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Seifert surfaces of alternating knots

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Bachelor thesis, BSc in Mathematics

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Budapest, 2024

Contents

Acknowledgements

I would like to thank my supervisor, Alexander Kubasch for introducing me to this captivating field of mathematics. I am immensely grateful for his patience, and all the time and effort he put into his clear and concise explanations. I thoroughly enjoyed our meetings all semester, especially trying to figure out the proofs together. I would also like to thank my friends and family for their support and for believing in me.

Preface

What is a knot? Intuitively we can think of it as a knotted piece of string, made of rubber, with the ends glued together. We do not distinguish between a knot and its deformations. But how can we tell whether two knots are the deformations of the same knot or not? Oftentimes, we cannot.

There are some properties called *knot invariants* that can help us tell knots apart. These properties of a knot never change, no matter how we deform the knot. So if a knot invariant differs on two knots, then those two knots cannot be the same.

In Chapter 1 we discuss the basic concepts of knot theory mainly based on [1] and [5]. In Chapter 2 and 3 we introduce surfaces and then more specifically Seifert surfaces. The trivial knot is the only knot that arises as the boundary of an embedded disc. Seifert's theorem provides us with an algorithm, however, showing that every knot arises as the boundary of an embedded surface. It is hence natural to use the complexity (i.e. genus) of such a surface to measure how complicated a knot is. In the Appendix at the end of the thesis we collected a number of knots together with their Seifert surfaces obtained by applying Seifert's algorithm to them.

After the necessary preparations, we can finally present our main theorem, regarding the connection between an alternating knot and its Seifert surfaces.

Alternating knots are a certain set of knots which are the simplest in a sense. The main theorem of this thesis claims that Seifert's algorithm gives us a Seifert surface of minimal genus when applied to an alternating knot.

The theorem was proven by Richard H. Cowell and Kunio Murasugi independently in 1958, with the use of Alexander polynomials. The proof we present in this thesis roughly follows the proof published by David Gabai in 1986 [3]. The Master's thesis of Rasmus Hedegaard [4] helped us a lot in re-enacting Gabai's proof. Both [3] and [4] give a proof for the theorem, however, we have found a case that neither of them discusses. We conclude the thesis by providing a proof that covers all cases.

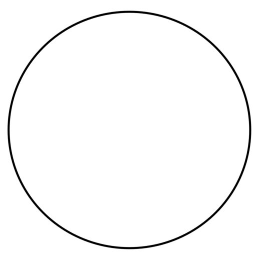
Chapter 1

Introduction to knot theory

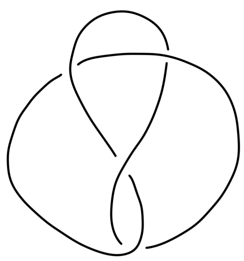
1.1 Knots

Definition 1.1.1. A *knot* K is an equivalence class of continuous embeddings $k\colon S^1 \to S^3$ where $k_1 \sim k_2$ if there exists an orientation preserving homeomorphism from $(S^3, \text{im}(k_1))$ to $(S^3, \text{im}(k_2))$.

Remark 1.1.2. Knots can be visualized by $K = \text{im}(k) \subset S^3$. (Figure 1.1.1)



(a) The unknot.



(b) The figure-8-knot.

Figure 1.1.1: Examples of knots.

Definition 1.1.3. An *oriented knot* is a knot together with a choice of orientation.

Remark 1.1.4. Oriented knots can be visualized by $K = \text{im}(k) \subset S^3$ and small arrows to indicate the orientation. Figure 1.1.2 shows a visualization of the oriented trefoil knot.

Definition 1.1.5. A knot K_1 is said to be *equivalent* to a knot K_2 if K_1 and K_2 represent the same equivalence class of continuous embeddings $k\colon S^1 \to S^3$ as defined in Definition 1.1.1. We define the equivalence of oriented knots analogously.

Remark 1.1.6. We could also define knots to be in $\mathbb{R}^3$. As it makes no difference whether we use S^3 or $\mathbb{R}^3$, we will use them interchangeably.

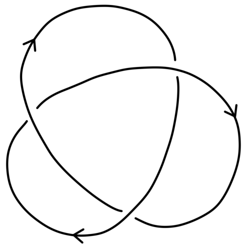


Figure 1.1.2: The oriented trefoil knot.

The definition of knots does not exclude so called *wild knots* for which an example is shown in Figure 1.1.3. In order to avoid these kinds of knots, we will define *tame knots*. Tame knots can be defined in the following three equivalent ways:

Definition 1.1.7. If a knot K_1 is equivalent to a knot K_2 whose image in S^3 is the union of a finite number of line segments, we say that K_1 is a *tame knot*.

Definition 1.1.8. A knot K is a *tame knot*, if for every $p \in K$ there exists an open neighbourhood U_p of p such that $(U_p, U_p \cap K)$ is homeomorphic to $(\mathbb{R}^3, \mathbb{R})$.

Definition 1.1.9. A knot K is a *tame knot* if it is equivalent to a $\mathcal{C}^\infty$ embedding of S^1 into S^3.

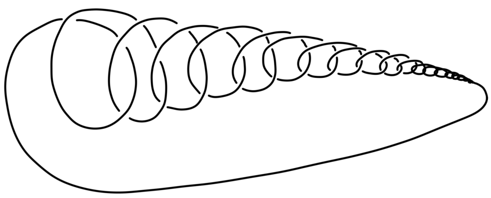


Figure 1.1.3: A wild knot.

Although it is not complicated to see the equivalence of these three definitions, we will not prove it. From now on we will only work with tame knots and we will refer to them simply as knots.

Definition 1.1.10 (Connected sum of two knots)**.** Suppose that K_1 and K_2 are two oriented knots in S^3 that are separated by an embedded S^2. Form the *connected sum* $K_1 \# K_2$ of K_1 and K_2 as follows. First choose an oriented rectangular disc R with boundary ∂R composed of four oriented arcs $\{e_1, e_2, e_3, e_4\}$ such that $K_1 \cap R = -e_1 \subset K_1$ and $K_2 \cap R = -e_3 \subset K_2$, and the separating sphere intersects R in a single arc and

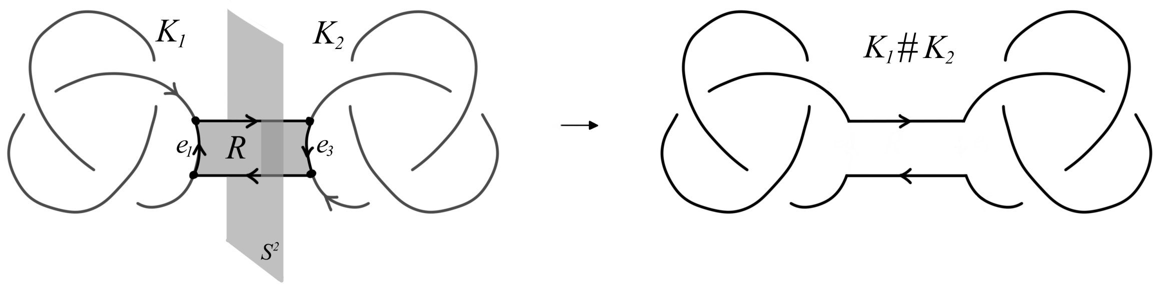


Figure 1.1.4: Connected sum of two knots.

intersects e_2 and e_4 in a single point each. Then define $K_1 \# K_2$ as

$$K_1 \# K_2 = (K_1 \setminus e_1) \cup e_2 \cup e_4 \cup (K_2 \setminus e_3).$$

The resulting knot does not depend on the chosen band R. [6]

Definition 1.1.11. We say that a knot K is a *prime knot* if for any decomposition of K as a connected sum of two knots, exactly one of the factors is the unknot.

Definition 1.1.12. We call the image of a knot $K \subset \mathbb{R}^3$ under a linear map $p\colon \mathbb{R}^3 \to \mathbb{R}^2$ a *projection* of the knot.

Definition 1.1.13. We say that a projection of a knot is a *regular knot projection* if the following two things apply:

(1) no three points of the knot project to the same point,

(2) if a point in the projection is the image of two points in the knot, the images of the two strands of the knot containing the two points intersect each other transversally.

Transversality means that at an intersection point the two tangent vectors of the projection are linearly independent. Figure 1.1.5 shows a transversal intersection on the left and a non-transversal intersection on the right, which we wish to avoid.

Figure 1.1.5: A transversal and a non-transversal intersection.

Claim 1.1.14. *For every knot K there exists a regular projection of K.*

We do not prove this claim.

Definition 1.1.15. Take a regular projection of a knot K. If two points of K map to the same point in this projection, we call the image of these two points a *crossing point* of the projection.

Take a crossing point of a regular projection. If we only look at the projection, we no longer know how the preimage of the crossing point (which is composed of two points) was positioned in the knot, which one was above the other one. To avoid losing this information, we slightly modify the regular projection in small neighbourhoods of the crossing points by "breaking" one of the lines corresponding to the strands of the knot. We do this in a way that the modified projection represents which strand is above the other in the knot as shown in Figure 1.1.6.

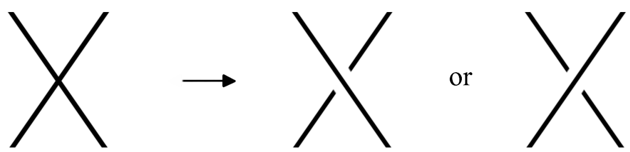


Figure 1.1.6: Modifying the projection in a small neighbourhood of a crossing point.

Definition 1.1.16. The *diagram* D of a knot K is a regular projection of K modified as described in the previous paragraph. The small neighbourhoods of the crossing points we modified are called the *crossings* of the diagram.

Figure 1.1.1 shows two knot diagrams, one with 0 and one with 4 crossings.

Definition 1.1.17. Let $\mathcal{K}$ be the set of knots. A *knot invariant* is a map I from $\mathcal{K}$ to any other structure (often $\mathbb{N}$). This implies that for any knot K_1, if K_1 is equivalent to a knot K_2 then $I(K_1) = I(K_2)$.

The knot invariant which we will discuss the most in this thesis is the genus of a knot K, $g(K)$. We will define this in Chapter 3. Until then, here are some examples of knot invariants.

Definition 1.1.18. The *crossing number* is a knot invariant $cr\colon \mathcal{K} \to \mathbb{N}$. Given all diagrams of a knot K, $cr(K)$ is the number of crossings in a diagram with the minimal amount of crossings.

Definition 1.1.19. We define the *stick number* $s(K)$ of a knot K to be the least amount of line segments needed to form an image of K as their union. This is an invariant of the knot.

Some other examples of knot invariants, which we will not discuss are 3-colourability, the unknotting number, the bridge number and the Alexander polynomial.

For a brief moment think of the knot as a loop of a string on a table. It is evident that we can move around the string without changing the knot, but we cannot cut open and glue the ends back togerher. Moving the string around, however, can change the diagram and the number of crossings in the diagram of the knot. It would only seem appropriate to have something in knot theory that describes this ”moving around” of the string.

Definition 1.1.20. The *Reidemeister moves* provide us ways to change the relations between the crossings of a diagram of a knot by changing the diagram itself. There are three Reidemeister moves:

(1) untwisting or twisting a strand of the knot thus reducing or increasing the number of crossings by one (Figure 1.1.7),

(2) removing or adding two crossings as shown in Figure 1.1.8,

(3) moving a strand of the knot from one side of a crossing to the other (Figure 1.1.9).

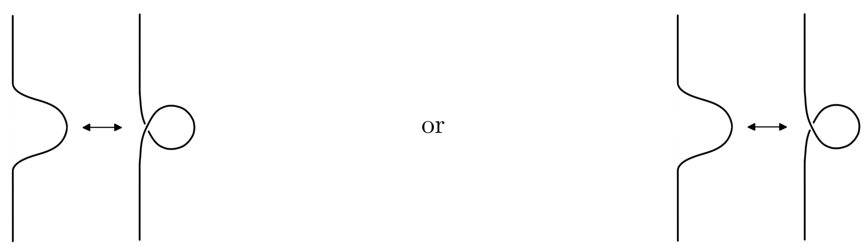


Figure 1.1.7: The first Reidmeister move.

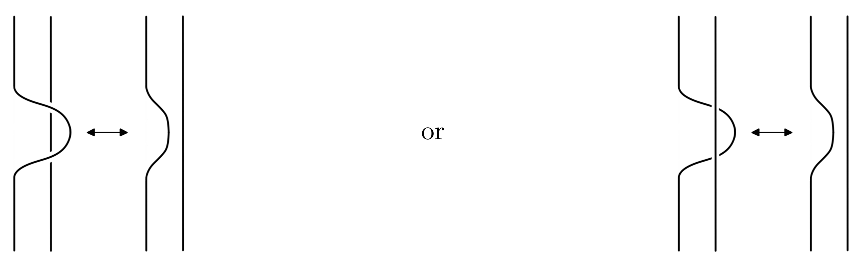


Figure 1.1.8: The second Reidmeister move.

Claim 1.1.21 (Reidemeister)**.** *Two knots K_1 and K_2 are equivalent if and only if there exists a series of Reidemeister moves and planar isotopies such that after applying them to a diagram of K_1 we obtain a diagram of K_2.*

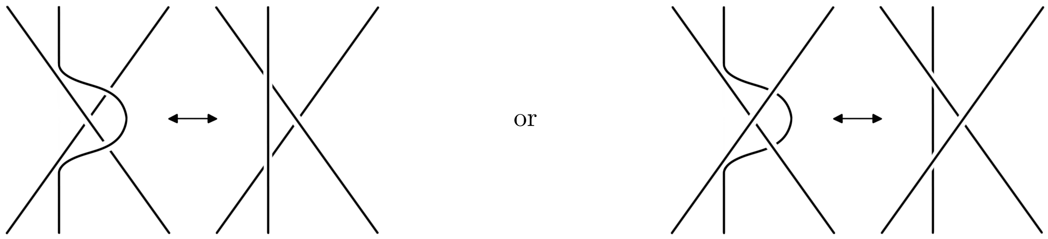
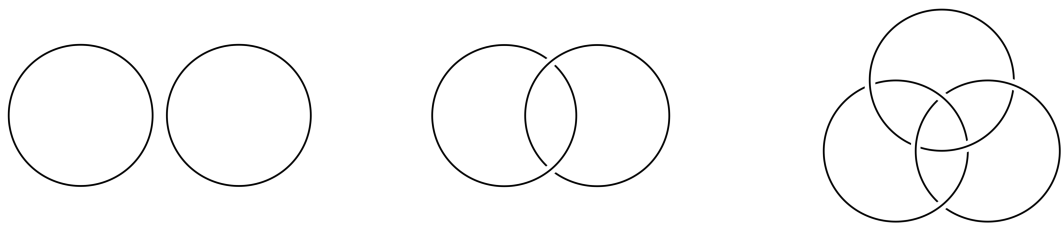


Figure 1.1.9: The third Reidmeister move.

1.2 Links

Definition 1.2.1. A *link* is a finite, ordered collection of knots that do not intersect each other. Or equivalently, a continuous embedding $l\colon S^1 \sqcup \ldots \sqcup S^1 \to S^3$. Each knot K_i is said to be a *component* of the link.

Definition 1.2.2. An *oriented link* is a link together with a choice of orientation on each component of the link.



(a) The unlink. (b) Hopf link. (c) Borromean rings.

Figure 1.2.1: Examples of links.

The disjoint union of two unknots, shown in Figure 1.2.1a is called the unlink. Other basic examples of links are the Hopf link (Figure 1.2.1b) and the Borromean rings (Figure 1.2.1c). The disjoint union of n unknots (Figure 1.2.2) is called the n-component unlink.

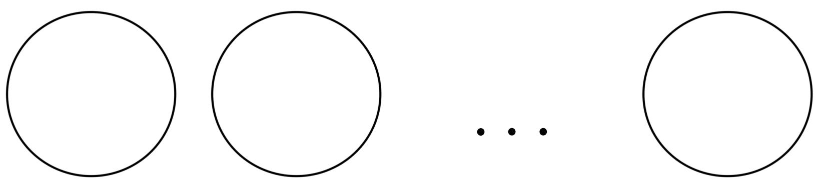


Figure 1.2.2: n-component unlink

Definition 1.2.3. We say that a link $L = \{K_1, K_2, \dots, K_m\}$ is *equivalent* to a link $L' = \{K'_1, K'_2, \dots, K'_n\}$ if

(1) $m = n$,

(2) there exists an orientation-preserving homeomorphism $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$, such that $\varphi(K_1) = K'_1, \varphi(K_2) = K'_2, \dots, \varphi(K_m) = K'_n$.

Definition 1.2.4. A diagram of a link is said to be *alternating* if it has crossings that alternate between over and under as one travels around the components of the link in a fixed direction. An *alternating link* is a link that has an alternating diagram.

For example, the diagrams shown in Figures 1.1.1 and 1.2.1 are all alternating diagrams.

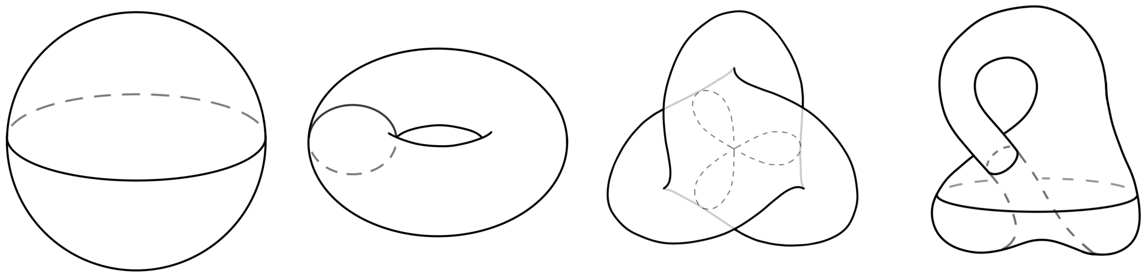
Chapter 2

Surfaces

2.1 Closed surfaces

Definition 2.1.1. A topological space X is a *surface* if it is a two-manifold, meaning that it is a T_2 and M_2 space in which every point has an open neighbourhood homeomorphic to an open subset of $\mathbb{R}^2$.

Some important examples for surfaces are the 2-sphere (Figure 2.1.1a), the torus (Figure 2.1.1b), the real projective plane (Figure 2.1.1c) and the Klein bottle (Figure 2.1.1d). Boy's surface is an immersion of the real projective plane into a 3-dimensional space. More about this construction can be found in [2].



(a) S^2 (b) $\mathbb{T}^2$ (c) Boy's surface ($\mathbb{RP}^2$) (d) Klein bottle

Figure 2.1.1: Examples of surfaces.

Definition 2.1.2. A *closed surface* is a surface which is compact.

Take a finite collection of pairwise disjoint triangles (together with their interiors) in the plane such that all of their sides are the same length. Now form a topological space S in the following way: Every side in a triangle is identified with exactly one other side in another triangle. This defines a graph G with the corners of the triangles as vertices and the sides as edges. It is easy to see that S is compact. If S is a closed surface, we say that G is a triangulation of S and S is a triangulated surface.

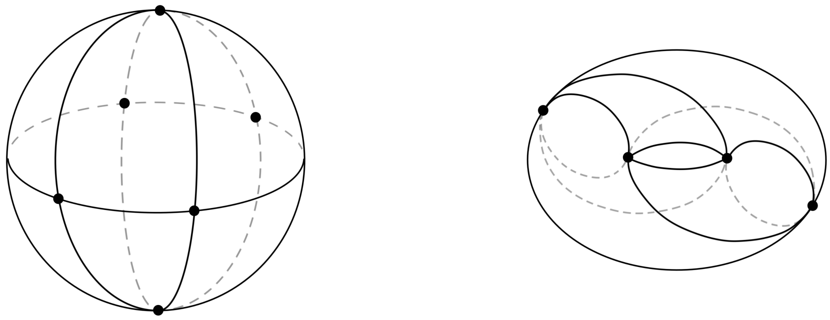


Figure 2.1.2: A triangulation of S^2 and $\mathbb{T}^2$.

Theorem 2.1.3. *Every closed surface S is homeomorphic to a triangulated surface.*

We will not prove this theorem as it falls outside of the scope of this thesis. However, a proof can be found in [7]. This essentially means that every closed surface has a triangulation.

Figure 2.1.2 shows a triangulation of the 2-sphere and a triangulation of the torus.

We can talk about the orientability of closed surfaces. Intuitively, we call a closed surface orientable if it has two sides. Now we will define this more precisely.

The orientation of a triangle is defined by an orientation of the boundary of the triangle as shown in Figure 2.1.3.

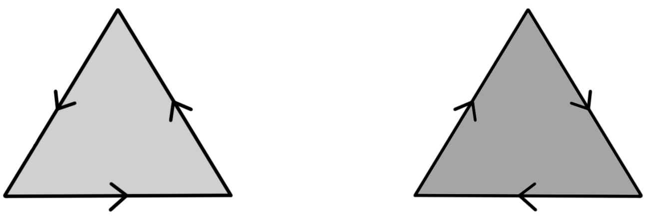


Figure 2.1.3: Orientation of a triangle.

Definition 2.1.4. A closed surface S is *orientable* if there exists a triangulation G of S such that the triangles in G can be oriented as follows. For each edge in G the orientations of the two neighbouring triangles are opposite on the edge as shown in Figure 2.1.4. An orientable surface has exactly two orientations. An *oriented* surface is an orientable surface together with a choice of orientation.

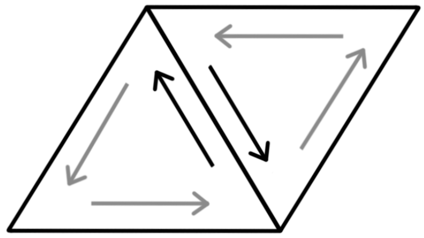


Figure 2.1.4: Edge in triangulation.

Definition 2.1.5. Take a triangulation of a closed surface and let V, E and F be the number of vertices, edges and faces in this triangulation. We define the *Euler characteristic of the triangulation* to be $V - E + F$.

Claim 2.1.6. *If we take two different triangulations of the same surface, the Euler characteristics of these two triangulations will be equal.*

We will not prove this. However, from this follows that the Euler characteristic only depends on the surface, not on which triangulation we use so from now on we will only talk about the *Euler characteristic of a surface.* We will denote the Euler characteristic of the surface S by $\chi(S)$.

In Figure 2.1.2 we see two examples of triangulations of surfaces. On the left side we see a triangulation of S^2 with 6 vertices, 12 edges and 8 faces, from which $\chi(S^2) = 2$ follows. On the right side we see a triangulation of $\mathbf{T}^2$ with 4 vertices, 12 edges and 8 faces, which determines $\chi(\mathbb{T}^2)$ to be 0.

Claim 2.1.7. *Let S and T be closed surfaces. If $S \cong T$ then $\chi(S) = \chi(T)$.*

This follows from the definition.

Definition 2.1.8 (Connected sum of closed surfaces)**.** Let S and T be closed surfaces. Now take $S \setminus \mathring{D}$ and $T \setminus \mathring{D}$ and let φ be a homeomorphism between $\partial(S \setminus \mathring{D})$ and $\partial(S \setminus \mathring{D})$. Such homeomoprhism exists as these are both homeomorphic to S^1. The *connected sum* of S and T is the closed surface

$$S\#T = ((S \setminus \mathring{D}) \sqcup (T \setminus \mathring{D}))/\sim$$

where $t \sim \varphi(t)$ for every $t \in \partial(S \setminus \mathring{D})$.

Figure 2.1.5 depicts the connected sum of two tori.

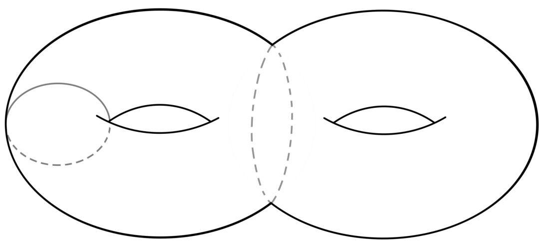


Figure 2.1.5: $\mathbb{T}^2 \# \mathbb{T}^2$

Claim 2.1.9. *If S and T are closed surfaces then $\chi(S\#T) = \chi(S) + \chi(T) - 2$.*

Proof. Since a disc and a triangle are homeomorphic, we can construct $S\#T$ by cutting out a triangle from a triangulation of each surface and gluing the two together along the triangles' boundaries, so that the vertices of the cut out triangle on S fall onto the vertices of the cut out triangle on T. Now take the triangulation of $S\#T$ we get as the union of the triangulations of S and T we took in the first place. Since we glued 6 vertices pairwise together, $V_{S\#T} = V_S + V_T - 3$. Similarly, $E_{S\#T} = E_S + E_T - 3$. As for the faces, we cut out one from each of the triangulations of S and T, so $F_{S\#T} = F_S + F_T - 2$. From these follows that

$$\chi(S\#T) = V_S + V_T - 3 - (E_S + E_T - 3) + F_S + F_T - 2 = \chi(S) + \chi(T) - 2.$$

□

2.2 Classification of closed surfaces

Example 2.2.1. The connected sum of g tori is an orientable surface called the *genus g surface* Σ_g. We call g the *genus* of Σ_g.

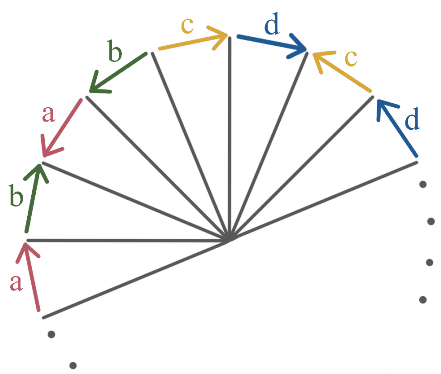


Figure 2.2.1: Standard triangulation of a genus g surface.

Claim 2.2.2. $\chi(\Sigma_g) = 2 - 2g$.

Proof. This follows from either Claim 2.1.9 and induction by g or the triangulation of Σ_g shown in Figure 2.2.1. □

Example 2.2.3. The connected sum of k real projective planes is a non-orientable surface N_k. The Klein bottle (Figure 2.1.1d) is the connected sum of two real projective planes.

Claim 2.2.4. $\chi(N_k) = 2 - k$.

Proof. This follows from either Claim 2.1.9 and induction by k or the triangulation of N_k shown in Figure 2.2.2. □

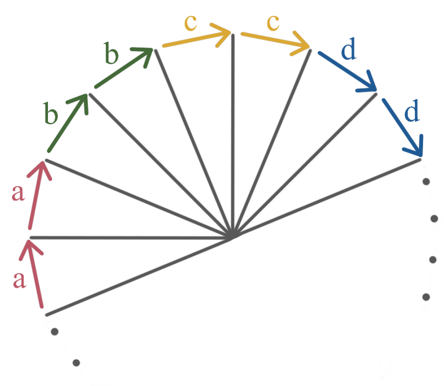


Figure 2.2.2: Standard triangulation of $N_k = \mathbb{RP}^2\#\ldots\#\mathbb{RP}^2$.

Theorem 2.2.5. *Every connected closed surface is homeomorphic to either a genus g surface or the connected sum of some real projective planes.*

We do not prove this theorem.

Corollary 2.2.6. *From Theorem 2.2.5 and Examples 2.2.1 and 2.2.3 follows that for every connected surface S and T, $S \cong T$ if and only if either both S and T are orientable or both are non-orientable and $\chi(S) = \chi(T)$.*

Remark 2.2.7. As the Euler characteristic of a connected closed orientable surface of genus g is $2 - 2g$, the higher the Euler characteristic, the lower the genus.

Chapter 3

Seifert surfaces

3.1 Surfaces with boundary

Definition 3.1.1. A topological space X is called a *surface with boundary* if it is a T_2 and M_2 space in which every point has an open neighbourhood homeomorphic to an open subset of the closed half-plane.

Definition 3.1.2. Let S be a surface with boundary. We say that $\partial S \subset S$ is the boundary of S if $p \in \partial S$ if and only if every homeomorphsim between an open neighbourhood of p and an open set of the closed upper half plane p is mapped to the line $y = 0$.

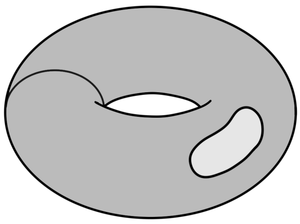


Figure 3.1.1: Torus with one boundary component.

Claim 3.1.3. *If S is a compact surface with boundary then ∂S is a compact 1-dimensional manifold.*

Claim 3.1.4. *If M is a compact 1-dimensional manifold, then $M \cong S^1 \sqcup \ldots \sqcup S^1$.*

Remark 3.1.5. We can define a triangulation (thus the orientability and the Euler characteristic) of a compact surface S with boundary as we did for closed surfaces, with the addition that ∂S must be covered with edges and vertices of the triangulation.

Proposition 3.1.6. *Let S and T be compact connected surfaces with boundary. S and T are homeomorphic if and only if the following three things apply:*

(1) either both S and T are orientable or both are non-orientable,

(2) $\chi(S) = \chi(T)$,

(3) S and T have the same number of boundary components.

Proof. It is easy to see that if $S \cong T$ then the three things listed apply.

Now for each boundary component of S take a disc which is bounded by said boundary component and is disjoint from S and all the other discs we have chosen so far. Let $\hat{S}$ be the surface we obtain by gluing these open discs to S along the boundary components of S. We obtain the surface $\hat{T}$ for T analogously.

It is easy to see that $\chi(\hat{S}) = \chi(S)+l$ where l is the number of boundary components of S. Evidently, $\chi(\hat{T}) = \chi(T) + l$, hence $\chi(\hat{S}) = \chi(\hat{T})$. By Corollary 2.2.6 the result follows. □

If S is an orientable surface with boundary then an orientation of S induces an orientation on ∂S by assigning the orientation of the edges that cover ∂S in a triangulation of S to ∂S as shown in Figure 3.1.2.

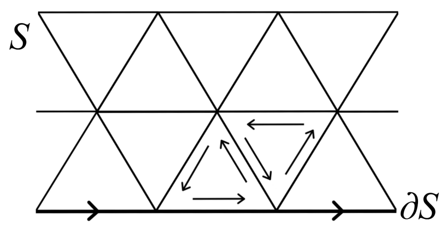


Figure 3.1.2: Orientation of the boundary of a surface S.

Remark 3.1.7. Vice versa, if the boundary of an orientable surface is oriented, it gives us an orientation of the surface in the following way: take a triangulation of the surface and let the orientation of the boundary determine the orientation of the triangles around the boundary, thus determining the orientation of every triangle in the triangulation.

From now on we will only work with orientable surfaces with boundary.

3.2 Seifert surfaces

Definition 3.2.1. Let K be a knot. A *Seifert surface* for K is an embedding $i\colon S \to \mathbb{R}^3$, where S is a connected oriented surface with boundary, such that $i(\mathring{S}) \subset \mathbb{R}^3 \backslash K$, $i(\partial S) = K$ and the orientation of K coincides with the induced orientation of ∂S.

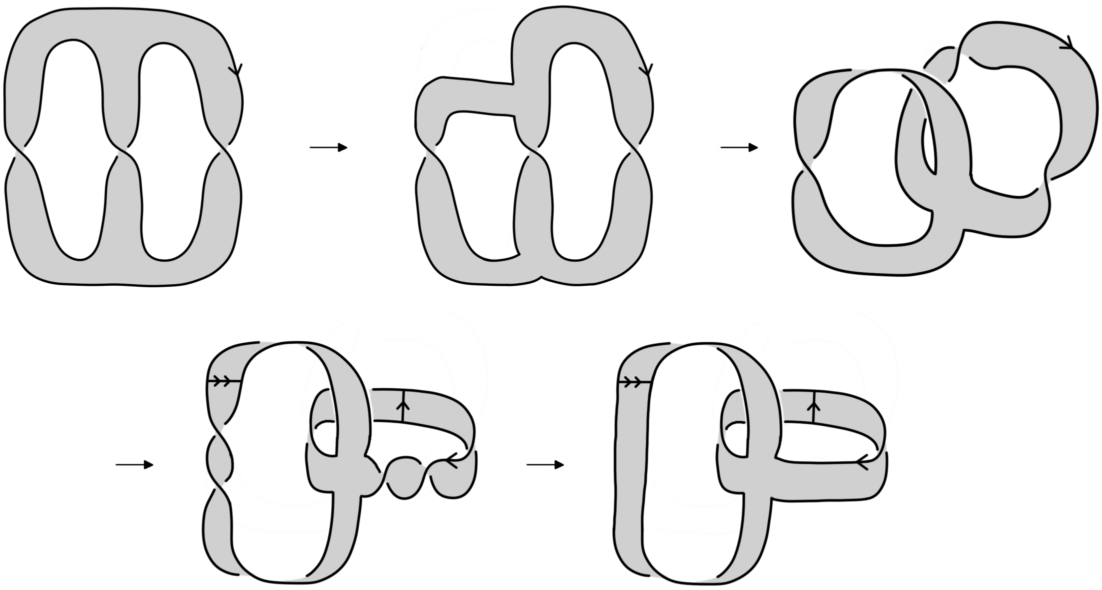


Figure 3.2.1: A Seifert surface for the trefoil knot.

In Figure 3.2.1 we take a Seifert surface for the trefoil knot and show that it is actually a torus with a one component boundary. It is fairly easy to see that the boundary of the surface in the first picture is the trefoil knot. We slowly isotope this into the surface shown in the second to last picture by moving it around in the 3-dimensional space. In the last step, we cut the surface open along the two lines marked with arrows, untwist both of the bands two times and then glue the ends back together so that the arrows are pointing in the right directions. We can do this as these two surfaces are homeomorphic. What we obtain after this procedure is the torus with one boundary component.

Definition 3.2.2. Let L be a link. A *Seifert surface* for L is an embedding $i\colon S \to \mathbb{R}^3$, where S is an oriented surface with boundary, such that $i(\mathring{S}) \subset \mathbb{R}^3 \setminus L$, $i(\partial S) = L$, the orientation of L coincides with the induced orientation of ∂S and there are no disjoint closed surfaces in S.

Remark 3.2.3. In this definition we allow S to be composed of multiple connected components, except for disjoint closed 2-spheres.

Theorem 3.2.4 (Seifert)**.** *Every link in S^3 admits a Seifert surface.*

Proof. Suppose that D is a diagram of a link L. To prove this theorem, we are going to use Seifert's algorithm:

(1) At every crossing point of D, let us cut the strings of L and reconnect them based on the orientation of the link as shown in Figure 3.2.2. As the result of this, D has now decomposed into disjoint simple closed curves called Seifert circles.

(2) We may now span these circles by pairwise disjoint embedded discs in $\mathbb{R}^3$. To be able to do this, we might have to lift or push down some of the circles in our $\mathbb{R}^3$ space.

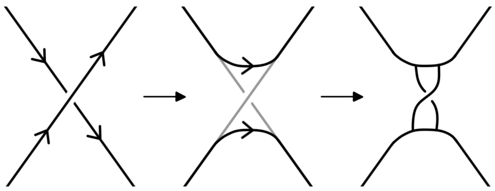


Figure 3.2.2: Oriented resolution in Seifert's algorithm.

(3) Finally, for each crossing of the original diagram we connect the discs with twisted bands as shown in Figure 3.2.2.

As a result of this algorithm, we get an oriented surface S such that the boundary of this surface is the link L and the orientation of S is the orientation induced by L (as described in Remark 3.1.7). □

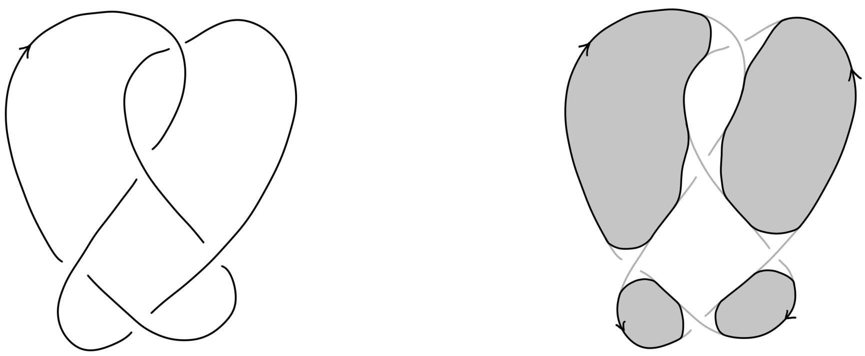


Figure 3.2.3: Cutting and reconnecting at the crossings to get Seifert circles.

In Figure 3.2.3 and Figure 3.2.4 we show Seifert's algorithm on an alternating knot, thus getting a Seifert surface of said knot.

In Figure 3.2.1 we showed a Seifert surface of genus 1 for the trefoil knot. It is clear that we can artificially increase the genus of a Seifert surface, Figure 3.2.5 shows an example. For this reason, we wish to find a Seifert surface for a knot K that is of minimal genus.

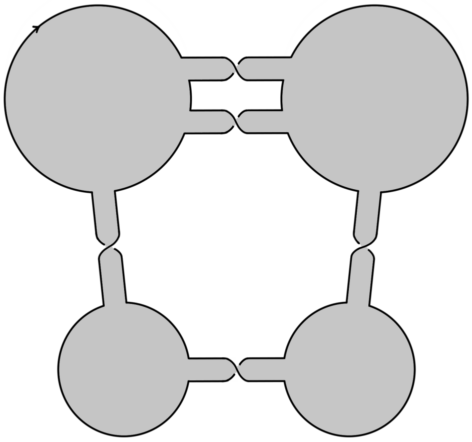


Figure 3.2.4: After connecting the circles with twisted bands we get the Seifert surface.

Definition 3.2.5. The *genus* of a knot K is an invariant of the knot, $g(K)$, defined by the genus of a Seifert surface of minimal genus for K.

Remark 3.2.6. In the case of links, it is not obvious how one can define the genus of a link. So instead of a minimal genus, we will look for a Seifert surface of maximal Euler characteristic. By Remark 2.2.7, this serves the same purpose as the minimal genus for knots.

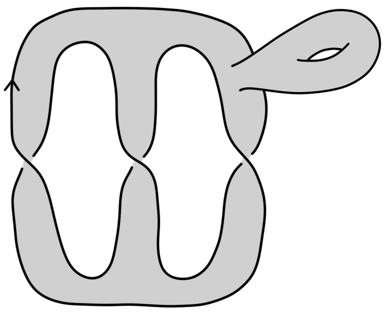


Figure 3.2.5: A Seifert surface of genus 2 of the trefoil knot.

Note that in the case of links, the Seifert surface of maximal Euler characteristic can differ based on the orientation of the link components. Examples of this can be found in the appendix.

Theorem 3.2.7. *For any knots K_1 and K_2 in S^3, $g(K_1 \# K_2) = g(K_1) + g(K_2)$.*

Proof. We first show that $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$. Let S_1 and S_2 be Seifert surfaces of minimal genus for K_1 and K_2 and define R as in the definition of the connected sum of two knots (Definition 1.1.10). It is easy to check that

$$S = S_1 \cup R \cup S_2$$

is a Seifert surface for $K_1 \# K_2$ and the genus of S is $g(K_1) + g(K_2)$.

Now we proceed to show that $g(K_1 \# K_2) \geq g(K_1)+g(K_2)$. Let T_0 be an arbitrary Seifert surface for $K_1 \# K_2$ and take an S^2 that separates K_1 and K_2 in S^3. By transversality we may assume that $T_0 \cap S^2$ is composed of a finite number of simple closed curves and arcs, where the endpoints of the arcs fall onto $K_1 \# K_2$. As $|(K_1 \# K_2) \cap S^2| = 2$, there is only one arc in $T_0 \cap S^2$. Each simple closed curve in $T_0 \cap S^2$ bounds a disc in S^2. We will perform a series of surgeries on T_0 in order to remove all the simple closed curves in $T_0 \cap S^2$.

Let C be a simple closed curve in $T_0 \cap S^2$ such that the disc that C bounds in S^2 does not contain another component of $T_0 \cap S^2$. Such curve C exists. By cutting T_0 open along C and gluing in a disc on each side of S^2 we obtain a surface T_1 with $T_1 \cap S^2$ having fewer components than $T_0 \cap S^2$. It is easy to check that $\chi(T_1) > \chi(T_0)$ applies.

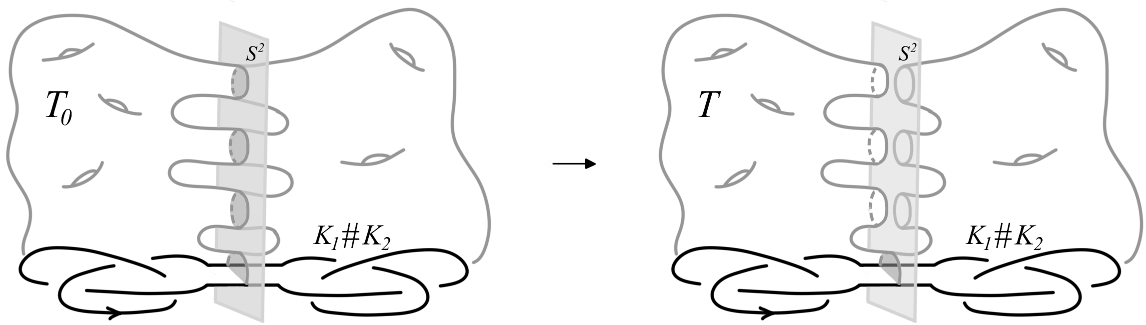


Figure 3.2.6: Performing a finite number of surgeries..

After a finite number of iterations we obtain a Seifert surface T for $K_1 \# K_2$ such that $\chi(T) > \chi(T_0)$ and $T \cap S^2$ is just one arc. It is a simple matter to check that if the genus of T is g then $g \geq g(K_1) + g(K_2)$. As T_0 was an arbitrary Seifert surface, the result follows. □

Proposition 3.2.8. $g(K) = 0$ *if and only if* K *is the unknot.*

Proof. If K is the unknot, a Seifert surface of minimal genus is a disc, which is of Euler characteristic 1. This means that the genus of this Seifert surface is 0, and there is no surface of negative genus.

Now assume that $g(K) = 0$. This means that there exists an embedding f of D^2 into $\mathbb{R}^3$ such that $\partial(f(D^2)) = K$. By slightly perturbing f we can obtain a $\mathcal{C}^\infty$ embedding of D^2 thus we may assume that f is $\mathcal{C}^\infty$. If f is $\mathcal{C}^\infty$ then for every $p \in f(\mathring{D}^2)$ there exists an open neighbourhood U_p of p such that $(U_p, f(\mathring{D}^2) \cap U_p) \cong (\mathbb{R}^3, \mathbb{R}^2)$.

There exists a small disc $D^2_\varepsilon \subset D^2$ such that $f(D^2_\varepsilon) \subset U_p \cap f(D^2)$. $K = \partial(f(D^2))$ is equivalent to $\partial(f(D^2_\varepsilon))$ therefore K is equivalent to a knot lying in $\mathbb{R}^2 \subset \mathbb{R}^3$ hence K is the unknot. □

Corollary 3.2.9. *Let $\mathcal{K}$ be the set of knots. With the connected sum as an operation, $(\mathcal{K}, \#)$ is not a group.*

Proof. We show that no element of $\mathcal{K}$ has an inverse.

Suppose that for a knot K_1 exists a knot K_2 such that $K_1 \# K_2$ is the unknot, and assume that K_1 is not the unknot. From Theorem 3.2.7 follows that

$$g(K_1) + g(K_2) = g(K_1 \# K_2) = 0.$$

This is only possible if both $g(K_1)$ and $g(K_2)$ are 0, meaning that both K_1 and K_2 are the unknot. This contradicts our hypothesis, showing that K_1 has no inverse in $(\mathcal{K}, \#)$. □

There is another very important corollary of Theorem 3.2.7, about prime knots.

Corollary 3.2.10. *Let K be a knot. If $g(K) = 1$ then K is a prime knot.*

Proof. Assume that $K = K_1 \# K_2$ where neither K_1 nor K_2 are the unknot, so both $g(K_1) \geq 1$ and $g(K_1) \geq 1$ are true. Now from Theorem 3.2.7 follows that

$$g(K) = g(K_1 \# K_1) = g(K_1) + g(K_2) \geq 2$$

which contradicts the assumption that $g(K) = 1$ □

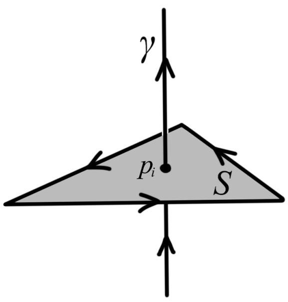
Chapter 4

Disjoint Seifert surfaces

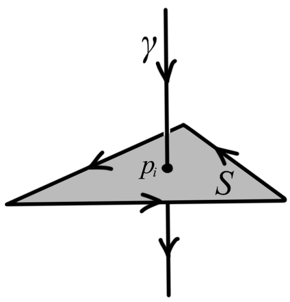
The lemma we prove in this section plays a crucial part in the proof of the main theorem of this thesis. However, to prove the lemma, we must use arguments for which the theory and reasoning falls outside the scope of this thesis. We will denote these arguments by "Black Box".

Black Box 4.0.1. *Let T and S be Seifert surfaces for a link L. By transversality we may assume that after perturbing T and S slightly, $T \cap S$ is the disjoint union of a finite number of simple closed curves and arcs, where the endpoints of the arcs fall onto L. Using tools such as the long exact sequence of homology groups, Poincaré-Lefschetz duality, and the universal coefficient theorem, we may even assume that $T \cap S$ contains no arcs.*

Definition 4.0.2. Take an oriented link L, a Seifert surface S for L and an oriented closed curve $\gamma\colon S^1 \to S^3 \setminus L$. We may assume that γ intersects S transversally, hence $\gamma \cap S = \{p_1, \ldots, p_n\}$ is a finite set. We define $\operatorname{ind}(\gamma \cap S, p_i)$ for every $i \in \{1, \ldots, m\}$



(a) $\operatorname{ind}(\gamma \cap S, p_i) = +1$



(b) $\operatorname{ind}(\gamma \cap S, p_i) = -1$

Figure 4.0.1: We define $\operatorname{ind}(\gamma \cap S, p_i)$ via the right hand rule.

to be ± 1 via the right hand rule as depicted in Figure 4.0.1. We say that the *algebraic* *intersection number of* γ *and* S is

$$\langle \gamma\,, S \rangle = \sum_{i=1}^{n} \mathrm{ind}(\gamma \cap S,\, p_i).$$

Black Box 4.0.3. *$\langle \gamma\,, S \rangle$ is well defined in the sense that if the oriented closed curves γ_1 and γ_2 are homotopic in $S^3 \setminus L$ then $\langle \gamma_1\,, S \rangle = \langle \gamma_2\,, S \rangle$.*

Black Box 4.0.4. *For homological reasons it follows that if T is another Seifert surface for L then $\langle \gamma\,, S \rangle = \langle \gamma\,, T \rangle$.*

Lemma 4.0.5. *If L is an oriented link in S^3 and S is a Seifert surface of L which is not of maximal Euler characteristic, then there exists a Seifert surface T for L such that $\chi(T) > \chi(S)$ and $\mathring{T} \cap \mathring{S} = \emptyset$.*

Proof. By Black Box 4.0.1 we may assume that $S \cap T$ consists of a finite number of pairwise disjoint simple closed curves. If $\mathring{S} \cap \mathring{T} = \emptyset$ then the lemma follows. Now we show that we may assume that neither of the connected components of $S \cap T$ bounds a disc in S, nor in T. This is to make sure that later on in this proof we do not accidentally create disjoint 2-spheres which would artificially increase the Euler characteristic of the Seifert surface.

Let C be a simple closed curve in $S \cap T$. If C bounds a disc in S then by cutting T open along C and gluing in a disc on each side of S we obtain a surface T' with $T' \cap S$ having fewer components than $T \cap S$ (Figure 4.0.2). It is easy to check that $\chi(T') > \chi(T)$ thus $\chi(T') > \chi(S)$ applies.

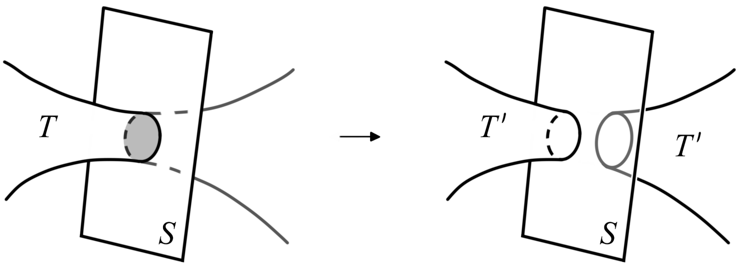


Figure 4.0.2: If C bounds a disc in S.

If C bounds a disc in T but does not bound a disc in S we can create a Seifert surface T' by taking a copy of S, cutting it open along C, gluing in a disc on each side of T and then pushing it slightly away from S based on the orientation of S (Figure 4.0.3). This new surface T' has the property that $\chi(T') = \chi(S) + 2$ and $\mathring{S} \cap \mathring{T}' = \emptyset$.

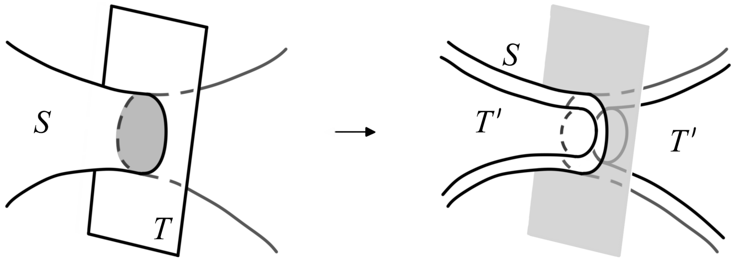


Figure 4.0.3: If C bounds a disc in T but not in S.

With these surgeries we can remove all components of $S \cap T$ that bound a disc in either S, T or both, hence we may assume that neither of the connected components of $S \cap T$ bounds a disc in S, nor in T.

For compactness reasons $L \cup S \cup T$ divides S^3 into a finite number of connected open sets (regions) such that $S^3 \setminus (L \cup S \cup T) = \{J_0, J_1, \dots, J_m\}$.

Take an $x_0 \in S^3 \setminus (L \cup S \cup T)$ and for every $x \in S^3 \setminus (L \cup S \cup T)$ let γ be an oriented path from x_0 to x. We define

$$\phi_0 \colon S^3 \setminus (L \cup S \cup T) \to \mathbb{Z} \text{ by}$$
$$\phi_0(x) = \langle \gamma \,, S \rangle - \langle \gamma \,, T \rangle.$$

We argue that ϕ_0 is well defined in the sense that $\phi_0(x)$ does not depend on which oriented path γ we choose. Take two oriented paths γ and η from x_0 to x. From Black Box 4.0.4 follows that $\langle \gamma - \eta \,, S \rangle = \langle \gamma - \eta \,, T \rangle$ as $\gamma - \eta$ is a closed curve. By definition we get

$$\langle \gamma \,, S \rangle - \langle \gamma \,, T \rangle = \langle \eta \,, S \rangle - \langle \eta \,, T \rangle$$

which leads us to the conclusion that ϕ_0 is well defined.

It is evident that for every $i \in \{0, \dots, m\}$, ϕ_0 is constant on J_i. We show that we can choose x_0 so that $\phi_0 \geq 0$. Since $\operatorname{im}(\phi_0) \subset \mathbb{Z}$ is a finite set, there exists a region J_k which minimizes ϕ_0. Let us choose x_0 from J_k. It is a simple matter to check that with this choice of x_0, $\phi_0 \geq 0$.

We claim that if $\mathring{S} \cap \mathring{T}' \neq \emptyset$ then $\max \phi_0 \geq 2$. Since S and T intersect each other transversally, a small neighbourhood of $S \cap T$ can be visualized locally as in Figure 4.0.4.

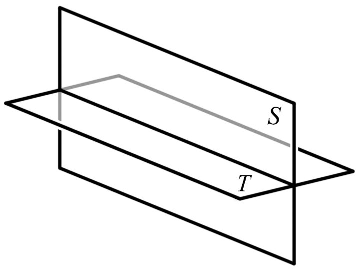


Figure 4.0.4: $S \cap T$ locally.

Locally $S \cap T$ divides S^3 into four regions. In Figure 4.0.5 a, b, c and d are the values of ϕ_0 in these regions. It is easy to check that these quadrants must contain at least three different numbers which leads us to the conclusion that $\max \phi_0 \geq 2$.

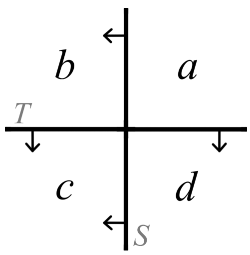


Figure 4.0.5: The values of ϕ_0 locally around $S \cap T$.

Now let J be a region which maximizes ϕ_0. In order to obtain a new Seifert surface T' we separate the following two cases:

$$\text{If } \chi(\overline{J} \cap S) \geq \chi(\overline{J} \cap T) \text{ then } T' = (T \setminus (\overline{J} \cap T)) \cup (\overline{J} \cap S), \tag{4.1}$$

$$\text{If } \chi(\overline{J} \cap S) < \chi(\overline{J} \cap T) \text{ then } T' = (S \setminus (\overline{J} \cap T)) \cup (\overline{J} \cap T), \tag{4.2}$$

where $\overline{J}$ denotes the closure of J. In both cases we perturb T' slightly if necessary to assure that T' and S are again in a position described in Black Box 4.0.1.

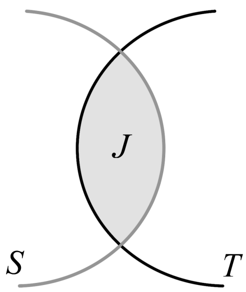


Figure 4.0.6: A region J which maximizes ϕ_0.

Note that the way we illustrate this in Figures 4.0.6, 4.0.7 and 4.0.8 is not entirely representative as J can be immensely complicated.

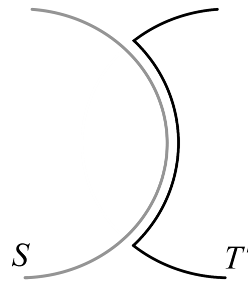


Figure 4.0.7: If $\chi(\overline{J} \cap S) \geq \chi(\overline{J} \cap T)$.

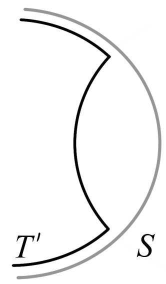


Figure 4.0.8: If $\chi(\overline{J} \cap S) < \chi(\overline{J} \cap T)$.

We argue that if we ever get to the second case (4.2), the lemma follows. Since we obtained T' by copying S, changing a part of S to something that is disjoint from S and then pushing this modified copy slightly away from S based on the orientation of S, $\mathring{S} \cap \mathring{T'} = \emptyset$ applies. $\chi(T') > \chi(S)$ as

$$\chi(S) = \chi(\overline{J} \cap S) + \chi(S \setminus (\overline{J} \cap S)) < \chi(\overline{J} \cap T) + \chi(S \setminus (\overline{J} \cap S)) = \chi(T').$$

So in this case T' is a Seifert surface for L such that $\chi(T') > \chi(S)$ and $\mathring{S} \cap \mathring{T'} = \emptyset$.

Now let us take a look at the first case (4.1). Similarly to the second case, it is easily seen that

$$\chi(T) = \chi(\overline{J} \cap T) + \chi(T \setminus (\overline{J} \cap T)),$$

so from $\chi(\overline{J} \cap S) \geq \chi(\overline{J} \cap T)$ it follows

$$\chi(T') = \chi(\overline{J} \cap S) + \chi(T \setminus (\overline{J} \cap T)) \geq \chi(T) > \chi(S).$$

If $\mathring{S} \cap \mathring{T'} = \emptyset$ then the lemma follows. If $\mathring{S} \cap \mathring{T'} \neq \emptyset$, we examine how ϕ_0 changes when T' takes the place of T. Define the map $\phi_1 \colon S^3 \setminus (L \cup S \cup T') \to \mathbb{Z}$ as we defined ϕ_0 but with T' instead of T.

We may assume that J_i are arranged so that $\phi_0(J_0) \leq \phi_0(J_1) \leq \ldots \leq \phi_0(J_m)$ where J_0 contains x_0 and $J_m = J$. For every $i \in \{0, 1, \ldots, m-1\}$, $\phi_0 = \phi_1$ in J_i. When T' took the place of T we merged J with some neighbouring regions and the value of ϕ_0 was maximal in J therefore $\phi_1 < \phi_0$ in J. By this procedure we either achieved that $\max \phi_1 < \max \phi_0$ or we strictly decreased the number of ϕ_0-maximal regions. It follows inductively that after a finite number of iterations we obtain a Seifert surface T such that $\chi(T) > \chi(S)$ and $\mathring{T} \cap \mathring{S} = \emptyset$. $\square$

Chapter 5

Seifert surfaces of alternating links

Seifert's algorithm gives us a Seifert surface for any link L, however nothing guarantees that the surface we obtain by applying the algorithm to an arbitrary diagram of L will be of maximal Euler characteristic. Let us take the unknot as an example. In Figure 5.0.1 we apply Seifert's algorithm to a diagram of the unknot. The surface we get is the torus

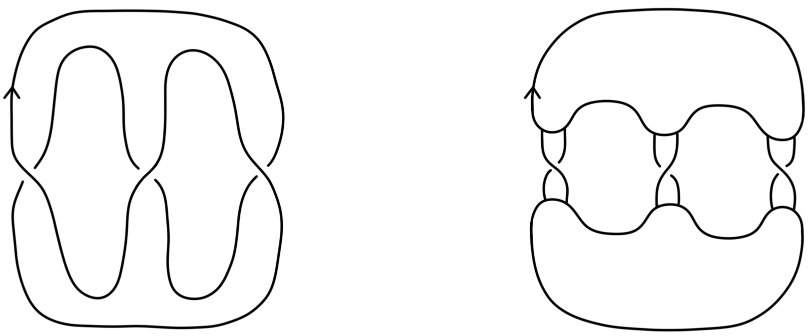


Figure 5.0.1: Applying Seifert's algorithm to a non-alternating diagram of the unknot.

with one boundary component even though we know that the genus of the unknot is 0. Notice that the diagram we worked with is not alternating which leads us to the main theorem of this thesis.

Theorem 5.0.1. *Let L be an oriented link in S^3, and let S be the surface obtained from Seifert's algorithm by applying it to an alternating diagram of L. Then S is a Seifert surface of maximal Euler characteristic.*

Proof. We prove the theorem by induction by the number of crossings.

For a link with a diagram of zero crossings, Seifert's algorithm gives us a finite number of disjoint discs. We prove that this is a maximal Euler characteristic Seifert surface for L. If L has zero crossings, then L is an m-component unlink. Let S be a Seifert surface for L

and $S_1, S_2, \ldots, S_k$ be the connected components of S so that $S = S_1 \sqcup S_2 \sqcup \ldots \sqcup S_k$. For every $i \in \{1, 2, \ldots, k\}$, the boundary of S_i is a disjoint union of k_i circles ($\partial(S_i) = S_1^1 \sqcup \ldots \sqcup S_{k_i}^1$), and

$$\chi(S_i) = 2 - 2g_i - k_i,$$

where g_i is the genus of S_i. Since S is a Seifert surface for L, S does not have disjoint closed components which leads to the conclusion that for every i, $k_i \geq 1$. For every K_j connected component of L there is exactly one component S_i of S such that $K_j \subset \partial S_i$, meaning that

$$\sum_{i=1}^{k} k_i = n.$$

From these follows that $k \leq m$. Now we calculate the Euler characteristic of S from the Euler characteristics of the S_i:

$$\chi(S) = \chi(S_1 \sqcup S_2 \sqcup \ldots \sqcup S_k) = \sum_{i=1}^{k} \chi(S_i) = \sum_{i=1}^{k} (2 - 2g_i - k_i).$$

Now using what we know about the sum of k_i, that $k \leq m$ and that for every i, $g_i \geq 0$ we get that

$$\chi(S) = \sum_{i=1}^{k} (2 - 2g_i - k_i) \leq 2k - m \leq 2m - m = m.$$

Knowing that by applying Seifert's algorithm to L we get a surface with Euler characteristic m, we see that for a diagram with zero crossings the theorem is true.

Now we assume that the theorem is true for all links with alternating diagrams of n crossings or less. Let L be a link with an alternating diagram of $n + 1$ crossings. We show that by applying Seifert's algorithm to this diagram of L we get a maximal Euler characteristic Seifert-surface.

Let S be the Seifert surface we got by applying Seifert's algorithm to the alternating diagram of $n+1$ crossings of L and suppose that S is not of maximal Euler characteristic. Now S is composed of discs and twisted bands.

We define a graph G_S with the Seifert circles of S as its vertices and edges that correspond to the twisted bands connecting the Seifert circles. We show that we can assume that G_S has no leaves. If a Seifert circle C represents a leaf, it has exactly one twisted band connected to it. Now we can untwist said band by turning C over and then merge the two Seifert circles connected by the now untwisted band, as shown in Figure 5.0.2.

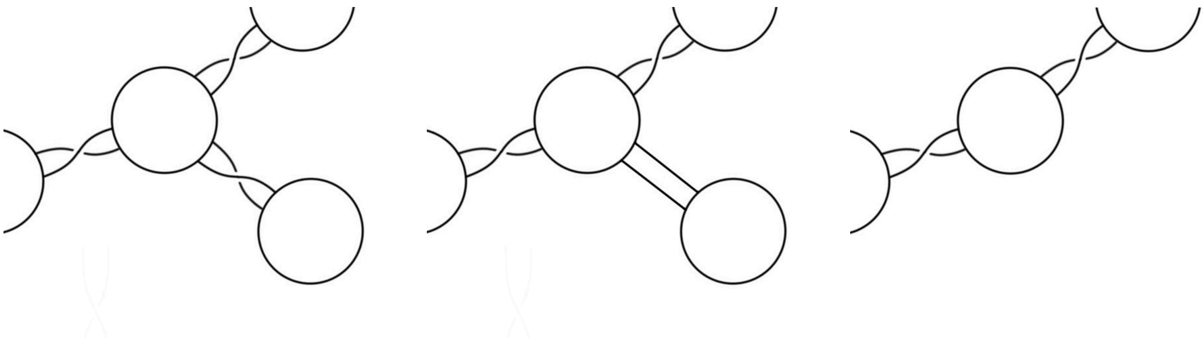


Figure 5.0.2: Removing a leaf.

The diagram we have now is still an alternating diagram of L and by doing this we do not change the Euler characteristic of S. This corresponds to removing a leaf and the edge connected to it from G_S. By repeating this procedure, we can get to an alternating diagram of L such that when applying Seifert's algorithm to it, the corresponding graph of the surface we got has no leaves. Untwisting bands this way leads to a diagram with less than $n+1$ crossing, for which the induction hypothesis is true.

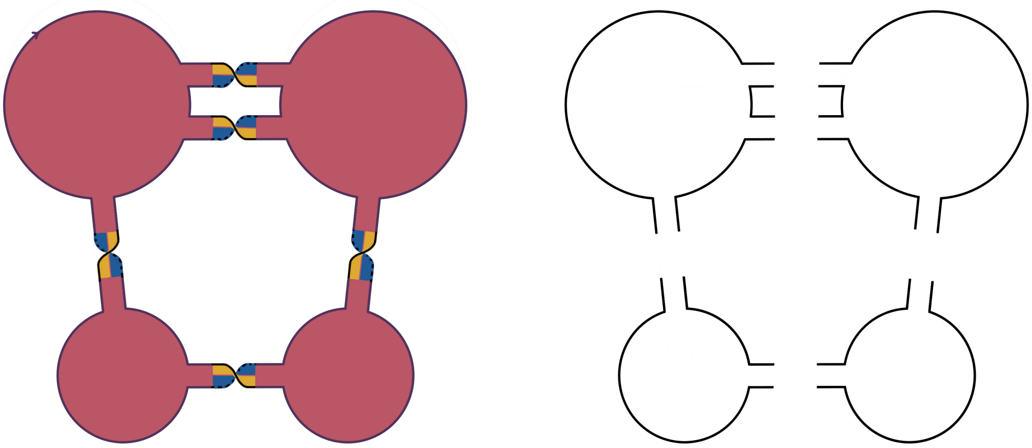
From this follows that when G_S is a forest, the Euler characteristic of S is m where m is the number of connected components in G_S, so S is in fact a Seifert surface of maximal Euler characteristic. Hence from now on we may assume that G_S contains a cycle.

We are going to prove the theorem in two steps. In the first step, we are going to prove it for the cases where all Seifert circles are unnested (meaning that S can be isotoped to lie in S^2 in a way that the Seifert circles bound pairwise disjoint disks in S^2). In the second step, we are going to prove the theorem for cases where there are nested Seifert circles in S.

Step 1. *All Seifert circles are unnested.*

Since all the Seifert circles in S are unnested, the way we defined G_S gives us a planar embedding of the graph. Therefore, we can look at S as being embedded in S^2 except for small neighbourhoods of the crossings. Figure 5.0.3a shows what we mean by this embedding, the parts of the surface coloured yellow (the lightest gray in monochrome) are above, the parts coloured blue (the darkest gray in monochrome) are below and the parts coloured red lie in S^2.

Near every crossing one of the strings is below and one is above S^2. Since L is alternating, when traveling around the knot, between any two adjacent crossings we have



(a) S in S^2 (b) $L \cap S^2$

Figure 5.0.3: S embedded in S^2 except for small neighbourhoods of the crossings.

to pass trough S^2. At the moment $L \cap S^2$ consists of some arcs, as shown in Figure 5.0.3b. Now we perturb S slightly in a small neighbourhood of ∂S in the following way: we either lift it up or lower it down, which results in ∂S intersecting S^2 in $2n+2$ points. The result of this perturbation is shown in Figure 5.0.4 where as before, the parts of the surface coloured yellow are above, the parts coloured blue are below and the parts coloured red lie in S^2. Figure 5.0.5 shows $S \cap S^2$ before and after we perturb S.

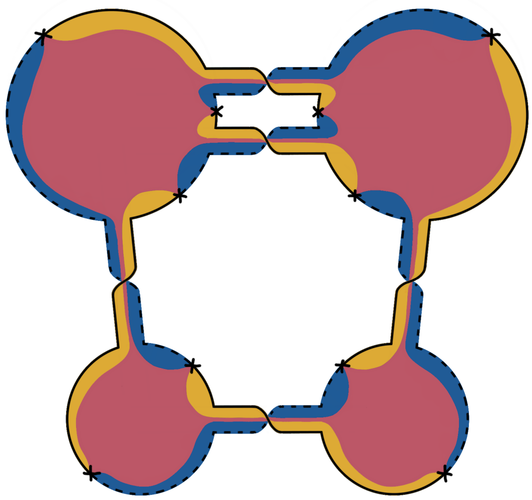


Figure 5.0.4: S in S^2 after perturbing.

When perturbing S we also isotope L so that $\partial S = L$ still applies. Now L intersects S^2 in finitely many points, each point corresponding to an arc in Figure 5.0.3b. These points are marked with an **x** in Figure 5.0.4. For every Seifert circle, the number of twisted bands around the circle equals the number of points where L intersects S^2 on the boundary of the circle.

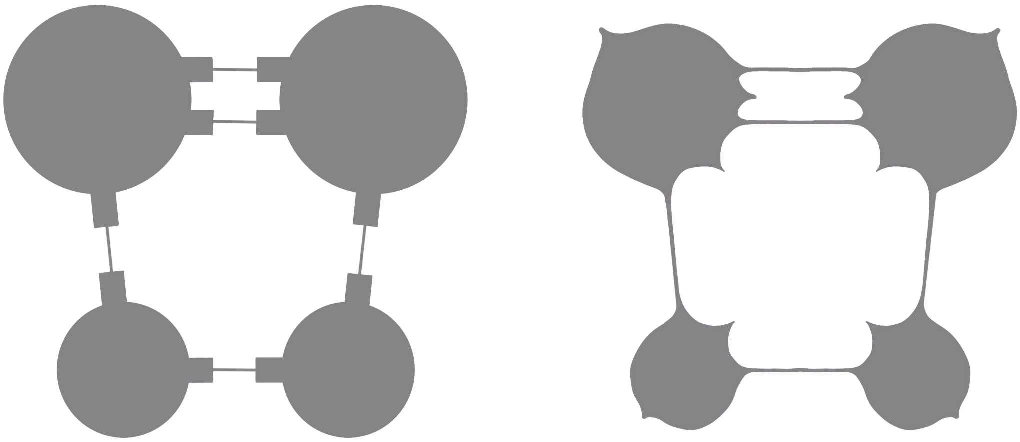


Figure 5.0.5: $S \cap S^2$ before and after we perturb S.

Because we assumed that S is not of maximal Euler characteristic, by Lemma 4.0.5 there exists a be a Seifert surface T for L such that $\chi(T) > \chi(S)$ and $\mathring{T} \cap \mathring{S} = \emptyset$. After a small isotopy, we may assume that T and S^2 have transversal intersection. Since $L \cap S^2$ consists of $2n+2$ points and $\partial T = L$, $T \cap S^2$ consists of $n+1$ arcs and some finite set of simple closed curves. By performing surgeries on T as described in the proof of Lemma 4.0.5, thus removing the circles from $T \cap S^2$, we may assume that T intersects S^2 in $n+1$ arcs.

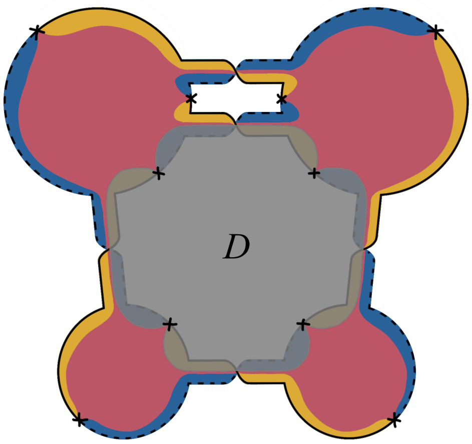


Figure 5.0.6: A connected component D in $S^2 \setminus S$.

Now take a look again at the planar embedding of G_S we described earlier. If we take a connected component D in $S^2 \setminus S$, the Seifert circles and twisted bands bounding D correspond to the vertices and edges of an innermost cycle in G_S (Figure 5.0.6). D is homeomorphic to a disc and we also know that $\partial D \subseteq S^2 \cap S$. Since S is orientable we know that there must be an even number of twisted bands around D, thus L must intersect ∂D in some $2k$ points. It follows that $T \cap D$ consists of k arcs.

We say that an arc in $T \cap D$ is innermost if it runs between two neighbouring points of $\partial D \cap L$. It is easy to check that $T \cap D$ contains at least one innermost arc and if $|\partial D \cap L| \geq 4$, there must be at least two innermost arcs in $T \cap D$.

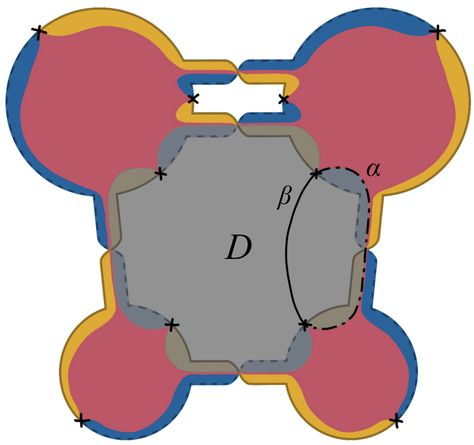
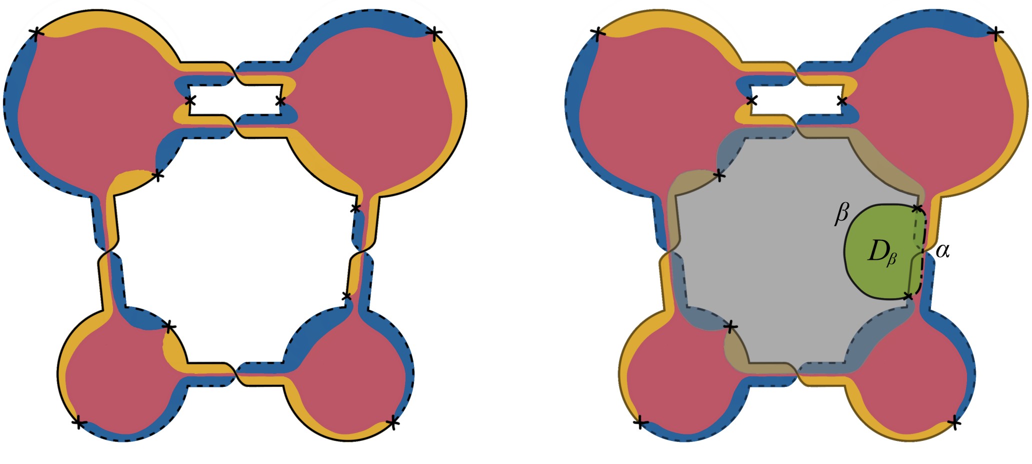


Figure 5.0.7: α and β.

Take an innermost arc β in $T \cap D$, and an arc α in $S \cap \overline{D}$ that has the same endpoints as β (Figure 5.0.7). Now we are going to cut S open along α, in the following way. We perturb S slightly so that $\partial S = L$ intersects S^2 as shown in Figure 5.0.8a. Our goal with this is the two endpoints of α and β to be on one twisted band. While doing this, we perturb T as well, so that $\partial T = L$ still applies. We call the part of D that is bounded by $\alpha \cup \beta$ now D_β (Figure 5.0.8b). From now on we no longer look at S and L as being



(a) Moving the intersection points.

(b) α and β now.

Figure 5.0.8: Perturbing S to move the endpoints of the arcs.

embedded in S^2. We take the twisted band that contains α and twist it once, thus moving the twist further up on the band (Figure 5.0.9). Now we see that α runs across the band, so we cut S open along α. By doing this, we obtain a new surface S', and we call the

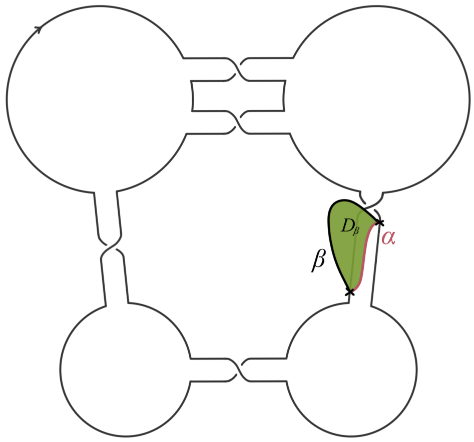


Figure 5.0.9: Moving the twist further up on the band.

boundary of this new surface L'. From the way we constructed S', it is evident that L' has a trivial crossing where L had the crossing we cut open. By undoing this trivial crossing using the first Reidemeister move, the new diagram of L' is still alternating, and S' is now exactly the Seifert surface for L' that we get by applying Seifert's algorithm to this new

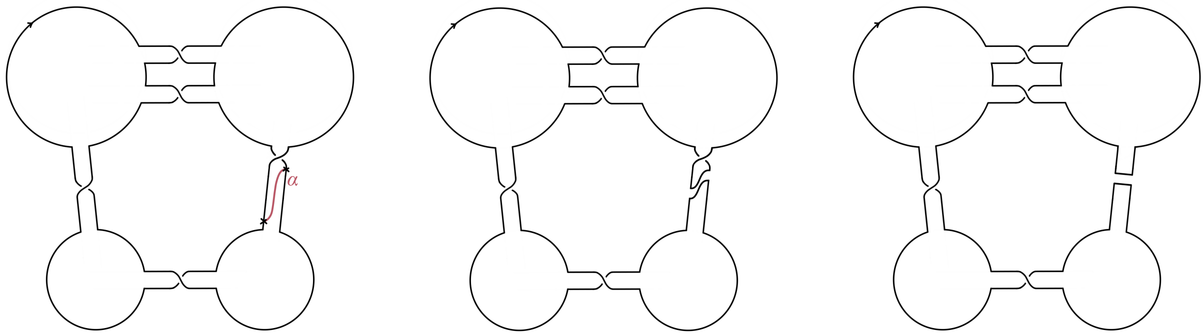


Figure 5.0.10: Cutting S open and undoing a trivial crossing.

diagram. It is easy to check that $\chi(S') = \chi(S) + 1$. This diagram of L' has n crossings, thus from the induction hypothesis follows that S' is of maximal Euler characteristic.

Parallel to this, we cut T open along β and glue in a copy of D_β on each side of the cut as shown in Figure 5.0.11. The surface T' we obtain this way has L' as its boundary and

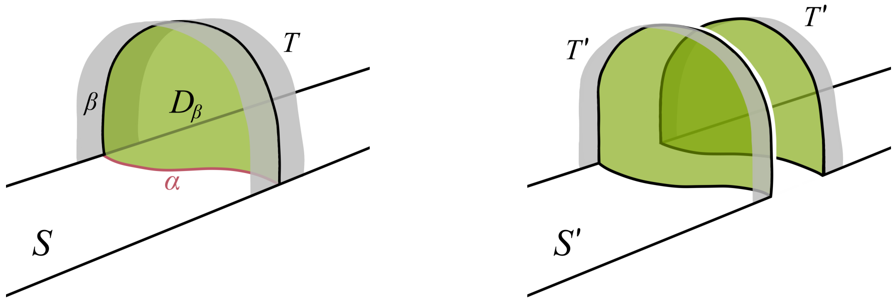


Figure 5.0.11: Creating T'.

$\chi(T') = \chi(T) + 1$ applies. From $\chi(T) > \chi(S)$ follows that $\chi(T') > \chi(S')$. This contradicts the induction hypothesis, since S' and T' are both Seifert surfaces for the link L' and S' is of maximal Euler characteristic. Thus we have proven that S is a Seifert surface of maximal Euler characteristic for L.

Step 2. *There are nested Seifert circles in S.*

For this step, we will arrange the Seifert circles of S into levels. The Seifert circles which are not contained by any other Seifert circles (so they are the "maximal" ones in a sense) go on the first level. On the second level there are the circles which are only contained by the ones on the first level. On the third level the ones that are only contained by the ones on the first and the second level, and so on. The way we arranged the Seifert circles guarantees that twisted bands will only go within a level or between neighbouring levels. An example for this is shown in Figure 5.0.12 where C_0 is on the second level, C_1, C_2 and C_3 are on the third.

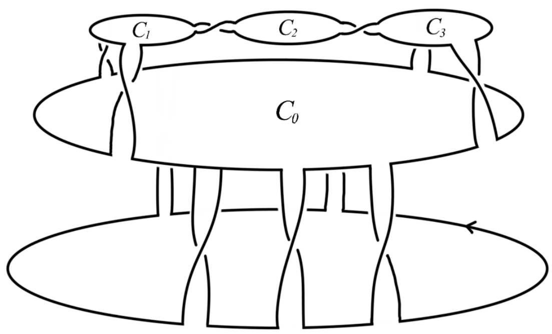


Figure 5.0.12: Arranging Seifert circles into levels.

Take a Seifert circle C_0 from the second highest level and let $C_1, \ldots, C_m$ be the Seifert circles C_0 contains. Since C_0 is on the second highest level, $C_1, \ldots, C_m$ are on the highest, meaning that they do not contain any other Seifert circles. Now take the part of S that is bounded by $C_0, C_1, \ldots, C_m$ and the twisted bands going between them and isotope them to lie in an S^2 except for small neighbourhoods of the crossings. (We can view this as $C_0, C_1, \ldots, C_m$ spanning a 2-sphere as shown in Figure 5.0.13.)

Now we perturb this part of S slightly along with L similarly to Step 1 with the addition that we choose all twisted bands leaving from C_0 that are not connected to any Seifert circles in $\{C_1, \ldots, C_m\}$ to go outside of S^2. We can achieve this by choosing the

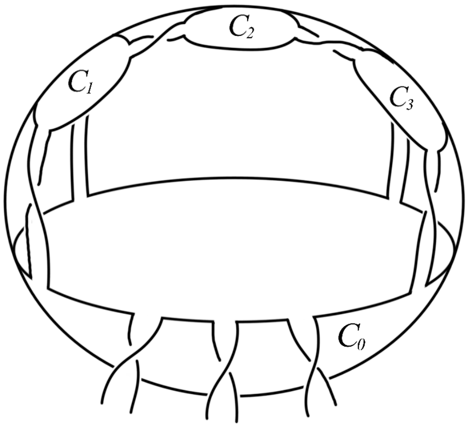


Figure 5.0.13: C_0, C_1, C_2 and C_3 in S^2.

intersection points of L and S^2 wisely. Figure 5.0.14 depicts this, where yellow represents the parts of the knot that are outside of S^2 and blue the parts that are inside.

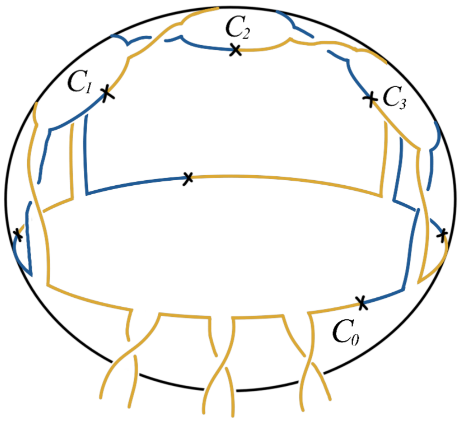


Figure 5.0.14: L after we perturb S.

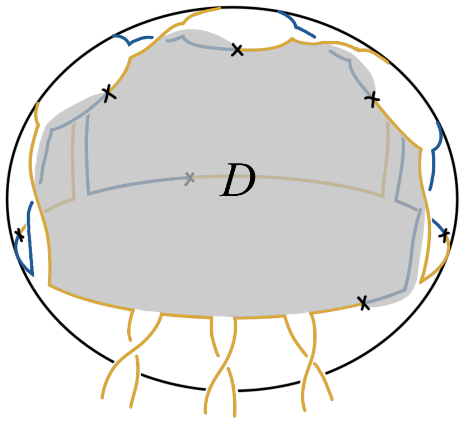


Figure 5.0.15: A connected component D in $S^2 \setminus S$.

If there are at least two twisted bands between C_0 and $\{C_1, \ldots, C_m\}$, take a connected component D in $S^2 \setminus S$ (Figure 5.0.15) and do as in Step 1 with one additional rule: if there are twisted bands in L leaving from C_0 between the endpoints of the arc β and $|\partial D \cap L| \geq 4$ then choose another innermost arc β' and apply the procedure (described in Step 1) to β' instead of β. If $|\partial D \cap L| = 2$ then there are two arcs in $S \cap \overline{D}$ with the same endpoints as β, let us choose α to be the one that does not go around the twisted bands leaving from C_0. This is shown in Figure 5.0.16.

If there is only one twisted band between C_0 and $\{C_1, \ldots, C_m\}$ then untwist it by turning $\{C_1, \ldots, C_m\}$ upside down as a whole, similarly to the leaf removal we described

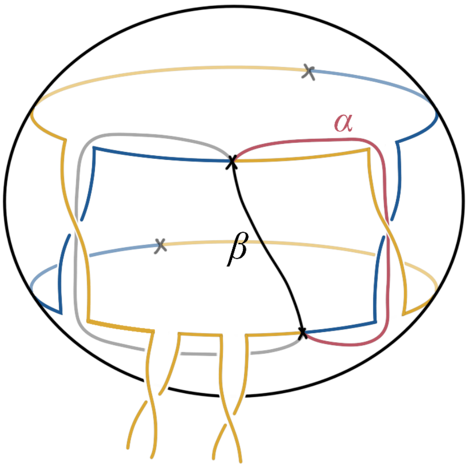


Figure 5.0.16: If $|\partial D \cap L| = 2$ then we choose α as shown here.

at the beginning of this proof (Figure 5.0.2), but do not merge this time. This removes a crossing in L, giving us an alternating link diagram with n crossings. This completes the proof of the theorem. $\square$

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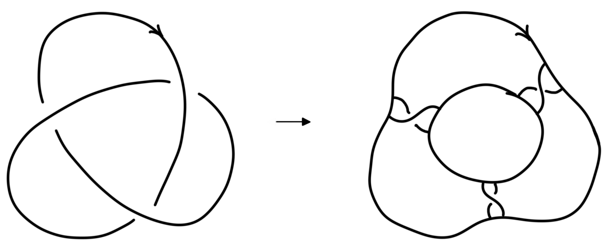
Appendix

In this appendix we collected a number of alternating knots and links with their Seifert surfaces obtained from Seifert' algorithm and computed their Euler characteristic.



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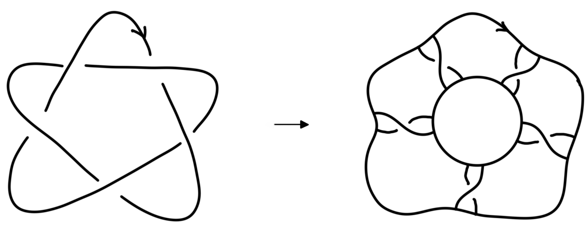

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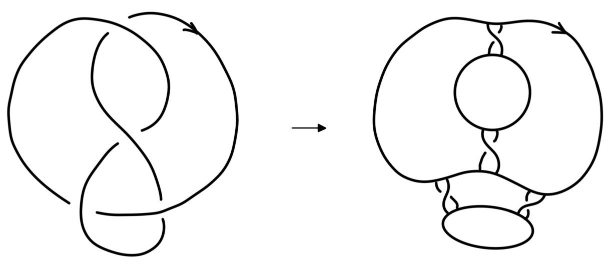

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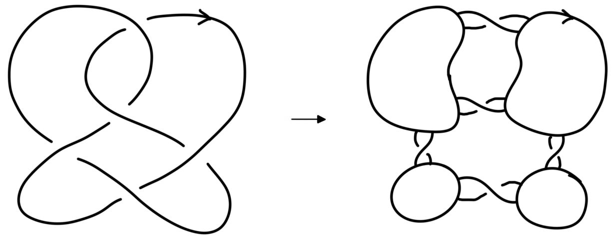

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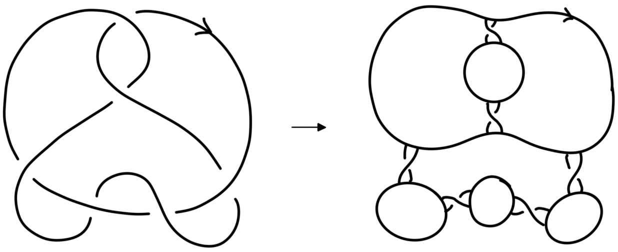

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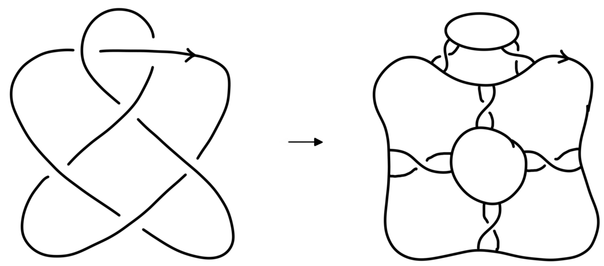

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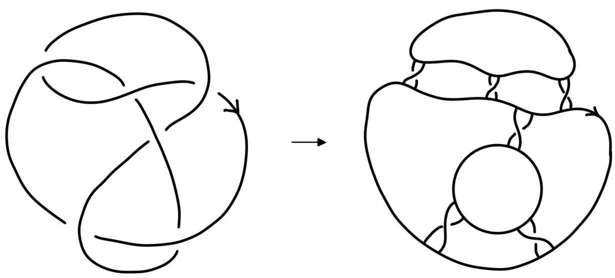

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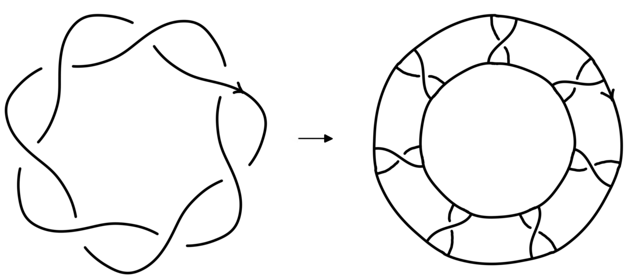

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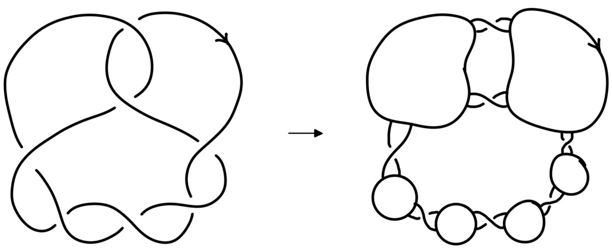

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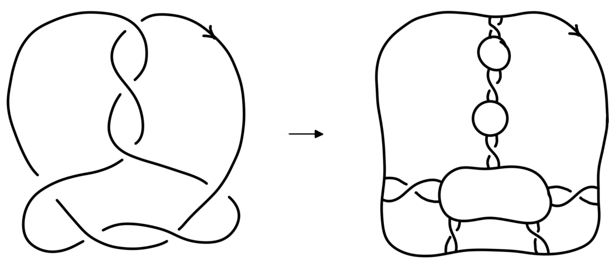

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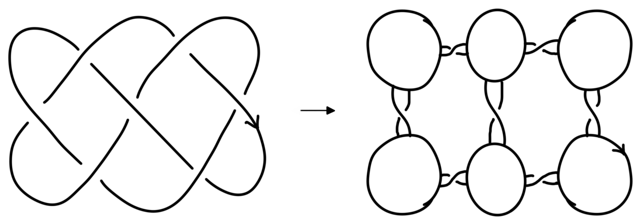

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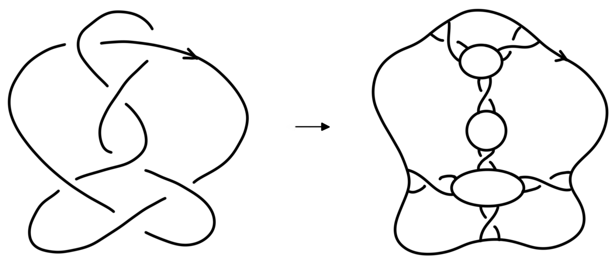

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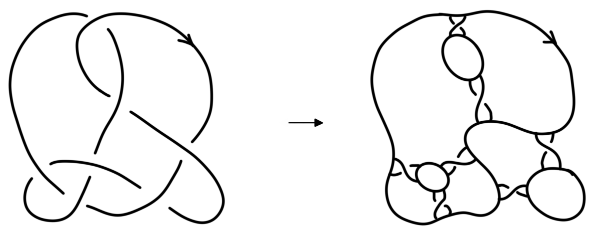

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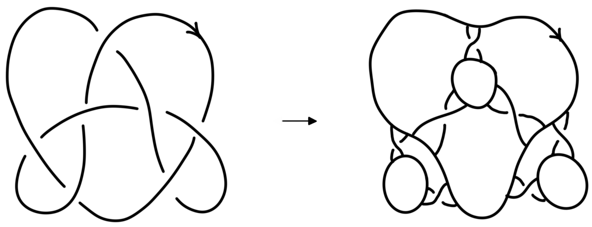

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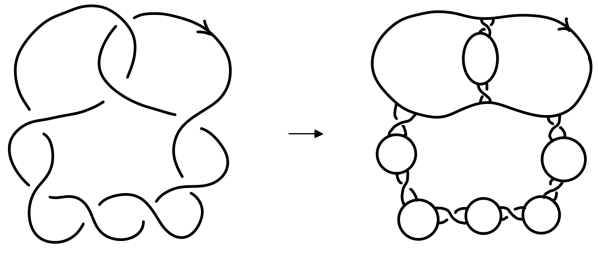

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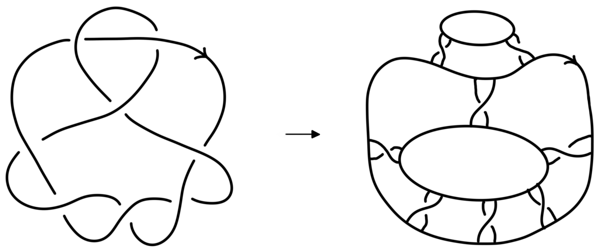

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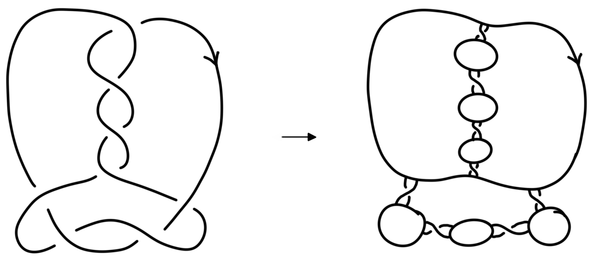

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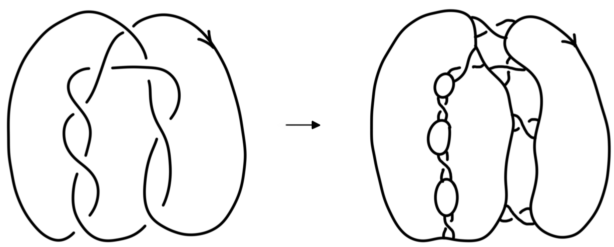

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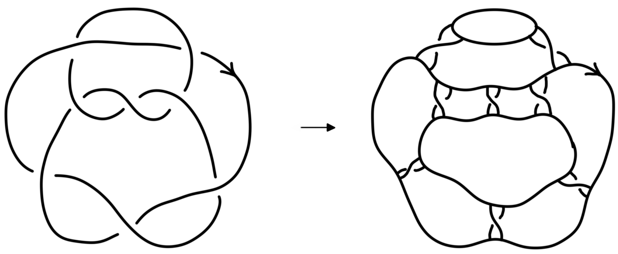

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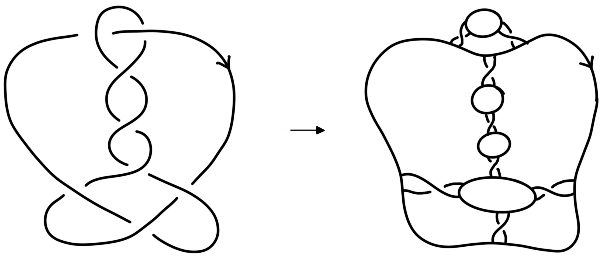

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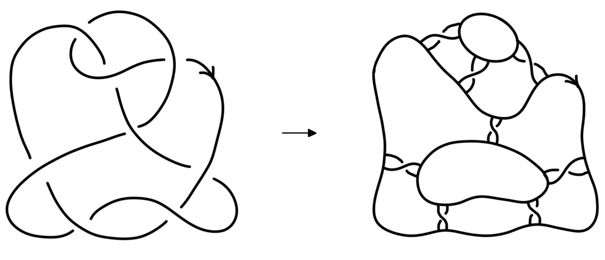

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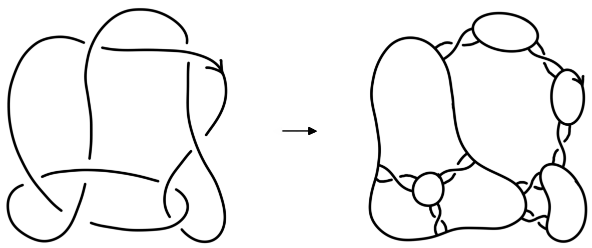

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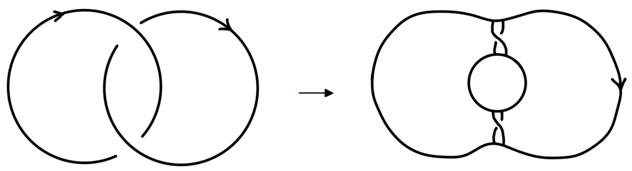

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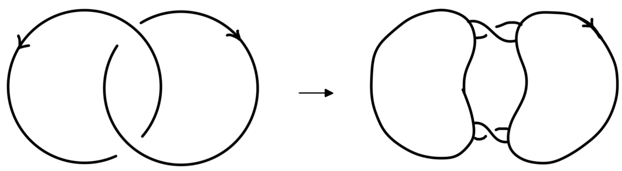

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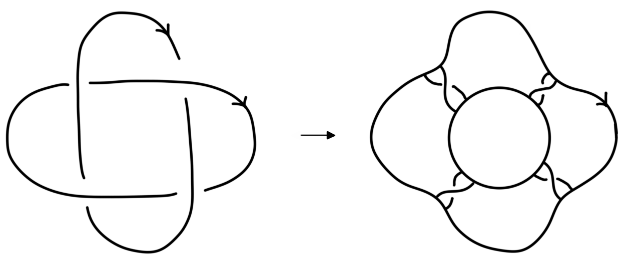

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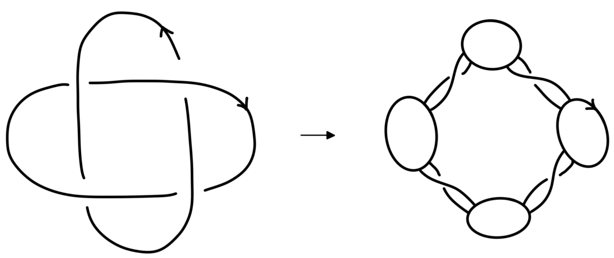

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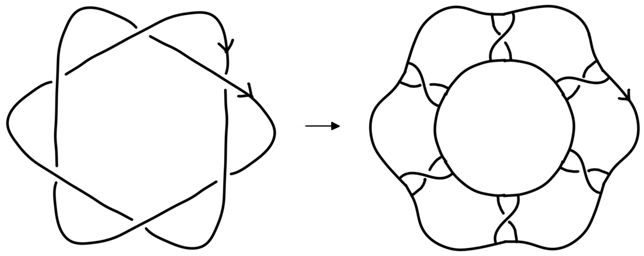

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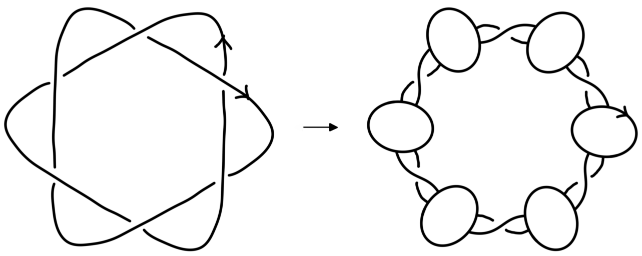

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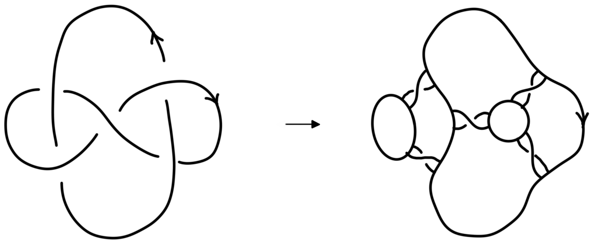

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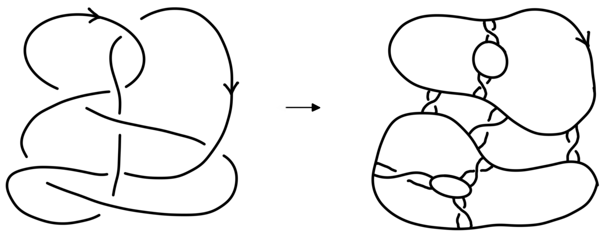

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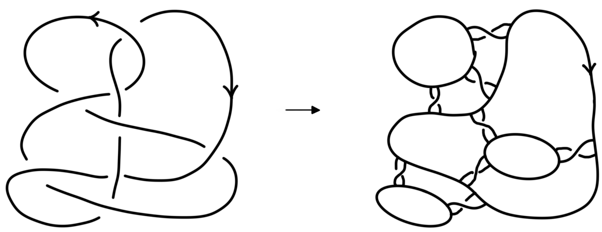

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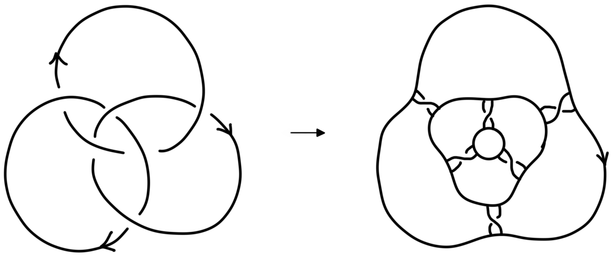

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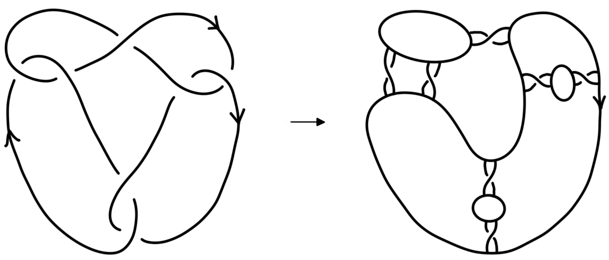

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NYILATKOZAT

Név: Somorjai Noémi Anna

ELTE Természettudományi Kar, szak: matematika BSc

NEPTUN azonosító: GQGQ4F

Szakdolgozat címe:
Seifert surfaces of alternating knots

A **szakdolgozat** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

Budapest, 2024. június 5-én

a hallgató aláírása