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ELEMENTARY SUBMODELS IN VARIOUS CHAPTERS OF SET THEORY

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Contents

Chapter 1

Introduction

The aim of this thesis to explain how to use elementary submodels to prove new theorems or to simplify old proofs in various areas of set theory, for example in infinite combinatorics.

This thesis consists of two main parts. In the first section (Chapter 3) we introduce the classical methods for proving two famous theorems of Erdos-Rado and in the second section (Chapters 4-6) we present the technique of using elementary submodels with the necessary preliminaries to understand them fully. The two parts do not depend on each other and can be read independently.

In Chapter 2 we lay out some preliminaries which is necessary for the thesis. In Chapter 3 we introduce the main concepts we will use in the first part and investigate some of their properties: cofinalities, club sets and stationary sets. Then, we showcase the main theorem for the two Erdos-Rado proofs, the Fodor's pressing down lemma. It says that a regressive function is necessarily constant on a significantly large subset of the domain. The first Erdos-Rado theorem we prove is about the existence of a Δ-system and the second is a Ramsey style question, which tells us about a large monochromatic set that we can always select when coloring a sufficiently large graph.

Onto the second part, in Chapter 4 we introduce the sets H_θ for every cardinal θ. They will be useful in Chapter 5, where we define elementary substructures and apply the theory to get elementary submodels from the sets H_θ. We first introduce important notions from logic, we define structures, terms and first-order formulas and precisely show what we mean by the value of a term and the truth of a formula. Then, we prove theorems about the existence of elementary submodels and investigate some of its interesting properties we can use later. In Chapter 6 we show some interesting applications of elementary submodels in topology, infinite combinatorics and in the last chapter we revisit the well-known Δ-system lemma and we prove it using elementary submodels.

Chapter 2

Preliminaries

We assume that the reader is familiar with basic notions and definitions from set theory and topology.

In some introductory set theory classes they define an ordinal as the equivalence class of well-ordered sets. We now use the Von Neumann definition of ordinals which means that we define every ordinal as a particular set that (canonically) represents the class. Thus, an ordinal number will be a well-ordered set; and it can be shown by transfinite induction that every well-ordered set will be order-isomorphic to exactly one ordinal number. This definition will imply that each ordinal is the set of all smaller ordinals. Formally, a set A is an ordinal if and only if A is well-ordered with respect to $\in$ and every element of A is also a subset of A. For example, the first few Von Neumann ordinals are:

$$\begin{aligned}
&0 = \{\} = \emptyset \\
&1 = \{0\} = \{\emptyset\} \\
&2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\} \\
&3 = \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}
\end{aligned}$$

If we consider an ordering $<$ on A, we view the set $<$ as a subset of $A \times A$. A pair $(x, y) \in <$ if and only if $x < y$. The expression $x \leq y$ is a shorthand for $(x, y) \in <$ or $x = y$.

Definition 2.0.1. *We say that the ordered sets $(A, <)$ and $(B, <)$ are isomorphic if there exists a function $f : A \to B$ which is an order-preserving bijection.*

Definition 2.0.2. *Let $(A, <)$ be a well-ordered set. The order type $tp((A, <))$ of the set $(A, <)$ is the unique Von Neumann ordinal α such that $(\alpha, \in)$ is isomorphic to*

$(A, <)$. *If the ordering is obvious from the context we usually write* $tp(A)$.

Notation 2.0.3. *We will denote the cardinality of the real line (continuum) with* $\mathfrak{c}$.

Notation 2.0.4. *For a cardinal* κ *we will denote the successor cardinal of* κ *with* κ^+.

Definition 2.0.5. *For a set* x*, let* $\bigcup x := \{z \mid \exists y\,(z \in y \wedge y \in x)\}$.

Notation 2.0.6. *A set* x *is called transitive if for all* $y \in x$ *we have that* $y \subseteq x$.

In other words if for all $z \in y \in x$ we have that $z \in x$, hence the name *transitive*. We will use the following notation as well:

Notation 2.0.7. *For cardinals* κ, λ *the set* $[\kappa]^\lambda := \{A \subseteq \kappa \big| |A| = \lambda\}$.

If not indicated otherwise the symbols $\lambda, \kappa, \mu, (\kappa_\xi)$ will denote cardinal numbers.

Chapter 3

Infinite combinatorics

3.1 Cofinalities

Definition 3.1.1. *Let $(A,<)$ be an ordered set. A subset $B \subseteq A$ is called unbounded or cofinal if for every $x \in A$ there exists $y \in B$ such that $y \geq x$.*

Theorem 3.1.2. *(Hausdorff) Let $(A,<)$ be an ordered set. There exists $B \subseteq A$ such that B is unbounded, $<\restriction_{B\times B}$ is a well-ordering of B and $tp(B) \leq |A|$.*

Proof. Let us consider a $\prec$ well-ordering of A with $tp(A) = |A|$. We will construct a subset $B \subseteq A$ such that B is unbounded and $<\restriction_{B\times B} = \prec\restriction_{B\times B}$. Let the definition of the set B be as follows: $B := \{x \mid x < y \Rightarrow x \prec y\}$ Suppose for a contradiction that B is not unbounded: $\exists a \in A\ \forall b \in B\ b < a$. Then the set

$$S := \{a \in A \mid \forall b \in B\ b < a\}$$

is not empty, so there exists a $\prec$-minimal element $a \in S$. We claim that $a \in B$. Suppose for a contradiction that $a \notin B$, hence there exists $y \in A$ such that $a < y$, but $a \succ y$. This means that $y \in S$ as well and a is not $\prec$-minimal. This concludes that $a \in B$ so $a < a$, which is a contradiction. □

Definition 3.1.3. *Let $(A,<)$ be an ordered set. Then*

$$\mathrm{cf}((A,<)) := \min\Big\{|B| \,\Big|\, B \subseteq A,\ B \textit{ is unbounded and } <\restriction B \times B \textit{ is a well-ordering of } B\Big\}.$$

For brevity, we denote $\mathrm{cf}((A,<))$ by $\mathrm{cf}(A)$ if the ordering is obvious from the context.

This definition is correct, because Theorem 3.1.2 guarantees the existence of at least one set B according to the given conditions. Note that for an ordered set A,

$\mathrm{cf}(A)$ is always a cardinal.

Definition 3.1.4. *Let α be a limit ordinal. We call α regular if and only if $\mathrm{cf}(\alpha) = \alpha$, and singular otherwise.*

Note that for a singular ordinal α, the cardinal $\mathrm{cf}(\alpha) < \alpha$. It is easy to see the following fact.

Fact 3.1.5. *If α is a regular ordinal, then α is a cardinal.*

Proposition 3.1.6. *Let α be an ordinal. Then $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$.*

Proof. We know, that $\mathrm{cf}(\mathrm{cf}(\alpha)) \leq \mathrm{cf}(\alpha)$. Now we show the inequality in the other direction. Let $\{\alpha_i \mid i \in \mathrm{cf}(\alpha)\}$ be an increasing unbounded subset of α of type $\mathrm{cf}(\alpha)$. Suppose that $I \subseteq \mathrm{cf}(\alpha)$ is an unbounded subset of $\mathrm{cf}(\alpha)$. Then, $\{\alpha_i \mid i \in I\}$ is an unbounded subset of α, hence $\mathrm{cf}(\mathrm{cf}(\alpha)) \geq \mathrm{cf}(\alpha)$. □

Remark 3.1.7. *The proposition above is true not just for cardinals, but for all ordered sets $(A, <)$ as well.*

The proposition above shows that for an ordinal α, the cardinal $\mathrm{cf}(\alpha)$ is regular.

Proposition 3.1.8. *Let κ be an infinite cardinal. Then the cardinal κ^+ is regular.*

Proof. Suppose for a contradiction, that $\mathrm{cf}(\kappa^+) < \kappa^+$. This means, that $\mathrm{cf}(\kappa^+) \leq \kappa$, so there exists a $(\beta_i)_{i<\kappa}$ unbounded subset of κ^+, where of course $\beta_i < \kappa^+$, so $|\beta_i| \leq \kappa$. Now, the union $\bigcup\limits_{i<\kappa} \beta_i$ is equal to κ^+, so we have written κ^+ as a union of κ many sets, each of which size $\leq \kappa$, which gives a contradiction. □

Proposition 3.1.9. *Let κ be an infinite cardinal. Then*

$$\mathrm{cf}(\kappa) = \min\{\lambda \mid \sum_{\xi<\lambda} \kappa_\xi = \kappa, \kappa_\xi < \kappa\}.$$

Proof. Let $\tau := \min\{\lambda \mid \sum\limits_{\xi<\lambda} \kappa_\xi = \kappa, \kappa_\xi < \kappa\}$. First, it is easy to see that $\mathrm{cf}(\kappa) \geq \tau$. Let's consider an unbounded subset $C \subseteq \kappa$ such that $|C| = \mathrm{cf}(\kappa)$. I claim that the sum of the elements of C is κ. It can not be smaller than κ since for every $\alpha < \kappa$ the sum $\sum\limits_{\alpha\in C} |\alpha|$ contains an ordinal bigger then α. It is also not bigger then κ because $\sum\limits_{\alpha\in C} |\alpha| \leq \mathrm{cf}(\kappa) \cdot \kappa \leq \kappa^2 = \kappa$.

Now, suppose for a contradiction that $\mathrm{cf}(\kappa) > \tau$. Let $A := \{\kappa_\xi \mid \sum\limits_{\xi<\tau} \kappa_\xi = \kappa\}$. Of course $|A| = \tau < \mathrm{cf}(\kappa)$ so A is bounded in κ. This means that there exists $\beta < \kappa \quad \forall \xi : \ \kappa_\xi \leq \beta$. Hence, $\kappa = \sum\limits_{\xi<\tau} \kappa_\xi \leq \tau \cdot |\beta| = \max(\tau, |\beta|) < \kappa$ which finishes the proof. □

3.2 Stationary sets

From now on let $\kappa \geq \omega_1$ be a **regular cardinal**. In this chapter we define the stationary and club sets and introduce some properties of them.

Definition 3.2.1. *A set $Z \subseteq \kappa$ is a closed set if and only if for every regular $\lambda < \kappa$ and for every strictly increasing sequence of type λ, the limit of the sequence is in Z.*

Definition 3.2.2. *A set $E \subseteq \kappa$ is a club set if and only if it is closed and unbounded.*

Definition 3.2.3. *A set $S \subseteq \kappa$ is a stationary set if and only if for $\forall E \subseteq \kappa$ club set $E \cap S \neq \emptyset$.*

Proposition 3.2.4. *Let $S \subseteq \kappa$ be a stationary set. Then, $|S| = \kappa$.*

Proof. Suppose for a contradiction that $|S| < \kappa$. This means that S can't be unbounded in κ by the regularity of κ. Let $\alpha := \sup S < \kappa$. The set $[\alpha + 1, \kappa) \subseteq \kappa$ is a club set, because it is closed and unbounded, but $[\alpha + 1, \kappa) \cap S = \emptyset$, contradicting that S is stationary. □

Proposition 3.2.5. *Fix a cardinal $\lambda < \kappa$. Let $\langle E_\alpha \subseteq \kappa, \alpha \in \lambda \rangle$ be club sets. Then $\bigcap_\alpha E_\alpha$ is a club set. In other words, the intersection of fewer than κ many club sets is also a club set.*

Proof. First, the intersection of closed sets is closed. To show that the intersection is unbounded, fix $\beta < \kappa$. We will find an element in the intersection, which is bigger than β. Consider an enumeration $\{\gamma_\alpha \mid \alpha \in \lambda\}$ of λ with type λ such that every ordinal $\gamma \in \lambda$ occurs exactly λ times. We can do this because $\lambda \cdot \lambda = \lambda$. We view these ordinals as the indices of the sets E_α. We construct a sequence of type λ with transfinite recursion. Let the first element β_0 in the sequence be an element from the first set E_{γ_0} according to the enumeration such that it is at least β. We can do this, because the sets are unbounded. Let's choose the second element from the second set E_{γ_1} such that it is bigger then the ordinal β_1. We can continue this procedure, in each iteration selecting a bigger element than the previous one from the appropriate set, because κ is regular and $\lambda < \kappa = \mathrm{cf}(\kappa)$, so the chosen ordinals can not form an unbounded subset at any point in the transfinite recursion. During the process from every set E_α we select an element λ many times so the limit of this sequence $\{\beta_\alpha \mid \alpha \in \lambda\}$ will be in each E_α since they are closed. Thus, the limit is in $\bigcap_\alpha E_\alpha$ and it is of course bigger then β. We have shown that $\bigcap_\alpha E_\alpha$ is unbounded. □

Proposition 3.2.6. *Let $S \subseteq \kappa$ be a stationary set and let C be a club set. Then, the set $S \cap C$ is stationary.*

Proof. Suppose for a contradiction that the set $S' := S \cap E$ is not stationary. Then there exists an F club set, for which $S' \cap F = \emptyset$, hence $S \cap (E \cap F) = \emptyset$, but $E \cap F$ is a club set because of Proposition 3.2.5, so this contradicts the fact that S'' is stationary. □

Proposition 3.2.7. *Let $S \subseteq \kappa$ be a stationary set. If we cut S into the disjoint union of $\lambda < \kappa$ many sets $\{S_i \mid i \in \lambda\}$, then at least one of them will be stationary.*

Proof. Suppose for a contradiction that none of the sets $\{S_i \mid i \in \lambda\}$ are stationary. Then there exists $\{E_i \mid i \in \lambda\}$ club sets, such that $E_i \cap S_i = \emptyset$ for all $i \in \lambda$. The set $E := \bigcap_{i \in \lambda} E_i$ is a club set because of Proposition 3.2.5 and $E \cap S = \emptyset$, contradicting the fact that S is stationary. □

Remark 3.2.8. *One might think of the club sets as having full measure and stationary sets like sets of positive measure in some sense.*

Remark 3.2.9. *Solovay's Theorem says that if κ is regular uncountable, then any stationary set in κ can be partitioned into κ-many pairwise disjoint stationary sets.*

Definition 3.2.10. *Let $\langle A_\alpha \subseteq \kappa \mid \alpha < \kappa \rangle$ be a sequence of subsets of κ. Then their diagonal intersection is defined as*

$$\underset{\alpha<\kappa}{\Delta} A_\alpha := \{\delta \mid \alpha < \delta \implies \delta \in A_\alpha\}.$$

Proposition 3.2.11. *Let $\langle A_\alpha \subseteq \kappa \mid \alpha < \kappa \rangle$ be club sets. Then their diagonal intersection is a club set.*

Proof. We first show that the diagonal intersection is closed. Fix an increasing sequence in the diagonal intersection: $\{\delta_\gamma\}_{\gamma<\mu}$ where μ is a regular cardinal less than κ. Let δ be the limit of the sequence. Let's fix a $\gamma < \delta$. There exists $\beta < \mu$ such that $\gamma < \delta_\beta$. We know that $\delta_\beta \in \underset{\alpha<\kappa}{\Delta} A_\alpha$, hence $\delta_\beta \in A_\gamma$. Again, $\delta_{\beta+1} \in \underset{\alpha<\kappa}{\Delta} A_\alpha$, so $\delta_{\beta+1} \in A_\gamma$. The set A_γ is closed, so the limit of the sequence $\delta_\beta, \delta_{\beta+1} \ldots$ which is δ is in A_γ concluding that $\delta \in \underset{\alpha<\kappa}{\Delta} A_\alpha$.

We now show that the diagonal intersection is unbounded. Fix $\beta < \kappa$, we need to find δ' such that $\beta < \delta' \in \underset{\alpha<\kappa}{\Delta} A_\alpha$. Let $\delta_0 > \beta$ be an arbitrary ordinal. By Proposition 3.2.5 the set $\bigcap_{\delta<\delta_0} A_\delta$ is unbounded, so there exists $\delta_1 > \delta_0$, such that $\delta_1 \in \bigcap_{\delta<\delta_0} A_\delta$. Again, by Proposition 3.2.5 we know that $\bigcap_{\delta<\delta_1} A_\delta$ is unbounded, so for

δ_1 there exists $\delta_2 > \delta_1$ such that $\delta_2 \in \bigcap\limits_{\delta<\delta_1} A_\delta$. This means that $\delta_2 \in \bigcap\limits_{\delta<\delta_0} A_\delta$ also, because this is a smaller intersection. We can continue this process by induction. Now, we have concluded that there is a sequence such that:

$$\delta_1, \delta_2, \delta_3, \dots \in \bigcap_{\delta<\delta_0} A_\delta.$$
$$\delta_2, \delta_3, \dots \in \bigcap_{\delta<\delta_1} A_\delta.$$
$$\delta_3, \dots \in \bigcap_{\delta<\delta_2} A_\delta.$$

Let δ' be the limit (supremum) of this sequence. Because these intersections are closed, we conclude that $\delta' \in \bigcap\limits_{\delta<\delta_n} A_\delta$ for all $n \in \omega$. It is clear that $\delta' > \beta$. Now we show, that $\delta' \in \underset{\alpha<\kappa}{\Delta} A_\alpha$. By definition we need that for all $\gamma < \delta' \implies \delta' \in A_\gamma$. Fix a $\gamma < \delta'$ and an n such that $\gamma < \delta_n$. We know that $\delta' \in \bigcap\limits_{\delta<\delta_n} A_\delta$ thus $\delta' \in A_\gamma$ concluding that the diagonal intersection is unbounded. □

3.3 Classical proofs of two Erdos-Rado theorems

We now have the tools necessary to introduce the key lemma of this section, the so called Fodor's pressing down lemma.

Definition 3.3.1. *Let $S \subseteq \kappa$ be a set. We call a function $f : S \to \kappa$ regressive if $f(\alpha) < \alpha$ for all $\alpha \in S \setminus \{0\}$.*

Lemma 3.3.2. *(Fodor's pressing down lemma) Let $S \subseteq \kappa$ be a stationary set and $f : S \to \kappa$ be a regressive function. Then, there exists $S' \subseteq S$ stationary, such that $f \restriction_{S'}$ is constant.*

Proof. Suppose for a contradiction that $\forall \gamma \in \kappa$ the set $A_\gamma := f^{-1}(\gamma)$ is not stationary. Then for all $\gamma \in \kappa$ there exists a club set E_γ such that $A_\gamma \cap E_\gamma = \emptyset$. Let

$$E := \Big(\underset{\alpha<\kappa}{\Delta} E_\alpha\Big) \setminus \{0\},$$

which is also a club set by Proposition 3.2.11, hence the intersection $S \cap E \neq$ is not empty. Choose $\delta \in S \cap E$ and let $\beta := f(\delta)$, so $\delta \in A_\beta$. This shows, that $\delta \notin E_\beta$. On the other hand, $\delta \in E$ and $\beta < \delta$ because f is regressive, so by the definition of the diagonal intersection, $\delta \in E_\beta$, a contradiction. □

The following Lemma and Proposition will also be useful in the next proof (Theorem 3.3.6).

Lemma 3.3.3. *Let $\kappa > \omega$ be a regular cardinal and let $g : \kappa \to \kappa$ be a function. Then, the set $E := \{\gamma \mid (\alpha < \gamma \Rightarrow g(\alpha) < \gamma) \wedge (\gamma < \kappa)\}$ is a club set in κ.*

Proof. We first show that E is closed. Consider a regular ordinal $\lambda < \kappa$ and a strictly increasing sequence of type λ:

$$\{x_\alpha \mid \alpha < \lambda\} \subseteq E.$$

Let $x := \sup x_\alpha$. We have to show that $\delta < x \Rightarrow g(\delta) < \gamma$. Fix an ordinal $\delta < x$. Now there exists an α such that $\delta < x_\alpha$, hence $g(\delta) < x_\alpha < x$. This concludes that $x \in E$.

To show that E is unbounded, fix $\delta < \kappa$: we will show that E has an element above δ. Let's construct a sequence of type ω in the following way: $\alpha_0 := \max(\delta, \sup\{g(\beta) \mid \beta \leq \delta\}$ and $\alpha_{i+1} := \max(\alpha_i, \sup\{g(\beta) \mid \beta \leq \alpha_i\}$ for all $i \in \omega$. Now $\delta \leq \alpha_0 < \alpha_1 < \cdots \to \alpha$. We will show that $\alpha \in E$, concluding that E is unbounded. To do this, let's fix an ordinal $\beta < \alpha$. There exists $n \in \omega$ such that $\beta < \alpha_n$, hence $g(\beta) < \alpha_{n+1} < \alpha$ by the definition of α_{n+1}. □

Proposition 3.3.4. *The set $S^{c^+}_{\omega_1} := \{\alpha < c^+ \mid \mathrm{cf}(\alpha) = \omega_1\}$ is stationary in c^+.*

Proof. Let's consider a club set $C \subseteq \kappa$. Of course $c \geq \omega_1$, so $c^+ > \omega_1$. This means that there exists a sequence of type ω_1 as a subset of C. The limit point s of this sequence is in C, because it is closed, but also $s \in S$, because it has cofinality ω_1. This means that $S \cap C \neq \emptyset$ for all club sets C so S is indeed stationary. □

Definition 3.3.5. *We call a family of sets $\mathcal{H}$ a Δ-system, if $\exists S$ such that $\forall F \neq F' \in \mathcal{H}$ we have that $F \cap F' = S$. The set S is called the root of the Δ-system.*

Theorem 3.3.6. *(Erdos-Rado) Let $|\mathcal{H}| = c^+$ such that $|F| \leq \omega$ for every $F \in \mathcal{H}$. There exists $\mathcal{H}' \subseteq \mathcal{H}$ such that $|\mathcal{H}|' = c^+$ and $\mathcal{H}'$ is a Δ-system.*

Proof. We know that $|\bigcup \mathcal{H}| = c^+$, because $c^+ \cdot \omega = c^+$. Let's consider the set $S := S^{c^+}_{\omega_1} = \{\alpha < c^+ \mid \mathrm{cf}(\alpha) = \omega_1\}$ as in Proposition 3.3.4. We know that S is stationary in c^+ and by Proposition 3.1.8 the cardinal c^+ is regular, so by Proposition 3.2.4 we get that $|S| = c^+$. Using this, we can view $\mathcal{H}$ as $\mathcal{H} = \{F_\alpha \mid \alpha \in S\}$. Let us consider the following sets: let $B_\alpha := [0, \alpha) \cap F_\alpha$ and let $C_\alpha := [\alpha, c^+) \cap F_\alpha$; of course, $F_\alpha = B_\alpha \sqcup C_\alpha$.

Consider the function $f(\alpha) := \sup B_\alpha$ which we claim that it is regressive. It was given that F_α is countable, so $|B_\alpha| \leq \omega$ as well meaning that the supremum can not reach α, since α has cofinality ω_1. This means that $f(\alpha) = \sup B_\alpha < \alpha$. Fodor's lemma (3.3.2) guarantees that there exists an ordinal γ and a stationary subset $S' \subseteq S$ such that $\forall \alpha \in S' : \quad f(\alpha) = \gamma$. Note that S was constructed in a way that all of its elements α are less than c^+. The function f is regressive so $f(\alpha) < c^+$ as well. The ordinal γ is equal to $f(\alpha)$ for an $\alpha \in S$, so we can conclude that $|\gamma| \leq c$. Let's examine that how many ways can the sets B_α look like for $\alpha \in S'$. We know that for $\alpha \in S'$ the sets $B_\alpha \subseteq \gamma$ and the sets B_α are countable, so let's count the number of countable subsets of γ! We claim that

$$|\{A \mid A \subseteq \gamma, |A| \leq \aleph_0\}| \leq c.$$

We have c options for choosing every element of $A \subseteq \gamma$, so in total there are at most $c^{\aleph_0} = c$ different sets.

Thus, if $\alpha \in S'$ the set B_α can look at most c many ways. Using this we can divide S' into at most c many parts based on how does the respective sets B_α look like below γ. Using Proposition 3.2.7, we get that there exists a stationary $S'' \subseteq S'$ such that $\forall \alpha, \beta \in S'' : \ B_\alpha = B_\beta$. This will be the root of the Δ-system. Again, from Proposition 3.2.4 we know, that $|S''| = c^+$. We are not done yet, there could be problems with the intersections of C_α and C_β even if $\alpha, \beta \in S''$. We will thin out S'' into $S''' \subseteq S''$ in a way that C_α and C_β will be disjoint if $\alpha \neq \beta \in S'''$.

Let us consider the function $g(\alpha) := \sup C_\alpha$ if $\alpha \in S$, and let $g(\alpha) := 0$ otherwise. Now, Lemma 3.3.3 guarantees, that $E := \{\gamma \mid \alpha < \gamma \Rightarrow g(\alpha) < \gamma\}$ is a club set. From Proposition 3.2.6 we know that the set $S''' := S'' \cap E$ is stationary. This means that $|S'''| = c^+$ as well (Proposition 3.2.4 again).

We still need to see, that the sets C_α and C_β are disjoint if $\alpha \neq \beta \in S'''$. Suppose that $\alpha < \beta$. Because $\beta \in E$ it follows from the definition of E that $\sup C_\alpha = g(\alpha) < \beta \leq \inf C_\beta$, hence they are really disjoint. The sets $\{F_\alpha \mid \alpha \in S'''\}$ form a Δ-set with root $F_\alpha \cap [0, \alpha]$ which is the same for all $\alpha \in S'''$. □

For the next theorem, let us introduce the so-called arrow notation $\kappa \to [\lambda]^\varrho_\mu$. In general, the left-hand side of the arrow denotes the base set of the coloring. On the right-hand side, the upper index is the dimension (are we coloring singletons, pairs, triples?), the lower index the number of colors and inside the bracket you see the size or order-type of the monochromatic sets we can always select. If the arrow is crossed over $\kappa \nrightarrow [\lambda]^\varrho_\mu$ that means there is a coloring witnessing that there are no monochromatic sets of that particular size. When coloring pairs from a set we usually view it as coloring the edges of the complete graph with the elements of the set as the vertices. For example the finite Ramsey theorem says that for every $k \in \omega$ there exists $R(k,k) \in \omega$ such that $R(k,k) \to [k]^2_2$. We know that $R(3,3) = 6$ hence using this notation $6 \to [3]^2_2$.

Remark 3.3.7. *We will give another proof of Theorem 3.3.6 in Theorem 6.3.3 using elementary submodels.*

Theorem 3.3.8. *(Erdos-Rado)*

$$c^+ \to [\omega_1]^2_\omega$$

This means that if we color the edges of the complete graph of size c^+ with ω colors, we can select a monochromatic set of size ω_1.

Proof. Let's fix a coloring $F : [c^+]^2 \to \omega$. Let's view the c^+ many vertices as the elements of $S := \{\alpha < c^+ \mid \mathrm{cf}(\alpha) = \omega_1\}$. We can do this because Proposition 3.3.4 tells us that S is stationary so $|S| = c^+$ by Proposition 3.2.4. Now, with the use of transfinite recursion, we will define monochromatic sets for all $\alpha \in S$ and for all colors $i \in \omega$. For this, fix $\alpha \in S$ and $i \in \omega$.

Choose a vertex $x_0^{\alpha,i}$ such that the edge between $x_0^{\alpha,i}$ and α is of color i and $x_0^{\alpha,i} < \alpha$. If this can be done, choose $x_1^{\alpha,i}$ such that $x_0^{\alpha,i} < x_1^{\alpha,i} < \alpha$ and all the edges between these vertices are of color i. After this, choose $x_2^{\alpha,i}$ such that $x_0^{\alpha,i} < x_1^{\alpha,i} < x_2^{\alpha,i} < \alpha$ and these four vertices form a monochromatic set of color i. Let's continue this with transfinite recursion. We require that $F(\{x_\eta^{\alpha,i}, x_\xi^{\alpha,i}\}) = i$ and $F(\{x_\eta^{\alpha,i}, \alpha\}) = i$ and $x_\eta^{\alpha,i} < x_\xi^{\alpha,i} < \alpha$ for every $\forall \eta < \xi$. If we can continue this process for ω_1 many steps then we have found a monochromatic set of size ω_1 so we are done. Suppose for a contradiction that this process terminates for all $\alpha \in S$ and for all $i \in \omega$ before ω_1 many steps. This means that there exists $\xi(\alpha, i)$ such that $x^{\alpha,i}_{\xi(\alpha,i)}$ is not definable and $\xi(\alpha, i) < \omega_1$.

Now for all $\alpha \in S$ let us define the following set:

$$B_\alpha = \Big\langle \langle x_\eta^{\alpha,i} \mid \eta < \xi(\alpha, i)\rangle \mid i \in \omega \Big\rangle.$$

Consider the function $f(\alpha) := \sup\{x_\eta^{\alpha,i} \mid \eta < \xi(\alpha, i), i \in \omega\}$. We claim that it is regressive. Since $\xi(\alpha, i)$ is a countable ordinal we have that $|\xi(\alpha, i)| \leq \aleph_0$ concluding that $\Big|\{x_\eta^{\alpha,i} \mid \eta < \xi(\alpha, i), i \in \omega\}\Big| \leq \aleph_0^2 = \aleph_0$. We also know that $\mathrm{cf}(\alpha) = \omega_1$ so this means that $\sup\{x_\eta^{\alpha,i} \mid \eta < \xi(\alpha, i), i \in \omega\} < \alpha$ concluding that f is regressive. Now, Theorem 3.3.2 guarantees, that there exists an ordinal γ and a stationary subset $S' \subseteq S$ such that $f(\alpha) = \gamma$ holds for every $\alpha \in S'$. Again, by Proposition 3.2.4 we get that $|S'| = c^+$.

Let's examine that how many different B_α could exist if $\alpha \in S'$ (that is $\sup B_\alpha = \gamma$). First, fix a color i. The possible values for the ordinal $\xi(\alpha, i)$ are the ordinals $\delta \in \omega_1$. For each $\delta \in \omega_1$ we will find that how many different strictly increasing sequences of type δ exist given that the supremum is at most γ. If we add these up for all $\delta \in \omega_1$ we will surely get an upper bound for the number of different strictly increasing sequences $(a_\eta)_{\eta < \xi(\alpha,i)}$. First, let's convince ourselves that $|\gamma| \leq c$. Note that S was constructed in a way that all of its elements α are less than c^+. The function f is regressive so $f(\alpha) < c^+$ as well. The ordinal γ is equal to $f(\alpha)$ for an $\alpha \in S$, so we can conclude that $|\gamma| \leq c$. Instead of counting the number of strictly increasing sequences of type δ with supremum at most γ, we will count the number of sequences of type δ with all elements being at most γ. Since there are at most c elements below γ, we have γ choices for all elements of the sequence so we have $c^{|\delta|}$ many different sequences. We know that the ordinal δ is countable because it is less than ω_1 so $c^{|\delta|} = c$. This also means that there are at most c many strictly increasing sequences of type δ with supremum at most γ. If we add this up for all $\delta \in \omega_1$, we get that there are at most $\omega_1 \cdot c \leq c^2 = c$ many strictly increasing sequences of type $\xi(\alpha, i)$. There are $\aleph_0$ many colors altogether so for every $\alpha \in S'$ the set B_α can look at most $c^{\aleph_0} = c$ many ways. This divides the set S' into at most c pieces; in every piece the respective B_α sets are the same.

Using Proposition 3.2.7 if we cut S' into these at most c many pieces, there will be an $S'' \subseteq S'$ which is stationary. Again, by Proposition 3.2.4 we get that $|S''| = c^+$. Choose two elements $\alpha < \beta \in S''$ and let $i := F(\alpha, \beta)$. Now, since $B_\alpha = B_\beta$ we get that $\xi(\beta, i) = \xi(\alpha, i)$ and $x_\eta^{\beta,i} = x_\eta^{\alpha,i}$ for every $\eta < \xi(\alpha, i)$ hence α is connected with color i to the vertices $\{x_\eta^{\beta,i} \mid \eta < \xi(\beta, i)\}$ and to β as well. Thus, we can extend the monochromatic set defined by β and i with α. This is a contradiction. $\square$

Remark 3.3.9. *Theorem 3.3.8 is a special case of the following theorem with $\kappa = \omega$:*

$$(2^\kappa)^+ \to [\kappa^+]^2_\kappa$$

Chapter 4

The sets H_θ

Later it will be useful for us to generate sufficiently large subsets of the universe, which we do here by constructing the sets H_θ for every infinite cardinal θ. Let's start with defining $\bigcup^n x$ for every $n \in \omega$.

Definition 4.0.1. *Let* $\bigcup^0 x := x$ *and for* $n \geq 1$ $\bigcup^n x := \bigcup \bigcup^{n-1} x$.

Definition 4.0.2. *The transitive closure of a set* x *is* $tc(x) = \bigcup_{n \in \omega} \bigcup^n x$.

Proposition 4.0.3. *For a set* x *the transitive closure* $tc(x)$ *is transitive and for all transitive sets* t *for which* $x \subseteq t$ *the set* $tc(x)$ *is the smallest transitive set:* $tc(x) \subseteq t$.

Proof. Let $z \in y \in tc(x)$. Then for some $n \in \omega$, $y \in \bigcup^n tc(x)$, meaning $z \in \bigcup \bigcup^n x = \bigcup^{n+1} x \subseteq tc(x)$, hence $z \in tc(x)$, concluding that $tc(x)$ is a transitive set.

Now, let t be a transitive set, containing x. We will prove by induction on n that $\bigcup^n x \subseteq t$. The base case holds, since $x = \bigcup^0 x \subseteq t$. Now assume $\bigcup^n x \subseteq t$. Then, $\bigcup^{n+1} x = \bigcup \bigcup^n x \subseteq \bigcup t$. But, since t is transitive $\bigcup t \subseteq t$, hence $\bigcup^{n+1} x \subseteq t$. This completes the proof. $\square$

Proposition 4.0.4. *The transitive closure for a set* a *is* $tc(a) = a \cup \bigcup_{b \in a} tc(b)$

Proof. First we see that $tc(a) \supseteq a \cup \bigcup_{b \in a} tc(b)$, because if $b \in a$ then of course $b \subseteq \bigcup a$ so $\bigcup^n b \subseteq \bigcup^{n+1} a$ as well, meaning that $tc(b) \subseteq tc(a)$.

For the other containment we will use Proposition 4.0.3. For this we need the set a to be a subset of the right hand side and we need the right hand side to be transitive. Of course a is a subset of the right hand side, and for proving the transitivity, let us choose an element b from the right hand side. If b is an element of a then the corresponding $tc(b)$ set contains every element of b. If b is an element of $\bigcup_{b \in a} tc(b)$ then of course there exists $b' \in a$ such that $b \in tc(b')$ and because the set $tc(b')$ is transitive (Proposition 4.0.3) all elements of b are in $tc(b')$ as well. $\square$

Definition 4.0.5. *The hereditary cardinality of a set x is $|tc(x)|$.*

Definition 4.0.6. *For a cardinal θ we define $H_\theta := \{x \mid |tc(x)| < \theta\}$.*

In other words, H_θ is the collection of sets of hereditary cardinality $< \theta$. It is not at all trivial that H_θ is a set, we will prove it in Proposition 4.0.14. We defined the sets H_θ for finite cardinals too, but from now on we will only use the sets H_θ for infinite cardinals θ. So, for example, for $H_{\aleph_0}$: every $a \in H_{\aleph_0}$ is finite, moreover every element $b \in a$ is finite, and so on.

More generally, we know that for H_θ, every $a \in H_\theta$ has less than θ elements, moreover every element $b \in a$ has less than θ elements, because otherwise $\bigcup a$ would have at least θ elements thus contradicting the fact that $tc(a) \supseteq \bigcup a$ has less then θ elements. For regular cardinals the following proposition holds:

Proposition 4.0.7. *Let $\theta > \aleph_0$ be a regular cardinal. Then, $H_\theta = \{x \mid \forall n \in \omega : |\bigcup^n x| < \theta\}$.*

Proof. Clearly, $H_\theta \subseteq \{x \mid |\bigcup^n x| < \theta\}$ for all $n \in \omega$. For the other containment, choose x such that $|\bigcup^n x| < \theta$ for all $n \in \omega$. By Proposition 3.1.9, the sum of fewer than θ many, smaller than θ cardinals is less then θ, so $|tc(x)| = \left|\bigcup\limits_{n\in\omega} \bigcup^n x\right| \leq \sum_{n\in\omega} |\bigcup^n x| < \theta$, so x is indeed in H_θ. □

Remark 4.0.8. *The proposition above is not true for singular cardinals (really, the problem is only with singular cardinals of cofinality ω, for example $\aleph_\omega$). Let x be a set such that $|\bigcup^n x| = \aleph_n$. For example, let $x = \omega \cup \{\omega_1\} \cup \{\{\omega_2\}\} \cup \ldots$. Then $|tc(x)| = \aleph_0 + \aleph_1 + \cdots = \aleph_\omega$ so $x \notin H_{\aleph_\omega}$ but $|\bigcup^n x| < \aleph_\omega$ for all $n \in \omega$.*

Proposition 4.0.9. *For every cardinal θ the set H_θ is transitive.*

Proof. Choose $a \in H_\theta$. This means that $|tc(a)| < \theta$. Since Proposition 4.0.4 tells us that $tc(a) = a \cup \bigcup\limits_{b\in a} tc(b)$ we get that $|tc(b)| \leq |tc(a)| < \theta$ for every $b \in a$, hence $b \in H_\theta$. □

To prove that H_θ is a set for every $\theta \in CARD$, first we will define the sets V_α for every $\alpha \in ON$.

Definition 4.0.10. *Let $V_0 := \emptyset$. For every $\alpha \in ON$ let's define $V_{\alpha+1} := \mathcal{P}(V_\alpha)$. For limit ordinals α let $V_\alpha := \bigcup_{\beta<\alpha} V_\beta$.*

It is clear that V_α is a set for every $\alpha \in ON$.

Definition 4.0.11. *For every set x let us define the rank of a set x as $rk(x) := \min\{\alpha \mid x \in V_{\alpha+1}\}$.*

Proposition 4.0.12. *For every set x, the ordinal $rk(x) = \sup\{rk(y) + 1 \mid y \in x\}$.*

Proof. Let $\delta := \sup\{rk(y) + 1 \mid y \in x\}$. First we will prove that $rk(x) \leq \delta$. By definition $rk(y) < \delta$ for all $y \in x$. This means that $y \in V_\delta$ for all $y \in x$, meaning $x \subseteq V_\delta$. Thus, $x \in V_{\delta+1}$, hence $rk(x) \leq \delta$. Clearly, $x \in V_{rk(x)+1} = \mathcal{P}(V_{rk(x)})$, thus $x \subseteq V_{rk(x)}$, hence for every $y \in x$, $y \in V_{rk(x)}$. This means that $rk(y) < rk(x)$ for every $y \in x$ so $\sup\{rk(y) + 1 \mid y \in x\} \leq rk(x)$ which is what we needed for the other direction. □

Proposition 4.0.13. *Let y be a transitive set. Then the set $H := \{rk(z) \mid z \in y\}$ is an ordinal.*

Proof. Let $\alpha := \sup\{rk(z) \mid z \in y\}$. We need to prove that $\beta \in H$ for all $\beta \in \alpha$. Suppose for a contradiction that $\exists \beta \in \alpha \setminus H$. We will construct an infinite strictly decreasing sequence of ordinals, which is not possible. Since $\beta < \alpha = \sup\{rk(z) \mid z \in y\}$, there exists $z_0 \in y$ and $\delta_0 > \beta$ such that $rk(z_0) = \delta_0$.

Using Proposition 4.0.12, $rk(z_0) = \sup\{rk(a) + 1 \mid a \in z_0\}$, if $rk(z_0) > \beta$ is a limit ordinal, we can choose $z_1 \in z$ such that $rk(z_1) > \beta$. If $rk(z_0)$ is a successor ordinal, then either $\beta + 1 = rk(z_0)$ or $\beta + 1 < rk(z_0)$. If it was the first case scenario, then by Proposition 4.0.12 there exists $a \in z_0$ such that $rk(a) = \beta$. Since y is transitive, $a \in y$ as well, but this is a contradiction, since a witnesses that $\beta \in H$. This means, that $\beta + 1 < rk(z_0)$, so again by Proposition 4.0.12 we can choose $z_1 \in z_0$ such that $rk(z_1) = \delta_1 > \beta$ and $\delta_0 > \delta_1$.

We can repeat this procedure by choosing $z_2 \in z_1$ with $rk(z_2) = \delta_2 > \beta$ and $\delta_2 < \delta_1$. With this, the sequence $(\delta_i)_{i\in\omega}$ is an infinite decreasing sequence of ordinals which is a contradiction. □

Proposition 4.0.14. *For every cardinal θ the class $H_\theta \subseteq V_\theta$, hence H_θ is a set.*

Proof. Choose a set $x \in H_\theta$. By definition, we know that $|tc(x)| < \theta$. Using Proposition 4.0.13, the set $\{rk(z) \mid z \in tc(x)\}$ is an ordinal, since $tc(x)$ is transitive. This set has at most $|tc(x)|$ elements, so $tp(\{rk(z) \mid z \in tc(x)\} = \sup\{rk(z) \mid z \in tc(x)\} < \theta$. Since θ is a limit ordinal, $\sup\{rk(z) + 1 \mid z \in tc(x)\} < \theta$ holds as well, but from this $\sup\{rk(z)+1 \mid z \in x\} < \theta$ follows, since $x \subseteq tc(x)$. Proposition 4.0.12 tells us, that $rk(x) < \theta$, meaning $x \in V_\theta$. This is what we wanted to show. □

Remark 4.0.15. *A crucial property of the sets H_θ is Theorem 5.2.5.*

Chapter 5

Formal introduction to elementary submodels

5.1 Logic

Our goal is to introduce the technique of using elementary submodels. This chapter provides the necessary knowledge from logic for this. This chapter is mainly based on [4]. First we define the *first order languages* which are collections of *first order formulas*. The alphabet of the first order languages consists of two disjoint subsets: the logic symbols and the non-logic symbols. The logic symbols are common to all languages, these are the following:

( ) ,	brackets, comma
$\neg$	negation
$\vee$	or
$\exists$	existential symbol
$x_0, x_1, \ldots$	infinitely many different symbols, these are the variable symbols
$=$	equation symbol

The non-logic symbols can also be divided into two disjoint parts, namely into function symbols and relation symbols.

Definition 5.1.1. *We call a triple $t = \langle F, R, \tau \rangle$ a similarity type if F and R are disjoint from each other and from the logic symbols and τ is a function which assigns a non-negative integer to all elements of $F \cup R$ and for every $r \in R$ the number $\tau(r) > 0$.*

If $t = \langle F, R, \tau \rangle$ is a similarity type then the elements of F are the function symbols and the elements of R are the relation symbols and τ tells us that a function symbol and a relation symbol is of how many variable. If for a function symbol f the arity $\tau(f) = 0$ we call f a constant symbol. By the cardinality of t we mean the cardinal $|F| + |R|$ and we denote it by $|t|$. The set $F \cup R$ along with the logic symbols gives the alphabet for the first order language of type t, we denote this by Σ_t. We denote the set of all finite sequences with elements from Σ_t with Σ_t^*.

To define the first order formulas first we need to define the terms.

Definition 5.1.2. *Let $t = \langle F, R, \tau \rangle$ be a similarity type. The set of terms (or expressions) $E(t)$ of type t is the smallest subset of Σ_t^* for which the following holds:*

(i) for all variable symbols we have that $x \in E(t)$

(ii) for all function symbols $f \in F$ if $\tau(f) = 0$ then $f \in E(t)$

(iii) for all function symbols $f \in F$ if $\tau(f) = n > 0$ and $k_0, k_1, \ldots k_{n-1} \in E(t)$ terms then $f(k_0, k_1, \ldots k_{n-1}) \in E(t)$.

The definition is correct, because the conditions $(i) - (iii)$ hold for Σ_t^* and if two subsets of it satisfy the conditions then their intersection does too. After this, we can define the formulas.

Definition 5.1.3. *Let $t = \langle F, R, \tau \rangle$ be a similarity type. The set of first order formulas $F(t)$ of type t is the smallest subset of Σ_t^* for which the following holds:*

(i) for all relation symbols r if its arity is $\tau(r) = n$ and $k_0, k_1, \ldots k_{n-1} \in E(t)$ are terms then $r(k_0, k_1, \ldots k_n) \in F(t)$

(ii) for all terms $k_0, k_1 \in E(t)$ the sequence of symbols $k_0 = k_1 \in F(t)$.

(iii) for all formulas $\varphi, \psi \in F(t)$ the sequence of symbols $(\varphi) \vee (\psi) \in F(t), \neg(\varphi) \in F(t)$ and for all variable symbols x the sequence of symbols $\exists x\,(\phi) \in F(t)$.

We can argue that the definition is correct similar to the argument given for the terms. We call a formula *atomic* if it is of the form $r(k_0, k_1, \ldots k_{n-1})$ or $k_0 = k_1$. Let the set of the atomic formulae be F_0 and let

$$F_{i+1} := F_i \cup \{(\varphi) \vee (\psi),\ \neg(\varphi),\ \exists x\,(\varphi) :\ \varphi, \psi \in F_i\}.$$

We claim that $F := \bigcup\{F_i \mid i \in \omega\} = F(t)$. First, the set F satisfies (i)-(iii) from the definition. On the other hand, with induction on i we can easily see that if (i)-(iii) hold for some subset $A \subseteq \Sigma_t^*$ then $F_i \subseteq A$ so $F \subseteq A$ as well. This means that F is indeed the smallest subset of Σ_t^* for which (i)-(iii) holds.

Sometimes we want to prove a statement for all formulae. We can do this with the following proposition.

Theorem 5.1.4. *(Formula induction) Suppose that a statement Ψ is true for the atomic formulae and if Ψ is true for φ and for ψ then it is true for $(\varphi) \vee (\psi)$, for $\neg(\varphi)$ and for $\exists x\,(\varphi)$. Then Ψ is true for every formula.*

Proof. Let the set of the atomic formulae be F_0, and for $i \in \omega$ we define the sets F_{i+1} as above:

$$F_{i+1} := F_i \cup \{(\varphi) \vee (\psi),\ \neg(\varphi),\ \exists x\,(\varphi) :\ \varphi, \psi \in F_i\}.$$

With induction on i we can conclude that Ψ is true for every formula in F_i thus it is true for every formula in $\bigcup\{F_i \mid i \in \omega\}$ which is exactly the set $F(t)$. □

This means that we can prove a statement for all formulae by first proving it for atomic formulae, then for $(\varphi) \vee (\psi)$ for $\neg(\varphi)$ and finally for $\exists x\,(\varphi)$.

We introduce more logic symbols which can be used to build formulas. For all terms k_0, k_1, all formulas φ, ψ and all variable symbols x we consider the following sequence of symbols as formulas:

(i) $k_0 \neq k_1$

(ii) $(\varphi) \rightarrow (\psi)$

(iii) $(\varphi) \wedge (\psi)$

(iv) $(\varphi) \leftrightarrow (\psi)$

(v) $\forall x(\varphi)$

These are really just shorthands for the existing logic symbols, for example $(\varphi) \leftrightarrow (\psi)$ is a shorthand for $((\varphi) \rightarrow (\psi) \wedge ((\psi) \rightarrow (\varphi))$ and $(\varphi) \rightarrow (\psi)$ is a shorthand for $(\neg(\varphi)) \vee (\psi)$.

We call a non-empty set with functions and relations on it a *structure*. For example all the groups are structures. The elements of the group form the base set and there are three functions: one with arity zero which represents the identity element, one with arity one which gives the inverse and one with arity two which gives the product for two elements.

A similarity type t automatically gives us function and relation symbols so we can call a structure of type t if for all function and relation symbols of t there exists a function and a relation of the same arity in the structure.

Definition 5.1.5. *Let $t = \langle F, R, \tau \rangle$ be a similarity type. A pair $\mathfrak{A} = \langle A, I \rangle$ is a structure of type t if A is a non-empty set and I is a function defined on $(F \cup R)$ such that for a function symbol $f \in F$ the interpretation $I(f)$ is a function of $\tau(f)$ variables on A and for a relation symbol $r \in R$ the interpretation $I(r)$ is a relation of $\tau(r)$ variables on A.*

We call the set A the base set (or underlying set) of the structure $\mathfrak{A}$ and the function I is the interpretations of the function and relation symbols. Usually if we denote a structure by a gothic letter, its underlying set is denoted by the corresponding latin capital letter. We denote the interpretations of the function symbol f and the relation symbol r in the structure $\mathfrak{A}$ by $f_{\mathfrak{A}}$ and $r_{\mathfrak{A}}$.

The above defined formulae of type t are statements about structures of type t. If a formula holds we say that it is *true* and if does not hold we say that it is *false*. We can decide whether a formula is true or false with recursion: $r(k_0, k_1, \ldots k_{n-1})$ is true if and only if the terms $k_0, k_1, \ldots k_{n-1}$ are in relation $r_{\mathfrak{A}}$; $k_0 = k_1$ is true if and only if the value of the two terms are the same; $\neg\varphi$ is true if and only if φ is false; $\varphi \vee \psi$ is true if and only if at least one of φ and ψ is true; finally $\exists x\,(\phi)$ is true if and only if there is an element of the structure $a \in A$ that if we replace x by a the resulting formula is true. But we can not even calculate the values of expressions until we say which variable symbols correspond to which elements of the structure. Let us write this down formally.

Definition 5.1.6. *Let $\mathfrak{A}$ be a structure of any type. Then the evaluation of the variable symbols is a function e which assigns an element of the base set of the structure A for every variable symbol x.*

Definition 5.1.7. *Consider an evaluation e over $\mathfrak{A}$ and let $a \in A$. The evaluation $e\,(x/a)$ denotes the evaluation e' over $\mathfrak{A}$ for which $e'\,(x) = a$ and for all other variable symbols $y \neq x$ the evaluation $e'\,(y) = e\,(y)$.*

First we define what we mean by the value of a term.

Definition 5.1.8. *Let $k \in E(t)$ be a term, let $\mathfrak{A}$ be a structure of type t, and let e be an evaluation of the variable symbols under $\mathfrak{A}$. We denote the value of the term k with the evaluation e with $k_{\mathfrak{A}}\,(e)$ and we define it by the hierarchy how the term was made by other terms:*

(i) if the term k is the variable symbol x then its value will be given by the evaluation e, that is $k_{\mathfrak{A}}\,(e) = e(x)$;

(ii) *if the term k is a function symbol $f \in F$ such that $\tau(f) = 0$ then the interpretation $f_{\mathfrak{A}}$ of the function symbol f is an element of A, that is $k_{\mathfrak{A}}(e) = f_{\mathfrak{A}}$;*

(iii) *otherwise, the term k is of the form $f(k_0, k_1, \dots)$ where the function symbol $f \in F$ is of $n = \tau(f) > 0$ variables. We already know the values of the terms, let $(k_i)_{\mathfrak{A}}(e) = a_i \in A$ for $i \in n$. Then we define $k_{\mathfrak{A}}(e) = f_{\mathfrak{A}}(a_0, a_1, \dots, a_{n-1})$.*

Now we can define what we mean by a formula being true for an evaluation e. Consider a formula $\varphi \in F(t)$, a structure $\mathfrak{A} = \langle A, I\rangle$ of type t and an evaluation e. By $\mathfrak{A} \models \varphi(e)$ we mean that φ is *true* in the structure $\mathfrak{A}$ with the evaluation e. If this is not the case we write $\mathfrak{A} \not\models \varphi(e)$ meaning that φ is *false* with the evaluation e.

For a formula φ we define $\mathfrak{A} \models \varphi(e)$ with formula induction (Proposition 5.1.4).

Definition 5.1.9. (i) *For all relation symbols r which have arity n and for all terms $k_0, k_1, \dots k_{n-1} \in E(t)$ we write $\mathfrak{A} \models r(k_0, k_1, \dots k_{n-1})(e)$ if and only if the values of the terms $k_0, k_1, \dots k_{n-1}$ are in relation r, that is $\langle (k_0)_{\mathfrak{A}}(e), (k_1)_{\mathfrak{A}}(e), \dots (k_{n-1})_{\mathfrak{A}}(e)\rangle \in r_{\mathfrak{A}}$;*

(ii) *for all terms $k_0, k_1 \in E(t)$ we write $\mathfrak{A} \models k_0 = k_1(e)$ if and only if the values of k_0 and k_1 are the same: $(k_0)_{\mathfrak{A}}(e) = (k_1)_{\mathfrak{A}}(e)$;*

(iii) *we write $\mathfrak{A} \models \neg\varphi(e)$ if and only if $\mathfrak{A} \not\models \varphi(e)$;*

(iv) *we write $\mathfrak{A} \models (\varphi \vee \psi)(e)$ if and only if $\mathfrak{A} \models \varphi(e)$ or $\mathfrak{A} \models \psi(e)$;*

(v) *we write $\mathfrak{A} \models \exists x\, \varphi(e)$ if and only if there exists an element $a \in A$ that if we replace x by a the resulting formula is true, that is $\mathfrak{A} \models \varphi(e(x/a))$ for some $a \in A$.*

If for all evaluations e we get that $\mathfrak{A} \models \varphi(e)$ we say that φ is true in $\mathfrak{A}$ or that $\mathfrak{A}$ *models* φ and we denote this by $\mathfrak{A} \models \varphi$. For example, every group G models that $1 \cdot x = x \cdot 1$ but not all groups models the formula $x \cdot y = y \cdot x$.

Let $a_0, a_1, \dots a_{n-1} \in A$. We use the shorthand $\mathfrak{A} \models \varphi(a_0, a_1, \dots, a_{n-1})$ if $\mathfrak{A} \models \varphi(e)$ for all evaluations e where $e(x_0) = a_0, \dots e(x_{n-1}) = a_{n-1}$.

5.2 Elementary Submodels

The idea behind elementary submodels is fairly simple: given a large structure $\mathfrak{A}$, we would like to consider substructures $\mathfrak{B}$ which are smaller than $\mathfrak{A}$ but reflects basic properties of the original structure.

Let's consider an example. Suppose $\mathfrak{A}$ is some Euclidean space where its base set is $A = \mathbb{R}^n$ along with all the lines, planes, hyperplanes, etc. Now, we look for a smaller structure $\mathfrak{B}$ of the same type where we want that (i) $B \subseteq A$ meaning that in B there are fewer points, lines, planes, etc. and (ii) the elements of B satisfy the same relations as A.

That is, if two lines of $\mathfrak{B}$ meet in $\mathfrak{A}$ then we require that their unique intersection must be in $\mathfrak{B}$ as well. Similarly, for any three points in $\mathfrak{B}$, there is a (hyper)plane in $\mathfrak{B}$ that contains them (since there was one in $\mathfrak{A}$). If this hyperplane is not unique, then there could be ones which are in $\mathfrak{A}$ but not in $\mathfrak{B}$. How to achieve such a $\mathfrak{B}$? If there is only a small number of new objects definable from a given set of elements already in $\mathfrak{B}$ then we can throw in all those without dramatically increasing the size of our structure. Repeat this process and if we keep track of all objects and operations appropriately, we will end up with the desired substructure.

The general framework of elementary submodels will provide a tool which saves us from repeating the very same closure argument over and over and gives us the nicest substructures we can imagine, all in a single step. Let us write this down formally.

Definition 5.2.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be both structures of type t. We say that $\mathfrak{B}$ is a substructure of $\mathfrak{A}$ if the following holds:*

(i) $B \subseteq A$

(ii) for all n-variable function symbols $f_{\mathfrak{B}} = f_{\mathfrak{A}} \restriction B^n$

(iii) for all n-variable relation symbols $r_{\mathfrak{B}} = r_{\mathfrak{A}} \restriction B^n$.

According to this definition, a substructure always contains the interpretations of constant symbols of the larger structure and it is closed for functions. If $\mathfrak{B}$ is a substructure of $\mathfrak{A}$ we write $\mathfrak{B} \subseteq \mathfrak{A}$.

Definition 5.2.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be both structures of type t. We say that $\mathfrak{B}$ is an elementary substructure of $\mathfrak{A}$ if it is a substructure and for any valuation e over $\mathfrak{B}$ and for all formulae $\varphi \in F(t)$ the following holds: $\mathfrak{B} \models \varphi(e) \Leftrightarrow \mathfrak{A} \models \varphi(e)$. If $\mathfrak{B}$ is an elementary substructure of $\mathfrak{A}$ we write $\mathfrak{B} \preccurlyeq \mathfrak{A}$.*

The well-known downward Löwenheim-Skolem theorem guarantees the existence of many small elementary submodels of a given structure.

Theorem 5.2.3. *(Löwenheim-Skolem) Let $\kappa \geq |t| \cdot \omega$ and let $\mathfrak{A}$ be a structure of type t and $X \subseteq A$ with $|X| = \kappa$. Then there exists $\mathfrak{B} \preccurlyeq \mathfrak{A}$ such that $X \subseteq B$ with $|B| = \kappa$.*

Before we prove this we give a well manageable condition to determine when a submodel is an elementary submodel:

Lemma 5.2.4. *(Tarski-Vaught criteria) Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures of type t and let $\mathfrak{B} \subseteq \mathfrak{A}$. The structure $\mathfrak{B}$ is an elementary submodel of $\mathfrak{A}$ if and only if for all formulae $\varphi \in F(t)$ and for all evaluation e over $\mathfrak{B}$ if $\mathfrak{A} \models (\exists x\, \varphi)\,(e)$ holds then there exists $b \in B$ for which $\mathfrak{A} \models \varphi\,(e(x/b))$.*

Formulating the statement differently: for all elements $b_0, b_1, \ldots b_{n-1} \in B$ and all formulae $\varphi(y_0, y_1, \ldots y_{n-1}, z) \in F(t)$ if there exists an element $a \in A$ for which $\mathfrak{A} \models \varphi(b_0, b_1, \ldots b_{n-1}, a)$ then there exists $b \in B$ for which $\mathfrak{A} \models \varphi(b_0, b_1, \ldots b_{n-1}, b)$

Proof. First, suppose that $\mathfrak{B} \preccurlyeq \mathfrak{A}$ and $\mathfrak{A} \models (\exists x\, \varphi)\,(e)$. By the definition of an elementary substructure $\mathfrak{B} \models (\exists x\, \varphi)\,(e)$. Thus, there exists an element $b \in B$ such that $\mathfrak{B} \models \varphi\,(e(x/b))$. The evaluation $e(x/b)$ is also over $\mathfrak{B}$ so by using the definition of an elementary substructure again we get $\mathfrak{A} \models \varphi\,(e(x/b))$ which is what we wanted.

For the other direction we will show that for every formula $\varphi \in F(t)$ and for every evaluation e over $\mathfrak{B}$ the statements $\mathfrak{A} \models \varphi\,(e)$ and $\mathfrak{B} \models \varphi\,(e)$ are equivalent. We will do this by formula induction (Proposition 5.1.4). For atomic formulas this automatically holds because of the condition $\mathfrak{B} \subseteq \mathfrak{A}$. If we suppose that $\mathfrak{A} \models \varphi\,(e) \Leftrightarrow \mathfrak{B} \models \varphi\,(e)$ then it is true that $\mathfrak{A} \models \neg\varphi\,(e) \Leftrightarrow \mathfrak{B} \models \neg\varphi\,(e)$, because

$$\mathfrak{A} \models \neg\varphi\,(e) \Leftrightarrow \mathfrak{A} \not\models \varphi\,(e) \Leftrightarrow \mathfrak{B} \not\models \varphi\,(e) \Leftrightarrow \mathfrak{B} \models \neg\varphi\,(e).$$

We can see similarly that if $\mathfrak{A} \models \varphi\,(e) \Leftrightarrow \mathfrak{B} \models \varphi\,(e)$ and $\mathfrak{A} \models \psi\,(e) \Leftrightarrow \mathfrak{B} \models \psi\,(e)$ then $\mathfrak{A} \models (\varphi \vee \psi)\,(e) \Leftrightarrow \mathfrak{B} \models (\varphi \vee \psi)\,(e)$

Now we only have to show how to pass from φ to $\exists x\, \varphi$. If $\mathfrak{A} \models (\exists x\, \varphi)\,(e)$ then because of the conditions given we know that $\mathfrak{A} \models \varphi\,(e(x/b))$ for some $b \in B$. By induction we know that $\mathfrak{B} \models \varphi(e(x/b))$ hence $\mathfrak{B} \models (\exists x\, \varphi)(e)$.

For the other implication, if $\mathfrak{B} \models (\exists x\, \varphi)\,(e)$ then exists $b \in B$ such that $\mathfrak{B} \models \varphi\,(e(x/b))$. Using the condition of the induction again there exists an element $b \in B$ such that $\mathfrak{A} \models \varphi\,(e(x/b))$ concluding that $\mathfrak{A} \models (\exists x\, \varphi)\,(e)$. □

Now we can prove Theorem 5.2.3.

Proof. Suppose that for a formula $\psi(y_0, y_1, \ldots y_{n-1}, z) \in F(t)$ its free variables are those $n+1$ many indicated above. We will call a function $f : A^n \to A$ the Skolem-function corresponding to the formula $\exists z\,(\psi)$ in the structure A if $\mathfrak{A} \models \exists z\, \psi(a_0, a_1, \ldots, a_{n-1}, z)$ then $\mathfrak{A} \models \psi(a_0, a_1, \ldots, a_{n-1}, f(a_0, a_1, \ldots, a_n))$ for all $a_0, a_1, \ldots a_{n-1} \in A$.

A construction for the Skolem-function for ψ is to choose a well-ordering $<$ of the set A and let

$$f(a_0, a_1, \ldots a_{n-1}) = \begin{cases} \min\limits_{<} \{a \in A : \mathfrak{A} \models \psi(a_0, a_1, \ldots a_{n-1}, a)\}, & \text{if it is non-empty,} \\ \min\limits_{<} A & \text{otherwise.} \end{cases}$$

Now, for every formula φ of the form $\exists z\,(\psi)$ let us fix a Skolem-function f_φ. Let us define a sequence of increasing substructures $\langle \mathfrak{B}_i \mid i \in \omega \rangle$ of $\mathfrak{A}$ in the following way. Let $X_0 = X$ and $\mathfrak{B}_0$ the structure generated by X_0. If we have already defined $\mathfrak{B}_i$, let

$$\begin{aligned} X_{i+1} = B_i \cup \{f_\varphi(b_0, b_1, \ldots b_{n-1}) \mid b_0, \ldots b_{n-1} \in B_i \text{ and} \\ f_\varphi \text{ is one of the fixed Skolem-functions}\} \end{aligned}$$

and let $\mathfrak{B}_{i+1}$ be the substructure generated by X_{i+1}. Finally, let $\mathfrak{B} = \bigcup\limits_{i\in\omega} \mathfrak{B}_i$ which is the structure with base set $\bigcup\limits_{i\in\omega} B_i$ It is easy to see that $|\mathfrak{B}_i| = \kappa$ hence $|\mathfrak{B}| = \kappa$.

We still need to show that $\mathfrak{B} \preccurlyeq \mathfrak{A}$. It is clear by the construction that $\mathfrak{B} \subseteq \mathfrak{A}$. Let us check the Tarski-Vaught criteria (5.2.4). Let $\psi(y_0, y_1, \ldots y_{n-1}, z) \in F(t)$ and let $b_0, b_1, \ldots b_{n-1} \in B$ and $a \in A$ such that $\mathfrak{A} \models \psi(b_0, b_1, \ldots b_{n-1}, a)$. We need to find $b \in B$ such that $\mathfrak{A} \models \psi(b_0, b_1, \ldots b_{n-1}, b)$.

Of course, $\mathfrak{A} \models \exists z\, \psi(b_0, b_1, \ldots b_{n-1}, z)$. There must exist $i \in \omega$ such that $b_0, b_1, \ldots b_{n-1} \in B_i$ because $\mathfrak{B} = \bigcup\limits_{i\in\omega} \mathfrak{B}_i$. Thus by the definition of X_{i+1}

$$b = f_{\exists z\, \psi}(b_0, b_1, \ldots b_{n-1}) \in X_{i+1} \subseteq B$$

hence $\mathfrak{A} \models \psi(b_0, b_1, \ldots b_{n-1}, b)$ as we wanted. □

Note that Theorem 5.2.3 can only give us submodels from a set and the (set-theoretic) universe is a proper class, but sometimes we want to somehow model the whole universe. As a workaround, we will use sufficiently large subsets of the universe. Every proof uses only finitely many formulae so it is enough for us if we choose a sufficiently large subset of the universe which reflects the finitely many formulae we use in the proof. This will be the Reflection Principle.

Theorem 5.2.5. *(Reflection Principle) Given a finite set of formulae Σ and a cardinal ρ, there is a $\theta > \rho$ so that for any $\phi \in \Sigma$ and $a_0, \ldots, a_n \in H_\theta$:*

$$H_\theta \models \phi(a_0, a_1, \ldots, a_n) \textit{ if and only if } \phi(a_0, a_1, \ldots a_n) \textit{ is true in the universe.}$$

The proof is very similar to Theorem 5.2.3 and it can be found in [5, Theorem II. 5. 3].

The following observation is quite useful. Suppose $\phi(x_0, \ldots, x_{n-1}) \in \Sigma$ and θ are as in the Reflection Principle and we have $M \preccurlyeq H_\theta$. Then for all $a_0, \ldots, a_n \in M$:

$$M \models \phi(a_0, a_1, \ldots, a_n) \text{ if and only if } \phi(a_0, a_1, \ldots a_n) \text{ is true in the universe.} \qquad (*)$$

Thus, with respect to a given set of finitely many formulae, M looks like an elementary submodel of the universe. If a formula ϕ satisfies $(*)$ it is said to be *absolute* over M.

From now on we will take elementary submodels from the sets H_θ. We can view every H_θ as a structure of type t where t has no function symbols and has one relation symbol of arity two. The interpretation of this relation symbol in H_θ is the relation $\in$.

We now introduce some interesting properties of every elementary submodel of the sets H_θ.

Lemma 5.2.6. *Let $x, y, a_0, \ldots a_{n-1} \in M \preccurlyeq H_\theta$ with $\theta \geq \aleph_1$ and let $f \in M$ be a function with $x \in dom(f)$ then the following holds:*

1. $\langle x, y \rangle \in M$,
2. $x \cap y \in M$,
3. $x \cup y \in M$,
4. $f(x) \in M$,
5. $\omega \subseteq M$,
6. $\omega \in M$,
7. $\{a_0, a_1, \ldots a_{n-1}\} \in M$.

Proof. (1.) Note that the set theoretic definition of $\langle x, y\rangle$ is $\{x, \{x, y\}\}$. We will calculate the set $tc(\{x, \{x, y\}\}$ according to Proposition 4.0.4:

$$tc(\{x, \{x, y\}\}) = \{x, \{x, y\}\} \cup tc(x) \cup tc(\{x, y\}) = \{x, \{x, y\}\} \cup tc(x) \cup \{x, y\} \cup tc(y) =$$

$$= \{\{x, y\}, x, y\} \cup tc(x) \cup tc(y)$$

This means that $|tc(\{x, \{x, y\}\})| \leq 3 + |tc(x)| + |tc(y)|$. We know that $|tc(x)| < \theta$ and $|tc(y)| < \theta$ because they are in H_θ and since θ is an infinite cardinal we can conclude that $|tc(\{x, \{x, y\}\})| < \theta$ as well, meaning $\{x, \{x, y\}\} \in H_\theta$. Let's consider the following formula $\varphi(b, c)$:

$$\exists z \Big(a \in z \Leftrightarrow (a = b \vee a = \{b, c\})\Big)$$

Now, $H_\theta \models \phi(x, y)$, by the elementary property of M we can conclude that $M \models \phi(x, y)$, which means that $\{x, \{x, y\}\} \in M$.

We can use a similar approach for $x \cap y, x \cup y$ as well.

(2.)Using Proposition 4.0.3 we get that $tc(x \cap y) \subseteq tc(x)$ since $tc(x)$ is a transitive set which contains $x \cap y$. This means that $|tc(x \cap y)| < \theta$, hence $x \cap y \in H_\theta$ as well. We can define $x \cap y$ in M using the formula below:

$$\exists z \Big(a \in z \Leftrightarrow (a \in x \wedge a \in y)\Big).$$

(3.) For the union, let's use Proposition 4.0.3 again: $tc(x \cup y) \subseteq tc(x) \cup tc(y)$ because $tc(x) \cup tc(y)$ is transitive and it contains $x \cup y$. This means that $|tc(x \cup y)| \leq |tc(x)| + |tc(y)|$, so again, $|tc(x \cup y)| < \theta$ meaning that $x \cup y \in H_\theta$. The formula we use for defining $x \cup y$ in M is the following:

$$\exists z \Big(a \in z \Leftrightarrow (a \in x \vee a \in y)\Big).$$

(4.) Let's recall that the set theoretic definition of a function is that it is a set of ordered pairs $\langle x, y\rangle$ such that if $\langle x, y\rangle, \langle x, y'\rangle \in f \Rightarrow y = y'$ which we decode as $f(x) = y$. According to the conditions we have $x \in M$, let us define $y := f(x)$ which is the second element in the (unique!) ordered pair which contains x as first element. Ee want to define the y in M (as an element). Based on the above ideas this is fairly straightforward: if $x, f \in M$ we know that $f \in H_\theta$ and since $tc(y) \subseteq tc(f)$, we conclude that $y \in H_\theta$ as well. We can now define y in M as well with the following formula:

$$\exists z\ \Big(\langle x, z\rangle \in f\Big).$$

(5.) First we show that $\emptyset \in M$. Of course $\emptyset \in H_\theta$, so we only need to find a formula which defines the $\emptyset$ in M. For example: $\exists z\ \forall a(a \notin z)$. This is true in H_θ so it is also true in M, so we conclude that $\emptyset \in M$. With this, we can define any natural number using induction. First let us convince ourselves that $\omega \subseteq H_\theta$. This is true because for all $n \in \omega$, the cardinal $|tc(n)|$ is finite and θ is infinite. Now, for example to show that $1 \in M$, consider the following formula:

$$\exists z\ \Big(a \in z \Leftrightarrow a = 0\Big).$$

This is true in H_θ so it is true in M, concluding that $1 \in M$. For proving $2 \in M$ we can use the following formula:

$$\exists z \quad \Big(a \in z \Leftrightarrow (a = 0 \vee a = 1)\Big)$$

meaning $2 \in M$. We can repeat this for any $n \in \omega$ concluding that $\omega \subseteq M$.

(6.) Since every ordinal is a transitive set, $tc(\omega) = \omega$ so $\omega \in H_\theta$ indeed (because θ is uncountable by the conditions). We need to find a formula which defines ω in M as well. For example, the following works: 'there exists x such that x is the smallest limit ordinal'.

(7.) It is easy to see that $A = \{a_0, a_1, \ldots a_{n-1}\} \in H_\theta$, and the formula which defines the set A in M is

$$\exists x\ \Big(a \in x \Leftrightarrow (a = a_0 \vee a = a_1 \vee \cdots \vee a_{n-1})\Big).$$

□

Lemma 5.2.7. *Suppose that $M \preccurlyeq H_\theta$ is an elementary submodel for some $\theta \geq \aleph_2$ and $X \in M$.*

1. *If X is countable then $X \subseteq M$;*
2. *if $X \setminus M \neq \emptyset$ then X is uncountable;*
3. *if M is countable then $M \cap \omega_1$ is an initial segment of ω_1;*
4. *if M is countable and $X \subseteq \omega_1$ is uncountable then $X \cap M$ is an unbounded subset of $\omega_1 \cap M$.*

Proof. Note that (1) and (2) are equivalent so we will only prove (1). If X is countable then the formula

$$\phi(X) \equiv (\exists f : \omega \to X) f[\omega] = X$$

must hold in H_θ. Note, that in the above formula we use that $\omega \in M$, because we can only choose evaluations e above M. So, $\phi(X)$ must hold in M as well, thus we can pick $f : \omega \to X$ in M such that $f[\omega] = X$. Now, for any $n \in \omega$ both $n, f \in M$ so $f(n) \in M$ as well. This means that $X = f[\omega] = \{f(n) \mid n \in \omega\} \subseteq M$.

(3.) Consider $\beta \in M \cap \omega_1$. We know that $\beta \in \omega_1$ hence β is countable. By (1) we get that $\beta \subseteq M$ hence for any $\alpha \in \beta$ the ordinal $\alpha \in M$ concluding that $M \cap \omega_1$ is an initial segment.

(4.) Suppose for a contradiction that there exists $\alpha \in \omega_1 \cap M$ such that $x < \alpha$ for any $x \in X \cap M$. In turn,

$$M \models \forall x(x \in X \Rightarrow x < \alpha)$$

so the set H_θ must satisfy this as well. Hence, $x < \alpha$ for any $x \in X$ which contradicts the fact that X is uncountable. □

The following observation is quite useful as well and points to the fact that for elementary submodels M, the ordinal $M \cap \omega_1$ plays a critical role.

Proposition 5.2.8. *Let $\theta \geq \aleph_2$ and suppose that $M \preccurlyeq H_\theta$ is countable and $\alpha = M \cap \omega_1$. If $X \in M$ is a subset of ω_1 and $\alpha \in X$ then X is stationary.*

Conversely for any stationary set $S \subseteq \omega_1$ and $X \in H_\theta$ there is a countable $M \preccurlyeq H_\theta$ which contains X and $M \cap \omega_1 \in S$.

Proof. Take a club set $C \subseteq \omega_1$ such that $C \in M$. The set C must be uncountable because any countable set in ω_1 is bounded, since the supremum of countably many countable ordinals is countable. So by (4) from Lemma 5.2.7 the set $C \cap M$ is unbounded in $\alpha = \omega_1 \cap M$ hence $\alpha \in C$ because the set C is closed. Thus, $\alpha \in X$ and $\alpha \in C$ meaning that the intersection $X \cap C$ is not empty hence the formula $'X \cap C \neq \emptyset$ for any club set $C \subseteq \omega_1'$ is true in M. By the elementary property of M this is true in H_θ as well. Also, every club set $C \subseteq \omega_1$ is in H_θ because $|tc(C)| \leq |tc(\omega_1)| = |\omega_1| = \aleph_1$. Thus, this formula is absolute between the universe and H_θ, meaning that X is stationary as it has non-empty intersection with every club set.

For the other direction we claim that there is a continuous increasing sequence of countable models $M_\alpha \preccurlyeq H_\theta$ all containing X so that $\omega_1 \subseteq \bigcup\{M_\alpha \mid \alpha \in \omega_1\}$. We call a sequence of models *continuous* if for limit ordinals α the model $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$. Let us use the Theorem 3.3.2 repeatedly. Let M_0 be the submodel given by Theorem 3.3.2 such that $\{X, \emptyset\} \subseteq M_\alpha$. When defining $M_{\alpha+1}$ we want $X_{\alpha+1} = M_\alpha \cup \{X, \alpha+1\}$ to be a subset of $M_{\alpha+1}$. Note that $|X_{\alpha+1}| = \aleph_0$ so by Theorem 3.3.2 we get that $|M_{\alpha+1}| = \aleph_0$. For limit ordinals $\alpha \in \omega_1$ let M_α be $\bigcup_{\beta<\alpha} M_\beta$. This is a countable union so M_α is countable as well. It is easy to check that M_α is indeed an elementary submodel using Theorem 5.2.4. Note that at first sight it seems we would not necessarily have $\alpha \in M_\alpha$ which seems to cause a problem since we want $\omega_1 \subseteq \bigcup\{M_\alpha \mid \alpha \in \omega_1\}$ to hold. But, $M_{\alpha+1}$ contains $\alpha + 1$ and $M_{\alpha+1} \cap \omega_1$ is an ordinal by (3) from Lemma 5.2.7, so $\alpha \in M_{\alpha+1}$ holds as well.

Now, the set $\{M_\alpha \cap \omega_1 \mid \alpha \in \omega_1\}$ must be a club set. It is closed because the sequence M_α is continuous and it is unbounded, because $M_{\alpha+1}$ contains $\alpha + 1$ for every $\alpha \in \omega_1$. This means that the set $\{M_\alpha \cap \omega_1 \mid \alpha \in \omega_1\}$ has a non-empty intersection with S, thus there is an $\alpha \in \omega_1$ so that $\omega_1 \cap M_\alpha \in S$. □

We now give a new proof to Fodor's pressing down lemma (Theorem 3.3.2) for $\kappa = \omega_1$ using elementary submodels.

Theorem 5.2.9. *(Fodor's pressing down lemma) Let $S \subseteq \omega_1$ be stationary and $f : S \to \omega_1$ be regressive. Then there exists a stationary $S' \subseteq S$ such that $f\restriction_{S'}$ is constant.*

Proof. Pick some countable submodel $M \preccurlyeq H_{\aleph_2}$ such that $f \in M$ and $\alpha := M \cap \omega_1 \in S$ by the converse of Proposition 5.2.8. Note that $\varepsilon := f(\alpha) < \alpha$ so $\varepsilon \in M$ because of (3) from Lemma 5.2.7. In turn if we let $S' = f^{-1}(\varepsilon) \subseteq S$ then of course $\alpha \in S'$.To use Proposition 5.2.8 to show that S' is stationary we need to see that $S' \in M$. Since $f \in H_{\aleph_2}$ of course $f^{-1}(\varepsilon) \in H_{\aleph_2}$ too. We can define S' in M as well by the formula

$$\exists z\ (a \in z \Leftrightarrow f(a) = \varepsilon)$$

This means that S' is indeed stationary and by definition f is constant on S'. □

Elementary submodels are often used to cut a large structure X into smaller pieces. The next Theorem is from [3].

Theorem 5.2.10. *Suppose that $X \in H_\theta$ is of size κ and* $\operatorname{cf} \kappa = \mu$. *Then there is a sequence $(M_\alpha)_{\alpha<\mu}$ such that*

1. *$X \in M_\alpha \preccurlyeq H_\theta$ and $|M_\alpha| < \kappa$,*

2. *(continuity) for any limit $\beta < \mu$ the model $M_\beta = \cup\{M_\alpha \mid \alpha < \beta\}$,*

3. *$X \subseteq \bigcup\{M_\alpha \mid \alpha < \mu\}$.*

Moreover we can assume that $(M_\alpha)_{\alpha<\beta} \in M_\beta$.

We use the models M_α to write X as the increasing union of the sets $X \cap M_\alpha$. We often apply some inductive assumption to $X \cap M_\alpha$ or $X \cap M_{\alpha+1} \setminus M_\alpha$. Note that the sets $X \cap M_{\alpha+1} \setminus M_\alpha$ partition the set X because the sequence $(M_\alpha)_{\alpha<\mu}$ is continuous.

The next Lemma is from [1]:

Lemma 5.2.11. *Let θ, κ and λ be infinite cardinals such that $\kappa^\lambda = \kappa$ and $\lambda < \theta$. Then for every $A \subseteq H_\theta$ with $|A| \leq \kappa$ there is $M \preccurlyeq H_\theta$ such that $A \subseteq M$ with $|M| \leq \kappa$ and for every $x \subseteq M$ with $|x| \leq \lambda$ we have that $x \in M$.*

We omit the proofs which are again variants of the Löwenheim-Skolem closure argument.

Chapter 6

Applications of elementary submodels

6.1 Topology

In this section we will showcase some proofs with the help of elementary submodels. This proofs are from [1].

First, let us recall a few basic notions from topology. A topological space is a pair $\langle X, \tau_X \rangle$, where τ_X is the topology of X meaning that the collection of open subsets of X. We often use τ instead of τ_X if it is obvious from the context and we sometimes omit τ fully if it is clear what the topology is on X. A topological space X is called Hausdorff if for any two distinct points $x, y \in X$ there are disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $y \in V$. The topology of a space X can be described by giving a base for the topology. A subset $B \subseteq \tau$ is a base for the topology τ of X if for all $x \in X$ and $U \in \tau$ for which $x \in U$ there is a $V \in B$ such that $x \in V \subseteq U$. If we do not require B to be a subset of τ, we get the notion of a network:

Definition 6.1.1. *A subset $N \subseteq P(X)$ is a network of X if for all $x \in X$ and $U \in \tau$ for which $x \in U$ there is a set $V \in N$ such that $x \in V \subseteq U$.*

A cardinal invariant of topological spaces is a mapping i assigning a cardinal $i(X)$ to each space X such that $i(X) = i(Y)$ if X and Y are homeomorphic. An easy example is the cardinality of a space. Clearly, any two spaces which are homeomorphic have the same cardinality. We now define two less trivial cardinal invariants.

Definition 6.1.2. *Let X be a topological space. Then $w(X)$ is the least cardinal κ*

such that X has a base of size κ. We call $w(X)$ the weight of X.

Definition 6.1.3. *Let X be a topological space. Then $nw(X)$ is the least cardinal κ such that X has a network of size κ. We call $nw(X)$ the network-weight of X.*

Remark 6.1.4. *Since every base of X is also a network of X, $w(X) \geq nw(X)$.*

In this section we prove a theorem of Arhangel'skii:

Theorem 6.1.5. *(Arhangel'skii) Let X and Y be compact Hausdorff topological spaces and let $f : X \to Y$ be continuous and onto. Then $w(Y) \leq w(X)$.*

The following lemma shows us that if we replace weights with network-weights it gets much easier:

Lemma 6.1.6. *Let X and Y be topological spaces and let $f : X \to Y$ be continuous and onto. Then $nw(Y) \leq nw(X)$.*

Proof. Let N be a network for X. We will show that $N' = \{f[V] : V \in N\}$ is a network for Y. Let $U \in Y$ be open and non-empty and choose $y \in U$. Choose $x \in X$ such that $f(x) = y$. Since f is continuous, $f^{-1}[U]$ is open and clearly $x \in f^{-1}[U]$. Thus, there exists $V \in N$ such that $x \in V \subseteq f^{-1}[U]$. Now, $f[V] \in N'$ and $y \in f[V] \subseteq U$. □

Theorem 6.1.5 follows from the next lemma, which shows that for compact Hausdorff spaces the weight and network-weight are the same. We give a proof of this fact using elementary submodels.

Lemma 6.1.7. *Let X be a compact Hausdorff topological space. Then $w(X) = nw(X)$.*

Proof. Let N be a network of X with $\kappa := |N| = |nw(X)|$ and let τ be the topology of X. Let θ be large enough for H_θ to contain X and τ as elements and such that all those finitely many formulae are absolute over H_θ that we want to be absolute in this following proof. We could write down these formulae, but it is not necessary, since the Reflection Principle (5.2.5) says that suitable θ exists for any finite set of formulae.

Now, using Theorem 5.2.3 pick $M \preccurlyeq H_\theta$ such that $N \cup \{N, X, \tau\} \subseteq M$ and $|M| = \kappa$. We claim that $\tau \cap M$ is a base for X which would be enough, because then the base would have cardinality at most κ, meaning $w(X) \leq nw(X)$.

Pick a non-empty $U \in \tau$ and let $x \in U$. If we prove that there exists $W \in \tau \cap M$ such that $x \in W \subseteq U$ that would show that $\tau \cap M$ indeed contains a base. For any

$y \in X \setminus U$ there are disjoint open sets U_y, V_y such that $x \in U_y$, and $y \in V_y$. Since N is a network, there are sets $A_y, B_y \in N$ such that $x \in A_y \subseteq U_y$ and $y \in B_y \subseteq V_y$.

Let us consider the following formula:

$$\varphi\,(u, v, t, a, b) = (\exists u, v \in t) \cap (u \cap v \neq \emptyset) \wedge (a \subseteq u) \wedge (b \subseteq v)$$

The sets $u = U_y$ and $v = V_y$ witness that $\varphi\ (u, v, \tau, A_y, B_y)$ is true in the universe. This means that we can choose large enough θ such that $H_\theta \models \varphi\ (u, v, \tau, A_y, B_y)$. Since $\tau, A_y, B_y \in M$ and M is an elementary submodel we get that $M \models \varphi(u, v, \tau, A_y, B_y)$ as well.

Let $U'_y, V'_y \in M$ be according to the formula: $U'_y, V'_y \in \tau$, the intersection $U'_y \cap V'_y$ is empty, $A_y \subseteq U'_y$ and $B_y \subseteq V'_y$. Clearly $X \setminus U \subseteq \bigcup_{y \in X \setminus U} V'_y$. The space X is compact and $X \setminus U$ is a closed subset so $X \setminus U$ is compact as well. This means that there exists a finite set $F \subseteq X \setminus U$ such that $X \setminus U \subseteq \bigcup_{y \in F} V'_y$. We will prove that $\bigcap_{y \in F} U'_y$ is an element of M. This would finish the proof since $x \in \bigcap_{y \in F} U'_y \subseteq U$, meaning $\tau \in M$ contains a base.

Suppose $|F| = n$ and let $\{U'_y \mid y \in F\} = \{U_1, U_2, \ldots, U_n\}$. Let $\phi(z, x_1, x_2, \ldots x_n)$ be the formula saying that the elements of z are precisely those, which are elements of $x_1, x_2, \ldots x_n$ that is $x \in z \Leftrightarrow (x \in x_1) \wedge (x \in x_2) \wedge \cdots \wedge (x \in x_n)$. Now, $\phi(W, U_1, U_2, \ldots, U_n)$ is true in H_θ if $W = \bigcap_{1 \leq i \leq n} U_i$. Again, we can choose large enough θ such that $H_\theta \models \phi(W, U_1, U_2, \ldots, U_n)$, meaning $M \models \phi(W, U_1, U_2, \ldots, U_n)$, hence we can conclude that the set $\bigcap_{y \in F} U'_y = \bigcap_{1 \leq i \leq n} U_i$ is an element of M, thus finishing the proof. $\square$

Remark 6.1.8. *The finitely many formulae we wanted H_θ to reflect from the universe are the φ and the ϕ formulas defined above.*

Corollary 6.1.9. *It is an interesting consequence of Theorem 6.1.5 that the continuous Hausdorff image of compact metrizable space is metrizable, which can be proven easily in the following way:*

(i) X is compact metrizable so it has a countable base,

(ii) Y has a countable base also by Theorem 6.1.5,

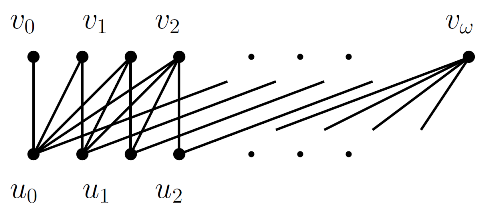
(iii) As Y is also normal (from compact Hausdorff again) so the Urysohn metrization theorem says that Y is metrizable.

6.2 Graphs with uncountable chromatic number

In this section I follow [3]. The next group of applications is about the following question: given a graph with large chromatic number, what can we say about its subgraphs? Is it true that certain cycles, paths or say highly connected graphs must embed into every graph with large enough chromatic number? P. Erdos showed that for any finite k, l there exists a graph with chromatic number k which contains no cycles of length $\leq l$. So, the chromatic number can be arbitrary large, while the l-neighbourhood of any vertex must form a tree. Now, trees have chromatic number two, so we see that there are graphs with arbitrary large chromatic number which locally have the smallest possible chromatic number.

Quite interestingly the above result does not extend to graphs with uncountable chromatic number. In fact the following theorem holds where $H_{\omega,\omega+1}$ denotes the so called infinite half graph: the vertices are $\{u_k : k \in \omega\} \cup \{v_k : k \in \omega + 1\}$ and the edges are $u_k v_l$ where $k \leq l \leq \omega$.

Theorem 6.2.1. *(A. Hajnal and P. Komjath, 1984) Any graph G of uncountable chromatic number must contain the graph $H_{\omega,\omega+1}$.*



In particular, all even cycles and actually all finite bipartite graphs must appear in any graph of uncountable chromatic number. We mention that the lack of finite half-graphs of a given size also has various consequences on the structure and regularity properties of a graph [6].

Proof. Suppose that G is a counterexample of minimal size κ. Let us denote the set of the vertices of G with V and the set of the neighbours of a vertex v with $N(v)$. Take a sequence of elementary submodels $(M_\alpha)_{\alpha<\mathrm{cf}(\kappa)}$ each of size less than κ and containing G using Proposition 5.2.10. Let $V_0 := V \cap M_1$ and for $\alpha \geq 1$ let $V_\alpha := V \cap M_{\alpha+1} \setminus M_\alpha$. Note that if one of the graphs V_α has uncountable chromatic number, then by the minimality of G it must contain the graph $H_{\omega,\omega+1}$, but then G must contain it as well, hence we can conclude that the graphs V_α have countable chromatic numbers.

We claim that for every α and $v \in V_\alpha$ the set $N(v) \cap M_\alpha$ is finite. Suppose for a contradiction that $\{u_n \mid n \in \omega\}$ is an infinite subset of $N(v) \cap M_\alpha$. The set $N(u_k)$ is in M_α because we can define it with a formula M_α. By (2) from Lemma 5.2.6 we get that the set $\bigcap_{k<n} N(u_k)$ is in M_α as well for any $n \in \omega$. Using (2) from Lemma 5.2.7 we get that the set $\bigcap_{k<n} N(u_k)$ must be uncountable for every $n \in \omega$ because v is an element of the intersection and $v \notin M_\alpha$. So, we can select pairwise distinct $v_n \in \bigcap_{k<n} N(u_k)$. Now, $\{u_k \mid k \in \omega\} \cup (\{v_k \mid k \in \omega\} \cup \{v\})$ is a copy of $H_{\omega,\omega+1}$ in G which is a contradiction.

We showed that for every α and $v \in V_\alpha$ the set $N(v) \cap M_\alpha$ is finite, hence we can glue the colorings of the graphs V_α together as they all have countable chromatic number. First, let us fix a coloring $g_\alpha : V_\alpha \to \omega$ for each graph V_α. We will give a coloring $g : V \to \omega \times \omega$, where $g(v) := \langle g_\alpha(v), h(v)\rangle$ and $v \in V_\alpha$. We will define the function $h_\alpha(v)$ now. First, let $h(v) = 0$ for $v \in V \cap M_1$, so the coloring is $g(v) := \langle g_0(v), 0\rangle$ for $v \in V \cap M_1$. Next, suppose that $g \restriction V \cap M_\alpha$ is already defined. For a vertex $v \in V_\alpha$ the set $N(v) \cap M_\alpha$ is finite, let us define $h(v) := \min\{\omega \setminus h[N(v) \cap M_\alpha]\}$. This definition ensures that there is no conflict between the colours on $V \cap M_\alpha$ and V_α. This finishes the construction of a good colouring g and the proof of the theorem is done. □

We saw that all even cycles must embed into any graph with uncountable chromatic number. However, finitely many odd cycles can be avoided: the simplest examples are the shift graphs [8]. Choose a cardinal κ and a natural number n. We will define a graph $SH_n(\kappa)$, the vertices of the shift graph are $[\kappa]^n$. The vertices a and b will be connected if and only if $a_0 < a_1 = b_0 < a_2 = b_1 < \dots a_{n-1} = b_{n-2} < b_{n-1}$. For the $n = 2$ case it is not too hard to see that $SH_2(\kappa)$ cannot contain a triangle. In fact, $SH_n(\kappa)$ contains no odd cycles of length at most $2n-1$. Moreover by choosing κ large enough the chromatic number of $SH_n(\kappa)$ can be made arbitrary large.

On the other hand, it was also shown that in any graph of uncountable chromatic number, all but finitely many odd cycles must appear.

Theorem 6.2.2. *(C. Thomassen, 1983, [7]) Any graph G of uncountable chromatic number must contain odd cycles of all but finitely many lengths.*

Proof. We can assume that G is connected, otherwise take a connected component of uncountable chromatic number. Fix a vertex x then run a Breadth First Search from that vertex. This will partition the vertices V into $(V_m)_{m\in\omega}$, where $v \in V_m$ if and only if the shortest path from x to v has exactly m edges. Note that this can be done because there is no transfinite path (the definition of a connected component is that all two vertices are connected with a path consisting of **finitely** many edges).

Now, because we partitioned V into ω many parts, some V_m most have uncountable chromatic number: if all of them could be colored by countably many colors then we could tie them together resulting in a countable good coloring of V. Fix an m for which V_m has uncountable chromatic number. We will find all odd cycles of length at least $2m+1$.

For any $k \in \omega$ we can find a copy H of the complete bipartite graph $K_{k,k}$ in V_m. Let uv be an edge in H and take paths P, P' of length m from x to u and v respectively. Note that $P \cup P'$ does not have edges in V_m and contains a path from u to v of even length $l \leq 2m$.

Moreover in H we can connect u and v in paths of length $1, 3, 5, \dots$ up to $2k-1$. Connecting those two we get odd cycles of length $l+1, l+3, \dots l+2k-1$. We can do this for any $k \in \omega$ hence the proof is done. □

6.3 Revisiting the Δ-system lemma

In this section I follow and [2] and [3]. The Δ-system lemma is one of the most cited results in set theory, ubiquitous in forcing arguments, topological proofs and Ramsey results. As a reminder here is the definition of a Δ-system.

Definition 6.3.1. *We call a family of sets $\mathcal{H}$ a Δ-system, if $\exists S$ such that $\forall F \neq F' \in \mathcal{H}$ we have that $F \cap F' = S$. The set R is called the root of the Δ-system.*

Theorem 6.3.2. *(Δ-system lemma) Every uncountable family of finite sets $\mathcal{F}$ contains an uncountable Δ-system.*

Proof. We can assume without loss of generality that we work with subsets of ω_1 hence take $\mathcal{F} \subseteq [\omega_1]^{<\omega}$. Pick a countable $M \preccurlyeq H_{\aleph_2}$ so that $\mathcal{F} \in M$. Fix any $b \in \mathcal{F} \setminus M$ and let $r = b \cap M$. Let us consider the set

$$E = \{a \in \mathcal{F} \mid r \subseteq a\}.$$

Of course $E \in H_{\aleph_2}$ because $|tc(E)| \leq |tc(\mathcal{F})| \leq \aleph_1$. The set E is also an element of M, because there is a formula which defines it (note that $r \in M$ by (7) from Lemma 5.2.6 since $|r| < \omega$ and all elements of r are in M):

$$\exists E\ (a \in E \Leftrightarrow (a \in \mathcal{F} \wedge r \subseteq a)).$$

Moreover, $b \in E \setminus M$ so E must be uncountable by (2) from Lemma 5.2.7. We will find our uncountable Δ-system in E with root r. Let us take a maximal subfamily E_0 of E in M which satisfies that $\{a \setminus r \mid a \in E_0\}$ is pairwise disjoint (that is, E_0 is a Δ-system with root r). By the elementarity property of M we get that this E_0 is also a maximal subfamily in $H_{\aleph_2}$, since we can define the set E_0 with a formula.

We claim that E_0 must be uncountable. Suppose for a contradiction that it is countable. By (1) from Lemma 5.2.7 we get that $E_0 \subseteq M$ as well. On the other hand $E_0 \subseteq M$ so $b \notin M$, hence the set $E_0 \cup \{b\}$ is a proper extension of E_0 in $H_{\aleph_2}$. We will check that $E_0 \cup \{b\}$ is a Δ-system. Of course, $r \subseteq b$, so we need to check that the set $\{a \setminus r \mid a \in \{E_0 \cup b\}\}$ is pairwise disjoint. We already know that $E_0 \subseteq M$, but applying (1) from Lemma 5.2.7 to the elements of E_0 we get that every element of E_0 is a subset of M. Since $b \cap M = r$, we know that b can only intersect the elements of M in r. This contradicts the fact that E_0 was maximal in $H_{\aleph_2}$ so indeed E_0 is an uncountable Δ-system. □

Now we revisit Theorem 3.3.6 and prove it with elementary submodels.

Theorem 6.3.3. *Every family $\mathcal{A} = \{A_\alpha \mid \alpha < c^+\} \subseteq [c^+]^\omega$ contains a Δ-system of size c^+.*

Proof. The proof will be really similar to the proof of Theorem 6.3.2, but we need to choose the submodel M more carefully. We want to pick $M \preccurlyeq H_\theta$ such that $M \cap c^+ \in c^+$, the cardinality of M is c and $[M]^\omega \subseteq M$. We can construct a continuous, increasing sequence of models $(M_\alpha)_{\alpha<\omega_1}$ each of size c such that when constructing the models $M_{\alpha+1}$ we ensure that the model $M_{\alpha+1}$ contains $\{\mathcal{A}\} \cup M_\alpha \cup \sup\{c^+ \cap M_\alpha\} \cup [M_\alpha]^\omega \cup \{c\}$ which is of cardinality c so we can choose $M_{\alpha+1}$ with cardinality c. By taking $M := \bigcup_{\alpha\in\omega_1} M_\alpha$ it can be easily seen that $M \cap c^+ \in c^+$ and $[M]^\omega \subseteq M$.

From now on, we can proceed essentially the same as in Theorem 6.3.2. Pick a $b \in \mathcal{A} \setminus M$ and let $r := b \cap M$. We have $r \in M$, since $[M]^{<\omega} \subseteq M$ by (7) from Lemma 5.2.6 and $[M]^\omega \subseteq M$ as we created M. Choose a subfamily $\mathcal{B} \in M$ such that $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{B}$ is maximal among the Δ-systems with root r. By elementarity, $\mathcal{B}$ is also maximal in the universe. We claim that $\mathcal{B}$ has cardinality c^+. If it would have cardinality at most c then we would have $\mathcal{B} \subseteq M$ because there exists a surjective function from c to $\mathcal{B}$ in M so we can define the elements $B \in \mathcal{B}$ (we need that $c \in M$ and that $c \subseteq M$). This means that $B \subseteq M$ would hold for every $B \in \mathcal{B}$ by (1) from Lemma 5.2.7. Thus, the family $\mathcal{B} \cup \{b\}$ would be a proper extension to $\mathcal{B}$ because it is a Δ-system with root r by the same reasoning as in the proof of Theorem 6.3.2. □

More generally we have the following theorem which can be easily proved with appropriately modifying the previous proof. It can be found in [2, Theorem 4.2].

Theorem 6.3.4. *If $\mathcal{A}$ is a family of finite sets such that $\kappa = |\mathcal{A}|$ is an uncountable regular cardinal then $\mathcal{A}$ contains a Δ-system of size κ.*

Chapter 7

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Szakdolgozat címe:
Elemi részmodellek használata a halmazelmélet területein

A **szakdolgozat** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

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a hallgató aláírása