain equivalence classes of F (or equivalently, if $E \supseteq F$ as subsets of $\mathbb{N}^2$). Define the nth *approximation* to some $E \in \mathcal{E}_\infty$ as $r_n(E) = E|_{p_n(E)}$.

For each approximation $a \in \mathcal{AE}_\infty$, which is now an equivalence relation on a natural number $p_n(E) = \{0, 1, \dots, p_n(E) - 1\}$, we define $\mathrm{dom}(a)$ to be this natural number. The natural finitization of $\leq$ on $\mathcal{E}_\infty$ is the following: $a \leq_{\mathrm{fin}} b$ if $\mathrm{dom}(a) = \mathrm{dom}(b)$ and a is coarser then b.

It is easy to check that axioms 1., 2. and 3. are satisfied, and though it is less straightforward, the corresponding pigeonhole principle holds too. Since $\mathcal{E}_\infty$ is a closed subspace of $\mathcal{AR}_\infty^{\mathbb{N}}$, the Abstract Ellentuck Theorem gives the following result.

Theorem 3.2.9 (Carlson-Simpson)**.** *The space* $(\mathcal{E}_\infty, \leq, r)$ *is a topological Ramsey space.*

Corollary 3.2.10 (Dual Silver Theorem)**.** *Suppose that c is a finite Souslin-measurable coloring of the family* $\mathcal{E}_\infty$*. Then there exists* $E \in \mathcal{E}_\infty$ *such that the family* $\mathcal{E}_\infty|_E$ *of all* $F \in \mathcal{E}_\infty$ *coarser than* E *is monochromatic.*

Remark. Aside the Ellentuck topology on $\mathcal{E}_\infty$ generated by basic open sets of form $[a, E]$ and the metrizable topology induced from $\mathcal{AE}_\infty^{\mathbb{N}}$, the Dual Silver Theorem remains true for a third topology as well, namely, the one inherited from $2^{(\mathbb{N}^2)}$ by identifying members of $\mathcal{E}_\infty$ with subsets of $\mathbb{N}^2$.

For $k \in \mathbb{N}$, let $\mathcal{E}_k = \mathcal{E}_k(\mathbb{N})$ be the collection of all equivalence relations on $\mathbb{N}$ with exactly k classes. It is natural to identify $\mathcal{E}_k$ with the subset of $k^{\mathbb{N}}$ which consists of surjective maps f with the property that, to avoid the distinction of equivalence classes, it satisfies $\min(f^{-1}(i)) < \min(f^{-1}(j))$ for $i < j < k$. Note that with this identification, $\mathcal{E}_k$ is an open subset of $k^{\mathbb{N}}$.

The Dual Silver Theorem has the following (not immediate) corollaries:

Corollary 3.2.11 (Dual Ramsey Theorem)**.** *Suppose that c is a finite Baire measurable coloring of* $\mathcal{E}_k$ *(relative to the topology inherited from* $k^{\mathbb{N}}$*). Then there exists* $E \in \mathcal{E}_\infty$ *such that the family* $\mathcal{E}_k|_E$ *of all* $F \in \mathcal{E}_k$ *coarser than* E *is monochromatic.*

For $k, m \in \mathbb{N}$, let $\mathcal{E}_k(m)$ be the collection of all equivalence relations on m with exactly k classes. Similarly to the way the Galvin-Prikry theorem implies the classical Ramsey theorem, we obtain the following result from the Carlson-Simpson theorem.

Corollary 3.2.12 (Finite Dual Ramsey Theorem)**.** *For every* $k, l, n \in \mathbb{N}$ *there exists* $m \in \mathbb{N}$ *such that for every l-coloring of* $\mathcal{E}_k(m)$ *there is an equivalence relation* $E \in \mathcal{E}_n(m)$ *such that the family* $\mathcal{E}_k(m)|_E$ *of all* $F \in \mathcal{E}_k(m)$ *coarser than* E *is monochromatic.*

3.3 Ramsey and hyperfiniteness

Call a set an *Ellentuck cube* if it is of the form $[a, A]$ for some $a \in [\mathbb{N}]^{<\mathbb{N}}$ and $A \in [\mathbb{N}]^{\mathbb{N}}$ with $\max(a) < \min(A)$, and call it a *pure Ellentuck cube*, if it is of the form $[\emptyset, A](= [A]^{\mathbb{N}})$ for some $A \in [\mathbb{N}]^{\mathbb{N}}$.

Theorem 3.3.1 (Mathias [Mat77], Soare [Soa69])**.** *Suppose that* E *is a countable Borel equivalence relation on* $[\mathbb{N}]^{\mathbb{N}}$*. Then there exists a pure Ellentuck cube* $[A]^{\mathbb{N}}$ *where* $E \subseteq \mathbb{E}_0$*, consequently* E *is hyperfinite on* $[A]^{\mathbb{N}}$*.*

Thus from Ramsey theoretic point of view, every countable Borel equivalence relation canonizes to a hyperfinite one on a positive set. Still, notice the difference from Baire category: investigating hyperfiniteness is not pointless Ramsey theoretically, as equivalence relations could still be able to perform a complex behaviour on Ramsey co-null sets.

We show a proof based on section 8.3. in [KSZ13]. Note that although the proof uses the Mathias forcing, this can be eliminated using "fusion" arguments, i.e., similar ones as in the proof of Lemma 3.1.7.

Definition 3.3.2. Identify the elements of $[\mathbb{N}]^{\mathbb{N}}$ with their increasing enumeration, define the maps S_{ev} and S_{odd} by

$$S_{\mathrm{ev}}(\{n_0, n_1, n_2, \ldots\}) = \{n_0, n_2, n_4 \ldots\},$$

$$S_{\mathrm{odd}}(\{n_0, n_1, n_2, \ldots\}) = \{n_1, n_3, n_5 \ldots\},$$

Proposition 3.3.3. *Let $A \in [\mathbb{N}]^{\mathbb{N}}$ and $f : [A]^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ be a Borel function such that $f(C) \setminus C$ is infinite for every $C \in [A]^{\mathbb{N}}$. Then there exists a set $B \in [A]^{\mathbb{N}}$ such that $f(C) \setminus B$ is infinite for every $C \in [B]^{\mathbb{N}}$.*

Proof. For $B \in [\mathbb{N}]^{\mathbb{N}}$, denote $B_{\mathrm{ev}} = S_{\mathrm{ev}}(B)$ and $B_{\mathrm{od}} = S_{\mathrm{od}}(B)$. Partition $[A]^{\mathbb{N}}$ into two Borel parts: $X = \{C : \ f(C_{\mathrm{ev}} \cap C_{\mathrm{od}}$ is infinite$\}$, and $Y = \{C : \ f(C_{\mathrm{ev}} \cap C_{\mathrm{od}})$ is finite$\}$. By the Galvin-Prikry theorem there exists $B' \in [A]^{\mathbb{N}}$ such that $[B']^{\mathbb{N}}$ is fully contained in X or Y. We claim that $B = B'_{\mathrm{ev}}$ is suitable.

Indeed, if $[B']^{\mathbb{N}} \subseteq X$, then for each $C \in [B]^{\mathbb{N}}$ exists some $C' \in [B']^{\mathbb{N}}$ with $C'_{\mathrm{ev}} = C$ and $C'_{\mathrm{od}} \subseteq B'_{\mathrm{od}}$. As $f(C) \setminus B \supseteq f(C) \cap B'_{\mathrm{od}} = f(C'_{\mathrm{ev}}) \cap B'_{\mathrm{od}} \supseteq f(C'_{\mathrm{ev}}) \cap C'_{\mathrm{od}}$, and the latter is infinite, $f(C) \setminus B$ is infinite as well.

Finally, in the case of $[B']^{\mathbb{N}} \subseteq Y$, it suffices to show that $f(C) \cap B'$ is finite for every $C \in [B]^{\mathbb{N}}$, as it implies that $f(C) \setminus B' \subseteq f(C) \setminus B$ is infinite. If $f(C) \cap B'$ was infinite for some C, then take $C' \in [B']^{\mathbb{N}}$ such that $C'_{\mathrm{ev}} = C$ and for any two successive element of C, choose the unique element of C' between them to be from $f(C) \cap B'$, if possible. Then $f(C'_{\mathrm{ev}}) \cap C'_{\mathrm{od}}$ would be infinite, contradicting $C' \in Y$. $\square$

For the rest of the proof, which uses forcing, we drop the convention of notating infinite subsets of $\mathbb{N}$ with capital letters.

Definition 3.3.4. The *Mathias forcing* is the poset P of all Ellentuck cubes $p = [a_p, b_p]$ ordered by inclusion: $q \leq p$ if $a_p \subseteq a_q$, $b_q \subseteq b_p$, and $a_q \setminus a_p \subseteq b_p$.

Proposition 3.3.5. *Let $f : [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ be a Borel function from the ground model. Then the Mathias forcing forces the following: either $f(\dot{x}_{gen}) \setminus \dot{x}_{gen}$ is finite, or there is a ground model superset $d \supseteq \dot{x}_{gen}$ such that $f(\dot{x}_{gen}) \setminus d$ is infinite.*

Proof. Suppose that a condition p forces that $\dot{f}(\dot{x}_{\mathrm{gen}}) \setminus \dot{x}_{\mathrm{gen}}$ is infinite. Then the set $\{x \in p : f(x) \setminus x$ is infinite$\}$ contains an Ellentuck cube, so by thinning out p we can assume that it is equal to one. Then by Proposition 3.3.3. we can thin it out even further to find $d \in [\mathbb{N}]^{\mathbb{N}}$ such that all elements in the cube have f-images with infinitely many elements outside of d, hence the second part of the clause is satisfied. $\square$

Proposition 3.3.6. *Let $b \subseteq \mathbb{N}$ be an M-generic real for Mathias forcing. Then for e_0, e_1 infinite sets $M[e_0] = M[e_1]$ if and only if the symmetric difference $e_0 \bigtriangleup e_1$ is finite.*

Proof. An infinite subset of a Mathias generic real is again Mathias generic (see [Mat77]), thus e_0 and e_1 are Mathias generic over M. If $e_0 \bigtriangleup e_1$ is finite, then they construct each other, and the models $M[e_0]$ and $M[e_1]$ are the same. For the opposite direction, suppose that $e_0 \setminus e_1$ is infinite, we will show that $e_0 \notin M[e_1]$. Indeed, if $e_0 \in M[e_1]$, then $e_0 = f(e_1)$ for some Borel function $f \in M$. Proposition 3.3.5. then implies the existence of an infinite set $d \in M$ such that $e_1 \subseteq d$ and $f(e_1) \setminus d$ is infinite. As b is Mathias generic over M with already infinite intersection with d, the set $b \setminus d$ must be finite. But then $f(e_1) \setminus b$ must be infinite, which contradicts $f(e_1) = e_0 \subseteq b$. $\square$

Proof of Theorem 3.3.1. By Theorem 2.1.10., E arises as the orbit equivalence relation of a Borel action of a countable group Γ. Let M be a countable elementary submodel of a large structure containing the group action, and let $b \subseteq \mathbb{N}$ be a Mathias generic real over M. Then for any $e \in [b]^{\mathbb{N}}$ and $e'Ee$ we have that $e' \in M[e]$, since $e' = \gamma(e)$ for some $\gamma \in \Gamma$. Then by Proposition 3.3.6., we have that $e\mathbb{E}_0 e'$. Thus $E|_{[b]^{\mathbb{N}}} \subseteq \mathbb{E}_0$. $\square$

Remark. [KSZ13] also proves a canonization result for the Millikan space (another topological Ramsey space), showing that CBERs are hyperfinite on positive sets. Wang and Panagiatopulos recently showed that CBERs are smooth on Carlson-Simpson cubes [PW22]. Thus we can formulate the following question:

Question 3.3.7. Is it true that every CBER on a tXopological Ramsey space is hyperfinite on a cube?

4 Non-hypersmoothness and acyclicity

In this section we present several novel results and constructions. These are motivated by the prior work of Vidnyánszky [Vid22] and Todorčević and Vidnyánszky [TV21]. The complete behavior of our examples is yet to be understood and we hope that they could be a basis for a future research.

4.1 An example for non-hypersmoothness

As we seen in Section 3.3., all countable Borel equivalence relations on $[\mathbb{N}]^{\mathbb{N}}$ are hyperfinite on $[x]^{\mathbb{N}}$ for some $x \in [\mathbb{N}]^{\mathbb{N}}$. On the other hand, equivalence relations with uncountable classes may behave in a much more complicated manner for Ramsey theoretic reasons.

Definition 4.1.1. We say that $x, y \in [\mathbb{N}]^{\mathbb{N}}$ are *almost disjoint*, if $x \cap y$ has finite cardinality.

Theorem 4.1.2 (Vidnyánszky, [Vid22])**.** *Let E be an equivalence relation on $[\mathbb{N}]^{\mathbb{N}}$ such that for every $x \in [\mathbb{N}]^{\mathbb{N}}$*

(1) there exist almost disjoint $y, z \in [x]^{\mathbb{N}}$ with $xEyEz$ and

(2) $E|_{[x]^{\mathbb{N}}}$ has uncountably many equivalence classes, then E is not hypersmooth.

Example 4.1.3 ([Vid22])**.** Let S be the shift map as in Definition 3.1.11., and S_{odd} as in Definition 3.3.2.

Denote by G the graph induced by S and S_{odd}, and consider the Borel equivalence relation E_G. For arbitrary $x \in [\mathbb{N}]^{\mathbb{N}}$, (1) is satisfied by $y = S_{\text{odd}}(x)$ and $z = S_{\text{odd}} \circ S(x)$, as $S_{\text{odd}}(x) \cap S_{\text{odd}} \circ S(x) = \emptyset$.

To check (2), define $x \ll y$ if for each m there exists k such that $|x \cap (y_k, y_{k+1})| > m$ (where $y = \{y_0, y_1, \ldots\}$). Clearly, if $x \ll y$ and $\{x, x'\} \in G$, then $x' \ll y$. Now for arbitrary $\{x^n : n \in \mathbb{N}\} \subseteq [x]^{\mathbb{N}}$ choose $y \subset x$ such that $x^n \ll y$ for all n. This means that y is not E_G-equivalent with any x^n, consequently, $E|_{[x]^{\mathbb{N}}}$ has uncountably many equivalence classes. Therefore, E_G is not hypersmooth.

The proof of Theorem 4.1.2. depends on the following theorem, which uses several Ramsey theoretic canonization results.

Theorem 4.1.4. *Assume that $f_n : [\mathbb{N}]^{\mathbb{N}} \to \left(2^{\mathbb{N}}\right)^{\mathbb{N}}$ is a Borel map for each $n \in \mathbb{N}$. Then there exist $x \in [\mathbb{N}]^{\mathbb{N}}$ and a countable set C such that for every $y, z \in [x]^{\mathbb{N}}$ almost disjoint and $n \in \mathbb{N}$ we have that $(f_n(x) = f_n(y)) \Rightarrow f_n(y) \in C$.*

Proof of Theorem 4.1.2. Assume the contrary, and let φ be a Borel reduction from E to $\mathbb{E}_1$. Let $f_n = p^n \circ \varphi$ where p is the shift map on $\left(2^{\mathbb{N}}\right)^{\mathbb{N}}$ defined by $p((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}}$. Then for yEz we have $f_n(y) = f_n(z)$ for some n as $\varphi(y)\mathbb{E}_1\varphi(z)$.

Choose x and C as in Theorem 4.1.4 for the sequence $(f_n)_{n\in\mathbb{N}}$. It suffices to show that $\varphi([x]^{\mathbb{N}})$ is contained in countably many $\mathbb{E}_1$-classes, as this contradicts (2). Take the countable set

$$C' = \{(x_n)_{n\in\mathbb{N}} : \exists m(\forall n < m\ x_n = 0 \land (x_m, x_{m+1} \ldots) \in C)\}.$$

For arbitrary $x' \in [x]^{\mathbb{N}}$ our first assumption gives $y, z \in [x']^{\mathbb{N}}$ almost disjoint such that $x'EyEz$. Thus $f_n(y) = f_n(z)$ for some n, hence $\varphi(x') \in [C']_{\mathbb{E}_1}$. □

Shani (personal communication) pointed out that, to prove that E is not hypersmooth, Cohen forcing methods can be used as well. See e.g. [AS21] and [LZ20] for some examples.

A motivation for finding additional examples for functions which generate a non-hypersmooth equivalence relation together with the shift map is the following.

Definition 4.1.5. A set $B \subseteq [\mathbb{N}]^{\mathbb{N}}$ is *dominating*, if for every $f \in [\mathbb{N}]^{\mathbb{N}}$ there exists some $g \in B$ which dominates f, that is, $g_n \geq f_n$ except for finitely many $n \in \mathbb{N}$, where $f = \{f_0, f_1, \ldots\}$ and $g = \{g_0, g_1, \ldots\}$ are the increasing enumerations of f and g.

Equivalently, a set $B \subseteq [\mathbb{N}]^{\mathbb{N}}$ is *non-dominating*, if there exists $f \in [\mathbb{N}]^{\mathbb{N}}$ such that for every $g \in B$ we have $g \leq^{\infty} f$, i.e., $g(n) \leq f(n)$ for infinitely many $n \in \mathbb{N}$.

Non-dominating sets have the advantage of being combinatorially well-behaving and at the same time still large, for example there are residual non-dominating sets, and also σ-compact sets are all non-dominating. In [TV21], Todorčević and Vidnyánszky showed that the set of Borel graphs with Borel chromatic number ≤ 3 is $\mathbf{\Sigma}^1_2$-complete using non-dominating sets, this proof makes use of the fact that the Borel chromatic number of the shift graph is infinite as we have seen in Proposition 3.1.12., however, on a non-dominating set, it is at most 3.

Proposition 4.1.6. *Let $C \subseteq [\mathbb{N}]^{\mathbb{N}}$ be non-dominating. Then $\chi_B(G_S|_C) \leq 3$.*

Proof. Take $f \in [\mathbb{N}]^{\mathbb{N}}$ such that for every $g \in B$ we have $g \leq^{\infty} f$. We can choose f with increasing enumeration $\{f_0, f_1, \ldots\}$ to be such that for every $g \in C$ we have $|[f_n, f_{n+1}] \cap g| \geq 2$ for infinitely many $n \in \mathbb{N}$. Define $c(g)$ to be the unique $j \in \{0, 1, 2\}$ such that $j \equiv k \mod 3$ where k is the smallest integer such that the first interval of the form $[f_n, f_{n+1}]$ which contains an element of $S^k(g)$ contains exactly 2. This is well-defined by our last observation, and it is easy to see that c is a Borel proper coloring of $G_S|_C$. $\square$

This motivates the question of whether there are equivalence relations, which are not simple on $[\mathbb{N}]^{\mathbb{N}}$, while become simpler, once we restrict to non-dominating sets.

Now we present another non-trivial example for an other equivalence relation which is non-hypersmooth by Theorem 4.1.2. It is not clear yet what can be stated about its behaviour on non-dominating sets.

Let f be a map on $\mathbb{N}$, and S_f be a map on $[\mathbb{N}]^{\mathbb{N}}$ defined in the following way: for an arbitrary $x = \{n_0, n_1, \ldots\} \in [\mathbb{N}]^{\mathbb{N}}$, define $k_0 < l_0 < k_1 < l_1 < \ldots$ recursively by $k_0 = 0$, $l_i = k_i + f(i) \cdot n_{k_i}$, and $k_i = l_{i-1} + n_{l_{i-1}}$. Let $S_f(x) = \bigcup_{i \in \mathbb{N}} \{n_{k_i}, n_{k_i+1}, \ldots, n_{l_i-1}\}$. In other words, we alternatingly keep and discard intervals from x, with the length of the interval depending on its first element. When applying this map, we lose a lot of information, however, on a non-dominating set, the extent of this loss is somewhat limited.

Denote by G_{S_f} the graph determined by S_f, consider $G = G_S \cup G_{S_f}$, and let $E = E_G$ be the connected component equivalence relation of G.

Proposition 4.1.7. *For every $x \in [\mathbb{N}]^{\mathbb{N}}$ there exist $y, z \in [x]^{\mathbb{N}}$, yEz such that $y \cap z = \emptyset$.*

Proof. For an arbitrary $x \in [\mathbb{N}]^{\mathbb{N}}$ we construct $x' \subseteq x$ and k_i^j, l_i^j ($j \in \{0, 1, 2\}, i \in \mathbb{N}$), satisfying the following conditions:

1. $k_0^j < l_0^j < k_1^j < l_1^j < \ldots$ for $j \in \{0, 1, 2\}$.

2. $k_i^0 < k_i^1 < l_i^0 < k_i^2 < l_i^1 < k_{i+1}^0 < l_i^2 < k_{i+1}^1$ for $i \in \mathbb{N}$.

3. $x' = \{m_0, m_1, \dots\} \cup \left(\bigcup_{i\in\mathbb{N}} \{m'_h : k_i^2 \leq h < l_i^1\}\right)$ so that $m_0 < m_1 < m_2 < \dots$ and $m_h < m'_h < m_{h+1}$.
4. $l_i^0 = k_i^0 + f(i) \cdot m_{k_i^0}$, and $k_{i+1}^0 = l_i^0 + m_{l_i^0}$ for $i \in \mathbb{N}$.
5. $l_i^1 = k_i^1 + f(i) \cdot m_{k_i^1}$, and $k_{i+1}^1 = l_i^1 + m_{l_i^1}$ for $i \in \mathbb{N}$.
6. $l_i^2 = k_i^2 + f(i) \cdot m'_{k_i^2}$, and $k_{i+1}^2 = l_i^2 + m_{l_i^2}$ for $i \in \mathbb{N}$.

This can be done recursively by choosing the elements of x' increasingly, and keeping track of k_i^j, l_i^j based on 4., 5. and 6. Throughout this construction, the only nontrivial condition is 2., which can be met by choosing $m_{l_i^j}$ large enough (namely, $m_{l_i^0} > f(i) \cdot m_{k_i^1} - (l_i^0 - k_i^1)$, $m_{l_i^1} > f(i) \cdot m'_{k_i^2} - (l_i^1 - k_i^2)$, and $m_{l_i^2} > f(i+1) \cdot m_{k_{i+1}^0} - (l_i^2 - k_{i+1}^0)$).

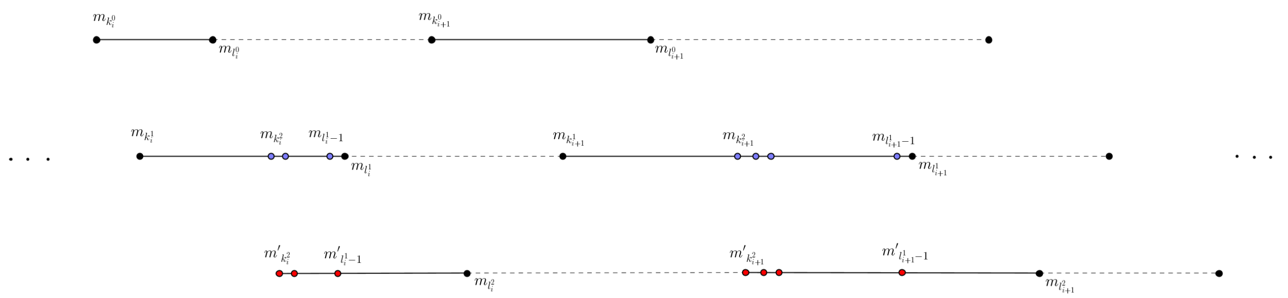

Figure 2: The disposition of elements in the proof of Proposition 4.1.7.

Let $y' = \{m_0, m_1, \dots\}$ and $z' = \left(y_1 \setminus \left(\bigcup_{i\in\mathbb{N}} \{m_h : k_i^2 \leq h < l_i^1\}\right)\right) \cup \left(\bigcup_{i\in\mathbb{N}} \{m'_h : k_i^2 \leq h < l_i^1\}\right)$. The conditions for k_i^0, l_i^0 imply $y'Ez'$, and our conditions also imply that

$$y := S_f \circ S^{k_0^1}(y') = \bigcup_{i\in\mathbb{N}} \{m_h : k_i^1 \leq h < l_i^1\} \text{ and}$$

$$z := S_f \circ S^{k_0^2}(z') = \left(\bigcup_{i\in\mathbb{N}} \{m'_h : k_i^2 \leq h < l_i^1\}\right) \cup \left(\bigcup_{i\in\mathbb{N}} \{m_h : l_i^1 \leq h < l_i^2\}\right)$$

are disjoint. $\square$

It is not so hard to check that E satisfies (2) of Theorem 4.1.2. as well, so indeed it is not hypersmooth.

4.2 Acyclicity with the shift map

As we have seen in Theorem 2.4.3. and Proposition 2.5.16., acyclicity can be an extremely useful property to establish non-hyperfiniteness, or more generally, non-hypersmoothness of equivalence relations. Since the shift map is a rather canonical object on $[\mathbb{N}]^{\mathbb{N}}$ (this intuition can be made precise using the work of Prömel and Voigt [PV85]), it is natural to consider maps that generate acyclic graphs with G_S. In this section, we investigate various constructions of maps, with keeping in mind the following test questions:

1. Are there natural maps f acyclic with the shift map?
2. Is the equivalence relation generated by $G_S \cup G_f$ hyperfinite on large, say, Ramsey co-null sets?
3. Is at least, G_f not smooth on Ramsey co-null sets?
4. How does $G_S \cup G_f$ behave on non-dominating sets?

Now we address question 1., and show an example for a Borel map $f : [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ such that the induced graph is acyclic together with the shift graph.

One immediate consequence of acyclicity with the shift map is, that almost identical (or precisely, E_{G_S}-equivalent) elements must not be mapped almost identically, as this would form a cycle containing two f-edges. This means that every element of any $x \in [\mathbb{N}]^{\mathbb{N}}$ must affect every suffix of $f(x)$. We achieve this in a way that readily implies acyclicity as well.

We identify $[\mathbb{N}]^{\mathbb{N}}$ with a co-countable subspace of $2^{\mathbb{N}}$ by taking the characteristic vector of an element of $[\mathbb{N}]^{\mathbb{N}}$. We also identify $2^{\mathbb{N}}$ with $4^{\mathbb{N}}$ by a $2^2 \to 4$ bijection.

Let $h' : 4^{<\omega} \to \mathbb{N}$ be an injective map, also take a $k' : \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ map such that for all $n, m, i, j \in \mathbb{N}$ $(k'(n))_i \neq (k'(m))_j$ (unless $n = m$ and $i = j$), and we also assume that for arbitrary $s \in 4^{<\omega}$ $h'(s) \neq (k'(n))_i$. For $n \in \mathbb{N}$, define $k(n) \in \{-1, +1\}^{\mathbb{N}}$ by setting the first $2^{(k'(n))_0}$ entries to $+1$, then setting the next $2^{(k'(n))_1}$ entries to -1, and so on. Similarly, for $a \in 4^{\mathbb{N}}$, define $h(a) \in \{-1, +1\}^{\mathbb{N}}$ by setting the first $2^{h'(a|0)}$ entries to $+1$, the next $2^{h'(a|1)}$ entries to -1, and so on.

We define $f : [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ in the following way: for $x = \{x_0, x_1, \ldots\} \in [\mathbb{N}]^{\mathbb{N}}$ first take $S(x)$, then convert it to an element $a = \{a_0, a_1, \ldots\} \in 4^{\mathbb{N}}$, then take the unique $b = \{b_0, b_1, \ldots\} \in 4^{\mathbb{N}}$ such that for every $i \in \mathbb{N}$

$$b_i \equiv a_i + (k(x_0))_i + (h(b))_i \mod 4.$$

Note that $(h(b))_i$ (which is the $(i+1)$th entry of $h(b)$) is determined by at most i digits of b, so indeed there is a unique b satisfying the defining equation. Then let $f(x)$ be $y \in [\mathbb{N}]^{\mathbb{N}}$ corresponding to b.

Proposition 4.2.1. *Let G_S and G_f be the graphs on $[\mathbb{N}]^{\mathbb{N}}$ generated by S and f. Then $G_S \cup G_f$ is acyclic.*

Proof. Suppose that on the contrary there is a cycle C in $G_S \cup G_f$, take an edge of C which is connecting x and $f(x)$ for some $x \in [\mathbb{N}]^{\mathbb{N}}$. (Such an edge exists, since G_S is acyclic.) Choose i such that in $4^{\mathbb{N}}$ the ith entry of x is only changed by translations with ± 1. Take $j > i$ such that when applying f to x, k switches between -1 and $+1$ on the jth and $(j+1)$th entries. As x is on the cycle C, there must be another edge of C from y to $f(y)$ where k or h switches between -1 and $+1$ at the exact same place. As there are infinitely many entries where k changes its value, there must be j_1 and j_2 such that this other change arises twice in h or twice in k at the same edge, but this would imply that $|j_1 - j_2|$ has two different binary forms, which is a contradiction. $\square$

It is not clear whether the equivalence relation obtained from G_f is smooth or not. However, we show that it is not only smooth on a Ramsey-conull set, but even it coincides with the equality relation on one, answering question 3. in this specific case. Our proof also imply that the equivalence relation induced by $G_S \cup G_f$ is hyperfinite on the same Ramsey co-null set, since it coincides there with $\mathbb{E}_0$, this means that in this case the answer to question 2. is positive as well.

Proposition 4.2.2. *Let E_f be the equivalence relation on $[\mathbb{N}]^{\mathbb{N}}$ whose classes are the connected components of G_f. Then E_f is trivial on a Ramsey co-null Borel set.*

Proof. We will show that E_f is trivial on $H := \{x \in [\mathbb{N}]^{\mathbb{N}} : (\nexists n, m \in x, n \neq m : |n-m| < 100)\}$. Suppose that there exists $x_1, x_2 \in H$ such that $x_1 E_f x_2$. Take an arbitrary edge $(y, f(y))$ on the path P connecting x_1 and x_2. Choose i such that in $4^{\mathbb{N}}$ the ith entry of y is only changed by translations with ± 1. If $(k(y_0))_j \neq (k(y_0))_{j+1}$ for $j > i$, then there exists an edge $(z, f(z))$ on P such that for some j', $|j'-j| \leq 2$ when applying f to z one of k or h also switches its value. As $|j'-j|$ can have only finitely many values, there must be j_1, j_1', j_2, j_2' such that $j_1' - j_1 = j_2' - j_2$, $(k(y_0))_{j_i} \neq (k(y_0))_{j_i+1}$, and k or h also switches between -1 and $+1$ at j_1' and j_2' at a fixed edge of P. But this would mean that $|j_2 - j_1| = |j_2' - j_1'|$ has two different binary forms. $\square$

Remark. We know that $\mathbb{E}_0 \leq_B E_f$, consequently E_f itself is not smooth.

Note that the map above does not respect Ramsey-nullness, since it changes the density of subsets of $\mathbb{N}$ quite radically. A natural direction is to consider only *Ramsey preserving* maps, i.e. maps such that a subset of $[\mathbb{N}]^{\mathbb{N}}$ is Ramsey co-null exactly when its preimage is Ramsey co-null.

Question 4.2.3. Suppose that $f : [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ is a Ramsey preserving Borel map such that $G_S \cup G_f$ is acyclic. Is there a Ramsey co-null set, where the associated equivalence relation E_{G_f} is smooth?

Related to question 2., assume that we are given a map f such that the generated graph $G_S \cup G_f$ is acyclic. We would like to understand, what properties must f have, so that the generated equivalence relation is hyperfinite.

Now we try to adapt the idea from Theorem 2.4.3. Suppose that it was hyperfinite, witnessed by $E = \bigcup_{n \in \mathbb{N}} E_n$ with an increasing sequence of finite equivalence relations $E_0 \subseteq E_1 \subseteq \ldots$. Let $G_n = G \cap E_n$, and for each $x \in [\mathbb{N}]^{\mathbb{N}}$ define $H^n_{S,x}$ to be the set of points which are reachable from x in G_n with a path starting with a shift-edge, and likewise $H^n_{f,x}$ to be the the set of those reachable in G_n with a path starting with an f-edge. Define $\varphi : [\mathbb{N}]^{\mathbb{N}} \to 2^{\mathbb{N}}$ by

$$(\varphi(x))(n) = \begin{cases} 1 & |H^n_{f,x}| > |H^n_{S,x}| \\ 0 & \text{otherwise} \end{cases}.$$

Observe that for $x E_S y$ and n large enough one cannot have $1 = (\varphi(x))(n) = (\varphi(y))(n)$. In other words, identifying $\varphi(x)$ with the subset of $\mathbb{N}$ whose characteristic function is $\varphi(x)$, φ maps almost identical subsets of $\mathbb{N}$ to almost disjoint ones. A naturally occurring problem is how maps with this property look like, and whether they can be canonized in some sense. A specific question is the following:

Question 4.2.4. Suppose that a Borel map $\varphi : [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ maps almost identical subsets of $\mathbb{N}$ to almost disjoint ones. Is there $x \in [\mathbb{N}]^{\mathbb{N}}$ such that $\varphi\left([x]^{\mathbb{N}}\right)$ is an almost disjoint family (i.e. any two distinct elements are almost disjoint)?

We show that the answer is negative. Choose an injective Borel map $h : [\mathbb{N}]^{<\mathbb{N}} \to \mathbb{N}$ such that for all $s \in [\mathbb{N}]^{<\mathbb{N}}$ we have $h(s) > |s|$. As usual, identify the elements of $[\mathbb{N}]^{\mathbb{N}}$ with their increasing enumeration, and for $x = \{x_0, x_1, \ldots\}$ define

$$\varphi(x) = \{x_{h(\{x_0,\ldots x_n\})^2} : n \in \mathbb{N}\}.$$

Proposition 4.2.5. *Suppose that* $0 < |x \triangle y| < \omega$*, then* $|\varphi(x) \cap \varphi(y)| < \omega$*.*

Proof. Let $x = \{x_0, x_1, \ldots\}$ and $y = \{y_0, y_1, \ldots\}$. As x and y are almost identical, suppose that there is some $k \in \mathbb{N}$ with $x_n = y_{n+k}$ for n large enough. But this contradicts the fact that $\{h(\{x_0, \ldots x_n\})^2 : n \in \mathbb{N}\}$ and $\{h(\{y_0, \ldots y_n\})^2 + k : n \in \mathbb{N}\}$ are almost disjoint sets. □

Proposition 4.2.6. *For an arbitrary* $x \in [\mathbb{N}]^{\mathbb{N}}$*, the set family* $\varphi\left([x]^{\mathbb{N}}\right)$ *is not almost disjoint.*

Proof. We construct $y, z \in [x]^{\mathbb{N}}$ in such a way that $|\varphi(y) \cap \varphi(z)| = \omega$. Firstly, let $y_0 = z_0 = x_0$. Suppose that we have already given $y \cap (n+1)$ and $z \cap (n+1)$ such that $n = y_k = z_l$ for some k, l. Observe that we defined h such that $h(\{y_0, \ldots y_k\})^2 > k$, choose $y_{k+1}, \ldots, y_{h(\{x_0,\ldots x_k\})^2-1}$ and similarly $z_{l+1}, \ldots, z_{h(\{x_0,\ldots x_l\})^2-1}$ arbitrarily. Then take $m \in x$ such that m is larger then the previously defined values, and define $y_{h(\{x_0,\ldots x_k\})^2} = z_{h(\{x_0,\ldots x_l\})^2} = m$. This way y and z have infinitely many common elements. □

Now, towards question 4., we show an other example for a map which generates an acyclic graph together with the shift graph, and also has some connections with non-dominating families.

Similarly to the previous construction for acyclicity, let $h' : [\mathbb{N}]^{<\omega} \to \mathbb{N}$ be an injective map, also take a $k' : \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ map such that we have that $(k'(n))_i \neq (k'(m))_j$ for all $n, m, i, j \in \mathbb{N}$ (unless $n = m$ and $i = j$), and we also assume that for arbitrary $s \in [\mathbb{N}]^{<\omega}$ we have $h'(s) \neq (k'(n))_i$. For $n \in \mathbb{N}$, define $k(n) \in \{-1, +1\}^{\mathbb{N}}$ by setting the first $2^{(k'(n))_0}$ entries to $+1$, then setting the next $2^{(k'(n))_1}$ entries to -1, and so on. Similarly, for $x \in [\mathbb{N}]^{\mathbb{N}}$ with increasing enumeration $x = \{x_0, x_1 \ldots\}$ define $h(x) \in \{-1, +1\}^{\mathbb{N}}$ by setting the first $2^{h'(\emptyset)}$ entries to -1, the next $2^{h'(\{x_0\})}$ entries to $+1$, the next $2^{h'(\{x_0,x_1\})}$ entries again to -1 and so on.

We assume that neither $k'(n)_0$ or $h'(\emptyset)$ are 0 or 1, observe that this way the first 4 entries of $f'(x) = k(x_0) + h(x)$ are 0. Define $f : [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ by

$$f(x) = \{x_{n+1} + f'(x_{n+1}) : \ n \in \mathbb{N}\}.$$

Proposition 4.2.7. *The graph* $G_S \cup G_f$ *is acyclic on the Ramsey co-null set* $\{x = \{x_0, x_1, \ldots\} : \lim_{n\to\infty}(x_{n+1} - x_n) = \infty\}$*.*

Proof. Observe that $\lim_{n\to\infty} x_{n+1} - x_n = \infty$ implies that for n large enough we have that the nth smallest entry of $f(x)$ is derived from the $(n+1)$st smallest entry of x. The rest of the argument is essentially the same as in the proof of Proposition 4.2.1. □

Now consider a non-dominating set B and let y be such that $S^k(x) \leq^\infty x$ for every $k \in \mathbb{N}$ and $x \in B$. Moreover, by thinning out y if necessary, we can assume that for every $x \in B$ we have $|[y_n, y_{n+1}] \cap x| \geq 2$ for infinitely many $n \in \mathbb{N}$. Let $B' \subseteq B$ consist of those $x \in B$ such that the first interval of the form $[y_n, y_{n+1}]$ which contains an element of x contains exactly 2. This way, B' is a G_S-independent set with all $x \in B$ having infinitely many shift-translates in B'. Also, for each $x \in B'$, the cardinality $|\{x' \in B' : \exists k S^k(x') = x\}|$ is obviously finite. This provides a way to see that G_S is hyperfinite on B, define $B'_n = \{x \in B' : |\{x' \in B' : \exists k S^k(x') = x\}| \leq n\}$ and consider the increasing finite equivalence relations

$$E_n = \{(x, x') : x = x' \vee (\exists k, l \leq n)\ S^k(x) = S^l(x') \in B'_n\}.$$

The sole fact that G_S is hyperfinite on B is not surprising at all, as G_S is already hyperfinite on the whole $[\mathbb{N}]^{\mathbb{N}}$, but this might suggest a way to handle $G_S \cup G_f$.

Proposition 4.2.8. *For a suitable witness of non-domination y, the set B' defined the same way will be $(G_S \cup G_f)$-independent.*

Proof. Indeed, as we noted before, the first 4 entries of $f'(x)$ is 0 for any $x \in [\mathbb{N}]^{\mathbb{N}}$, thus $f(x)$ will contain exactly one element in $[y_n, y_{n+1}]$ if $[y_n, y_{n+1}] \cap x = \{x_0, x_1\}$. If we thin out a witness y to satisfy that for each $x \in B$ with increasing enumeration $\{x_0, x_1, \ldots\}$ we have $y_n > x_{2n} + 2n$, then for every such n there is $k \leq n$ with $S^k(x) \in B'$, since the first entries of $f'(x)$ being 0 imply that if $[y_m, y_{m+1}] \cap x \supseteq \{x_0, x_1, x_2\}$ then $[y_m, y_{m+1}] \cap f(x) \supseteq \{x_1, x_2\}$. $\square$

One can argue similarly to the previous proof that there are infinitely many $n \in \mathbb{N}$ with $f^n(x) \in B'$.

Note that for $x \in B'$, there are clearly finitely many other elements of B', from where one can reach x using only S and f (and not their inverse). Of course this does not imply hyperfiniteness, as two connected elements of B' may not have a common successor reachable only by S and f. Nonetheless, these observations may be a line of attack for proving that the equivalence relation generated by $G_S \cup G_f$ is not too complex on a non-dominating set.

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