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Rényi's parking problem with random sets

MSc thesis

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Contents

Introduction

Rényi's parking problem concerns the following random process: given an interval of length x, we repeatedly place intervals, or "cars" of unit length with position chosen uniformly from the remaining space until no more cars can fit. The higher dimensional analogue, where, for instance, balls or cubes are placed instead of intervals, has various practical applications, such as in the kinetic theory of liquids, or the packing density of solid objects in a container.

The one-dimensional problem was first investigated by Rényi [19], who proved that if the expected number of cars is $M(x)$, then there is a constant $C \approx 0,7476$ such that $M(x)/x \to C$ as $x \to \infty$. He also gave the analytic formula

$$C = \int_0^\infty \exp\left(-2\int_0^t \frac{1-e^{-u}}{u}\,du\right)dt,$$

and additionally proved that $M(x) = Cx + C - 1 + O(x^{-n})$ for all n. This was later improved by Dvoretzky and Robbins [6] to

$$M(x) = Cx + C - 1 + O\left(\left(\frac{2e}{x}\right)^{x-3/2}\right).$$

These results are presented in Chapter 1. Dvoretzky and Robbins also proved that the number of cars is asymptotically normal, furthermore, they gave an estimate for the variance. The asymptotic normality was independently proved by Mannion [11].

Palásti [17] conjectured that in the two-dimensional case, when unit squares are placed in an $x \times y$ rectangle, the expected number of squares $M(x, y)$ has the asymptotic $M(x, y) \sim C^2xy$ as $x, y \to \infty$, where C is the one-dimensional parking constant. This conjecture is still open, however, numerical results [4] suggest that it is false.

Several variants of the parking problem have been a topic of research. An iterated version of Rényi's problem is discussed in [10]. One-dimensional variants where the intervals have random length bounded from below were investigated in [1, 14, 15]. A review of some results about the continuous problem can be found in [20]. A discrete version of the problem has also been considered. Page [16] showed that if there are n points in a row and in each attempt, a pair of adjacent points is chosen uniformly out of the remaining points, then after this process terminates, the expected ratio of chosen points tends to $1 - e^{-2}$ as $n \to \infty$. Some other discrete models were studied in [2, 5, 7, 18].

In this thesis, we consider a variant of the parking problem where intervals of random length are placed. In this model, we make infinitely many attempts placing compact intervals in some bounded open interval U. In each attempt, we pick a starting point of the interval I uniformly in U, then we independently choose the (possibly zero) length of I from some fixed distribution μ_0. If the interval I obtained this way is contained in U and disjoint from all the previously placed intervals, then it is placed down, otherwise, we do nothing in that attempt. It is easy to see that when μ_0 is a Dirac measure on a positive real number, this model is equivalent to Rényi's original problem.

The main goal of Chapter 2 is to answer the following question proposed by M. Abért:

Question. For what distributions μ_0 do the placed intervals exhaust U in Lebesgue measure?

It is relatively easy to see that if $\mu_0((0,r)) = 0$ for some $r > 0$, then U is almost never exhausted. We will see that this is the case also when $\mu_0(\{0\}) > 0$. Somewhat surprisingly, it turns out that except for these two cases, the Lebesgue measure of U is exhausted no matter what μ_0 is. More precisely, the following zero-one law is true:

Theorem 1. *If either $\mu_0(\{0\}) > 0$ or $\mu_0((0,r)) = 0$ for some $r > 0$, then the uncovered part of U has positive Lebesgue measure almost surely. Otherwise, the uncovered part has zero Lebesgue measure almost surely.*

Instead of looking at the one-dimensional case, we will focus on a higher dimensional generalization, from which the one-dimensional variant follows as a special case. In this generalized model, U is taken to be an open set of finite Lebesgue measure in the Euclidean space $\mathbb{R}^d$. Instead of the length distribution μ_0, we take a measure μ on the space of compact sets containing the origin (more precisely, we take μ to be a Borel measure on the space of compact sets equipped with the Hausdorff metric). In each attempt, we proceed as follows: we choose a set X with distribution μ and independently choose a point $Y \in U$ uniformly. Then we take the set $Z = X + Y$ and place it down if it is contained in U and disjoint from all the previously placed sets. Note that by the assumption that μ is concentrated on sets containing the origin, it is always true that $Y \in Z$.

We will see that if the family of sets that μ is concentrated on satisfies the conditions defined in Section 2.2, then U is exhausted almost surely. In particular, the following special case of the main theorem is true:

Theorem 2. *Suppose that μ is concentrated on a family of compact sets that contains finitely many sets up to similarity, each of which has positive Lebesgue measure. If there is an $r > 0$ such that μ a.e. set has diameter at least r, then the placed sets almost never exhaust U in Lebesgue measure. Otherwise, U is exhausted almost surely.*

For example, if we place compact balls with positive radius chosen from some fixed distribution, then U is exhausted if and only if the radius is unbounded from below. We will also see that if the radius is zero with positive probability, then U is not exhausted. More generally, we will show the following theorem:

Theorem 3. *Suppose that $\mu(\{\{0\}\}) > 0$. Then U is almost never exhausted in Lebesgue measure.*

It is clear that in the one-dimensional case, combining Theorems 2 and 3 yields Theorem 1.

A further generalization of these results will be proved. Instead of choosing the translation uniformly, it will be chosen from some probability distribution ν concentrated on U. This allows us to pick a random point from $\mathbb{R}^d$ as well. We will see that if ν and the Lebesgue measure on U are absolutely continuous with respect to each other and $\lambda(U) < \infty$, then Theorems 2 and 3 still hold. Furthermore, in the case $\lambda(U) = \infty$, Theorem 2 holds with the additional assumption that each set is the union of convex sets with non-empty interiors, or there is a uniform bound on the diameters. We do not know whether Theorem 3 holds when $\lambda(U) = \infty$.

The main idea of the proof is that instead of trying to determine the probability that a given point x is covered, we can show that the Lebesgue density of the union of placed sets is positive at x. If we take a ball $B(x, r)$, then we can look at the first set Z that intersects this ball. If the diameter of Z is large compared to r, then the conditions in Section 2.2 imply that either the measure in a neighborhood is bounded from below, or x is contained in some set of arbitrarily small measure. If the diameter of Z is small compared to r, then a lower bound can be given for the conditional probability that Z is not too small. A Borel–Cantelli argument shows that if the diameter can be arbitrarily small, then Z is infinitely often not too small, giving a lower bound on the density using the condition on Z.

In Chapter 3, we return to the one-dimensional case with the goal of determining the Hausdorff dimension of the uncovered part of the unit interval U. During the parking process, each time an interval is placed, it splits a maximal uncovered interval into two pieces. If the length of an interval can be arbitrarily small, then this gives a binary tree structure analogous to the Cantor set. Using a theorem of Mauldin and Williams [12] about a certain type of random recursive construction, we will first determine the Hausdorff dimension when $F(t) = t^\alpha$ for some $\alpha > 0$, where F is the cumulative distribution function of the interval length. We will see that the Hausdorff dimension of the complement is $s(\alpha)$ almost surely for some $0 < s(\alpha) < 1$. Furthermore, $s(\alpha)$ can be expressed in terms of the beta function: it is the unique solution of the implicit equation

$$2(\alpha + 1)\mathrm{B}(\alpha + 1, s(\alpha) + 1) = 1.$$

The main result in Chapter 3 is a condition for determining the Hausdorff dimension in the general case. We will show the following theorem:

Theorem 4. *Let $F(t)$ be the cumulative distribution function of the interval length. Assume that $F(t) > 0$ for $t > 0$. If there is an $\varepsilon > 0$ such that $F(t)/t^\alpha$ is increasing (resp. decreasing) on $(0, \varepsilon)$, then the Hausdorff dimension of the complement at most (resp. at least) $s(\alpha)$ almost surely.*

The proof of this theorem is based on constructing a coupling with the case $F(t) = t^\alpha$ and using the result from [12] to derive the bounds. Finally, we will construct an example where the complement has Hausdorff dimension 0 almost surely and another example where the complement has Hausdorff dimension 1, but zero Lebesgue measure almost surely.

Previous results. When the intervals can be arbitrarily small, this model was first studied by Coffman, Mallows and Poonen [9] in the case when μ_0 is uniform. Generalizing their results, Baryshnikov and Gnedin [3] investigated the case when the cumulative distribution function of μ_0 is $F(t) = t^\alpha$ for $\alpha > 0$. They showed that for $\beta \geq 0$, the expected sum of βth powers of the gap lengths after N attempts is asymptotically $c(\beta)N^{(s(\alpha)-\beta)/(\alpha+1)}$ for some constant $c(\beta)$, where $s(\alpha)$ is the solution of the equation above. Substituting $\beta = 1$, it follows from this result that the intervals placed exhaust U in Lebesgue measure almost surely. The main contribution of this thesis is solving the problem of exhaustion for every distribution μ_0 in one dimension and generalizing the result to higher dimensions. However, unlike [3], our proof does not give explicit estimates for the Lebesgue measure. The method of our proof is also completely different, as the proofs in [3, 9] are based on solving a recursive integral equation similar to that used in Rényi's original proof.

The equation for the Hausdorff dimension $s(\alpha)$ was also obtained in [3]. The dimension in the case $\alpha = 1$ was computed in [12], though their model was not directly related to the parking problem. The contribution of this thesis to the problem of determining the Hausdorff dimension is the condition in Theorem 4. While the previous results are directly applicable only to $F(t) = t^\alpha$ due to the requirement of stochastic self-similarity, this theorem can be used for many choices of F. Our proof depends crucially on the previous result by constructing a coupling with the case $F(t) = t^\alpha$.

Notations

$U(x, y)$	uniform distribution on the interval (x, y)
$\Gamma(x)$	gamma function
$\mathrm{B}(x, y)$	beta function
$\lceil x \rceil$	ceiling function
$\mathbb{N}$	set of natural numbers $\{0, 1, 2, \dots\}$
$\mathcal{K}(\mathbb{R}^d)$	non-empty compact subsets of $\mathbb{R}^d$
$B(x, r)$	open ball of radius r centered in x
$\lvert A \rvert$	cardinality of set A
χ_A	indicator function of set A
$\operatorname{int} A$	interior of set A
∂A	boundary of set A
$\overline{A}$	closure of set A
$\operatorname{diam} A$	diameter of set A
A^*	finite sequences made from A
λ	Lebesgue measure
$\operatorname{supp} \nu$	support of measure ν
$\sigma(\mathcal{F})$	σ-algebra generated by $\mathcal{F}$
$\mathcal{F}\rvert_A$	restriction of the σ-algebra to $A \in \mathcal{F}$
$(X \mid \mathcal{F})$	conditional distribution of X given $\mathcal{F}$
$\lvert \sigma \rvert$	length of sequence σ
$\sigma\rvert_n$	initial segment of length n
$\sigma \leq \rho$	σ is an initial segment of ρ
$\sigma < \rho$	σ is a strict initial segment of ρ
$\dim_{\mathrm{H}} A$	Hausdorff dimension of A
$\mathcal{H}^s(A)$	s-dimensional Hausdorff measure of A
$\mathcal{H}^s_\delta(A)$	s-dimensional Hausdorff premeasure of A

1. The one-dimensional parking problem

1.1 The parking constant

We will present Rényi's [19] derivation for the parking constant in the one-dimensional case. None of the results in this section are new.

Given an $x > 0$, we consider the following problem on the interval $(0, x)$:

Definition 1.1.1. Let $\{Y_i\}_{i=1}^{\infty}$ be i.i.d. variables with distribution $U(0, x)$. We define the interval $I_i = [Y_i, Y_i + 1]$. The index set $S \subseteq \{1, 2, \dots\}$ is defined recursively: $i \in S$ if $Y_i < x - 1$ and for all $j < i$ such that $j \in S$, $I_i \cap I_j = \emptyset$.

Note that it does not matter whether the intervals are defined as open or closed, since the probability that any two endpoints coincide is zero.

Let ν_x be the distribution of $|S|$ when the interval has length x. One of the main questions first investigated by Rényi is determining the expected number of intervals $M(x) = \mathbb{E}(\nu_x)$. He proved the following theorem about M:

Theorem 1.1.2 (Rényi)**.** *There is a constant C such that $M(x)/x \to C$ as $x \to \infty$. The constant C can be expressed as*

$$C = \int_0^{\infty} \exp\left(-2 \int_0^t \frac{1 - e^{-u}}{u}\, du\right) dt \approx 0.7476.$$

Before we begin the proof, notice that $M(x) = 0$ for $0 < x \leq 1$. Also, $M(x) = 1$ for $1 < x < 2$, since $Y_i < x$ with positive probability and at most one interval can fit.

Lemma 1.1.3. *If $x > 0$ and $x \neq 1$, then $xM'(x + 1) + M(x + 1) = 2M(x) + 1$.*

Proof. Consider the parking process on the interval $(0, x+1)$. Since the attempts are independent, we may assume that $1 \in S$. Conditioning on this event, Y_1 is uniform in $(0, x)$. If $t = Y_1$, then I_1 splits the interval $(0, x + 1)$ into intervals the intervals $(0, t)$ and $(t + 1, x + 1)$. It is easy to check that given t, $|\{i \in S \,|\, Y_i \subseteq (0, t)\}|$ has the same distribution as ν_y and similarly, the number of intervals in $(t + 1, x + 1)$ has distribution ν_{t-x}. Therefore,

$$M(x + 1) = \mathbb{E}\big(\nu_{x+1}\big) = \mathbb{E}(\mathbb{E}(|S| \,|\, t)) = 1 + \mathbb{E}\big(\mathbb{E}(\nu_t \,|\, t) + \mathbb{E}(\nu_{x-t} \,|\, t)\big) =$$
$$= 1 + \frac{1}{x} \int_0^x \big(M(t) + M(x - t)\big)\, dt = 1 + \frac{2}{x} \int_0^x M(t)\, dt.$$

Multiplying by x, we obtain

$$xM(x+1) = 2\int_0^x M(t)\,dt + x. \tag{1.1.1}$$

This integral equation implies that M is continuous on the interval $(1,\infty)$. By the fundamental theorem of calculus, $\int_0^x M$ is differentiable on $(1,\infty)$, so M is differentiable on $(2,\infty)$. It is also clear that $M(x) = 1$ for $1 < x < 2$, hence M is also differentiable on $(1,2)$. The statement follows by differentiating Eq. (1.1.1). □

For $s > 0$, the Laplace transform of M is defined as

$$\varphi(s) = \int_0^\infty M(x)e^{-sx}\,dx. \tag{1.1.2}$$

The convergence of this integral is clear, since $0 \le M(x) \le x$ for every $x \ge 0$.

Lemma 1.1.4. $\lim_{s\to 0^+} s^2\varphi(s) = C$.

Proof. Let $w(s) = e^s\varphi(s)$. Since $M(x) = 0$ for $0 \le x \le 1$, it is clear that

$$\int_0^\infty M(x+1)e^{-sx}\,dx = \int_0^\infty M(x)e^{-s(x-1)}\,dx = e^s\varphi(s) = w(s). \tag{1.1.3}$$

It follows from Lemma 1.1.3 that $M'(x+1) = O(1)$. Therefore, $\int_0^\infty xM'(x+1)e^{-xs}\,dx$ is continuous as a function of s. Consequently,

$$\begin{aligned}\int_0^\infty xM'(x+1)e^{-xs}\,dx &= \frac{d}{ds}\left(\int_0^s\int_0^\infty xM'(x+1)e^{-xt}\,dx\,dt\right) = \\ &= \frac{d}{ds}\left(\int_0^\infty\int_0^s xM'(x+1)e^{-xt}\,dt\,dx\right) = \\ &= -\frac{d}{ds}\left(\int_0^\infty M'(x+1)e^{-xs}\,dx\right) = \\ &= -\frac{d}{ds}\left(\left[M(x+1)e^{-xs}\right]_{x=0^+}^\infty + s\int_0^\infty M(x+1)e^{-xs}\,dx\right) = \\ &= -\frac{d}{ds}(-1 + se^s\varphi(s)) = -w(s) - sw'(s).\end{aligned} \tag{1.1.4}$$

Taking the Laplace transform of the equation in Lemma 1.1.3, we obtain

$$\int_0^\infty xM'(x+1)e^{-xs}\,dx + \int_0^\infty M(x+1)e^{-xs}\,dx = 2\int_0^\infty M(x)e^{-xs}\,dx + \int_0^\infty e^{-xs}.\,dx$$

Substituting Eqs. (1.1.2) to (1.1.4), we get the equation

$$-w(s) - sw'(s) + w(s) = 2\varphi(s) + \frac{1}{s},$$

hence,

$$sw'(s) + 2w(s)e^{-s} = -\frac{1}{s}. \tag{1.1.5}$$

We will solve this differential equation using variation of constants. First, we find a particular solution of the homogeneous equation:

$$\begin{aligned} s w_0'(s) + 2w_0(s)e^{-s} &= 0 \\ (\log w_0(s))' = \frac{w_0'(s)}{w_0(s)} &= -\frac{2e^{-s}}{s} \\ \log w_0(s) &= 2\int_s^\infty \frac{e^{-u}}{u}\,du \\ w_0(s) &= \exp\left(2\int_s^\infty \frac{e^{-u}}{u}\,du\right) \end{aligned}$$

Now let $w(s) = c(s)w_0(s)$. Substituting into Eq. (1.1.5), we get the equation

$$\begin{aligned} sc'(s)w_0(s) + sc(s)(w_0'(s) + 2w_0e^{-s}) &= -\frac{1}{s} \\ sc'(s)w_0(s) &= -\frac{1}{s} \\ c(s) &= A + \int_s^\infty \frac{dt}{t^2 w_0(t)}. \end{aligned}$$

The convergence of this integral is easy to see, since $w_0 \geq 1$. It follows that the general solution for w is of the form

$$w(s) = Aw_0(s) + \int_s^\infty \frac{w_0(s)}{t^2 w_0(t)}\,dt$$

for some constant A. Since $M(x) \leq x$, we can see that

$$0 \leq w(s) \leq e^s \int_0^\infty xe^{-xs}\,dx = \left[-\frac{xe^{-xs}}{s}\right]_{x=0}^\infty + \int_0^\infty \frac{e^{-xs}}{s}\,dx = \frac{1}{s^2},$$

hence $\lim_{s\to\infty} w(s) = 0$. It is also clear that $\lim_{s\to\infty} w_0(s) = e^0 = 1$. Therefore,

$$\lim_{s\to\infty} w(s) = A + \lim_{s\to\infty} \int_s^\infty \frac{dt}{t^2 w_0(t)} = A,$$

consequently, $A = 0$.

An easy calculation shows that

$$\begin{aligned} s^2 w(s) &= \int_s^\infty \frac{s^2 w_0(s)}{t^2 w_0(t)}\,dt = \int_s^\infty \frac{s^2}{t^2} \exp\left(2\int_s^t \frac{e^{-u}}{u}\,du\right) dt = \\ &= \int_s^\infty \exp\left(-2[\log u]_{u=s}^t + 2\int_s^t \frac{e^{-u}}{u}\,du\right) = \\ &= \int_s^\infty \exp\left(-2\int_s^t \frac{1-e^{-u}}{u}\,du\right) dt. \end{aligned}$$

Since $s^2\varphi(s) = e^{-s}s^2 w(s)$, it remains to show that $s^2 w(s) \to C$ as $s \to 0^+$. We will assume that $s \leq 1$. Note that if $u \geq 1$, then

$$\frac{1-e^{-u}}{u} \geq \frac{1-e^{-1}}{u} \geq \frac{3}{5u},$$

which yields the bound

$$\exp\left(-2\int_s^t \frac{1-e^{-u}}{u}\,du\right) \le \exp\left(-2\int_1^t \frac{3}{5u}\,du\right) = e^{-6/5\cdot\log t} = t^{-6/5}$$

for $t \ge 1$. On the other hand, the integrand of the outer integral is clearly bounded from above by 1. Since $\min\bigl(1, t^{-6/5}\bigr)$ is integrable on $(0,\infty)$, we can apply the dominated convergence theorem:

$$\begin{aligned}\lim_{s\to 0^+} s^2 w(s) = \lim_{s\to 0^+} \int_0^\infty \chi_{\{t>s\}} \exp\left(-2\int_s^t \frac{1-e^{-u}}{u}\,du\right) dt &= \\ = \int_0^\infty \left(\lim_{s\to 0^+} \chi_{\{t>s\}} \exp\left(-2\int_s^t \frac{1-e^{-u}}{u}\,du\right)\right) dt &= \\ = \int_0^\infty \exp\left(-2\int_0^t \frac{1-e^{-u}}{u}\,du\right) dt = C.\end{aligned}$$

This calculation also shows that the constant C is finite. □

To finish the proof, Rényi used the following variant of the Hardy–Littlewood Tauberian theorem:

Theorem 1.1.5 ([8, Theorem 108])**.** *Suppose that* $\alpha : (0,\infty) \to \mathbb{R}$ *is increasing and for some* $\beta > 0$ *and* C*,*

$$\lim_{s\to 0^+} s^\beta \int_0^\infty e^{-sx}\,d\alpha(x) = C.$$

Then

$$\lim_{x\to\infty} \frac{\alpha(x)}{x^\beta} = \frac{C}{\Gamma(\beta+1)},$$

where Γ *is the gamma function.*

By Lemma 1.1.4, applying this theorem to $\alpha(x) = \int_0^x M(t)\,dt$ and $\beta = 2$ yields

$$\lim_{x\to\infty} \frac{1}{x^2} \int_0^x M(t)\,dt = \frac{C}{2}.$$

Using Eq. (1.1.1), we can conclude that

$$\lim_{x\to\infty} \frac{M(x)}{x} = \lim_{x\to\infty} \frac{M(x+1)}{x} = \lim_{x\to\infty} \frac{2}{x^2} \int_0^x M(t)\,dt = C.$$

In the next section, an elementary way of finishing the proof will also be shown.

1.2 Estimate of the expectation

In this section, an estimate for $M(x)$ due to Dvoretzky and Robbins [6] is proved. While they also gave estimates for higher moments of ν_x, a simplified version of their proof is presented here, giving only the estimate for $M(x)$.

Theorem 1.2.1 ([6, Theorem 3])**.** *There exists a constant* C_1 *such that*

$$M(x) = C_1 x + C_1 - 1 + O\bigg(\Big(\frac{2e}{x}\Big)^{x-3/2}\bigg).$$

Proof. Let $f(x) = M(x) + 1$. It follows from Eq. (1.1.1) that for $x > 0$,

$$f(x+1) = \frac{2}{x}\int_0^x f(t)\,dt. \tag{1.2.1}$$

For $x \le y$, this implies that

$$f(y+1) = \frac{2}{y}\int_0^x f(t)\,dt + \frac{2}{y}\int_x^y f(t)\,dt = \frac{x}{y}f(x+1) + \frac{2}{y}\int_x^y f(t)\,dt. \tag{1.2.2}$$

Notice that the function $x \mapsto x+1$ is also a solution of Eq. (1.2.1). A similar calculation shows that

$$y + 2 = \frac{x}{y}(x+2) + \frac{2}{y}\int_x^y (t+1)\,dt. \tag{1.2.3}$$

Let

$$I_x = \inf_{x\le t\le x+1} \frac{f(t)}{t+1}, \quad S_x = \sup_{x\le t\le x+1} \frac{f(t)}{t+1}.$$

For $0 < x, y$, it follows from Eqs. (1.2.2) and (1.2.3) that

$$f(y+1) - I_x(y+2) = \frac{x}{y}(f(x) - I_x(x+1)) + \frac{2}{y}\int_x^y (f(t) - I_x(t+1))\,dt \ge 0 + 0 = 0,$$

hence

$$I_{x+1} = \inf_{x\le y\le x+1} \frac{f(y+1)}{y+2} \ge \inf_{x\le y\le x+1} \frac{I_x(y+2)}{y+2} = I_x,$$

and similarly, $S_x \le S_{x+1}$.

Furthermore, for $0 < x \le y \le x+2$,

$$f(y+1) - f(x+1) = \frac{x-y}{y}f(x+1) + \frac{2}{y}\int_x^y f(t)\,dt = O(1)$$

as $x \to \infty$, since $f(x) = O(x)$. Consequently,

$$\frac{f(y+1)}{y+2} - \frac{f(x+1)}{x+2} = \frac{f(y+1) - f(x+1)}{y+2} + f(x+1)\Big(\frac{1}{y+2} - \frac{1}{x+2}\Big) = O\Big(\frac{1}{x}\Big).$$

Therefore, $S_x - I_x = o(1)$. Furthermore, for $x \le y \le x+1$, $I_x - I_y = o(1)$ and $S_x - S_y = o(1)$. As a result, the limit $C_1 = \lim_{x\to\infty} I_x = \lim_{x\to\infty} S_x$ exists.

Let $f^*(x) = f(x) - C_1(x+1)$ and fix an integer $n \ge 2$. Note that $I_n \le C_1 \le S_n$, hence f^* takes on both non-negative and non-positive values on the interval $[n, n+1]$. The continuity of f on $(1, \infty)$ implies that for some $n \le y \le n+1$, $f^*(y) = 0$. Clearly, f^* also satisfies Eq. (1.2.1) and consequently Eq. (1.2.2), which shows that for $n \le x \le n+1$,

$$f^*(x+1) = \frac{y}{x}f^*(y) + \frac{2}{y}\int_y^x f^*(t)\,dt = \frac{2}{y}\int_y^x f^*(t)\,dt.$$

Let $T_x = \sup_{x \le y \le x+1} |f^*(x)|$. The previous equation implies that

$$T_{n+1} \le \frac{2}{y} \int_n^{n+1} T_n \le \frac{2}{n} T_n.$$

By induction,

$$T_n \le \frac{2^{n-1}}{(n-1)!} T_2$$

for $n \ge 2$. Hence, by Stirling's formula,

$$T_{n-1} \le \frac{2^{n-2} n(n-1)}{n!} T_2 = O\left(\frac{2^n n^2}{\sqrt{n}(n/e)^n} \right) = O\left(\frac{2e}{n} \right)^{n-3/2}.$$

As the function $x \mapsto (2e/x)^{x-3/2}$ is decreasing for large x, we can conclude that

$$|f^*(x)| \le T_{\lceil x \rceil - 1} = O\left(\frac{2e}{\lceil x \rceil} \right)^{\lceil x \rceil - 3/2} = O\left(\frac{2e}{x} \right)^{x-3/2}.$$

Expanding the definitions of f^* and f, we obtain the estimate to be proved. □

Finally, we will prove that $C_1 = C$, giving an alternative proof for $M(x)/x \to C$.

Corollary 1.2.2. $\lim_{s \to 0^+} s^2 \varphi(s) = C_1$.

Proof. First, notice that

$$s^2 \int_0^\infty e^{-sx} x \, dx = s^2 \left[-\frac{e^{-sx} x}{s} + \frac{e^{-sx}}{s^2} \right]_{x=0}^\infty = 1.$$

By Theorem 1.2.1, $M(x) = C_1 x + O(1)$, hence,

$$s^2 \varphi(s) = s^2 \int_0^\infty e^{-sx} M(x) \, dx = C_1 + O(1) \, s^2 \int_0^\infty e^{-sx} \, dx = C_1 + O(s).$$

□

2. The parking problem with random sets

Assume that μ is a Borel probability measure on the space of non-empty compact sets $\mathcal{K}(\mathbb{R}^d)$ with the topology induced by the Hausdorff metric. We will additionally assume that $0 \in X$ for μ a.e. set X. Furthermore, let $U \subseteq \mathbb{R}^d$ be a non-empty open set and let ν be a Borel probability measure on U. The parking process is defined analogously to the one-dimensional case:

Definition 2.0.1. Let $\{X_i\}_{i=1}^{\infty}$ and $\{Y_i\}_{i=1}^{\infty}$ be independent sequences of i.i.d. variables, where $X_i \sim \mu$ and $Y_i \sim \nu$. We define the set $Z_i = X_i + Y_i$. The index set $S \subseteq \{1, 2, \dots\}$ is defined recursively: $i \in S$ if $Z_i \subseteq U$ and for all $j < i$ such that $j \in S$, $Z_i \cap Z_j = \emptyset$. Finally, let $A = \bigcup_{i \in S} Z_i$.

The measurability of Z_i is clear, since the map $\mathcal{K}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathcal{K}(\mathbb{R}^d), (X, Y) \mapsto X + Y$ is continuous. It is also easy to check that the set $\{(Z, Z') \in \mathcal{K}(\mathbb{R}^d) \times \mathcal{K}(\mathbb{R}^d) \mid Z \cap Z' = \emptyset\}$ is open, implying the measurability of the event $\{i \in S\}$ for every i.

2.1 Singleton sets with positive probability

We will first show that if Z_i is a singleton set with positive probability, then $\lambda(U \setminus A) > 0$ almost surely.

Proposition 2.1.1. *The maps $Z \mapsto \lambda(Z)$ and $Z \mapsto \nu(Z)$ with domain $\mathcal{K}(\mathbb{R}^d)$ are Borel measurable.*

Proof. It is enough to prove upper semi-continuity of the maps. Let $Z \in \mathcal{K}(\mathbb{R}^d)$ such that $\lambda(Z) < t$ for some t. Since Z is closed, clearly $Z = \bigcap_{n=1}^{\infty} Z_{1/n}$, where $Z_{1/n}$ is the open $1/n$-neighborhood of Z. The compactness of Z implies that $Z_{1/n}$ is bounded, therefore $\lambda(Z_{1/n})$ is finite. By measure continuity, there is an n such that $\lambda(Z_{1/n}) < t$. Suppose that $d_{\mathrm{H}}(Z, Z') < 1/n$ for some Z', where d_{H} denotes the Hausdorff metric. Then $Z' \subseteq Z_{1/n}$, hence $\lambda(Z') \leq \lambda(Z_{1/n}) < t$. This proves the upper semi-continuity.

For ν, the proof is the same, except that the measure continuity follows from the finiteness of ν. □

Lemma 2.1.2. *Assume that $\mu(\{\{0\}\}) = p_0 > 0$. Let $\mathcal{F}_m = \sigma(X_1, \dots, X_m, Y_1, \dots, Y_m)$ and suppose that $\mathcal{F}_m$ is given. Let $A_m = \bigcup_{i \leq m, i \in S} Z_i$, clearly A_m is $\mathcal{F}_m$-measurable. Also, let $X \sim \mu$*

and $Y \sim \nu$ be independent variables and let $Z = X + Y$. If $\nu(A_m) < 1$, then

$$\mathbb{P}(\nu(A) = 1 \,|\, \mathcal{F}_m) \le \frac{\mathbb{P}(\nu(Z) > 0, Z \subseteq U \setminus A_m)}{p_0(1 - \nu(A_m))}.$$

Proof. Let $i > m$. We will first give an upper bound on $\mathbb{P}(i \in S \,|\, \mathcal{F}_m, X_i, Y_i)$.

Suppose that $m < j < i$ is such that $X_j = \{0\}$ and $Y_j \in Z_i$. If $j \in S$, then $Y_j \in Z_i \cap Z_j$. Otherwise, there is a $k < j$ for which $Y_j \in Z_k$, hence $Y_j \in Z_i \cap Z_k$. In both cases $i \notin S$. Therefore, a necessary condition for $i \in S$ is that there is no $m \le j < i$ such that $X_j = \{0\}$ and $Y_j \in Z_i$. Another necessary condition is that $Z_i \subseteq U \setminus A_m$. It follows from the independence that

$$\begin{aligned}
&\mathbb{P}(i \in S \,|\, \mathcal{F}_m, X_i, Y_i) \le \\
&\le \mathbb{P}\big((\nexists\, m < j < i : X_j = \{0\} \wedge Y_j \in Z_i), Z_i \subseteq U \setminus A_m \,\big|\, \mathcal{F}_m, X_i, Y_i\big) = \\
&= \prod_{m=j+1}^{i-1} \big(1 - \mathbb{P}(X_j = \{0\}, Y_j \in Z_i \,|\, X_i, Y_i)\big) \cdot \chi_{\{Z_i \subseteq U \setminus A_m\}} = \\
&= (1 - p_0\, \nu(Z_i))^{i-m-1} \chi_{\{Z_i \subseteq U \setminus A_m\}}.
\end{aligned}$$

It is easy to see that the conditional distribution of Z_i given $\mathcal{F}_m$ is the same as that of Z. Therefore,

$$\begin{aligned}
\mathbb{E}\big(\nu(A \setminus A_m) \,\big|\, \mathcal{F}_m\big) &= \mathbb{E}\Big(\sum_{i=m+1}^{\infty} \chi_{\{i \in S\}} \nu(Z_i) \,\Big|\, \mathcal{F}_m\Big) = \\
&= \sum_{i=m+1}^{\infty} \mathbb{E}\big(\chi_{\{i \in S\}} \nu(Z_i) \,\big|\, \mathcal{F}_m\big) = \\
&= \sum_{i=m+1}^{\infty} \mathbb{E}\big(\mathbb{P}(i \in S \,|\, \mathcal{F}_m, X_i, Y_i)\, \nu(Z_i) \,\big|\, \mathcal{F}_m\big) \le \\
&\le \sum_{i=m+1}^{\infty} \mathbb{E}\big((1 - p_0\, \nu(Z_i))^{i-m-1} \chi_{\{Z_i \subseteq U \setminus A_m\}} \nu(Z_i) \,\big|\, \mathcal{F}_m\big) = \\
&= \sum_{i=m+1}^{\infty} \mathbb{E}\big((1 - p_0\, \nu(Z))^{i-m-1} \chi_{\{Z \subseteq U \setminus A_m\}} \nu(Z)\big) = \\
&= \mathbb{E}\Big(\sum_{i=m+1}^{\infty} (1 - p_0\, \nu(Z))^{i-m-1} \chi_{\{Z \subseteq U \setminus A_m\}} \nu(Z)\Big).
\end{aligned}$$

If $\nu(Z) = 0$, then every term of the sum is zero. Otherwise,

$$\sum_{i=m+1}^{\infty} (1 - p_0\, \nu(Z))^{i-m-1} \nu(Z) = \frac{\nu(Z)}{p_0\, \nu(Z)} = \frac{1}{p_0}.$$

Consequently,

$$\mathbb{E}\big(\nu(A \setminus A_m) \,\big|\, \mathcal{F}_m\big) \le \mathbb{E}\left(\frac{\chi_{\{\nu(Z) > 0, Z \subseteq U \setminus A_m\}}}{p_0}\right) = \frac{\mathbb{P}\big(\nu(Z) > 0, Z \subseteq U \setminus A_m\big)}{p_0}.$$

Finally, we can apply Markov's inequality:

$$\mathbb{P}\big(\nu(A) = 1 \,\big|\, \mathcal{F}_m\big) = \mathbb{P}\big(\nu(A \setminus A_m) = 1 - \nu(A_m) \,\big|\, \mathcal{F}_m\big) \leq$$
$$\leq \frac{\mathbb{E}\big(\nu(A \setminus A_m) \,\big|\, \mathcal{F}_m\big)}{1 - \nu(A_m)} \leq \frac{\mathbb{P}\big(\nu(Z) > 0, Z \subseteq U \setminus A_m\big)}{p_0(1 - \nu(A_m))}.$$

□

Theorem 2.1.3. *Suppose that $\mu(\{\{0\}\}) > 0$, $\nu \ll \lambda$ and $\lambda(U) < \infty$. Then $\lambda(U \setminus A) > 0$ almost surely.*

Proof. Fix an $\varepsilon > 0$. If $\lambda(U \setminus A) = 0$, then there exists a minimal k such that $\lambda(U \setminus A_k) < \varepsilon$. We will show that

$$\mathbb{P}\big(\lambda(U \setminus A) = 0 \,\big|\, k = m\big) \leq \frac{\mathbb{P}(0 < \lambda(X) < \varepsilon)}{p_0}.$$

for every m.

Since A_m is a compact subset of the open set U, it follows that $U \setminus A_m$ is a non-empty open set, therefore $\lambda(U \setminus A_m) > 0$. If $\nu(A_m) = 1$, then for every $i > m$, $Y_i \in A_m$ almost surely, which implies that $i \notin S$. Consequently, $A = A_m$ and $\lambda(U \setminus A) = \lambda(U \setminus A_m) > 0$. Hence, it is enough to consider the case when $\nu(A_m) < 1$. Note that $\{k = m\}, \{\nu(A_m) < 1\} \in \mathcal{F}_m$. Further conditioning on $\mathcal{F}_m$, then applying Lemma 2.1.2, we obtain

$$\begin{aligned}
&\mathbb{P}\big(\nu(A) = 1 \,\big|\, k = m, \nu(A_m) < 1, \mathcal{F}_m\big) \leq \\
&\leq \frac{\mathbb{P}(\nu(Z) > 0, Z \subseteq U \setminus A_m)}{p_0(1 - \nu(A_m))} \leq \\
&\leq \frac{\mathbb{P}(0 < \lambda(Z) \leq \lambda(U \setminus A_m), Y \in U \setminus A_m)}{p_0(1 - \nu(A_m))} \leq \\
&\leq \frac{\mathbb{P}(0 < \lambda(X) < \varepsilon, Y \in U \setminus A_m)}{p_0(1 - \nu(A_m))} = \\
&= \frac{\mathbb{P}(0 < \lambda(X) < \varepsilon)\mathbb{P}(Y \in U \setminus A_m)}{p_0(1 - \nu(A_m))} = \\
&= \frac{\mathbb{P}(0 < \lambda(X) < \varepsilon)}{p_0}.
\end{aligned}$$

As this inequality is true for all m, it follows from measure continuity that

$$\mathbb{P}(\lambda(U \setminus A) = 0) \leq \frac{\mathbb{P}(0 < \lambda(X) < \varepsilon)}{p_0} \overset{\varepsilon \to 0}{\longrightarrow} 0.$$

□

Remark. When ν is uniform on U, we obtain Theorem 3 as a special case.

Generally, it is not true that $\nu(A) < 1$ almost surely. For example, if $\operatorname{supp} \nu \subseteq U$ is compact, then μ can be chosen such that $\operatorname{supp} \nu \subseteq Z_1 \subseteq U$ with positive probability. This clearly implies that $\mathbb{P}(\nu(A) = 1) > 0$.

2.2 The density and neighborhood conditions

Definition 2.2.1. A family of compact sets $\mathcal{D}$ satisfies the *density condition* if there is an $\varepsilon > 0$ such that for every $X \in \mathcal{D}$, every $x \in X$ and $0 < r \leq \operatorname{diam} X$,

$$\lambda(X \cap B(x,r)) \geq \varepsilon r^d,$$

where $B(x,r)$ is the open ball of radius r centered around x.

It is easy to check that if $\mathcal{D}_1, \ldots, \mathcal{D}_n$ satisfy the density condition, then so does $\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_n$. It is also clear that the definition is invariant under similarities.

An example of a family satisfying the density condition are the rectangular boxes

$$\mathcal{B}_K = \left\{ \times_{i=1}^d [0,a_i] \,\middle|\, a_1, \ldots, a_d > 0; \forall i,j : a_i \leq K a_j \right\}.$$

for some $K \geq 1$.

Definition 2.2.2. A *shape* is an equivalence class of $\mathcal{K}(\mathbb{R}^d)$ modulo the similarities of $\mathbb{R}^d$.

Proposition 2.2.3. *Let $\mathcal{D} \subseteq K(\mathcal{R}^d)$ be a family containing finitely many shapes, each of which is the union of finitely many convex sets with non-empty interiors. Then $\mathcal{D}$ satisfies the density condition.*

Proof. We will first consider the case when $\mathcal{D} = \{C\}$ for some convex compact set C with non-empty interior. Choose a ball $B(x, r_0) \subseteq C$ with $r_0 > 0$. Suppose that $y \in C$ and $0 < r \leq \operatorname{diam} C$. Let $t = r/(\operatorname{diam} C + r_0) \in (0,1)$. It follows from the convexity assumption that

$$B(y + t(x-y), tr_0) = (1-t)y + tB(x, r_0) \subseteq C.$$

It is also clear from $|x - y| \leq \operatorname{diam} C$ that $y + t(x-y) \in \overline{B(y, t \operatorname{diam} C)}$, therefore

$$B(y + t(x-y), tr_0) \subseteq B(y, t \operatorname{diam} C + tr_0) = B(y,r).$$

Hence,

$$\lambda(C \cap B(y,r)) \geq \lambda(B(y + t(x-y), tr_0)) = r^d \lambda\left(B\left(0, \frac{r_0}{\operatorname{diam} C + r_0}\right)\right).$$

This means that the condition holds for $\varepsilon = \lambda\big(B\big(0, \frac{r_0}{\operatorname{diam} C + r_0}\big)\big)$.

Now let $\mathcal{D} = \{X\}$ for $X = C_1 \cup \ldots C_n$, where each C_i is convex and has non-empty interior. We know that for every C_i, there is an $\varepsilon_i > 0$ that satisfies the condition. Let

$$\varepsilon = \min_{1 \leq i \leq n} \varepsilon_i \left(\frac{\operatorname{diam} C_i}{\operatorname{diam} X} \right)^d.$$

Let $y \in X$ and $0 < r \leq \operatorname{diam} X$. Then there is an i such that $y \in C_i$. It follows from the density condition for $\{C_i\}$ that

$$\lambda(X \cap B(y,r)) \geq \lambda\left(C_i \cap B\left(y, r\frac{\operatorname{diam} C_i}{\operatorname{diam} X}\right)\right) \geq \varepsilon_i \left(\frac{r \operatorname{diam} C_i}{\operatorname{diam} X}\right)^d \geq \varepsilon r^d,$$

therefore $\{X\}$ also satisfies the density condition.

Now assume that for $X_1, \dots, X_n \in \mathcal{K}(\mathbb{R}^d)$, each $\{X_i\}$ satisfies the density condition with ε_i. It is easy to check that the condition holds for $\mathcal{D} = \{X_1, \dots, X_n\}$ with $\varepsilon = \min_{1 \le i \le n} \varepsilon$. Finally, since the definition is invariant under similarity, the same ε works if each set in $\mathcal{D}$ is similar to some X_i. $\square$

Definition 2.2.4. The family $\mathcal{N} \subseteq \mathcal{K}(\mathbb{R}^d)$ satisfies the *neighborhood condition* if following two properties hold:

(1) There is an $R > 0$ such that for every $X \in \mathcal{N}$, $\operatorname{diam} X \le R$.

(2) For every $\delta > 0$, there is an $\eta > 0$ such that for every $X \in \mathcal{N}$,

$$\lambda(\{x \in \mathbb{R}^d \,|\, 0 < d(x, X) < \eta \operatorname{diam} X\}) \le \delta \lambda(X),$$

where $d(x, X)$ denotes the Euclidean distance between x and X.

Remark. If $\mathcal{N}$ satisfies Condition (2), then for some constant $c > 0$, the inequality $\operatorname{diam}(X) \le c\lambda(X)^{1/d}$ holds for every $X \in \mathcal{N}$. Therefore, if $\lambda(U) < \infty$, then the diameter of a set $X \in \mathcal{N}$ that can possibly fit in U is automatically bounded. It follows that if μ is concentrated on $\mathcal{N}$, then $\mathcal{N}$ can be replaced by a family that also satisfies Condition (1). Hence, Condition (1) matters only in the case $\lambda(U) = \infty$.

Proposition 2.2.5. *Let $\mathcal{N} \subseteq \mathcal{K}(\mathbb{R}^d)$ be family such that $\sup_{X \in \mathcal{N}} \operatorname{diam} X < \infty$. If $\mathcal{N}$ contains finitely many shapes, each of which has positive Lebesgue measure, then $\mathcal{N}$ satisfies the neighborhood condition.*

Proof. It is easy to check that Condition (2) is invariant under similarity. This implies that it suffices to consider the case when $\mathcal{N}$ is finite.

Let $\delta > 0$ and $X \in \mathcal{N}$. Note that

$$\bigcap_{l=1}^{\infty} \{x \,|\, 0 < d(x, X) < \operatorname{diam} X / l\} = \emptyset.$$

Since X is bounded, the set $\{x \,|\, 0 < d(x, X) < \operatorname{diam} X / l\}$ is also bounded, so its Lebesgue measure is finite. By measure continuity and the assumption that $\lambda(X) > 0$, there is an l_X such that $\{x \,|\, 0 < d(x, X) < \operatorname{diam} X / l_X\} \le \delta\lambda(X)$. Finally, it easy to check that for $\eta = \min_{X \in \mathcal{N}} 1/l_X$, the condition holds. $\square$

Theorem 2.2.6. *Suppose that $\mathcal{D}$ satisfies the density condition, $\mathcal{N}$ satisfies the neighborhood condition and $\mu(\mathcal{D} \cup \mathcal{N}) = 1$. If there is an $r > 0$ such that $\mu(\{X \,|\, 0 < \operatorname{diam} X < r\}) = 0$, then $\lambda(U \setminus A) > 0$ almost surely.*

Proof. By the neighborhood condition, there is an $\eta > 0$ such that for every $X \in \mathcal{N}$,

$$\lambda(\{x \in \mathbb{R}^d \,|\, 0 < d(x, X) < \eta \operatorname{diam} X\}) \le \lambda(X).$$

Note that $0 \in X$ and $\operatorname{diam} X \geq r$ for μ a.e. X. Consequently,

$$\lambda(B(0,\eta r)) \leq \lambda(\{x \in \mathbb{R}^d \mid 0 \leq d(x,X) < \eta \operatorname{diam} X\}) \leq \lambda(X) + \lambda(X) = 2\lambda(X).$$

Therefore, $\lambda(X) \geq 1/2 \cdot \lambda(B(0,\eta r))$.

If $S = \emptyset$, then clearly $\lambda(A) = 0$. Otherwise, let $m = \min S$ and choose an $x \in \partial Z_m$. Since $Z_m \subseteq U$ is closed, it follows that $\lambda(B(x,1) \cap U \setminus Z_m) > 0$. Suppose for contradiction that $\lambda(U \setminus A) = 0$. Let $A_k = \bigcup_{i \leq k, i \in S} Z_i$. It follows from measure continuity that there is a $k \geq m$ such that

$$\lambda(B(x,1+R) \cap U \setminus A_k) < \min\Big(\frac{1}{2}\lambda(B(0,\eta r)),\, \varepsilon r^d,\, \lambda(B(x,1) \cap U \setminus Z_m)\Big),$$

where ε and R are the constants from Definitions 2.2.1 and 2.2.4. We may assume that $R \geq r$. It is easy to check that

$$\lambda(B(x,1) \cap U \cap A_k \setminus Z_m) = \lambda(B(x,1) \cap U \setminus Z_m) - \lambda(B(x,1) \cap U \setminus A_k) > 0,$$

in particular, $B(x,1) \cap U \cap A_k \setminus Z_m$ is non-empty. Since Z_m and $A_k \setminus Z_m$ are both closed, the connectedness of $B(x,1)$ implies that $B(x,1) \cap U \not\subseteq A_k$. It follows that $B(x,1) \cap U \setminus A_k$ is a non-empty open set, hence it has positive measure.

It is enough to show that $B(x,1) \cap A \setminus A_k$ is countable, which implies that $B(x,1) \cap U \setminus A$ has positive measure. If $\operatorname{diam} Z_i = 0$ for some i, then Z_i is a singleton, so such sets contribute only countably many points. Suppose for contradiction that $Z_i \in \mathcal{D} \cup \mathcal{N}$ is such that $\operatorname{diam} Z_i \geq r$, $Z_i \subseteq U \setminus A_k$ and $Z_i \cap B(x,1) \neq \emptyset$. Choose a $y \in Z_i \cap B(x,1)$. If $Z_i \in \mathcal{D}$, then

$$\lambda(Z_i \cap B(x,1+R)) \geq \lambda(Z_i \cap B(y,r)) \geq \varepsilon r^d > \lambda(B(x,1+R) \cap U \setminus A_k) \geq \lambda(Z_i \cap B(x,1+R)),$$

a contradiction. If $Z_i \in \mathcal{N}$, then $\operatorname{diam} Z_i \leq R$, which implies that $Z_i \subseteq \overline{B(y,R)} \subseteq B(x,1+R)$, therefore,

$$\lambda(Z_i \cap B(x,1+R)) = \lambda(Z_i) \geq \frac{1}{2}\lambda(B(0,\eta r)) > \lambda(B(x,1+R) \cap U \setminus A_k) \geq \lambda(Z_i \cap B(x,1+R)),$$

which is also a contradiction. □

2.3 Proof of the main theorem

We will use the notation $F(t) = \mu(\{X \mid \operatorname{diam} X < t\})$. The goal of this section is to prove the following theorem:

Theorem 2.3.1. *Suppose that $\mathcal{D}$ satisfies the density condition, $\mathcal{N}$ satisfies the neighborhood condition and $\mu(\mathcal{D} \cup \mathcal{N}) = 1$. If $F(t) > 0$ for every $t > 0$, $\mu(\{\{0\}\}) = 0$ and $\nu \ll \lambda$, then $\nu(A) = 1$ almost surely.*

Remark. In general, it is not true that $\lambda(A) = 1$ almost surely. For example, if U contains a ball $B(x,2r)$ such that $\nu(B(x,2r)) = 0$ and $\mu(\{X \mid \operatorname{diam} X < r\}) = 1$, then $A \cap B(x,r) = \emptyset$ almost surely.

In the whole section, we will assume that $F(t) > 0$ for every $t > 0$ and $\nu \ll \lambda$.

Let f be the density function of ν and let $L \subseteq U$ be the set of Lebesgue points of f in U where $f > 0$. By Lebesgue's differentiation theorem, λ a.e. $x \in \mathbb{R}^d$ is a Lebesgue point of f. It follows that

$$\mu(U \setminus L) = \int_{U \setminus L} f(x)\, d\lambda(x) = \int_{U \cap f^{-1}(0)} f(x)\, d\lambda(x) = 0,$$

therefore $\mu(L) = 1$.

Lemma 2.3.2. *For every $x \in L$ and $r > 0$ such that $B(x, r) \subseteq U$, there exists a $k \in S$ almost surely such that $Z_k \cap B(x, 2r) \neq \emptyset$.*

Proof. We will first prove that there is an i such that $Z_i \subseteq B(x, r)$ almost surely. Since Z_i are i.i.d., it is enough to show that $\mathbb{P}(Z_i \subseteq B(x, r)) > 0$. We can see that if $\operatorname{diam} X_i < r/2$ and $Y_i \in B(x, r/2)$, then $Z_i \subseteq B(x, r)$. Therefore,

$$\begin{aligned}\mathbb{P}(Z_i \subseteq B(x, r)) \geq \mathbb{P}(\operatorname{diam} X_i < r/2, Y_i \in B(x, r/2)) = \\ = F(r/2)\, \nu(B(x, r/2)) \geq \frac{1}{2} F(r/2)\, f(x)\, \lambda(B(x, r/2)) > 0.\end{aligned}$$

If $Z_i \subseteq B(x, r)$, then either $i \in S$ or there is a $j < i$ such that $j \in S$ and $Z_j \cap Z_i \neq \emptyset$, which implies that $Z_j \cap B(x, r) \neq \emptyset$. In both cases the needed k exists. □

Fix an $x \in L$. By the definition of L, there is an $r_0 > 0$ such that $B(x, r_0) \subseteq U$ and for every $r < r_0$,

$$\frac{f(x)}{2} \leq \frac{\nu(B(x, r))}{\lambda(B(x, r))} \leq 2f(x).$$

For an integer $n \geq 1$, let $r_n = r_0/2^n$.

To avoid difficulties later around measurability, we will choose $r_0 = 1/l$, where l is the smallest positive integer such that $B(x, r_0) \subseteq U$ and for every $n \geq 1$,

$$\frac{f(x)}{2} \leq \frac{\nu(B(x, r_n))}{\lambda(B(x, r_n))} \leq 2f(x).$$

For $n \geq 1$, let

$$k_n = \min\{k \in S \mid Z_k \cap B(x, 2r_n) \neq \emptyset\}.$$

Note that k_n exists almost surely by Lemma 2.3.2. Furthermore, let $d_n = 1$ if $\operatorname{diam} Z_{k_n} \geq r_n/2$, otherwise let $d_n = 0$.

Lemma 2.3.3. *For every $i \geq 1$,*

$$\mathbb{P}(d_i = 1 \mid d_1, \ldots, d_{i-1}) \geq \frac{1}{4 \cdot 3^d}\left(1 - \frac{F(r_{i+1})}{F(r_i)}\right).$$

Proof. Clearly, the sequence $\{k_n\}$ is monotonic. Set $k_0 = 0$. It is easy to see that there is a unique pair (m, j) such that $m \geq 1$, $i > j \geq 0$ and $k_i = k_{i-1} = \cdots = k_{j+1} = m > k_j$. Therefore, it is enough to prove the inequality by further conditioning on m and j. From now on, we will consider m and j to be fixed.

We can further condition on $\mathcal{F} = \sigma(X_1, \dots X_{m-1}, Y_1, \dots, Y_{m-1})$. It is easy to check that for $i < m$, the event $\{i \in S\}$ is in $\mathcal{F}$. It follows that given $k_j < m$, the variables $k_1, \dots, k_j$ are $\mathcal{F}|_{\{k_j<m\}}$-measurable, hence $d_1, \dots, d_j$ are also $\mathcal{F}|_{\{k_j<m\}}$-measurable. Therefore,

$$\begin{aligned}
&\mathbb{P}(d_i = 1 \,|\, d_1, \dots, d_{i-1},\, m,\, j,\, \mathcal{F}) = \\
&= \mathbb{P}(d_i = 1 \,|\, k_i = k_{i-1} = \dots = k_{j+1} = m > k_j,\, d_1, \dots, d_{i-1},\, \mathcal{F}) = \\
&= \mathbb{P}(d_i = 1 \,|\, k_i = k_{i-1} = \dots = k_{j+1} = m > k_j,\, d_{j+1}, \dots, d_{i-1},\, \mathcal{F}) = \\
&= \mathbb{P}(d_i = 1 \,|\, k_i = m,\, k_{j+1} \ge m > k_j,\, d_{j+1}, \dots, d_{i-1},\, \mathcal{F}).
\end{aligned}$$

If $\operatorname{diam} Z_m \ge r_i$, then $d_i = 1$, so we can further condition on $\operatorname{diam} Z_m < r_i$. Since $k_{j+1} = \dots = k_i = m$, this implies that $d_{j+1} = \dots = d_{i-1} = 0$. Consequently,

$$\begin{aligned}
&\mathbb{P}\big(d_i = 1 \,\big|\, k_i = m,\, k_{j+1} \ge m > k_j,\, \operatorname{diam} Z_m < r_i,\, d_{j+1}, \dots, d_{i-1},\, \mathcal{F}\big) = \\
&= \mathbb{P}\big(\operatorname{diam} Z_m \ge r_i/2 \,\big|\, m \in S,\, Z_m \cap B(x, 2r_i) \ne \emptyset,\, \operatorname{diam} Z_m < r_i,\, k_{j+1} \ge m > k_j,\, \mathcal{F}\big) = \\
&= \frac{\mathbb{P}(m \in S,\, Z_m \cap B(x, 2r_i) \ne \emptyset,\, r_i/2 \le \operatorname{diam} Z_m < r_i \,|\, k_{j+1} \ge m > k_j,\, \mathcal{F})}{\mathbb{P}(m \in S,\, Z_m \cap B(x, 2r_i) \ne \emptyset,\, \operatorname{diam} Z_m < r_i \,|\, k_{j+1} \ge m > k_j,\, \mathcal{F}).}
\end{aligned}$$

We will now give bounds for both the numerator and the denominator. For the numerator, notice that since $k_i \ge k_{j+1} \ge m$, $Z_l \cap B(x, 2r_i) = \emptyset$ for every $l < m$. It follows that if $Z_m \subseteq B(x, 2r_i)$, then $m \in S$. A sufficient condition for $Z_m \subseteq B(x, 2r_i)$ is that $\operatorname{diam} X_m < r_i$ and $Y_m \in B(x, r_i)$. It is easy to see that $\{k_{j+1} \ge m > k_j\} \in \mathcal{F}$. Since X_m, Y_m and $\mathcal{F}$ are independent, we obtain

$$\begin{aligned}
&\mathbb{P}(m \in S,\, Z_m \cap B(x, 2r_i) \ne \emptyset,\, r_i/2 \le \operatorname{diam} Z_m < r_i \,|\, k_{j+1} \ge m > k_j,\, \mathcal{F}) \ge \\
&\ge \mathbb{P}(Y_m \in B(x, r_i),\, r_i/2 \le \operatorname{diam} X_m < r_i \,|\, k_{j+1} \ge m > k_j,\, \mathcal{F}) \ge \\
&\ge \mathbb{P}(Y_m \in B(x, r_i))\, \mathbb{P}(r_i/2 \le \operatorname{diam} X_m < r_i) \ge \\
&\ge \nu(B(x, r_i))(F(r_i) - F(r_i/2)) \ge \frac{1}{2} f(x) \lambda(B(x, r_i))(F(r_i) - F(r_i/2)).
\end{aligned}$$

For the denominator, it is easy to check that if $\operatorname{diam} Z_m < r_i$ and $Z_m \cap B(x, 2r_i) \ne \emptyset$, then $Y_m \in B(x, 3r_i)$. Using the independence again,

$$\begin{aligned}
&\mathbb{P}(m \in S,\, Z_m \cap B(x, 2r_i) \ne \emptyset,\, \operatorname{diam} Z_m < r_i \,|\, k_{j+1} \ge m > k_j,\, \mathcal{F}) \le \\
&\le \mathbb{P}(Y_m \in B(x, 3r_i),\, \operatorname{diam} X_m < r_i \,|\, k_{j+1} \ge m > k_j,\, \mathcal{F}) = \\
&= \mathbb{P}(Y_m \in B(x, 3r_i))\, \mathbb{P}(\operatorname{diam} X_m < r_i) = \\
&= \nu(B(x, 3r_i)) F(r_i) \le 2 f(x) \lambda(B(x, 3r_i)) F(r_i) = 2 \cdot 3^d f(x) \lambda(B(x, r_i)) F(r_i).
\end{aligned}$$

Combining the inequalities yields the bound

$$\begin{aligned}
&\mathbb{P}(d_i = 1 \,|\, k_i = m,\, k_{j+1} \ge m > k_j,\, d_{j+1} = \dots = d_{i-1} = 0,\, \mathcal{F}) \ge \\
&\ge \frac{1/2 \cdot f(x) \lambda(B(x, r_i))(F(r_i) - F(r_i/2))}{2 \cdot 3^d f(x) \lambda(B(x, r_i))} = \frac{1}{4 \cdot 3^d} \cdot \frac{F(r_i) - F(r_i/2)}{F(r_i)} = \frac{1}{4 \cdot 3^d} \left(1 - \frac{F(r_i/2)}{F(r_i)}\right).
\end{aligned}$$

This concludes the proof. $\square$

Lemma 2.3.4. *Suppose that* $0 < x_n \leq 1$ *for every* $n \geq 1$*. Then* $\prod_{n=1}^{\infty} x_n = 0$ *for if and only if* $\sum_{n=1}^{\infty}(1 - x_n) = \infty$*.*

Proof. Let $\log 0 = -\infty$, this extends the logarithm function continuously to $[0, \infty)$. Assume that $\prod_{n=1}^{\infty} x_n = 0$. If $x_n < 1/2$ for infinitely many n, then the sum is clearly infinite. Since $x_n > 0$ for all n, finitely many factors do not affect convergence of the product to zero. Therefore, we may assume that $x_n \geq 1/2$ for all n. By concavity of logarithm, $\log x_n \geq -2\log 2 \cdot (1 - x_n)$ for all n. It follows that

$$-\infty = \log \prod_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \log x_n \geq -2\log 2 \sum_{n=1}^{\infty}(1 - x_n),$$

which implies that $\sum_{n=1}^{\infty}(1 - x_n) = \infty$.

For the other direction, we can use the inequality $\log x_n \leq x_n - 1$:

$$\log \prod_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \log x_n \leq \sum_{n=1}^{\infty}(x_n - 1) = -\infty.$$

□

Lemma 2.3.5. *Suppose that* $\mu(\{\{0\}\}) = 0$*. Then* $\limsup_{n\to\infty} d_n = 1$ *almost surely.*

Proof. It is enough to prove that for every j, $\mathbb{P}(\forall n \geq j : d_n = 0) = 0$. It follows from Lemma 2.3.3 that

$$\begin{aligned}
\mathbb{P}(\forall n \geq j : d_n = 0) &= \lim_{i\to\infty} \mathbb{P}(d_j = d_{j+1} = \cdots = d_i = 0) = \\
&= \lim_{i\to\infty} \prod_{n=j}^{i} \mathbb{P}(d_n = 0 \,|\, d_j = \cdots = d_{n-1} = 0) = \\
&= \prod_{n=j}^{\infty} \mathbb{P}(d_n = 0 \,|\, d_j = \cdots = d_{n-1} = 0) \leq \\
&\leq \prod_{n=j}^{\infty} \left(1 - \frac{1}{4 \cdot 3^d}\left(1 - \frac{F(r_{n+1})}{F(r_n)}\right)\right).
\end{aligned}$$

By Lemma 2.3.4, it is enough to prove that

$$\sum_{n=j}^{\infty} \frac{1}{4 \cdot 3^d}\left(1 - \frac{F(r_{n+1})}{F(r_n)}\right) = \infty,$$

or equivalently,

$$\sum_{n=j}^{\infty}\left(1 - \frac{F(r_{n+1})}{F(r_n)}\right) = \infty.$$

Applying the other direction of Lemma 2.3.4, it suffices to show that

$$0 = \prod_{n=j}^{\infty} \frac{F(r_{n+1})}{F(r_n)} = \lim_{i\to\infty} \prod_{n=j}^{\infty} \frac{F(r_{n+1})}{F(r_n)} = \lim_{i\to\infty} \frac{F(r_{i+1})}{F(r_j)} = \frac{\mu(\{X \mid \operatorname{diam} X = 0\})}{F(r_j)},$$

which is clear from the assumption $\mu(\{\{0\}\}) = 0$. □

Remark. The proof of Lemma 2.3.5 is essentially the same as that of the second Borel–Cantelli lemma, except that instead of independence, we have bounds on the conditional probabilities.

Lemma 2.3.6. *It is almost surely true that for μ a.e. x,* $\limsup d_n(x) = 1$.

Proof. It follows from Fubini's theorem and Lemma 2.3.5 that

$$\mathbb{E}\big(\nu(\{x \mid \limsup d_n(x) \neq 1\})\big) = \mathbb{E}\bigg(\int_L \chi_{\{\limsup d_n(x)\neq 1\}}\, d\nu\bigg) = \int_L \mathbb{P}(\limsup d_n(x) \neq 1)\, d\nu = 0,$$

therefore, $\nu(\{x \mid \limsup d_n(x) \neq 1\}) = 0$ almost surely.

However, to use Fubini's theorem, we must check that the integrand is measurable. Let $\Omega = \big(\mathcal{K}(\mathbb{R}^d)\times\mathbb{R}^d\big)^{\mathbb{N}}$ be the probability space. It suffices to show that the map $d_n : \Omega\times\mathbb{R}^d \to \{0,1\}$ is measurable.

For fixed $r_0 = 1/l$, the condition $B(x, r_0) \subseteq U$ is closed. For every n, the set

$$\left\{x \;\middle|\; \frac{f(x)}{2} \leq \frac{\nu(B(x,r_n))}{\lambda(B(x,r_n))} \leq 2f(x)\right\}$$

is measurable, since f is measurable and so is $\nu(B(x, r_n)) = \int \chi_{\{B(x,r_n)\}} f\, d\lambda$. Since l was chosen to be minimal, the measurability of r_0 follows easily.

Finally, for fixed r_0, n and m, the condition $Z_m \cap B(x, 2r_n) \neq \emptyset$ is open and the map $Z_m \mapsto \operatorname{diam} Z_m$ is continuous, implying the measurability of d_n. □

Proof (Theorem 2.3.1). It is clear that $X_i \in \mathcal{D}\cup\mathcal{N}$ for all i almost surely. By Lemma 2.3.6, it is almost surely true that for ν a.e. x, $\limsup d_n(x) = 1$. We will show that these two conditions imply that $\nu(A) = 1$.

Let

$$d(x) = \limsup_{r\to 0^+} \frac{\lambda(A \cap B(x,r))}{\lambda(B(x,r))}$$

By Lebesgue's density theorem, $d(x) = \chi_A(x)$ for λ a.e. x. By the assumption that $\nu \ll \lambda$, this implies that $d(x) = \chi_A(x)$ for ν a.e. x. Hence, it is enough to show that for ν a.e. x, $x \in A$ or $d(x) > 0$.

Fix a $y \in \mathbb{R}^d$ and also fix a $\delta > 0$. By the neighborhood condition, there is an η with the property that for every i such that $X_i \in \mathcal{N}$,

$$\lambda(\{x \in \mathbb{R}^d \mid 0 < d(x, Z_i) < \eta \operatorname{diam} Z_i\}) \leq \delta\lambda(Z_i).$$

We may assume that $\eta \leq 1$. Let

$$N_\delta = B(y,1) \cap \bigcup_{i\in S,\, X_i\in\mathcal{N}} \{x \in \mathbb{R}^d \mid 0 < d(x, Z_i) < \eta \operatorname{diam} Z_i\}$$

We will now show that if $x \in U \cap B(y,1)$ and $\limsup d_n(x) = 1$, then either $x \in A \cup N_\delta$ or $d(x) > 0$. Suppose that $x \in U \cap B(y,1)$ and $d_n(x) = 1$ for some n. Since $k_n \in S$, clearly $Z_{k_n} \subseteq A$.

First, consider the case $X_{k_n} \in \mathcal{D}$. Choose a point $z \in Z_{k_n} \cap B(x, 2r_i)$. Let ε be the constant from the density condition. Since $\operatorname{diam} Z_{k_n} \geq r_n/2$, we can use the density condition:

$$\lambda(Z_{k_n} \cap B(z, r_n/2)) = \lambda(X_{k_n} \cap B(z - Y_{k_n}, r_n/2)) \geq \varepsilon (r_n/2)^d.$$

As $B(z, r_n/2) \subseteq B(x, 4r_n) \subseteq B(x, 4r_n/\eta)$, we obtain the bound

$$\frac{\lambda(A \cap B(x, 4r_n/\eta))}{\lambda(B(x, 4r_n/\eta))} \geq \frac{\lambda(Z_{k_n} \cap B(z, r_n/2))}{(r_n/2)^d \lambda(B(x, 8/\eta))} \geq \frac{\varepsilon}{\lambda(B(x, 8/\eta))}.$$

Now assume that $X_{k_n} \in \mathcal{N}$. If $d(x, Z_{k_n}) = 0$, then $x \in A$. If $0 < d(x, Z_{k_n}) < \eta \operatorname{diam} Z_{k_n}$, then $x \in N_\delta$. Otherwise, $\eta \operatorname{diam} Z_{k_n} \leq d(x, Z_{k_n}) < 2r_n$, therefore $\operatorname{diam} Z_{k_n} \leq 2r_n/\eta$. This implies that $Z_{k_n} \subseteq B(x, 2r_n + 2r_n/\eta) \subseteq B(x, 4r_n/\eta)$. We know from $d_n(x) = 1$ that $\operatorname{diam} Z_{k_n} \geq r_n/2$, hence $\eta \operatorname{diam} Z_{k_n} \geq \eta r_n/2$. By the neighborhood condition,

$$\lambda(B(x, \eta r_n/2)) \leq \lambda(\{x \in \mathbb{R}^d \mid 0 \leq d(x, Z_i) < \eta \operatorname{diam} Z_i\}) \leq (1 + \delta)\lambda(Z_i).$$

Consequently,

$$\frac{\lambda(A \cap B(x, 4r_n/\eta))}{\lambda(B(x, 4r_n/\eta))} \geq \frac{\eta^{2d}}{8^d} \cdot \frac{\lambda(Z_{k_n})}{\lambda(B(x, \eta r_n/2))} \geq \frac{\eta^{2d}}{8^d(1+\delta)}.$$

Note that in both cases, we obtained a lower bound that does not depend on n. Since $d_n = 1$ for infinitely many n and $r_n \to 0$, we can conclude that if $x \notin A \cup N_\delta$, then $d(x) > 0$.

For ν a.e. x, $\limsup d_n(x) = 1$ by Lemma 2.3.6. It follows that for ν a.e. $x \in B(y, 1)$, either $d(x) > 0$ or $x \in A \cup N_\delta$. This is true for every $\delta > 0$, so for ν a.e. $x \in B(y, 1)$, either $d(x) > 0$ or $x \in A \cup \bigcap_{l=1}^\infty N_{1/l}$. Assuming that $\nu\big(\bigcap_{l=1}^\infty N_{1/l}\big) = 0$, this implies that for ν a.e. $x \in B(y, 1)$, $d(x) > 0$ or $x \in A$, then we are done, since $\mathbb{R}^d$ can be covered by countably balls of the form $B(y, 1)$. By the assumption $\nu \ll \lambda$, it suffices to prove that $\lambda\big(\bigcap_{l=1}^\infty N_{1/l}\big) = 0$.

The neighborhood condition implies that there is an R such that $\operatorname{diam} Z_i = \operatorname{diam} X_i \leq R$ whenever $X_i \in \mathcal{N}$. For every $x \in N_\delta$, there is an $i \in S$ such that $X_i \in \mathcal{N}$ and $d(x, Z_i) < \eta \operatorname{diam} Z_i \leq \operatorname{diam} Z_i \leq R$. Clearly, $Z_i \subseteq B(x, 2R) \subseteq B(y, 2R + 1)$. Combined with the disjointness of the sets $\{Z_i \mid i \in S\}$, this implies that

$$\begin{aligned}
\lambda(N_\delta) &\leq \lambda\Bigg(\bigcup_{\substack{i \in S,\, X_i \in \mathcal{N} \\ Z_i \subseteq B(x, 2R+1)}} \{x \in \mathbb{R}^d \mid 0 < d(x, Z_i) < \eta \operatorname{diam} Z_i\} \Bigg) \leq \\
&\leq \sum_{\substack{i \in S,\, X_i \in \mathcal{N} \\ Z_i \subseteq B(x, 2R+1)}} \lambda(\{x \in \mathbb{R}^d \mid 0 < d(x, Z_i) < \eta \operatorname{diam} Z_i\}) \leq \\
&\leq \sum_{\substack{i \in S,\, X_i \in \mathcal{N} \\ Z_i \subseteq B(x, 2R+1)}} \delta \lambda(Z_i) \leq \delta \lambda(B(x, 2R+1)).
\end{aligned}$$

Therefore, $\lambda(N_\delta) \to 0$ if $\delta \to 0$. ☐

Combining Theorems 2.1.3, 2.2.6 and 2.3.1, we obtain the following zero-one law:

Theorem 2.3.7. *Suppose that $\lambda(U) < \infty$, $\nu \ll \lambda$ and $\lambda|_U \ll \nu$. Also, assume that for some $\mathcal{D}, \mathcal{N} \subseteq \mathcal{K}(\mathbb{R}^d)$, $\mathcal{D}$ satisfies the density condition, $\mathcal{N}$ satisfies the neighborhood condition and $\mu(\mathcal{D} \cup \mathcal{N}) = 1$. If either $\mu(\{\{0\}\}) > 0$ or there is an $r > 0$ such that $\mu(\{X \mid 0 < \operatorname{diam} X < r\}) = 0$, then $\mathbb{P}(\lambda(U \setminus A) = 0) = 0$. Otherwise, $\mathbb{P}(\lambda(U \setminus A) = 0) = 1$.*

Remark. When ν is uniform, we can set $\mathcal{D} = 0$ and by Proposition 2.2.5 we can choose $\mathcal{N}$ to be a family containing finitely many shapes with positive Lebesgue measure. This gives Theorem 2 as a special case.

The zero-one law fails for $\mathbb{P}(\lambda(U \setminus A) = 0)$ when the condition $\lambda|_U \ll \nu$ is omitted. For example, consider the case when $d \geq 2$, $U = B(0, 2)$ and ν is uniform on $B(0, 2) \setminus B(0, 1)$. If μ is chosen such that the sets Z_i are either balls or spherical shells of the form $\overline{B(x, 2r)} \setminus B(x, r)$ (with bounded radii), then it follows from Proposition 2.2.5 that the neighborhood condition is satisfied (furthermore, an easy calculation shows that the density condition also holds for this family). The measure μ can be chosen so that with positive probability, $Z_1 = \overline{B(x, 2r)} \setminus B(x, r) \subseteq U$ and $B(x, r) \subseteq B(0, 1) \subseteq \overline{B(x, 2r)}$ for some x and r. Since all Z_i are connected, after placing Z_1, no set in S can intersect $B(x, r)$, since it is separated from the support of ν by Z_1. Therefore, $\lambda(U \setminus A) > 0$ in this case. On the other hand, μ can be chosen such that, additionally, $B(0, 1) \subseteq Z_1 \subseteq B(0, 2)$ with positive probability. After placing such Z_1, it follows from Theorem 2.3.1 that $\nu(A) = 1$ almost surely, which implies that $\lambda(U \setminus A) = 0$. As a result, $\mathbb{P}(\lambda(U \setminus A) = 0) \notin \{0, 1\}$ for this construction.

A similar construction shows that without the condition $\lambda|_U \ll \nu$, the zero-one law fails also for $\mathbb{P}(\nu(A) = 1)$, which is possibly different from $\mathbb{P}(\lambda(U \setminus A) = 0)$ in this case. Let $U = B(0, 2)$ and let ν be uniform on $B(0, 1)$. If we allow balls, spherical shells, plus singleton sets with positive probability, then μ can be chosen such that $\nu(Z_1) = 1$ with positive probability. We can choose μ such that additionally, there is a positive probability that $B(0, 1)$ covers the hole in the spherical shell $Z_1 \subseteq U$. After placing such Z_1, $\nu(A) < 1$ by Theorem 2.1.3.

3. Hausdorff dimension of the complement

We will consider a special case of the problem when $d = 1$, $U = (0, 1)$ and $\nu = \lambda|_U$. We will further assume that μ is the distribution of the interval $[0, x]$, where x has some distribution μ_0 defined on $[0, \infty)$. Clearly, μ has cumulative distribution function $F(t) = \mu_0([0, t)) = \mu(\{X \mid \operatorname{diam} X < t\})$.

The goal of this section is to give conditions for determining the Hausdorff dimension of the set $U \setminus A$. Using the notation $F(0^+) = \lim_{t\to 0^+} F(t) = \mu_0(\{0\})$, note that if $F(0^+) > 0$ or $F(\varepsilon) = 0$ for some $\varepsilon > 0$, then by Theorems 2.1.3 and 2.2.6, $\lambda(U \setminus A) > 0$ almost surely, hence $\dim_{\mathrm{H}}(U \setminus A) = 1$. For the whole chapter, we will assume that $F(0^+) = 0$ and $F(\varepsilon) > 0$ for every $\varepsilon > 0$.

3.1 The binary tree of intervals

Let $j \in \{1, 2, \dots\}$ and let I be a maximal interval of $(0, 1) \setminus \bigcup_{i<j, i\in S} Z_i$. Rather than working with the sequence $\{Z_i\}$, it is more convenient to consider the first interval in S contained in I. This first interval splits I into two smaller intervals, allowing us to define a binary tree of intervals.

Definition 3.1.1. Let $\sigma \in \{0, 1\}^*$. We define the index i_σ and the open interval I_σ recursively:

$$\begin{aligned} I_\emptyset &= (0, 1) \\ i_\sigma &= \min\{i \mid Z_i \subseteq I_\sigma\} \\ I_{\sigma 0} &= (\inf I_\sigma, \min Z_{i_\sigma}) \\ I_{\sigma 1} &= (\max Z_{i_\sigma}, \sup I_\sigma) \end{aligned}$$

Note that by Lemma 2.3.2, the assumption that $F(\varepsilon) - F(0^+) > 0$ for every ε implies that i_σ exists almost surely. It is easy to check that the intervals $\{Z_{i_\sigma}\}_{\sigma\in\{0,1\}^*}$ are disjoint, in particular, the indices $\{i_\sigma\}_{\sigma\in\{0,1\}^*}$ are all distinct.

Proposition 3.1.2. $S = \{i_\sigma \mid \sigma \in \{0, 1\}^*\}$.

Proof. We will prove by induction on k that $k \in S \leftrightarrow k \in \{i_\sigma \mid \sigma \in \{0, 1\}^*\}$.

Suppose for contradiction that $k \in S$, but $k \neq i_\sigma$ for every σ. We will define a sequence $b_1, b_2, \ldots \in \{0,1\}$ recursively such that $Z_k \subseteq I_{b_1 b_2 \ldots b_i}$ for all $i \geq 0$. By the definition of S, $Z_k \subseteq (0,1) = I_\emptyset$. Assume that $b_1, \ldots, b_i$ are already defined for some $i \geq 0$. By the minimality of $i_{b_1,\ldots,b_i}$, it follows that $i_{b_1 \ldots b_i} \leq k$. There cannot be equality here by the assumption, so the induction hypothesis implies that $i_{b_1 \ldots b_i} \in S$. Since k is also in S, this means that $Z_{i_{b_1 \ldots b_i}} \cap Z_k = \emptyset$. Note that Z_k is connected, consequently, $Z_k \subseteq I_{b_1 \ldots b_i b_{i+1}}$ for some $b_{i+1} \in \{0,1\}$. This way, we have defined a sequence $\{b_i\}_{i=1}^\infty$ satisfying $i_{b_1 \ldots b_i} \leq k$ for all i. This is a contradiction, since the indices $i_{b_1,\ldots,b_i}$ are all distinct.

Now assume that $k \notin S$, but $i_\sigma = k$ for some σ. Then $Z_k \subseteq I_\sigma \subseteq (0,1)$, so by the definition of S, there is a $j < k$ such that $j \in S$ and $Z_j \cap Z_k \neq \emptyset$. By the induction hypothesis, $j = i_\rho$ for some $\rho \in \{0,1\}^*$. Note that $i_\sigma \neq i_\rho$ (therefore, $\sigma \neq \rho$), so the disjointness implies that $Z_{i_\sigma} \cap Z_{i_\rho} = \emptyset$. Hence, $\emptyset \neq Z_k \cap Z_j = Z_{i_\sigma} \cap Z_{i_\rho} = \emptyset$, a contradiction. □

For $\sigma, \rho \in \{0,1\}^*$, we will use the notation $\sigma \leq \rho$ to denote that σ is an initial segment of ρ and similarly, $\sigma < \rho$ when σ is a strict initial segment of ρ.

For $\rho \in \{0,1\}^*$, let $l_\rho = \operatorname{diam} I_\rho$ and $z_\rho = \operatorname{diam} Z_{i_\rho}$. We define the σ-algebra

$$\mathcal{F}_\rho = \sigma\big(\{l_\gamma\}_{\gamma \leq \rho}, \{z_\gamma\}_{\gamma < \rho}\big).$$

It is clear from the definitions that $l_\rho = l_{\rho 0} + z_\rho + l_{\rho 1}$. Since l_ρ is $\mathcal{F}_\rho$-measurable, it follows easily that

$$\mathcal{F}_{\rho 0} = \sigma(\mathcal{F}_\rho, l_{\rho 0}, z_\rho) = \sigma(\mathcal{F}_\rho, l_{\rho 1}, z_\rho) = \mathcal{F}_{\rho 1}.$$

Definition 3.1.3. For $t > 0$, let $X \sim \mu_0$ and $Y \sim U(0,t)$ be independent variables. We define L_t as the conditional distribution

$$L_t \sim (X \mid X + Y < t).$$

Lemma 3.1.4. *Let $\sigma \in \{0,1\}^*$. Given $\mathcal{F}_\sigma$, z_σ has distribution L_{l_σ}. If we additionally know z_σ, then $0 \leq z_{\sigma 0} \leq l_\sigma - z_\sigma$ is uniformly distributed. Formally,*

$$\big(z_\sigma \,\big|\, \mathcal{F}_\sigma\big) \sim L_{l_\sigma}$$
$$\big(l_{\sigma 0} \,\big|\, \mathcal{F}_\sigma, z_\sigma\big) \sim U(0, l_\sigma - z_\sigma).$$

Proof. It suffices to prove the statement when further conditioning on $i_\sigma = k$ for each k, since i_σ has countably many possible values. Let $X_i = [0, x_i]$ and $\mathcal{G} = \sigma(x_1, \ldots, x_{k-1}, Y_1, \ldots, Y_{k-1})$. For every $\rho < \sigma$, it follows from the minimality of i_ρ that $i_\rho < i_\sigma = k$. It is easy to check that this implies that $\sigma(\mathcal{F}_\sigma, \{i_\sigma = k\})|_{\{i_\sigma = k\}} \subseteq \sigma(\mathcal{G}, \{i_\sigma = k\})|_{\{i_\sigma = k\}}$, furthermore, $\inf I_\sigma$ and $\sup I_\sigma$ are both $\sigma(\mathcal{G}, \{i_\sigma = k\})|_{\{i_\sigma = k\}}$-measurable. Also, assuming $i_\sigma = k$, we can see that $z_\sigma = x_k$ and $l_{\sigma 0} = Y_k - \inf I_\sigma$. Therefore, it is enough to show that

$$\big(x_k \,\big|\, \mathcal{G}, i_\sigma = k\big) \sim L_{l_\sigma}$$
$$\big(Y_k - \inf I_\sigma \,\big|\, \mathcal{G}, i_\sigma = k, x_k\big) \sim U(0, l_\sigma - x_k).$$

Let $y = Y_k - \inf I_\sigma$. Since $Z_{i_\sigma} \subseteq I_\sigma$ if and only if $\inf I_\sigma < Y_k \le Y_k + x_k < \sup I_\sigma$, clearly $i_\sigma = k$ if and only if $i_\sigma \ge k, 0 < y < l_\sigma$ and $x_k + y < l_\sigma$. Notice that $\{i_\sigma \ge k\} \in \mathcal{G}$. This means that it suffices to show that

$$\big(x_k \,\big|\, \mathcal{G},\, 0 < y < l_\sigma,\, x_k + y < l_\sigma\big) \sim L_{l_\sigma}$$
$$\big(y \,\big|\, \mathcal{G},\, 0 < y < l_\sigma,\, x_k + y < l_\sigma,\, x_k\big) \sim U(0, l_\sigma - x_k).$$

Since $\mathcal{G}$, x_k and Y_k are independent, it follows that given $\mathcal{G}$ and $0 < y < l_\sigma$, x_k and y are independent variables with distributions $x_k \sim \mu_0$ and $y \sim U(0, l_\sigma)$. The statement about the conditional distribution of x_k now reduces to the definition of L_t. For the conditional distribution of y, the conditional independence of x_k and y implies that

$$\big(y \,\big|\, \mathcal{G},\, 0 < y < l_\sigma,\, x_k + y < l_\sigma,\, x_k\big) \sim \big(y \,\big|\, x_k,\, l_\sigma,\, 0 < y < l_\sigma - x_k\big) \sim U(0, l_\sigma - x_k).$$

□

Lemma 3.1.5. *Let $(X, \mathcal{M}, \rho)$ be a probability space and let $A_1, A_2, \dots \in X$ be a sequence of i.i.d. variables with distribution ρ. Suppose that $P \in \mathcal{M}$ satisfies $0 < \rho(P) < 1$. Partition the sequence $\{A_i\}$ into two subsequences $\{A_j^{(0)}\}$ and $\{A_j^{(1)}\}$ as follows: if $A_i \in S$, then it is put in $\{A_j^{(0)}\}$, otherwise, it is put in $\{A_j^{(1)}\}$. Then $\{A_j^{(0)}\}$ and $\{A_j^{(1)}\}$ are independent (almost surely) infinite sequences of i.i.d. variables.*

Proof. It is clear that if $0 < \rho(P) < 1$, then $A_i \in P$ and $A_i \notin P$ both occur infinitely often almost surely.

By the uniqueness of the product measure, it suffices to prove that for every sequence of sets $\{E_j^{(0)}\}$ and $\{E_j^{(1)}\}$ in $\mathcal{M}$ and every n, the variables $\{E_j^{(b)}\}$ are independent for $b \in \{0, 1\}$ and $j \le n$. We will prove that

$$\mathbb{P}(\forall j \le n, b \in \{0,1\} : A_j^{(b)} \in E_j^{(b)}) = \prod_{j \le n} \frac{\rho(E_j^{(0)} \cap P)}{\rho(P)} \cdot \prod_{j \le n} \frac{\rho(E_j^{(1)} \setminus P)}{\rho(X \setminus P)}.$$

Setting all except one of the $E_i^{(b)}$ to X, we get that $\mathbb{P}(A_i^{(0)} \in E_i^{(0)}) = \rho(E_i^{(0)} \cap P)/\rho(P)$ and $\mathbb{P}(A_i^{(1)} \in E_i^{(1)}) = \rho(E_i^{(1)} \setminus P)/\rho(X \setminus P)$, from which the independence is clear.

Let $i_j^{(0)}$ be the jth smallest i such that $A_j \in P$ and similarly, let $i_j^{(1)}$ be the jth smallest i such that $A_j \notin P$. It is clear that the values $i_j^{(b)}$ are all distinct and $A_j^{(b)} = A_{i_j^{(b)}}$. Since the vector $(i_1^{(0)}, \dots, i_n^{(0)}, i_1^{(1)}, \dots, i_n^{(1)})$ has countably many possible values, it is enough to prove the equality above after conditioning on a particular value of this vector. Conditioning on this vector gives information only on which $A_i^{(b)}$ are is P, hence they are conditionally independent (but no longer identically distributed). Therefore,

$$
\begin{aligned}
&\mathbb{P}(\forall j \le n, b \in \{0,1\} : A_j^{(b)} \in E_j^{(b)} \mid i_1^{(0)}, \dots, i_n^{(0)}, i_1^{(1)}, \dots, i_n^{(1)}) = \\
&= \prod_{j \le n} \mathbb{P}\big(A_{i_j^{(0)}} \in E_j^{(0)} \bigm| A_{i_j^{(0)}} \in P\big) \cdot \prod_{j \le n} \mathbb{P}\big(A_{i_j^{(1)}} \in E_j^{(1)} \bigm| A_{i_j^{(1)}} \notin P\big) = \\
&= \prod_{j \le n} \frac{\rho(E_j^{(0)} \cap P)}{\rho(P)} \cdot \prod_{j \le n} \frac{\rho(E_j^{(1)} \setminus P)}{\rho(X \setminus P)},
\end{aligned}
$$

which completes the proof. □

Lemma 3.1.6. *Let $\sigma \in \{0,1\}^*$. Given $\mathcal{F}_{\sigma 0}$, the subtrees $\{l_{\sigma 0 \rho}\}_{\rho \in \{0,1\}^*}$ and $\{l_{\sigma 1 \rho}\}_{\rho \in \{0,1\}^*}$ are independent.*

Proof. Let $X'_i = X_{i+i_\sigma}$ and $Y'_i = Y_{i+i_\sigma}$. We will first show that given $\mathcal{F}_{\sigma 0}$, the pairs (X'_i, Y'_i) for $i \ge 1$ are i.i.d.. We can prove this by further conditioning on $i_\sigma = k$ for some k. Let $\mathcal{G} = \sigma(X_1, \dots, X_k, Y_1, \dots, Y_k)$. It is clear that $\sigma(\mathcal{F}_{\sigma 0}, \{i_\sigma = k\})|_{\{i_\sigma = k\}} \subseteq \mathcal{G}|_{\{i_\sigma = k\}}$. Since G and $\{(X_i, Y_i)\}_{i>k}$ are independent, it follows that given $\mathcal{F}_{\sigma 0}$ and $i_\sigma = k$, $\{(X'_i, Y'_i)\}_{i \ge 1}$ is a sequence of i.i.d. variables with distribution $\mu \times \nu$. Since this is true for every k and the distribution of the sequence does not depend on k, we can conclude that the distribution is the same when conditioning only on $\mathcal{F}_{\sigma 0}$.

It is easy to see that $\inf I_{\sigma 0}$ and $\sup I_{\sigma 0}$ are both $\mathcal{F}_{\sigma 0}$-measurable. Let $Z'_i = X'_i + Y'_i = Z_{i+i_\sigma}$. We can apply Lemma 3.1.5 to the sequence $\{Z'_i\}$ with $P = \{Z \mid Z \subseteq I_{\sigma 0}\}$, since we have seen in the proof of Lemma 2.3.2 that $Z_i \subseteq I_{\sigma 0}$ and $Z_i \subseteq I_{\sigma 1}$ both have positive probability. We get a partition of $\{Z'_i\}$ into independent subsequences $\{Z_j^{(0)}\}$ and $\{Z_j^{(1)}\}$ such that $Z_j^{(0)} \in P$ and $Z_j^{(1)} \notin P$ for every j. It follows from the minimality of i_σ that $\{l_{\sigma 0 \rho}\}_{\rho \in \{0,1\}^*}$ is determined by $\{Z_j^{(0)}\}$ and $\{l_{\sigma 1 \rho}\}_{\rho \in \{0,1\}^*}$ is determined by $\{Z_j^{(1)}\}$ (since $I_{\sigma 0} \cap I_{\sigma 1} = \emptyset$). The independence of the subsequences implies that $\{l_{\sigma 0 \rho}\}_{\rho \in \{0,1\}^*}$ and $\{l_{\sigma 1 \rho}\}_{\rho \in \{0,1\}^*}$ are also independent. □

Theorem 3.1.7. *Let $\{U_\sigma\}_{\sigma \in \{0,1\}^*}$ and $\{V_\sigma\}_{\sigma \in \{0,1\}^*}$ be independent sequences of i.i.d. variables uniform on $(0,1)$. We define the variables l'_σ and z'_σ for $\sigma \in \{0,1\}^*$ recursively:*

$$
l'_\emptyset = 1, \qquad z'_\sigma = F^{-1}_{L_{l'_\sigma}}(V_\sigma), \qquad l'_{\sigma 0} = U_\sigma (l'_\sigma - z'_\sigma), \qquad l'_{\sigma 1} = (1 - U_\sigma)(l'_\sigma - z'_\sigma),
$$

where $F^{-1}_{L_t}(p) = \inf\{x \mid F_{L_t}(x) \ge p\}$ is the generalized inverse of the cumulative distribution function $F_{L_t}(x) = \mathbb{P}(L_t < x)$. Then $\{l_\sigma\}_{\sigma \in \{0,1\}^*}$ and $\{l'_\sigma\}_{\sigma \in \{0,1\}^*}$ have the same distribution.

Proof. Denote by $|\rho|$ the length of $\rho \in \{0,1\}^*$. It suffices to prove that for every σ and n and every tree of borel sets $\{E_\sigma\}_{\sigma \in \{0,1\}^*}$, there is a Borel measurable function $f(t)$ such that for every $\mathbf{x} = \{x_\gamma\}_{\gamma \le \sigma}$ and $\mathbf{y} = \{y_\gamma\}_{\gamma < \sigma}$,

$$
\begin{aligned}
f(x_\sigma) &= \mathbb{P}\big(\forall |\rho| \le n : l_{\sigma\rho} \in E_{\sigma\rho} \bigm| \{l_\gamma\}_{\gamma \le \sigma} = \mathbf{x}, \{z_\gamma\}_{\gamma < \sigma} = \mathbf{y}\big) = \\
&= \mathbb{P}\big(\forall |\rho| \le n : l'_{\sigma\rho} \in E_{\sigma\rho} \bigm| \{l'_\gamma\}_{\gamma \le \sigma} = \mathbf{x}, \{z'_\gamma\}_{\gamma < \sigma} = \mathbf{y}\big).
\end{aligned}
$$

Since $l_\emptyset = l'_\emptyset = 1$, setting $\sigma = \emptyset$ and $x_\emptyset = 1$, we get $\{l_\rho\}_{|\rho|\le n} \sim \{l'_\rho\}_{|\rho|\le n}$ for all n, which implies that $\{l_\rho\} \sim \{l'_\rho\}$.

We will use induction on n. The case $n = 0$ is trivial since both conditional probabilities are $\chi_{\{x_\sigma \in E_\sigma\}}$.

Assume that $n \ge 1$. For $b \in \{0,1\}$, the induction hypothesis implies that there is a Borel measurable function $f_b(t)$ such that

$$\begin{aligned} f_b(t) &= \mathbb{P}\big(\forall|\rho| < n : l_{\sigma b\rho} \in E_{\sigma b\rho} \,\big|\, \{l_\gamma\}_{\gamma\le\sigma}, \{z_\gamma\}_{\gamma<\sigma}, l_{\sigma b} = t, z_\sigma\big) = \\ &= \mathbb{P}\big(\forall|\rho| < n : l'_{\sigma b\rho} \in E_{\sigma b\rho} \,\big|\, \{l'_\gamma\}_{\gamma\le\sigma}, \{z'_\gamma\}_{\gamma<\sigma}, l'_{\sigma b} = t, z'_\sigma\big). \end{aligned}$$

It follows from Lemmas 3.1.4 and 3.1.6 that

$$\begin{aligned} &\mathbb{P}\big(\forall|\rho| \le n : l_{\sigma\rho} \in E_{\sigma\rho} \,\big|\, \{l_\gamma\}_{\gamma\le\sigma}, \{z_\gamma\}_{\gamma<\sigma}\big) = \\ &= \mathbb{E}\Big(\mathbb{P}\big(\forall|\rho| \le n : l_{\sigma\rho} \in E_{\sigma\rho} \,\big|\, \mathcal{F}_{\sigma 0}\big) \,\Big|\, \mathcal{F}_\sigma\Big) = \\ &= \chi_{\{l_\sigma\in E_\sigma\}}\mathbb{E}\Big(\mathbb{P}\big(\forall|\rho| < n : l_{\sigma 0\rho} \in E_{\sigma 0\rho} \,\big|\, \mathcal{F}_{\sigma 0}\big)\mathbb{P}\big(\forall|\rho| < n : l_{\sigma 1\rho} \in E_{\sigma 1\rho} \,\big|\, \mathcal{F}_{\sigma 1}\big) \,\Big|\, \mathcal{F}_\sigma\Big) = \\ &= \chi_{\{l_\sigma\in E_\sigma\}}\mathbb{E}\big(f_0(l_{\sigma 0})f_1(l_{\sigma 1}) \,\big|\, \mathcal{F}_\sigma\big) = \\ &= \chi_{\{l_\sigma\in E_\sigma\}}\mathbb{E}\Big(\mathbb{E}\big(f_0(l_{\sigma 0})f_1(l_{\sigma 1}) \,\big|\, \mathcal{F}_\sigma, z_\sigma\big) \,\Big|\, \mathcal{F}_\sigma\Big) = \\ &= \chi_{\{l_\sigma\in E_\sigma\}}\mathbb{E}\left(\int_0^1 f_0\big(u(l_\sigma - z_\sigma)\big)f_1\big((1-u)(l_\sigma - z_\sigma)\big)\,du \,\middle|\, \mathcal{F}_\sigma\right) = \\ &= \chi_{\{l_\sigma\in E_\sigma\}}\mathbb{E}_{z\sim L_{l_\sigma}}\left(\int_0^1 f_0\big(u(l_\sigma - z)\big)f_1\big((1-u)(l_\sigma - z)\big)\,du\right). \end{aligned}$$

We will now do a similar calculation for $\{l'_\sigma\}$. It is well-known that for fixed t, $F^{-1}_{L_t}(V_\sigma)$ has distribution L_t. Consequently, given l'_σ, z'_σ has distribution $L_{l'_\sigma}$. Since $\{l'_\gamma\}_{\gamma\le\sigma}$ and $\{z'_\gamma\}_{\gamma<\sigma}$ depend only on $\{(U_\rho, V_\rho)\}_{\rho<\sigma}$, the independence implies that

$$\begin{aligned} &\mathbb{P}\big(\forall|\rho| \le n : l'_{\sigma\rho} \in E_{\sigma\rho} \,\big|\, \{l'_\gamma\}_{\gamma\le\sigma}, \{z'_\gamma\}_{\gamma<\sigma}\big) = \\ &= \mathbb{E}\Big(\mathbb{P}\big(\forall|\rho| \le n : l'_{\sigma\rho} \in E_{\sigma\rho} \,\big|\, \{l'_\gamma\}_{\gamma\le\sigma}, \{z'_\gamma\}_{\gamma<\sigma}, U_\sigma, z'_\sigma\big) \,\Big|\, \{l'_\gamma\}_{\gamma\le\sigma}, \{z'_\gamma\}_{\gamma<\sigma}\Big) = \\ &= \chi_{\{l'_\sigma\in E_\sigma\}}\mathbb{E}\big(f_0(l'_{\sigma 0})f_1(l'_{\sigma 1}) \,\big|\, \{l'_\gamma\}_{\gamma\le\sigma}, \{z'_\gamma\}_{\gamma<\sigma}\big) = \\ &= \chi_{\{l'_\sigma\in E_\sigma\}}\mathbb{E}\big(f_0\big(U_\sigma(l'_\sigma - z'_\sigma)\big)f_1\big((1-U_\sigma)(l'_\sigma - z'_\sigma)\big) \,\big|\, \{l'_\gamma\}_{\gamma\le\sigma}, \{z'_\gamma\}_{\gamma<\sigma}\big) = \\ &= \chi_{\{l'_\sigma\in E_\sigma\}}\mathbb{E}_{z\sim L_{l'_\sigma}}\left(\int_0^1 f_0\big(u(l'_\sigma - z)\big)f_1\big((1-u)(l'_\sigma - z)\big)\,du\right). \end{aligned}$$

Therefore,

$$f(t) = \chi_{\{t\in E_\sigma\}}\mathbb{E}_{z\sim L_t}\left(\int_0^1 f_0\big(u(t-z)\big)f_1\big((1-u)(t-z)\big)\,du\right)$$

satisfies the condition. □

Note that the tree $\{l_\sigma\}$ completely determines the set A. From now on, we can assume by Theorem 3.1.7 that $l_\sigma = l'_\sigma$ and $z_\sigma = z'_\sigma$.

3.2 Random recursive constructions

We will now determine the Hausdorff dimension in the special case when $F(t) = t^\alpha$ for some $\alpha > 0$. The main result used in the proof is the following theorem by Mauldin and Williams [12] about a random recursive construction:

Definition 3.2.1. Let $J \subseteq \mathbb{R}^d$ be a non-empty compact set with the property $J = \overline{\operatorname{int} J}$. A family of random compact sets $\mathbf{J} = \{J_\sigma\}_{\sigma \in \mathbb{N}^*}$, is called a *construction* if it satisfies the following properties:

(1) $J_\emptyset = J$ and for every $\sigma \in \mathbb{N}^*$, $J_\sigma \subseteq \mathbb{R}^d$ is either empty or similar to J.

(2) For every $\sigma \in \mathbb{N}^*$, the sets $J_{\sigma 0}, J_{\sigma 1}, \dots$ are subsets of J_σ with pairwise disjoint interiors.

(3) If $J_\sigma \neq \emptyset$, let $T_{\sigma i} = \operatorname{diam} J_{\sigma i} / \operatorname{diam} J_\sigma$ and $\tau_\sigma = (T_{\sigma 0}, T_{\sigma 1}, \dots)$ with the convention $\operatorname{diam} \emptyset = 0$. Then the vectors τ_σ are i.i.d..

Theorem 3.2.2 ([12, Theorem 1.1])**.** *Let* $\mathbf{J}$ *be a construction. Define the set*

$$K = \bigcap_{n=0}^{\infty} \bigcup_{\sigma \in \mathbb{N}^n} J_\sigma .$$

If $\mathbb{E}\big(|\{i \,|\, T_i \neq 0\}|\big) > 1$, *then* $\mathbb{P}(K \neq \emptyset) > 0$. *Furthermore, if* $K \neq \emptyset$, *then almost surely*

$$\dim_{\mathrm{H}} K = \min\big\{ s \geq 0 \,\big|\, \mathbb{E}\big(\textstyle\sum_{i=0}^{\infty} T_i^s\big) \leq 1 \big\} > 0.$$

To use this theorem, we will take $J = [0,1]$, $J_\sigma = \overline{I_\sigma}$ for $\sigma \in \{0,1\}^*$ and $J_\sigma = \emptyset$ for $\sigma \in \mathbb{N}^* \setminus \{0,1\}^*$. It is trivial that (1) and (2) are satisfied by this family.

Using Proposition 3.1.2, we can see that

$$U \setminus A = U \setminus \bigcup_{\sigma \in \{0,1\}^*} Z_{i_\sigma} = U \setminus \bigcup_{n=0}^{\infty} \bigcup_{|\sigma| < n} Z_{i_\sigma} = \bigcap_{n=0}^{\infty} \Big(U \setminus \bigcup_{|\sigma| < n} Z_{i_\sigma} \Big) = \bigcap_{n=0}^{\infty} \bigcup_{\sigma \in \{0,1\}^n} I_\sigma .$$

It follows immediately that $U \setminus A \subseteq K$. We can easily see that

$$K \setminus (U \setminus A) \subseteq \bigcup_{n=0}^{\infty} \bigcup_{\sigma \in \{0,1\}^n} (J_\sigma \setminus I_\sigma) = \bigcup_{n=0}^{\infty} \bigcup_{\sigma \in \{0,1\}^n} \partial I_\sigma ,$$

which is countable. Since countably many points do not affect the Hausdorff dimension, this shows that $\dim_{\mathrm{H}}(U \setminus A) = \dim_{\mathrm{H}} K$.

Theorem 3.2.3. *Suppose that* $F(t) = t^\alpha$ *for* $0 \leq t \leq 1$ *and some fixed* $\alpha > 0$. *Let* $s = s(\alpha) > 0$ *be the unique solution of the equation*

$$2(\alpha + 1)\mathrm{B}(\alpha + 1, s + 1) = 1,$$

where B *is the beta function. Then* $\dim_{\mathrm{H}}(U \setminus A) = s$ *almost surely.*

Proof. We will first check that the distribution L_t/t does not depend on $0 < t \le 1$. Let $X \sim \mu_0$ and $Y \sim U(0,t)$ be independent. For $0 < u \le 1$, we can see that

$$\mathbb{P}(L_t/t < u) = \mathbb{P}(X < tu \,|\, X+Y < t) = \frac{\mathbb{P}(X < tu, Y + X < t)}{\mathbb{P}(X+Y<t)} = \frac{\int_0^{tu}(t-x)/t \cdot dF(x)}{\int_0^t (t-x)/t \cdot dF(x)} =$$
$$= \frac{\int_0^{tu} \alpha(\alpha+1)(t-x)x^{\alpha-1}\,dx}{\int_0^t \alpha(\alpha+1)(t-x)x^{\alpha-1}\,dx} = \frac{t^{\alpha+1}\int_0^u \alpha(\alpha+1)(1-x)x^{\alpha-1}\,dx}{[(\alpha+1)tx^\alpha - \alpha x^{\alpha+1}]_{x=0}^t} =$$
$$= \int_0^u \alpha(\alpha+1)(1-x)x^{\alpha-1}\,dx,$$

which does not depend on t. Hence, $F_{L_t}^{-1}(v)/t = F_{L_t/t}^{-1}(v)$ depends only on v. Furthermore, the previous calculation shows that the density function of L_t/t is

$$f_{L_t/t}(x) = \alpha(\alpha+1)(1-x)x^{\alpha-1}.$$

The next step is to show that Property (3) holds in this case, or equivalently, the vectors

$$\left(\frac{l_{\sigma 0}}{l_\sigma}, \frac{l_{\sigma 1}}{l_\sigma}\right) = \left(U_\sigma\left(1 - \frac{z_\sigma}{l_\sigma}\right), (1-U_\sigma)\left(1 - \frac{z_\sigma}{l_\sigma}\right)\right)$$

are i.i.d.. This is clear, since $z_\sigma/l_\sigma = F_{L_{l_\sigma}}^{-1}(V_\sigma)/l_\sigma$ depends only on V_σ and the vectors (U_σ, V_σ) are i.i.d..

We will now calculate $\mathbb{E}\big((l_{\sigma 0}/l_\sigma)^s + (l_{\sigma 1}/l_\sigma)^s\big)$. Using the formula for the density function, we get

$$\mathbb{E}\left(\left(\frac{l_{\sigma 0}}{l_\sigma}\right)^s + \left(\frac{l_{\sigma 1}}{l_\sigma}\right)^s\right) = 2\int_0^1\int_0^1 u^s(1-x)^s \cdot \alpha(\alpha+1)(1-x)x^{\alpha-1}\,du\,dx =$$
$$= \frac{2\alpha(\alpha+1)}{s+1}\int_0^1 (1-x)^{s+1}x^{\alpha-1}\,dx =$$
$$= \frac{2\alpha(\alpha+1)\mathrm{B}(\alpha, s+2)}{s+1} = 2(\alpha+1)\mathrm{B}(\alpha+1, s+1).$$

It is trivial that $0 \in K$, so $K \ne \emptyset$. Theorem 3.2.2 implies that $\dim_{\mathrm{H}} K$ is almost surely the minimal s such that $2(\alpha+1)\mathrm{B}(\alpha+1, s+1) \le 1$. Note that the beta function is continuous and strictly decreasing in its second argument and furthermore, $2(\alpha+1)\mathrm{B}(\alpha+1,1) = 2$. This shows that $s > 0$ is the unique solution of the equation $2(\alpha+1)\mathrm{B}(\alpha+1, s+1) = 1$. □

Proposition 3.2.4. *The function $s(\alpha)$ is a continuous, strictly decreasing bijection from $(0,\infty)$ to $(0,1)$.*

Proof. Let $\Phi(\alpha, s) = 2(\alpha+1)\mathrm{B}(\alpha+1, s+1)$ for $0 \le \alpha$ and $0 < s \le 1$. It follows from the properties of the beta function that $\Phi(\alpha, s) = 2s\mathrm{B}(\alpha+2, s)$. This shows that Φ is continuous and strictly decreasing in both variables. Suppose that $0 < \alpha < \beta$. We can see that

$$\Phi(\alpha, s(\alpha)) = 1 = \Phi(\beta, s(\beta)) > \Phi(\alpha, s(\beta)),$$

which implies that $s(\alpha) > s(\beta)$. Hence, $s(\alpha)$ is strictly decreasing.

We have seen earlier that $s(\alpha) > 0$ for every $\alpha > 0$. A simple calculation shows that $\Phi(\alpha, 1) = 2/(\alpha+2) < 1$, therefore, $s(\alpha) < 1$ for every $\alpha > 0$. It remains to check that $s(\alpha)$ is a surjection onto $(0,1)$, since combined with monotonicity, this implies that $s(\alpha)$ is continuous. Let $0 < s < 1$. We can check that $\Phi(0, s) = 2/(s+1) > 1$ and $\lim_{\alpha\to\infty} \Phi(\alpha, s) = 0$. By the intermediate value theorem, there is an $\alpha > 0$ such that $\Phi(\alpha, s) = 1$, which implies that $s(\alpha) = s$. $\square$

Corollary 3.2.5. *For every $0 < s < 1$, there exists a μ_0 such that $\dim_{\mathrm{H}}(U \setminus A) = s$ almost surely.*

Proof. By Proposition 3.2.4, there is an α such that $s(\alpha) = s$. It follows from Theorem 3.2.3 that if $F(t) = t^\alpha$ for $0 \le t \le 1$, then $\dim_{\mathrm{H}}(U \setminus A) = s(\alpha) = s$ almost surely. $\square$

3.3 Bounds for the Hausdorff dimension

The goal of this section is to prove Theorem 4. The main idea is to define a coupling with the t^α case, then use the result from the previous section to obtain a bound.

Lemma 3.3.1. *It is almost surely true that*

$$\lim_{n\to\infty} \max_{\sigma\in\{0,1\}^n} l_\sigma = 0.$$

Proof. It is clear that $l_{\sigma 0} \le l_\sigma U_\sigma$ and $l_{\sigma 0} \le l_\sigma(1-U_\sigma)$. Since U_σ and $1-U_\sigma$ both have distribution $U(0,1)$, it follows from the independence that if $|\sigma| = n$, then

$$\mathbb{E}\big(l_\sigma^2\big) \le \big(\mathbb{E}(U(0,1)^2)\big)^n = \frac{1}{3^n}.$$

Fix an $\varepsilon > 0$. It easily follows from Markov's inequality that

$$\mathbb{P}\Big(\max_{\sigma\in\{0,1\}^n} l_\sigma \ge \varepsilon\Big) \le \sum_{\sigma\in\{0,1\}^n} \mathbb{P}\big(l_\sigma^2 \ge \varepsilon^2\big) \le \frac{2^n}{3^n\varepsilon^2}.$$

Since this bound is exponentially small, the Borel–Cantelli lemma implies that for large enough n, $\max_{\sigma\in\{0,1\}^n} l_\sigma < \varepsilon$. As this is true for every $\varepsilon > 0$, the statement follows. $\square$

Lemma 3.3.2. *Suppose that F and G are two cumulative distribution functions on $[0,\infty)$ that are positive on the interval $(0,\infty)$. Let L_t^F and L_t^G be the corresponding distributions for $\mu_0 \sim F$ and $\mu_0 \sim G$, respectively. If there is an $\varepsilon > 0$ such that F/G is increasing on $(0,\varepsilon)$, then for every $0 < u \le t \le \varepsilon$, $\mathbb{P}(L_t^F < u) \le \mathbb{P}(L_t^G < u)$.*

Proof. Let $X \sim F$ and $Y \sim U(0,t)$ be independent. It is clear from the definition of L_t that

$$\mathbb{P}(L_t^F < u) = \frac{\mathbb{P}(X < u, X+Y < t)}{\mathbb{P}(X+Y < t)} = \frac{\frac{1}{t}\int_0^t \mathbb{P}(X < u, X < t-x)\,dx}{\frac{1}{t}\int_0^t \mathbb{P}(X < x)\,dx} =$$

$$= \frac{\int_0^t F(\min(u, t-x))\,dx}{\int_0^t F(t-x)\,dx} = \frac{\int_0^t F(\min(u, x))\,dx}{\int_0^t F(x)\,dx}.$$

Taking reciprocals, we obtain the formula

$$\frac{1}{\mathbb{P}(L_t^F < u)} = \frac{\int_0^t F(x)\,dx}{\int_0^t F(\min(u,x))\,dx} = \frac{\int_0^t \big(F(\min(u,x)) + \chi_{\{u<x\}}(F(x)-F(u))\big)\,dx}{\int_0^t F(\min(u,x))\,dx} =$$
$$= 1 + \frac{\int_u^t (F(x)-F(u))\,dx}{\int_0^t F(\min(u,x))\,dx}.$$

A similar formula holds for L_t^G. Therefore, it suffices to show that

$$\frac{\int_u^t (F(x)-F(u))\,dx}{\int_0^t F(\min(u,x))\,dx} \geq \frac{\int_u^t (G(x)-G(u))\,dx}{\int_0^t G(\min(u,x))\,dx}.$$

Let $c = F(u)/G(u)$. The monotonicity of F/G implies that $F(x) \leq cG(x)$ for $x \leq u$ and $F(x) \geq cG(x)$ for $x \geq u$. Consequently,

$$\frac{\int_u^t (F(x)-F(u))\,dx}{\int_0^t F(\min(u,x))\,dx} \geq \frac{\int_u^t (cG(x)-cG(u))\,dx}{\int_0^t cG(\min(u,x))\,dx} = \frac{\int_u^t (G(x)-G(u))\,dx}{\int_0^t G(\min(u,x))\,dx}.$$

□

Theorem 3.3.3. *Let $\alpha > 0$. If there is an $\varepsilon > 0$ such that $F(t)/t^\alpha$ is increasing on $(0,\varepsilon)$, then $\dim_{\mathrm{H}}(U \setminus A) \leq s(\alpha)$ almost surely.*

Proof. Let $G(t) = t^\alpha$ for $0 \leq t \leq 1$. Define the variables $\tilde{l}_\sigma$ and $\tilde{z}_\sigma$ recursively by the formulae

$$\tilde{l}_\emptyset = 1, \qquad \tilde{z}_\sigma = F^{-1}_{L^G_{\tilde{l}_\sigma}}(V_\sigma), \qquad \tilde{l}_{\sigma 0} = U_\sigma(\tilde{l}_\sigma - \tilde{z}_\sigma), \qquad \tilde{l}_{\sigma 1} = (1-U_\sigma)(\tilde{l}_\sigma - \tilde{z}_\sigma).$$

Note that $\tilde{l}_\sigma$ is the diameter of J_σ in the case $\mu_0 \sim G$. By Lemma 3.3.1, $l_\sigma \geq \varepsilon$ occurs finitely many times, so there is a $c > 1$ such that for every $\sigma \in \{0,1\}^*$ and $b \in \{0,1\}$, if $l_\sigma \geq \varepsilon$, then $l_{\sigma b}/\tilde{l}_{\sigma b} \leq c$.

We will prove by induction on σ that $l_\sigma \leq c\tilde{l}_\sigma$ for every $\sigma \in \{0,1\}^*$. This is clearly true for $\sigma = \emptyset$. Now assume that $l_\sigma \leq c\tilde{l}_\sigma$ for some σ. We have to prove that $l_{\sigma b} \leq c\tilde{l}_{\sigma b}$ for $b \in \{0,1\}$. If $l_\sigma \geq \varepsilon$, then we are done by the choice of c. Otherwise, it follows from Lemma 3.3.2 that $F_{L^F_{l_\sigma}} \leq F_{L^G_{l_\sigma}}$. We have seen in the proof of Theorem 3.2.3 that $F^{-1}_{L^G_t}(V_\sigma)/t$ depends only on V_σ, therefore,

$$\frac{z_\sigma}{l_\sigma} = \frac{F^{-1}_{L^F_{l_\sigma}}(V_\sigma)}{l_\sigma} \geq \frac{F^{-1}_{L^G_{l_\sigma}}(V_\sigma)}{l_\sigma} = \frac{F^{-1}_{L^G_{\tilde{l}_\sigma}}(V_\sigma)}{\tilde{l}_\sigma} = \frac{\tilde{z}_\sigma}{\tilde{l}_\sigma}.$$

By the induction hypothesis,

$$l_{\sigma 0} = U_\sigma(l_\sigma - z_\sigma) = U_\sigma l_\sigma\left(1 - \frac{z_\sigma}{l_\sigma}\right) \leq U_\sigma l_\sigma\left(1 - \frac{\tilde{z}_\sigma}{\tilde{l}_\sigma}\right) \leq cU_\sigma \tilde{l}_\sigma\left(1 - \frac{\tilde{z}_\sigma}{\tilde{l}_\sigma}\right) = c\tilde{l}_{\sigma 0}.$$

A similar calculation shows that $l_{\sigma 1} \leq c\tilde{l}_{\sigma 1}$.

We will finish the argument based on the proof of [12, Theorem 1.3]. We have seen in the proof of Theorem 3.2.3 that the vectors $(\tilde{l}_{\sigma 0}/\tilde{l}_\sigma, \tilde{l}_{\sigma 1}/\tilde{l}_\sigma)$ are independent and $s = s(\alpha)$ satisfies the equation $\mathbb{E}\big((\tilde{l}_{\sigma 0}/\tilde{l}_\sigma)^s + (\tilde{l}_{\sigma 1}/\tilde{l}_\sigma)^s\big) = 1$. Let $S_{s,n} = \sum_{\sigma \in \{0,1\}^n} \tilde{l}_\sigma^s$. It follows from the previous equation that $\{S_{s,n}\}_{n=0}^\infty$ is a positive martingale, therefore it converges almost surely. Consequently, there is an $M > 0$ such that $S_{s,n} \leq M$ for every n.

Fix a $\delta > 0$. By Lemma 3.3.1, $l_\sigma < \delta$ whenever $n = |\sigma|$ is large enough. Note that $K \subseteq \bigcup_{\sigma \in \{0,1\}^n} J_\sigma$, hence,

$$\mathcal{H}^s_\delta(K) \leq \sum_{\sigma \in \{0,1\}^n} (\operatorname{diam} J_\sigma)^s = \sum_{\sigma \in \{0,1\}^n} l_\sigma^s \leq c^s \sum_{\sigma \in \{0,1\}^n} \tilde{l}_\sigma^s = c^s S_{s,n} \leq c^s M.$$

Taking the limit $\delta \to 0$, this implies that $\mathcal{H}^s(K) \leq c^s M < \infty$, therefore $\dim_{\mathrm{H}} K \leq s$. □

Corollary 3.3.4. *There exists a* μ_0 *such that* $\dim_{\mathrm{H}}(U \setminus A) = 0$ *almost surely.*

Proof. Let $F(t) = t^{\log(1/t)}$ for $0 < t \leq 1$. For every $\alpha > 0$, we can see that

$$\frac{F(t)}{t^\alpha} = t^{\log(1/t) - \alpha} = \exp\Big(-\log\frac{1}{t} \cdot \Big(\log\frac{1}{t} - \alpha\Big)\Big).$$

It is easy to check that this function is increasing on the interval $(0, e^{-\alpha})$. By Theorem 3.3.3, $\dim_{\mathrm{H}}(U \setminus A) \leq s(\alpha)$ almost surely. Since this is true for every $\alpha > 0$, we can conclude from Proposition 3.2.4 that $\dim_{\mathrm{H}}(U \setminus A) = 0$ almost surely. □

Before we begin the proof of the lower bound, we need the following lemma. The proof is based on ideas by Moran [13, Theorem III].

Lemma 3.3.5. *Let* $\{J_\sigma\}_{\sigma \in \mathbb{N}^*}$ *be a family of compact sets in* $\mathbb{R}^d$ *satisfying properties (1) and (2) of Definition 3.2.1. We will additionally assume that* $\lim_{k \to \infty} \max_{\sigma \in \mathbb{N}^k} \operatorname{diam} J_\sigma = 0$ *and there is an* $\varepsilon > 0$ *such that for every* $\sigma \in \mathbb{N}^*$ *and* $i \in \mathbb{N}$, *if* $J_{\sigma i} \neq \emptyset$, *then* $\operatorname{diam} J_{\sigma i} \geq \varepsilon \operatorname{diam} J_\sigma$. *Let* $K = \bigcap_{n=0}^\infty \bigcup_{\sigma \in \mathbb{N}^n} J_\sigma$ *and suppose that* $\mathcal{H}^s(K) < \infty$ *for some* $s \geq 0$. *Then there exists a* c *such that for every* $\delta > 0$, *there is a set* $\Sigma \subseteq \mathbb{N}^*$ *such that* $0 < \operatorname{diam} J_\sigma < \delta$ *for every* $\sigma \in \Sigma$, $K \subseteq \bigcup_{\sigma \in \Sigma} J_\sigma$ *and* $\sum_{\sigma \in \Sigma} (\operatorname{diam} J_\sigma)^s \leq c$.

Proof. Fix an $M > \mathcal{H}^s(K)$. By the definition of the Hausdorff metric, there are sets $C_1, C_2, \ldots$ such that $\operatorname{diam} C_i < \min(\delta, \operatorname{diam} J_\emptyset)$ for every i, $K \subseteq \bigcup_{i=1}^\infty C_i$ and $\sum_{i=1}^\infty (\operatorname{diam} C_i)^s < M$. By slightly enlarging the sets, we may assume that $0 < \operatorname{diam} C_i$ for every i. For each i, let

$$\Sigma_i = \big\{\sigma \in \mathbb{N}^* \setminus \{\emptyset\} \,\big|\, J_\sigma \cap C_i \neq \emptyset,\ \operatorname{diam} J_\sigma \leq \operatorname{diam} C_i,\ \operatorname{diam} J_{\sigma|_{|\sigma|-1}} > \operatorname{diam} C_i\big\},$$

where $\sigma|_p$ denotes the initial segment of length p.

We will first show that $C_i \cap K \subseteq \bigcup_{\sigma \in \Sigma_i} J_\sigma$. Let $x \in C_i \cap K$. Choose a k such that $\max_{\sigma \in \mathbb{N}^k} \operatorname{diam} J_\sigma < \operatorname{diam} C_i$. By the definition of K, there is a σ such that $|\sigma| = k$ and $x \in J_\sigma$. There is a minimal $0 < j \leq k$ such that $\operatorname{diam} J_{\sigma|_j} < \operatorname{diam} C_i$. Clearly, $x \in J_\sigma \subseteq J_{\sigma|_j}$, so it follows from the minimality of j that $\sigma|_j \in \Sigma_i$.

Next, we will give an upper bound on $|\Sigma_i|$. Note that by Property (2), if neither σ nor σ' is an initial segment of the other, then the interiors of J_σ and $J_{\sigma'}$ are disjoint. It is also clear that if $\sigma < \sigma'$, then at most one of σ and σ' can be in Σ_i. Therefore, the sets $\{\operatorname{int} J_\sigma\}_{\sigma \in \Sigma_i}$

are disjoint. Fix an $x \in C_i$, clearly $C_i \subseteq \overline{B(x, \operatorname{diam} C_i)}$. It follows from the definition of Σ_i that $J_\sigma \subseteq B(x, 2\operatorname{diam} C_i)$ every $\sigma \in \Sigma_i$. We can also see that if $\sigma \in \Sigma_i$, then $\operatorname{diam} J_\sigma \geq \varepsilon \operatorname{diam} J_{\sigma|_{|\sigma|-1}} \geq \varepsilon \operatorname{diam} C_i$. Therefore,

$$\lambda(B(x, 2\operatorname{diam} C_i)) \geq \lambda\left(\bigcup_{\sigma\in\Sigma_i} \operatorname{int} J_\sigma\right) = \sum_{\sigma\in\Sigma_i} \lambda(\operatorname{int} J_\sigma) = \sum_{\sigma\in\Sigma_i} \left(\frac{\operatorname{diam} J_\sigma}{\operatorname{diam} J_\emptyset}\right)^d \lambda(\operatorname{int} J_\emptyset) \geq$$
$$\geq |\Sigma_i| \left(\frac{\varepsilon \operatorname{diam} C_i}{\operatorname{diam} J_\emptyset}\right)^d \lambda(\operatorname{int} J_\emptyset).$$

Since $\lambda(\operatorname{int} J_\emptyset) > 0$, we obtain the bound

$$|\Sigma_i| \leq \frac{\lambda(B(x, 2\operatorname{diam} C_i))}{(\varepsilon \operatorname{diam} C_i / \operatorname{diam} J_\emptyset)^d \lambda(\operatorname{int} J_\emptyset)} = \frac{\lambda(B(0,2))}{(\varepsilon / \operatorname{diam} J_\emptyset)^d \lambda(\operatorname{int} J_\emptyset)}.$$

This bound does not depend on either δ or i, hence there is a constant N such that $|\Sigma_i| \leq N$ for every i.

It remains to show that $\Sigma = \bigcup_{i=1}^\infty \Sigma_i$ satisfies the condition. Clearly, $\operatorname{diam} J_\sigma < \delta$ for every $\sigma \in \Sigma$. Also,

$$K = \bigcup_{i=1}^\infty (K \cap C_i) \subseteq \bigcup_{i=1}^\infty \bigcup_{\sigma\in\Sigma_i} J_\sigma = \bigcup_{\sigma\in\Sigma} J_\sigma.$$

Finally,

$$\sum_{\sigma\in\Sigma} (\operatorname{diam} J_\sigma)^s \leq \sum_{i=1}^\infty \sum_{\sigma\in\Sigma_i} (\operatorname{diam} J_\sigma)^s \leq \sum_{i=1}^\infty N (\operatorname{diam} C_i)^s \leq NM.$$

□

We will need an additional lemma about a modified version of the construction in the case $F(t) = t^\alpha$. This lemma, though not stated separately, was proved more generally as part of the proof of [12, Theorem 3.6].

Lemma 3.3.6. *Let $0 < \alpha$, $0 < s' < s(\alpha)$ and $0 < \eta < 1$. Define $\tilde{l}_\sigma$ as in the proof of Theorem 3.3.3 (with $G(t) = t^\alpha$). For some $n > 0$, define the construction $\tilde{\mathbf{J}} = \{\tilde{J}_\sigma\}_{\sigma\in\{0,1\}^*}$ recursively:*

$$\tilde{J}_\emptyset = [0,1]$$
$$\tilde{J}_{\sigma 0} = \begin{cases} [\min \tilde{J}_\sigma, \min \tilde{J}_\sigma + \tilde{l}_{\sigma 0}] & \text{if } \tilde{J}_\sigma \neq \emptyset \text{ and } \tilde{l}_{\sigma 0}/\tilde{l}_\sigma \geq 1/n \\ \emptyset & \text{otherwise} \end{cases}$$
$$\tilde{J}_{\sigma 1} = \begin{cases} [\max \tilde{J}_\sigma - \tilde{l}_{\sigma 1}, \max \tilde{J}_\sigma] & \text{if } \tilde{J}_\sigma \neq \emptyset \text{ and } \tilde{l}_{\sigma 1}/\tilde{l}_\sigma \geq 1/n \\ \emptyset & \text{otherwise} \end{cases}$$

Let $\tilde{K} = \bigcap_{k=0}^\infty \bigcup_{\tilde{J}_\sigma \in \mathbb{N}^k}$. If n is large enough, then $\mathbb{P}(\tilde{K} = \emptyset) < \eta$, furthermore, if $\tilde{K} \neq \emptyset$, then $\dim_{\mathrm{H}} \tilde{K} > s'$ almost surely.

Proof. We will first check that $\tilde{\mathbf{J}}$ is indeed a construction. Property (1) is obvious. Property (2) is also clear, since $\tilde{l}_{\sigma 0} + \tilde{l}_{\sigma 1} = \tilde{l}_\sigma - \tilde{z}_\sigma \le \tilde{l}_\sigma$. Finally, Property (3) follows from the independence of the vectors $(\tilde{l}_{\sigma 0}/\tilde{l}_\sigma, \tilde{l}_{\sigma 1}/\tilde{l}_\sigma)$.

We have seen that the vectors $(\tilde{l}_{\sigma 0}/\tilde{l}_\sigma, \tilde{l}_{\sigma 1}/\tilde{l}_\sigma)$ are identically distributed. Since $\tilde{l}_{\sigma 0}$ and $\tilde{l}_{\sigma 1}$ are positive, it follows from measure continuity that there exists an n_0 such that

$$\mathbb{P}\left(\min\left(\frac{\tilde{l}_{\sigma 0}}{\tilde{l}_\sigma}, \frac{\tilde{l}_{\sigma 1}}{\tilde{l}_\sigma}\right) \ge \frac{1}{n_0}\right) > \frac{1}{\eta + 1}.$$

Let $\Phi(t) = \mathbb{E}\big((\tilde{l}_{\sigma 0}/\tilde{l}_\sigma)^t + (\tilde{l}_{\sigma 1}/\tilde{l}_\sigma)^t\big)$ and

$$\Phi_n(t) = \mathbb{E}\big(\chi_{\{\tilde{l}_{\sigma 0}/\tilde{l}_\sigma \ge 1/n\}}(\tilde{l}_{\sigma 0}/\tilde{l}_\sigma)^t + \chi_{\{\tilde{l}_{\sigma 1}/\tilde{l}_\sigma \ge 1/n\}}(\tilde{l}_{\sigma 1}/\tilde{l}_\sigma)^t\big).$$

It is clear that Φ is strictly decreasing, therefore, $\Phi(s') > \Phi(s) = 1$. Since $\tilde{l}_{\sigma b} > 0$, it follows from the bounded convergence theorem that $\Phi_n \to \Phi$ pointwise, in particular, there exists an n_1 such that $\Phi_n(s') > 1$ for $n \ge n_1$. From now on, we will assume that $n \ge \max(n_0, n_1)$. We can see that

$$\mathbb{E}\big(|\{b \in \{0,1\} \mid \tilde{J}_b \ne \emptyset\}|\big) = \mathbb{E}\big(\chi_{\{\tilde{l}_{\sigma 0}/\tilde{l}_\sigma \ge 1/n\}} + \chi_{\{\tilde{l}_{\sigma 1}/\tilde{l}_\sigma \ge 1/n\}}\big) \ge \Phi_n(s') > 1.$$

It follows from Theorem 3.2.2 that $\mathbb{P}(\tilde{K} = \emptyset) < 1$, furthermore, if $\tilde{K} \ne \emptyset$, then $\Phi_n(\dim_{\mathrm{H}} \tilde{K}) \le 1$. Since Φ_n is decreasing, this implies that $\dim_{\mathrm{H}} \tilde{K} > s'$.

It is easy to check that

$$\tilde{K} = \bigcap_{k=0}^{\infty} \bigcup_{\sigma \in \mathbb{N}^k} \tilde{J}_\sigma = \bigcup_{b \in \{0,1\}} \bigcap_{k=0}^{\infty} \bigcup_{\sigma \in \mathbb{N}^k} \tilde{J}_{b\sigma}.$$

Note that given J_b and $J_b \ne \emptyset$, the set $\bigcap_{k=0}^{\infty} \bigcup_{\sigma \in \mathbb{N}^k} \tilde{J}_{b\sigma}$ has, the same distribution as $\tilde{K}$, transformed by the similarity that takes $J_\emptyset = [0,1]$ to J_b. Let $C_i = \mathbb{P}\big(\chi_{\{\tilde{l}_{\sigma 0}/\tilde{l}_\sigma \ge 1/n\}} + \chi_{\{\tilde{l}_{\sigma 1}/\tilde{l}_\sigma \ge 1/n\}} = i\big)$ for $i \in \{0,1,2\}$, clearly $C_0 + C_1 + C_2 = 1$. It follows from the independence that $p = \mathbb{P}(\tilde{K} = \emptyset)$ satisfies the equation $p = \psi(p)$, where $\psi(x) = C_0 + C_1 x + C_2 x^2$. Since $n \ge n_0$, the choice of n_0 implies that $C_2 > 1/(\eta + 1)$. Consequently,

$$\psi(\eta) = C_0 + C_1\eta + C_2\eta^2 \le C_0 + C_1 + C_2\eta^2 = 1 - C_2(1 - \eta^2) < 1 - (1 - \eta) = \eta.$$

Notice that $\psi(p) = p$, $\psi(\eta) < \eta$ and $\psi(1) = 1$. Since $p < 1$ and ψ is strictly convex, this implies that $p < \eta < 1$, hence $\mathbb{P}(\tilde{K} = \emptyset) < \eta$. □

Theorem 3.3.7. *Let $\alpha > 0$. If there is an $\varepsilon > 0$ such that $F(x)/t^\alpha$ is decreasing on $(0, \varepsilon)$, then $\dim_{\mathrm{H}}(U \setminus A) \ge s(\alpha)$ almost surely.*

Proof. Fix $0 < s' < s = s(\alpha)$ and $0 < \eta < 1$. We will show that $\mathbb{P}(\dim_{\mathrm{H}} K < s') \le 2\eta$. Taking the limits $\eta \to 0$, then $s' \to s$, it follows that $\dim_{\mathrm{H}}(U \setminus A) = \dim_{\mathrm{H}} K \ge s$ almost surely.

It follows from Lemma 3.3.1 that there is an $n > 0$ almost surely such that $l_{\sigma b} \geq 1/n$ whenever $l_\sigma \geq \varepsilon$. By measure continuity, for large enough n,

$$\mathbb{P}\Big(\exists \sigma \in \{0,1\}^*, b \in \{0,1\} : l_\sigma, \geq \varepsilon, l_{\sigma b} < \frac{1}{n}\Big) \leq \eta.$$

By Lemma 3.3.6, if n is large enough, then $\mathbb{P}(\tilde{K} = \emptyset) < \eta$, furthermore, if $\tilde{K} \neq \emptyset$, then $\dim_{\mathrm{H}} \tilde{K} > s'$. Consequently,

$$\mathbb{P}\bigg(\dim_{\mathrm{H}} \tilde{K} \leq s' \vee \Big(\exists \sigma \in \{0,1\}^*, b \in \{0,1\} : l_\sigma \geq \varepsilon, l_{\sigma b} < \frac{1}{n}\Big)\bigg) \leq 2\eta.$$

It remains to prove that if $\dim_{\mathrm{H}} \tilde{K} > s'$ and $l_{\sigma b} \geq 1/n$ whenever $l_\sigma \geq \varepsilon$, then $\dim_{\mathrm{H}} K \geq s'$. First, assume that $l_\sigma < \varepsilon$ for some σ. It follows from Lemma 3.3.2 that $F_{L^F_{l_\sigma}} \geq F_{L^G_{l_\sigma}}$, where $G(t) = t^\alpha$ (note that the inequality is reversed compared to Theorem 3.3.3). Like in the proof of Theorem 3.3.3, this implies that

$$\frac{z_\sigma}{l_\sigma} = \frac{F^{-1}_{L^F_{l_\sigma}}(V_\sigma)}{l_\sigma} \leq \frac{F^{-1}_{L^G_{l_\sigma}}(V_\sigma)}{l_\sigma} = \frac{F^{-1}_{L^G_{\tilde{l}_\sigma}}(V_\sigma)}{\tilde{l}_\sigma} = \frac{\tilde{z}_\sigma}{\tilde{l}_\sigma},$$

We will check that for every $\sigma \in \{0,1\}^*$ and $b \in \{0,1\}$, if $\tilde{l}_{\sigma b} \geq \tilde{l}_\sigma/n$, then $l_{\sigma b} \geq l_\sigma/n$. This is clear if $l_{\sigma b} \geq 1/n$. If $l_{\sigma b} < 1/n$, then by the assumption, $l_\sigma < \varepsilon$, which implies that $z_\sigma/l_\sigma \leq \tilde{z}_\sigma/\tilde{l}_\sigma$. Therefore,

$$\frac{l_{\sigma 0}}{l_\sigma} = U_\sigma\left(1 - \frac{z_\sigma}{l_\sigma}\right) \geq U_\sigma\left(1 - \frac{\tilde{z}_\sigma}{\tilde{l}_\sigma}\right) = \frac{\tilde{l}_{\sigma 0}}{\tilde{l}_\sigma} \geq \frac{1}{n},$$

and a similar calculation shows that $l_{\sigma 1}/l_\sigma \geq 1/n$.

Next, we will prove by induction on σ that $\tilde{l}_\sigma \leq n l_\sigma$. This is clearly true for $\sigma = \emptyset$. Assume that $\tilde{l}_\sigma \leq n l_\sigma$, we need to prove that $\tilde{l}_{\sigma b} \leq n l_{\sigma b}$. If $l_\sigma \geq \varepsilon$, then we are done by the assumption that $l_{\sigma b} \geq 1/n$. Otherwise, $z_\sigma/l_\sigma \leq \tilde{z}_\sigma/\tilde{l}_\sigma$, hence,

$$\tilde{l}_{\sigma 0} = U_\sigma(\tilde{l}_\sigma - \tilde{z}_\sigma) = U_\sigma \tilde{l}_\sigma\left(1 - \frac{\tilde{z}_\sigma}{\tilde{l}_\sigma}\right) \leq U_\sigma \tilde{l}_\sigma\left(1 - \frac{z_\sigma}{l_\sigma}\right) \leq n U_\sigma l_\sigma\left(1 - \frac{\tilde{z}_\sigma}{\tilde{l}_\sigma}\right) = n l_{\sigma 0},$$

and similarly, $\tilde{l}_{\sigma 1} \leq n l_{\sigma 1}$.

Define the set

$$\hat{J}_\sigma = \begin{cases} J_\sigma & \text{if } \tilde{J}_\sigma \neq \emptyset \\ \emptyset & \text{otherwise,} \end{cases}$$

and let $\hat{K} = \bigcap_{k=0}^\infty \bigcup_{\sigma \in \mathbb{N}^k} \hat{J}_\sigma$. It is immediately clear that $\hat{K} \subseteq K$. If $\hat{J}_{\sigma b} \neq \emptyset$, then $\tilde{l}_{\sigma b}/\tilde{l}_\sigma \geq 1/n$, so $\operatorname{diam} \hat{J}_{\sigma b}/\operatorname{diam} \hat{J}_\sigma \geq 1/n$. Hence, $\{\hat{J}_\sigma\}$ satisfies the conditions of Lemma 3.3.5. Suppose for contradiction that $\dim_{\mathrm{H}} K < s'$. It follows that $\mathcal{H}^{s'}(\hat{K}) = 0$. Fix a $\delta > 0$. By Lemma 3.3.5, there is a $\Sigma \subseteq \{0,1\}^*$ such that $0 < \operatorname{diam} \hat{J}_\sigma < \delta$ for every $\sigma \in \Sigma$, $\hat{K} \subseteq \bigcup_{\sigma \in \Sigma} \hat{J}_\sigma$ and $\sum_{\sigma \in \Sigma} (\operatorname{diam} J_\sigma)^s \leq c$, where c does not depend on δ.

We will show that $\tilde{K} \subseteq \bigcup_{\sigma \in \Sigma} \tilde{J}_\sigma$. Let $x \in \tilde{K}$. By the definition of $\tilde{K}$, there is a sequence $\sigma_0, \sigma_1, \dots \in \{0,1\}^*$ such that $|\sigma_i| = i$ and $x \in J_{\sigma_i}$ for every i. Since $\tilde{z}_\rho > 0$ almost surely for every ρ, this sequence is unique, hence $\sigma_0 < \sigma_1 < \dots$. By compactness, there exists a $y \in \bigcap_{i=0}^{\infty} \hat{J}_{\sigma_i} \in \hat{K}$, as the sets $\hat{J}_{\sigma_i}$ are nested and non-empty. Choose a $\sigma \in \Sigma$ such that $y \in \hat{J}_\sigma$. It is easy to see that $z_\rho > 0$ almost surely for every ρ, which implies that the sets $\{J_\rho\}_{|\rho|=|\sigma|}$ are disjoint. Therefore, $\sigma = \sigma_{|\sigma|}$, consequently, $x \in \tilde{J}_\sigma$.

We know that $\tilde{l}_\sigma \leq n l_\sigma$ for every σ. Therefore, $\tilde{l}_\sigma \leq n l_\sigma < n\delta$ for every $\sigma \in \Sigma$. It follows that

$$\mathcal{H}^s_{n\delta}(\tilde{K}) \leq \sum_{\sigma \in \Sigma} \tilde{l}_\sigma^{s'} \leq n^{s'} \sum_{\sigma \in \Sigma} l_\sigma^{s'} \leq c.$$

Taking the limit $\delta \to 0$ yields the bound $\mathcal{H}^{s'}(\tilde{K}) \leq c$, since c does not depend on δ. This implies that $\dim_{\mathrm{H}} \tilde{K} \leq s'$, contradicting the assumption that $\dim_{\mathrm{H}} \tilde{K} > s'$. This shows that $\dim_{\mathrm{H}} K \geq s'$. □

Corollary 3.3.8. *There exists a μ_0 for which $\lambda(U \setminus A) = 0$ and $\dim_{\mathrm{H}}(U \setminus A) = 1$ almost surely.*

Proof. Let

$$F(t) = \begin{cases} \frac{1}{\log(1/t)} & \text{if } 0 < t < e^{-1} \\ 1 & \text{if } e^{-1} \leq t. \end{cases}$$

Clearly, $F(0^+) = 0$, so $\lambda(U \setminus A) = 0$ almost surely by Theorem 2.3.1.

Let $\alpha > 0$. A simple calculation shows that

$$\left(\log \frac{F(t)}{t^\alpha}\right)' = \left(\alpha \log \frac{1}{t} - \log\log \frac{1}{t}\right)' = -\frac{\alpha}{t} + \frac{1}{t\log(1/t)} = \frac{1}{t}\left(\frac{1}{\log(1/t)} - \alpha\right),$$

which is negative if $t < e^{-1/\alpha}$. It follows from Theorem 3.3.7 that $\dim_{\mathrm{H}}(U \setminus A) \geq s(\alpha)$ almost surely. By Proposition 3.2.4, this implies that $\dim_{\mathrm{H}}(U \setminus A) = 1$ almost surely. □

Remark. It is easy to construct distributions for which the conditions of Theorems 3.3.3 and 3.3.7 do not hold. However, for many "natural" continuous distributions, it is true that for every $\alpha > 0$, $F(t)/t^\alpha$ is monotonic in a neighborhood of zero. In this case, we can use Theorems 3.3.3 and 3.3.7 combined with the continuity of $s(\alpha)$ to find the s such that $\dim_{\mathrm{H}}(U \setminus A) = s$ almost surely.

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