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Hopf algebras and their applications in linguistics

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Preface

Hopf algebras are versatile algebraic structures. They first arose in the work of Heinz Hopf in the context of algebraic topology in the 1940s. Since then, they have appeared naturally and proven useful in several further areas of mathematics, especially group theory and combinatorics. Starting with the 1990s, Hopf algebras have found applications in theoretical physics as well, becoming a central tool in perturbative quantum field theory. The latest (and perhaps most surprising) emergence of a new application of these structures is as recent as 2023, when mathematical physicist M. Marcolli and linguists R. Berwick and N. Chomsky proposed ([7, 5, 6]) a potential application for Hopf algebras in linguistics, specifically in generative syntax. They suggest that Hopf algebras may constitute a suitable framework for the description of certain transformations of sentences. The aim of this thesis is to provide a concise introduction to Hopf algebras and to briefly present their use in linguistics.

Chapter 1

Hopf algebras in mathematics

The purpose of this chapter is to define Hopf algebras and present their fundamental properties, to the extent that is necessary in order to describe their applications in transformational grammar. After introducing some preliminaries, we will develop the basics of coalgebra and Hopf algebra theory. Some theorems are given without proof for purposes of brevity, but all proofs can be found in [1] or [3].

1.1 Preliminaries

Throughout the thesis, $\mathbb{K}$ denotes some fixed field.

Tensor products

First, we need to briefly introduce (or recall) the concept of the *tensor product* of two vector spaces U and V. Here we give an explicit construction for the tensor product $U \otimes V$, but the concept can also be defined – from a category theoretical point of view – via the *universal property*, which we will define shortly. In an effort to keep this thesis self-contained and accessible to anyone with a standard undergraduate math background, we take the concrete point of view.

Informally, the tensor product gives us a way to “multiply” elements of different vector spaces U and V over the same field $\mathbb{K}$.

Definition 1.1.1 (The tensor product)**.** Let

$$L = \mathbb{K}\{(u, v) : u \in U, v \in V\}$$

be the free vector space on $U \times V$ over $\mathbb{K}$, that is to say, the elements of L are (finite) formal linear combinations of pairs (u, v).

Let R be the linear subspace in L spanned by elements of the form

$$(u_1 + u_2, v) - (u_1, v) - (u_2, v)$$
$$(u, v_1 + v_2) - (u, v_1) - (u, v_2)$$
$$(\lambda u, v) - \lambda(u, v)$$
$$(u, \lambda v) - \lambda(u, v)$$

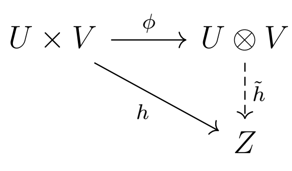
where $u, u_1, u_2 \in U$, $v, v_1, v_2 \in V$, and $\lambda \in \mathbb{K}$. Then the *tensor product* of U and V is the quotient space

$$U \otimes V = L/R,$$

and the image (equivalence class) of (u, v) in $U \otimes V$ is denoted $u \otimes v$.

The elements of $U \otimes V$ are therefore linear combinations of elements $u_i \otimes v_i$ for some $u_i \in U, v_i \in V$. Elements of the form $u \otimes v$ are called *pure tensors*. Note that not everything in $U \otimes V$ is a pure tensor! Some elements just cannot be expressed in this simple form.

Proposition 1.1.2 (The universal property)**.** *Let $\phi : U \times V \to U \otimes V$ be the function that maps (u, v) to $u \otimes v$. This map is clearly bilinear, by the construction of the tensor product. Now if $h : U \times V \to Z$ is any bilinear map, then there exists a* unique linear map $\tilde{h} : U \otimes V \to Z$ *such that $h = \tilde{h} \circ \phi$, in other words, $\tilde{h}$ makes the following diagram commutative:*



The proof follows directly from our construction. This is a crucial property of the tensor product – so crucial that in fact it is often used to *define* the tensor product of two vector spaces. In this approach, $U \otimes V$ is defined as the unique vector space together with a bilinear map ϕ which fulfills the universal property. Then we could forget about the explicit construction that proved its existence, and deduce every property of $U \otimes V$ from the universal property. We will make use of the universal property repeatedly, however, we still find it useful to think of the elements of the tensor product space as linear combinations of pure tensors, keeping in mind the bilinearity which arose from factorizing with the corresponding relations.

A consequence of the universal property is that we get a practical way of defining linear maps on $U \otimes V$: if we want a *linear* map $U \otimes V \to Z$, simply define a *bilinear* map h from $U \times V$ to Z, and the universal property will guarantee that h factors through $U \otimes V$, providing a suitable $\tilde{h}$. We will soon see two examples of this application.

Now let us summarize a few basic properties of tensor products without proof.

1. If U and V are finite-dimensional vector spaces, $\dim U = n$ and $\dim V = m$, then $U \otimes V$ is also finite-dimensional, and $\dim U \otimes V = n \cdot m$.

2. The tensor product is associative in the following sense:

   If U, V, W are vector spaces, then there exists a canonical isomorphism

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W).$$

   This allows us to identify the two and denote this space with the triple tensor product $U \otimes V \otimes W$.

3. The tensor product of vector spaces is commutative in the sense that there exists a canonical isomorphism

$$U \otimes V \cong V \otimes U.$$

   However, note that the tensor product of *vectors* is not commutative, i.e. $u \otimes v \neq v \otimes u$ in general.

Let us now suppose that we have two linear maps, $f : U \to U'$ and $g : V \to V'$. We want to define *their* tensor product, that is, a linear map from $U \otimes V$ to $U' \otimes V'$ induced by f and g. The universal property will provide this map for us. With the above notations, let $Z = U' \otimes V'$, and define $h : U \times V \to U' \otimes V'$ to be $h(u, v) = f(u) \otimes g(v)$. This map is bilinear due to the linearity of f and g and the bilinearity of the tensor product. Hence it induces a unique linear map from $U \otimes V$ to $U' \otimes V'$, which we denote by $f \otimes g$. If f and g are both injective or both surjective, then the same is true for $f \otimes g$.

Algebras

The main idea for an algebra is that we want a structure that is simultaneously a $\mathbb{K}$-vector space and a ring. Let us see how we can formalize this.

Let A be a ring with identity.

Definition 1.1.3 (The naïve definition of an algebra)**.** A is a $\mathbb{K}$-algebra if $\mathbb{K}$ is a subset of the center $Z(A)$ and $1_{\mathbb{K}} = 1_A$.

These are reasonable assumptions if we want A to behave like a vector space: the conditions guarantee that we can multiply elements of A by scalars in $\mathbb{K}$, and also that scalars commute with all elements of the ring.

Example 1.1.4. $A = \mathbb{K}[x]$ is a $\mathbb{K}$-algebra, since we can identify $\mathbb{K}$ with constant polynomials, which commute with all polynomials. The same is true for $\mathbb{K}[x_1, \ldots, x_n]$.

What about the ring of n-by-n matrices $\mathbb{K}^{n\times n}$? Is this a $\mathbb{K}$-algebra? Unfortunately, by our definition above, it is not: $\mathbb{K}$ is not technically a subset of $Z(\mathbb{K}^{n\times n})$, even though we feel that if we could somehow identify every scalar λ with the corresponding diagonal matrix λI, the conditions would be fulfilled. This motivates the following definition:

Definition 1.1.5 (The actual definition of an algebra)**.** A is a $\mathbb{K}$-algebra given a ring homomorphism $\eta : \mathbb{K} \to A$ such that $\eta(\mathbb{K}) \subseteq Z(A)$ and $\eta(1_{\mathbb{K}}) = 1_A$. η is called the *unit*.

Note that whenever η is a ring homomorphism on $\mathbb{K}$, it is automatically injective, so $\mathbb{K} \cong \eta(\mathbb{K})$. This definition does make A into a vector space, since we can now multiply elements of A by a scalar via $\lambda a := \eta(\lambda) \cdot a$.

Now $\mathbb{K}^{n\times n}$ is indeed a $\mathbb{K}$-algebra with $\eta : \lambda \mapsto \lambda I$.

Remark 1.1.6. Depending on how we define polynomials, we may also need a ring homomorphism η in the case of $\mathbb{K}[x]$ to embed $\mathbb{K}$ into $\mathbb{K}[x]$. If we define the elements of $\mathbb{K}[x]$ to be sequences with terms in $\mathbb{K}$ which are eventually zero, then $\eta : \lambda \mapsto (\lambda, 0, \ldots)$ serves this purpose. However, if we think of polynomials as formal expressions $a_0 + a_1x + \ldots + a_nx^n$, then it is very natural to think of the scalar λ and the constant polynomial λ as the same thing.

There exists an equivalent formulation of this definition in terms of *commutative diagrams*. This will be very useful for us here since Hopf algebras are most easily formulated using commutative diagrams.

Before we give the diagrammatic definition, a short remark is needed on *the tensor product of algebras*. If A and B are $\mathbb{K}$-algebras, then they can be regarded as $\mathbb{K}$-vector spaces (via η), hence their tensor product $A \otimes B$ is defined and is a vector space as well. We can define a multiplication on $A \otimes B$ by letting

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$$

and extending linearly. This product provides a ring structure on $A \otimes B$ (with identity element $1_A \otimes 1_B$), making it into a $\mathbb{K}$-algebra.

Definition 1.1.7 (The diagrammatic definition of an algebra)**.** An *(associative unital) algebra* over a field $\mathbb{K}$ is a vector space A over $\mathbb{K}$ equipped with linear maps $m : A \otimes A \to A$ (multiplication) and $\eta : \mathbb{K} \to A$ (unit) such that the following diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes id} & A \otimes A \\ {\scriptstyle id \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{m} & A \end{array} \qquad \begin{array}{ccccc} & \overset{\eta \otimes id}{\nearrow} & A \otimes A & \overset{id \otimes \eta}{\nwarrow} & \\ \mathbb{K} \otimes A & & \downarrow{\scriptstyle m} & & A \otimes \mathbb{K} \\ & \underset{\cong}{\searrow} & A & \underset{\cong}{\swarrow} & \end{array}$$

Here the intuition is that the first diagram describes the associativity of the multiplication map m, while the second diagram describes the same map η as above that provides a way to embed $\mathbb{K}$ into A.

A brief note is due on what $\cong$ means here. Every element of $\mathbb{K} \otimes A$ is a pure tensor, since

$$\sum_{i=1}^{n} \lambda_i \otimes a_i = \sum_{i=1}^{n} (\lambda_i \cdot 1_{\mathbb{K}}) \otimes a_i = \sum_{i=1}^{n} 1_{\mathbb{K}} \otimes \lambda_i a_i = 1_{\mathbb{K}} \otimes \sum_{i=1}^{n} \lambda_i a_i = 1_{\mathbb{K}} \otimes a.$$

Then, on the left side of the second diagram, $\cong$ is simply the canonical isomorphism $\tilde{h}$ from $\mathbb{K} \otimes A$ to A which sends $1_{\mathbb{K}} \otimes a$ to a. The existence of this map is guaranteed by the universal property: $h : \mathbb{K} \times A \to A, \quad (1_{\mathbb{K}}, a) \mapsto a$ is clearly an isomorphism which extends to $\tilde{h}$. An isomorphism between $A \otimes \mathbb{K}$ and A can be defined analogously.

Proposition 1.1.8. *Definitions (1.1.5) and (1.1.7) are equivalent.*

Proof. (1.1.7) $\implies$ (1.1.5): This direction follows from a mostly mechanical checking of the needed properties.

- <u>A is a ring</u>: Addition on A is given by the vector space structure. Define a multiplication on A by
$$a \cdot b = m(a \otimes b).$$
This product is associative since
$$(a \cdot b) \cdot c = m(a \otimes b) \cdot c = m(m(a \otimes b) \otimes c),$$
$$a \cdot (b \cdot c) = a \cdot m(b \otimes c) = m(a \otimes m(b \otimes c)),$$
and the first diagram states precisely that these two are equal. The identity element is $\eta(1_{\mathbb{K}})$ since
$$\eta(1_{\mathbb{K}}) \cdot a = m(\eta(1_{\mathbb{K}}) \otimes a) = a = a \cdot \eta(1_{\mathbb{K}})$$
holds by the second diagram. Distributivity is guaranteed by the bilinearity of the tensor product and the linearity of m:
$$a \cdot (b + c) = m(a \otimes (b + c)) = m(a \otimes b + a \otimes c) =$$
$$= m(a \otimes b) + m(a \otimes c) = a \cdot b + a \cdot c,$$
and right-distributivity can be checked the exact same way.

- <u>η is a ring homomorphism</u>: η clearly preserves addition due to its linearity. To see that it also preserves multiplication, notice that since $\eta(\mu) \in A$,
$$\eta(\lambda) \cdot \eta(\mu) = m(\eta(\lambda) \otimes \eta(\mu)) = m((\eta \otimes id)(\lambda \otimes \eta(\mu))) =$$
$$= m((\eta \otimes id)(1_{\mathbb{K}} \otimes \lambda\eta(\mu))) = \lambda\eta(\mu) = \eta(\lambda\mu)$$
holds by the second diagram and the linearity of η. $\eta(1_{\mathbb{K}}) = 1_A$ holds by definition.

- <u>$\eta(\mathbb{K}) \subseteq Z(A)$</u>: Using, again, the second diagram,
$$\eta(\lambda) \cdot a = m(\eta(\lambda) \otimes a) = \lambda a = m(a \otimes \eta(\lambda)) = a \cdot \eta(\lambda).$$

(1.1.5) $\implies$ (1.1.7): This direction is a little bit trickier since at one point we will need to invoke the universal property. Otherwise, we are practically doing the same as above but backwards.

- A is a vector space: Addition on A is given by the ring structure. Define multiplication by a scalar by

  $$\lambda a = \eta(\lambda) \cdot a.$$

  To verify the vector space axioms, notice that

  $$\begin{aligned}\lambda(\mu a) = \lambda(\eta(\mu) \cdot a) = \eta(\lambda) \cdot (\eta(\mu) \cdot a) = (\eta(\lambda) \cdot \eta(\mu)) \cdot a = \\ = \eta(\lambda\mu) \cdot a = (\lambda\mu)a\end{aligned}$$

  holds due to associativity in the ring and the fact that η is a ring homomorphism. Furthermore, note that

  $$1_{\mathbb{K}} a = \eta(1_{\mathbb{K}}) \cdot a = 1_A \cdot a = a.$$

  The two "distributivity" axioms follow from distributivity in the ring and again from η being a ring homomorphism:

  $$\begin{aligned}\lambda(a+b) = \eta(\lambda) \cdot (a+b) = \eta(\lambda) \cdot a + \eta(\lambda) \cdot b = \lambda a + \lambda b, \\ (\lambda + \mu)a = \eta(\lambda + \mu) \cdot a = (\eta(\lambda) + \eta(\mu)) \cdot a = \\ = \eta(\lambda) \cdot a + \eta(\mu) \cdot a = \lambda a + \mu a.\end{aligned}$$

- There exists a linear map m which satisfies the first diagram:

  We need to define a *linear* map on $A \otimes A$. The multiplication map $\cdot : A \times A \to A$ is bilinear since it is clearly additive in both arguments by distributivity, and

  $$(\lambda a) \cdot b = (\eta(\lambda) \cdot a) \cdot b = \eta(\lambda) \cdot (a \cdot b) = \lambda(a \cdot b).$$

  Therefore by the universal property, $\cdot$ induces a corresponding linear map m defined by

  $$m(a \otimes b) = a \cdot b.$$

To see that m does satisfy the property described by the first diagram, notice that the expressions

$$m(m(a \otimes b) \otimes c) = m((a \cdot b) \otimes c) = (a \cdot b) \cdot c$$
$$m(a \otimes m(b \otimes c)) = m(a \otimes (b \cdot c)) = a \cdot (b \otimes c)$$

are clearly equal, which is what we wanted.

- η is linear and satisfies the second diagram: η as a ring homomorphism clearly preserves addition. It preserves multiplication by a scalar, as

$$\eta(\lambda\mu) = \eta(\lambda) \cdot \eta(\mu) = \lambda\eta(\mu).$$

(Note that here we think of λ as a scalar and μ as an element of the vector space $\mathbb{K}$.) The diagram is satisfied since

$$m(\eta(1_{\mathbb{K}}) \otimes a) = m(1_A \otimes a) = 1_A \cdot a = a = m(a \otimes \eta(1_{\mathbb{K}})).$$

□

Example 1.1.9 (The group algebra)**.** Let G be a finite group, with the group multiplication simply written as juxtaposition. Consider the free vector space on G over $\mathbb{K}$ denoted by $\mathbb{K}G$, that is,

$$\mathbb{K}G = \left\{ \sum_{g \in G} \alpha_g g : \ \alpha_g \in \mathbb{K} \right\}$$

is the vector space of formal linear combinations of the elements of G, where addition and multiplication by a scalar are defined naturally. We can define multiplication (denoted $\cdot$ for readability, but will be referred to as $m_{\mathbb{K}G}$) on $\mathbb{K}G$ *using the group multiplication* in G in the following way: The set $\{1_{\mathbb{K}}g : \ g \in G\}$ forms a basis of $\mathbb{K}G$. We let $1_{\mathbb{K}}g \cdot 1_{\mathbb{K}}h = 1_{\mathbb{K}}gh$ for $g, h \in G$, and then extend linearly to the other elements of $\mathbb{K}G$. This product satisfies the ring axioms, thus $\mathbb{K}G$ is also a ring. Taking the canonical ring homomorphism $\eta_{\mathbb{K}G} : \mathbb{K} \to \mathbb{K}G$, $\lambda \mapsto \lambda 1_G$, we get an algebra structure $(\mathbb{K}G, m_{\mathbb{K}G}, \eta_{\mathbb{K}G})$ called the *group algebra*.

What is the *dual space* of $\mathbb{K}G$, i.e. the vector space of linear functionals on $\mathbb{K}G$? Linear maps $\mathbb{K}G \to \mathbb{K}$ correspond exactly to arbitrary functions $G \to \mathbb{K}$. Since for finite-dimensional vector spaces, V is isomorphic to V^*, here we obtain that

$\mathbb{K}G \cong \mathbb{K}^G = \{\phi : G \to \mathbb{K}\}$, and the map $f : g \mapsto \delta_g(x)$ (where δ_g is the Kronecker delta function on G), when extended linearly, defines an isomorphism of vector spaces between them.

$\mathbb{K}^G$ is also an algebra over $\mathbb{K}$: for instance, we can utilize the pointwise product of functions $\phi, \psi \in \mathbb{K}^G$, and let $m_{\mathbb{K}^G}(\phi \otimes \psi) = \phi \cdot \psi \in \mathbb{K}^G$ be the function $g \mapsto \phi(g)\psi(g)$. Note that this construction makes absolutely no use of the group structure. With unit $\eta_{\mathbb{K}^G} : \mathbb{K} \to \mathbb{K}^G$, $\eta_{\mathbb{K}^G}(1_{\mathbb{K}}) = \psi \equiv 1_{\mathbb{K}}$, the identically $1_{\mathbb{K}}$ functional on $\mathbb{K}G$, we obtain an algebra structure $(\mathbb{K}^G, m_{\mathbb{K}^G}, \eta_{\mathbb{K}^G})$.

Lie algebras

Definition 1.1.10. A *Lie algebra* is a vector space $\mathfrak{g}$ over $\mathbb{K}$, together with a map $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the *Lie bracket* such that

1. $[.,.]$ is bilinear,
2. $[.,.]$ is skew-symmetric, i.e. $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$, and
3. the *Jacobi identity* holds:
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$
for all $x, y, z \in \mathfrak{g}$.

Note that (2) readily implies that $[x, x] = 0$ for all $x \in \mathfrak{g}$.

Example 1.1.11. $\mathfrak{g} = \mathbb{R}^3$ with $[x, y] = x \times y$ (the cross product) is a Lie algebra.

Example 1.1.12. Any associative algebra A can be made into a Lie algebra by taking the Lie bracket to be
$$[x, y] = xy - yx$$
for all $x, y \in A$.

Proof. Bilinearity and skew-symmetry are evident from the definition. The Jacobi identity follows directly from the associativity of the algebra, since

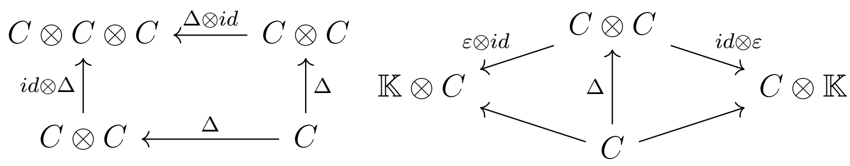
$$[x, [y, z]] = x(yz) - x(zy) - (yz)x + (zy)x$$
$$[y, [z, x]] = y(zx) - y(xz) - (zx)y + (xz)y$$
$$[z, [x, y]] = z(xy) - z(yx) - (xy)z + (yx)z.$$

□

1.2 Elementary coalgebra theory

In Definition 1.1.7 of the previous chapter, we saw that the defining properties of algebras can be captured by two commutative diagrams. Dualizing these diagrams (that is, flipping all the arrows), we get the following definition for a *coalgebra*:

Definition 1.2.1. A *(coassociative counital) coalgebra* over a field $\mathbb{K}$ is a vector space C over $\mathbb{K}$ equipped with linear maps $\Delta : C \to C \otimes C$ (comultiplication) and $\varepsilon : C \to \mathbb{K}$ (counit) such that the following diagrams commute:



Here $\Delta(c)$ is an element of $C \otimes C$, so it is of the form $\sum_i c_i^{(1)} \otimes c_i^{(2)}$. We will often use the following notation introduced by Sweedler [8]:

$$\Delta(c) = \sum_i c_i^{(1)} \otimes c_i^{(2)} =: c^{(1)} \otimes c^{(2)}.$$

Note that this is a slight abuse of notation, since $\Delta(c)$ is, most of the time, not a pure tensor. However, this abbrevation is analogous to the widely used Einstein summation convention in physics, and will be useful for our purposes.

Let us denote the map $(\Delta \otimes Id) \circ \Delta$ by Δ_3. Iterating Δ, we can recursively define the map

$$\Delta_n = (\Delta \otimes Id^{\otimes n-2}) \circ \Delta_{n-1}$$

with the notation $\Delta_2 = \Delta$. The coassociativity property guarantees that this is a unique map from C to $C^{\otimes n}$, i.e., for all $0 \leq i \leq n-2$,

$$\Delta_n = (Id^{\otimes i} \otimes \Delta \otimes Id^{\otimes n-i-2}) \circ \Delta_{n-1}.$$

This allows us to, using Sweedler notation, denote the elements of the image of the iterated coproduct by

$$\Delta_n(c) = c^{(1)} \otimes \ldots \otimes c^{(n)}.$$

Example 1.2.2. The ground field $\mathbb{K}$ is a coalgebra with the maps $\Delta = \varepsilon = Id : \mathbb{K} \to \mathbb{K} = \mathbb{K} \otimes \mathbb{K}$.

Proposition 1.2.3. *Let C be any coalgebra. Then its dual C^* is an algebra.*

Proof. The proof depends on the crucial fact that there is a canonical embedding of $C^* \otimes C^*$ into $(C \otimes C)^*$: if $\phi, \psi \in C^*$ are linear functionals on C, let the image of $\phi \otimes \psi$ be the linear map from $C \otimes C$ to $\mathbb{K}$ that sends $c \otimes d$ to $\phi(c)\psi(d)$. Composing this embedding with $\Delta' : (C \otimes C)^* \to C^*$, the dual map of Δ_C, defines a product on C^*. (Recall that the *dual map* of a linear map $T : U \to V$ is the linear map $T' : V^* \to U^*$ defined by $T'(f) = f \circ T$.) This product, by construction, can be written as the *convolution product*

$$\phi * \psi = m_{\mathbb{K}} \circ (\phi \otimes \psi) \circ \Delta_C$$

where $m_{\mathbb{K}}$ is multiplication in $\mathbb{K}$. As for the unit of C^* we need a linear map $\eta : \mathbb{K} \to C^*$, we can simply take $\eta(\lambda) = \lambda\varepsilon$. Associativity and the properties of the unit follow by duality. □

Remark 1.2.4*.* The converse is only true in finite dimensions: The dual of a *finite dimensional* algebra is a coalgebra.

Proof. We do the exact same construction as above but in reverse: if m is the multiplication map of the algebra A, then its dual m' is a a linear map from A^* to $(A \otimes A)^*$. If A is finite dimensional, then since $(A \otimes A)^*$ and $A^* \otimes A^*$ share the same dimension, they are isomorphic (this is definitely not guaranteed in infinite dimensions!), and the above embedding provides an isomorphism between them. Again, composing m' with this embedding gives a comultiplication map on A^*. $\varepsilon : A^* \to \mathbb{K} \cong \mathbb{K}^*$ is simply the dual map of $\eta : \mathbb{K} \to A$. □

Let us return to our main example from the last chapter, the group algebra. We will show that a coalgebra structure can also be defined on this.

Example 1.2.5 (The group coalgebra)**.** Recall that we have previously (in Example 1.1.9) introduced two ways of looking at the group algebra: either as $\mathbb{K}G$ with multiplication defined by the product in G or as the space $\mathbb{K}^G$ of functions from G to $\mathbb{K}$ with multiplication defined by the pointwise product of functions. Since we purposefully took a *finite* group G, our group algebra A is finite dimensional, so by the previous remark, its dual A^* is a coalgebra.

Let us first consider A as $\mathbb{K}G$ and describe the coalgebra structure thus induced on $(\mathbb{K}G)^* = \mathbb{K}^G$. The counit is simply the dual of the unit $\eta_{\mathbb{K}G}$,

$$\eta'_{\mathbb{K}G} = \varepsilon_{\mathbb{K}G} : (\mathbb{K}G)^* \to \mathbb{K}^*.$$

We may identify $\mathbb{K}$ with $\mathbb{K}^*$ via the isomorphism $\lambda \mapsto (1_{\mathbb{K}} \mapsto \lambda)$. Hence the above map – explicitly – is the linear map such that $\varepsilon_{\mathbb{K}G}(\delta_{1_G}) = 1$ and $\varepsilon_{\mathbb{K}G}(\delta_g) = 0$ for all $g \neq 1_G$. We get comultiplication similarly, via the dual map of $m_{\mathbb{K}G}$. $m'_{\mathbb{K}G}$ is a linear map from $(\mathbb{K}G)^*$ to $(\mathbb{K}G \otimes \mathbb{K}G)^* \cong (\mathbb{K}G)^* \otimes (\mathbb{K}G)^*$, where $\cong$ is given by the isomorphism $i : \delta_{x\otimes y} \mapsto \delta_x \otimes \delta_y$. What is this explicitly? If the image of δ_g (for some $g \in G$) under $m'_{\mathbb{K}G}$ is of the form $m'_{\mathbb{K}G}(\delta_g) = \sum\limits_{x,y\in G} \alpha_{x,y}\delta_{x\otimes y} \in (\mathbb{K}G \otimes \mathbb{K}G)^*$, then notice that, from the definition of the dual map, $\alpha_{x,y} = (m'_{\mathbb{K}G}(\delta_g))(x \otimes y) = \delta_g(m_{\mathbb{K}G}(x \otimes y)) = \delta(xy)$. Hence the only nonzero terms in the sum are the ones where $g = xy$, or equivalently, $y = x^{-1}g$. This shows that

$$\Delta_{\mathbb{K}^G}(\delta_g) = (i \circ m'_{\mathbb{K}G})(\delta_g) = i\left(\sum_{x\in G} \delta_{x\otimes x^{-1}g}\right) = \sum_{x\in G} \delta_x \otimes \delta_{x^{-1}g}.$$

Now let us take $A = \mathbb{K}^G$, and see what coalgebra structure $m_{\mathbb{K}^G}$ defines on $\mathbb{K}G$. The counit is the dual of $\eta_{\mathbb{K}^G}$,

$$\eta'_{\mathbb{K}^G} = \varepsilon_{\mathbb{K}^G} : \mathbb{K}G = \left(\mathbb{K}^G\right)^* \to \mathbb{K}^* = \mathbb{K}.$$

Explicitly, this is the trivial linear map such that $\varepsilon_{\mathbb{K}G}(g) = 1$ for all $g \in G$. Following the exact same construction as above, $m'_{\mathbb{K}^G} : \mathbb{K}G = \left(\mathbb{K}^G\right)^* \to \left(\mathbb{K}^G \otimes \mathbb{K}^G\right)^* \cong \left(\mathbb{K}^G\right)^* \otimes \left(\mathbb{K}^G\right)^* = \mathbb{K}G \otimes \mathbb{K}G$ is the dual map, where $\cong$ is given by $j : \delta_{\delta_g \otimes \delta_h} \mapsto g \otimes h$. $m'_{\mathbb{K}^G}(g)$ is of the form $\sum\limits_{x,y\in G} \beta_{x,y}\delta_{\delta_x\otimes\delta_y} \in \left(\mathbb{K}^G \otimes \mathbb{K}^G\right)^*$ with $\beta_{x,y} = (m'_{\mathbb{K}^G}(g))(\delta_x \otimes \delta_y) = m_{\mathbb{K}^G}(\delta_x \otimes \delta_y)(g) = \delta_x(g)\delta_y(g)$. Hence almost all terms are zero, and the sum simplifies to

$$\Delta_{\mathbb{K}G}(g) = \left(j \circ m'_{\mathbb{K}^G}\right)(g) = g \otimes g.$$

Note that this last construction – of $\Delta_{\mathbb{K}G}$ and $\varepsilon_{\mathbb{K}G}$ – did not use the group structure at all, meaning that if instead of a group G we take an arbitrary set S, we may construct a coalgebra structure on $\mathbb{K}S$ in the same fashion. This motivates the following definition:

Definition 1.2.6 (Group-like elements)**.** An element c of a coalgebra C is *group-like* if $\varepsilon(c) = 1$ and $\Delta(c) = c \otimes c$. The set of group-like elements is denoted $\Gamma(C)$.

We will see later that $\Gamma(C)$ is actually a group when C is a Hopf algebra.

Definition 1.2.7 (Primitive elements)**.** An element c of a coalgebra C is *primitive* if $\Delta(c) = c \otimes 1 + 1 \otimes c$.

Proposition 1.2.8 (Convolution)**.** *Let (A, m, η) be an algebra and (C, Δ, ε) be a coalgebra. Then the set $Hom(C, A)$ of linear maps from C to A is an associative algebra with unit $\eta \circ \varepsilon$ by the* convolution product

$$f * g = m \circ (f \otimes g) \circ \Delta.$$

Proof. First of all, $*$ is a linear map because it is a composition of linear maps.

Take $f, g, h \in Hom(C, A)$. Then

$$\begin{aligned}
((f * g) * h) &= m \circ ((f * g) \otimes h) \circ \Delta = \\
&= m \circ ((m \circ (f \otimes g) \circ \Delta) \otimes h) \circ \Delta = \\
&= m \circ (m \otimes Id_A) \circ (f \otimes g \otimes h) \circ (\Delta \otimes Id_C) \circ \Delta = \\
&= m_3 \circ (f \otimes g \otimes h) \circ \Delta_3 = \\
&= (f * (g * h))
\end{aligned}$$

where m_n and Δ_n denote the compositions of m and Δ $(n-1)$ times (maps $A^{\otimes n} \to A$ and $C \to C^{\otimes n}$ respectively). This proves the associativity of $*$.

As for the unit, we need to show that for all $f \in Hom(C, A)$,

$$f * (\eta \circ \varepsilon) = f = (\eta \circ \varepsilon) * f.$$

Indeed,

$$\begin{aligned}
f * (\eta \circ \varepsilon) &= m \circ (f \otimes (\eta \circ \varepsilon)) \circ \Delta = \\
&= m \circ (Id_A \otimes \eta) \circ (f \otimes Id_{\mathbb{K}}) \circ (Id_C \otimes \varepsilon) \circ \Delta = \\
&= f = (\eta \circ \varepsilon) * f.
\end{aligned}$$

To see why this is the case, and where we used the algebra and coalgebra structures of A and C, perhaps it is useful to look at what the above map does to an element in C. For any $c \in C$,

$$(f * (\eta \circ \varepsilon))(c) = (m \circ (Id_A \otimes \eta) \circ (f \otimes Id_{\mathbb{K}}))(((Id_C \otimes \varepsilon) \circ \Delta)(c))$$

which, by the unitary property of C, is equal to

$$(m \circ (Id_A \otimes \eta) \circ (f \otimes Id_{\mathbb{K}}))(c \otimes 1_{\mathbb{K}}) = (m \circ (Id_A \otimes \eta))(f(c) \otimes 1_{\mathbb{K}}).$$

Now, by the unitary property of A, this equals $f(c)$, which was what we wanted. □

Remark 1.2.9. The canonical map $A \otimes C^* \to Lin(C, A)$ is a morphism of algebras, and an isomorphism when A or C is finite dimensional.

Now we introduce the concept of *graded* algebras and coalgebras, which is fundamental in all of the examples we will consider later.

Definition 1.2.10. A vector space V is *graded* if it decomposes as a direct sum of vector spaces $V = \bigoplus_{n \in \mathbb{N}} V_n$. It is *reduced* if $V_0 = 0$.

An algebra A is *graded* if it is a graded vector space with the additional condition that the multiplication map m sends $A_k \otimes A_\ell$ to A_{k+l}. Similarly, a coalgebra C is *graded* if Δ sends C_n to $\bigoplus_{k+\ell=n} C_k \otimes C_\ell$.

The classical example of a graded algebra is the polynomial ring $\mathbb{K}[x]$, where the grading is provided by the degree of the polynomials.

Remark 1.2.11. Note that $\eta(1_{\mathbb{K}}) \in A_0$, because if η took $1_{\mathbb{K}}$ to A_n for some $n > 0$, then, since $m(a_k \otimes \eta(1_{\mathbb{K}})) = a_k$ for all $k \in \mathbb{N}$, $a_k \in A_k$, $m(A_k \otimes A_n) \subseteq A_{k+n}$ would not hold. Therefore the unit η maps $\mathbb{K}$ into A_0.

Dually, let us show that the counit map ε has to take everything of degree higher than zero to 0. Let us take an element $x \in C_n$ for some $n \geq 1$. Then its image under the coproduct is of the form $\Delta(x) = \sum_{k+\ell=n} c_k^{(1)} \otimes c_\ell^{(2)}$ with $c_k^{(i)} \in C_k$ and $c_\ell^{(i)} \in C_\ell$, $i = 1, 2$. By the counitary property (see the right side of the diagram below), $(id \otimes \varepsilon)(\Delta(x)) = \sum_{k+\ell=n} c_k^{(1)} \otimes \varepsilon\left(c_\ell^{(2)}\right)$ is in $C_n \otimes \mathbb{K}$, more precisely, is equal to $x \otimes 1_{\mathbb{K}}$. Hence we see that $c_n^{(1)} = x$ and $\varepsilon\left(c_0^{(2)}\right) = 1_{\mathbb{K}}$. (Furthermore, for all nonzero terms $c_k^{(1)} \otimes c_{n-k}^{(2)}$ $(0 \leq k < n)$, we have $\varepsilon\left(c_{n-k}^{(2)}\right) = 0$.) Now, looking at the left side of the diagram, we see that $\sum_{k+\ell=n} \varepsilon\left(c_k^{(1)}\right) \otimes c_\ell^{(2)} \in \mathbb{K} \otimes C_n$. What is important to us here is specifically that now we can conclude that $0 = \varepsilon\left(c_n^{(1)}\right) = \varepsilon(x)$, which was what we wanted.

$$\begin{array}{ccccc} & & \sum\limits_{k+\ell=n} c_k^{(1)} \otimes c_\ell^{(2)} & & \\ & \overset{\varepsilon\otimes id}{\swarrow} & \Big\uparrow \Delta & \overset{id\otimes\varepsilon}{\searrow} & \\ \mathbb{K}\otimes C_n \ni \sum\limits_{k+\ell=n} \varepsilon\left(c_k^{(1)}\right)\otimes c_\ell^{(2)} & & & & \sum\limits_{k+\ell=n} c_k^{(1)} \otimes \varepsilon\left(c_\ell^{(2)}\right) \in C_n\otimes\mathbb{K} \\ & \nwarrow & & \nearrow & \\ & & x \in C_n & & \end{array}$$

Definition 1.2.12. A graded (co)algebra is *connected* if (C_0) $A_0 = \mathbb{K}$.

Example 1.2.13 (The polynomial (co)algebra)**.** The polynomials $\mathbb{K}[x]$ naturally form an algebra. A coalgebra structure can be defined on $\mathbb{K}[x]$ by

$$\Delta(x^n) = (1\otimes x + x\otimes 1)^n = \sum_{k=1}^{n} \binom{n}{k} x^k \otimes x^{n-k},$$
$$\varepsilon(1) = 1, \quad \varepsilon(x) = 0.$$

This structure is a graded connected algebra and coalgebra.

1.3 Hopf algebras

Definition 1.3.1. A *bialgebra* is a 5-tuple $(B, m, \eta, \Delta, \varepsilon)$ such that

- (B, m, η) is an algebra;
- (B, Δ, ε) is a coalgebra;
- The algebra and coalgebra structures are 'compatible' in some sense: Δ and ε are morphisms of algebras, or equivalently, m and η are morphisms of coalgebras. This equivalence is best expressed diagrammatically: both conditions translate into the commutativity of the diagrams

$$\begin{array}{ccccc} B\otimes B & \xrightarrow{m} & B & \xrightarrow{\Delta} & B\otimes B \\ {\scriptstyle \Delta\otimes\Delta}\Big\downarrow & & & & \Big\uparrow{\scriptstyle m\otimes m} \\ B^{\otimes 4} & & \xrightarrow{Id\otimes T\otimes Id} & & B^{\otimes 4} \end{array}$$

$$\begin{array}{ccc} B\otimes B & \xrightarrow{m} & B \\ {\scriptstyle \varepsilon\otimes\varepsilon}\Big\downarrow & & \Big\downarrow{\scriptstyle \varepsilon} \\ \mathbb{K}\otimes\mathbb{K} & \xrightarrow{\cong} & \mathbb{K} \end{array} \quad \begin{array}{ccc} \mathbb{K} & \xrightarrow{\eta} & B \\ {\scriptstyle \cong}\Big\downarrow & & \Big\downarrow{\scriptstyle \Delta} \\ \mathbb{K}\otimes\mathbb{K} & \xrightarrow{\eta\otimes\eta} & B\otimes B \end{array} \quad \begin{array}{ccc} \mathbb{K} & \xrightarrow{\eta} & B \\ {\scriptstyle =}\Big\downarrow & & \Big\downarrow{\scriptstyle \varepsilon} \\ \mathbb{K} & \xrightarrow{=} & \mathbb{K} \end{array}$$

where $T(a\otimes b) = b\otimes a$ for $a, b\in B$.

Here the first and second (resp. third and fourth) diagrams express that m (resp. η) is a coalgebra morphism, while the first and third (resp. second and fourth) express that Δ (resp. ε) is an algebra morphism.

Example 1.3.2. $\mathbb{K}[x]$ with algebra and coalgebra structure as per Example 1.2.13 is a bialgebra.

Example 1.3.3. $(\mathbb{K}G, m_{\mathbb{K}G}, \eta_{\mathbb{K}G}, \Delta_{\mathbb{K}G}, \varepsilon_{\mathbb{K}G})$ forms a bialgebra, as shown by the commutativity of the following diagrams. (We have omitted the indices for readability.)

$$\begin{array}{ccccc}
g \otimes h & \xrightarrow{m} & gh & \xrightarrow{\Delta} & gh \otimes gh \\
{\scriptstyle \Delta\otimes\Delta}\downarrow & & & & \uparrow{\scriptstyle m\otimes m} \\
g \otimes g \otimes h \otimes h & & \xrightarrow{Id\otimes T\otimes Id} & & g \otimes h \otimes g \otimes h
\end{array}$$

$$\begin{array}{ccc}
g \otimes h & \xrightarrow{m} & gh \\
{\scriptstyle \varepsilon\otimes\varepsilon}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\
1_{\mathbb{K}} \otimes 1_{\mathbb{K}} & \xrightarrow{\cong} & 1_{\mathbb{K}}
\end{array}
\qquad
\begin{array}{ccc}
1_{\mathbb{K}} & \xrightarrow{\eta} & 1_{\mathbb{K}}1_G \\
{\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \Delta} \\
1_{\mathbb{K}} \otimes 1_{\mathbb{K}} & \xrightarrow{\eta\otimes\eta} & 1_G \otimes 1_G
\end{array}
\qquad
\begin{array}{ccc}
1_{\mathbb{K}} & \xrightarrow{\eta} & 1_{\mathbb{K}}1_G \\
{\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\
1_{\mathbb{K}} & \xrightarrow{=} & 1_{\mathbb{K}}
\end{array}$$

Remark 1.3.4 (Graded bialgebras). A bialgebra B is graded if it is graded both as an algebra and a coalgebra. From the previous section, we know that the unit η maps $\mathbb{K}$ to B_0, and ε is the null map on B_i for $i > 0$. It is connected if $B_0 \cong \mathbb{K}$. If B is a graded connected bialgebra, then η is the inclusion $\mathbb{K} = B_0 \hookrightarrow B$, and ε is the canonical projection from B to B_0.

Lemma 1.3.5. *(Connection between bialgebras and Lie algebras) The vector space of primitive elements (see Definition 1.2.7) of a bialgebra B, written $Prim(B)$, forms a Lie subalgebra of B for the Lie bracket $[x, y] = xy - yx$.*

Proof. Let $x, y \in Prim(B)$. Then, since Δ is an algebra map,

$$\begin{aligned}
\Delta([x, y]) &= \Delta(x)\Delta(y) - \Delta(y)\Delta(x) \\
&= (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) - (y \otimes 1 + 1 \otimes y)(x \otimes 1 + 1 \otimes y) \\
&= (xy - yx) \otimes 1 + 1 \otimes (xy - yx) = [x, y] \otimes 1 + 1 \otimes [x, y].
\end{aligned}$$

□

Now we are ready to define Hopf algebras. It follows from Proposition 1.2.8 that $End(B)$ of a bialgebra B is an algebra for the convolution product, with unit $\nu = \eta \circ \varepsilon$.

Definition 1.3.6. A *Hopf algebra* is a bialgebra H such that $Id \in End(H)$ has a left and right inverse S for the convolution product, called the *antipode*. That is,

$$\begin{aligned}(Id * S)(h) = m\left(\sum_i h_i^{(1)} \otimes S\left(h_i^{(2)}\right)\right) &= \eta(\varepsilon(h)) \\ = m\left(\sum_i S\left(h_i^{(1)}\right) \otimes h_i^{(2)}\right) &= (S * Id)(h)\end{aligned}$$

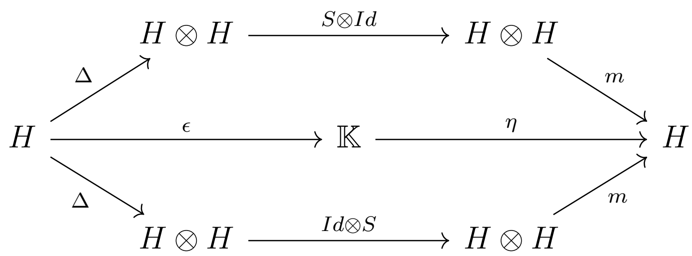
or, more compactly in Sweedler notation,

$$S\left(h^{(1)}\right)h^{(2)} = \nu(h) = h^{(1)}S\left(h^{(2)}\right)$$

for all $h \in H$.

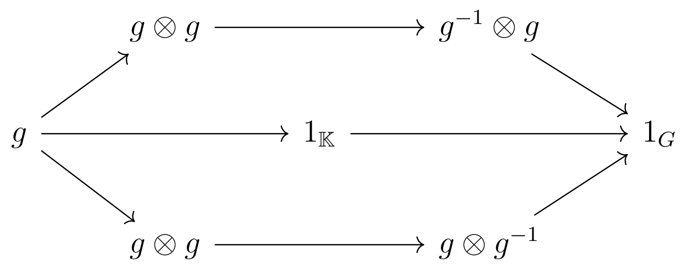
Remark 1.3.7. If Id has a left inverse S' and a right inverse S'', then the two coincide. This implies that the antipode, if it exists, is unique.

Visually, this definition can be expressed via the diagram below:



The intuition behind the antipode is that we think of it as somehow analogous to an inverse. To strengthen this intuition, let us check that in the case of the group bialgebra, the group inverse $S(g) = g^{-1}$ is indeed an antipode.

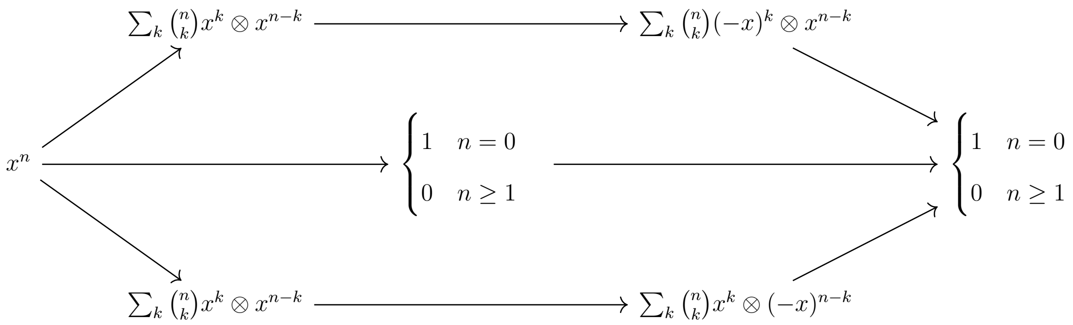
Example 1.3.8 (The group Hopf algebra)**.** Taking $S(g) = g^{-1}$, the above diagram becomes



for elements $g \in G$. Hence when extended linearly, S does satisfy the requirements for an antipode.

Let us also describe another relatively intuitive example:

Example 1.3.9 (The polynomial Hopf algebra)**.** Let $H = \mathbb{K}[x]$ with the bialgebra structure from 1.2.13. Then $S(x^n) = (-x)^n$ is an antipode, since the following diagram commutes:



Theorem 1.3.10 (Takeuchi)**.** *Let H be a graded connected bialgebra. Then H is a Hopf algebra with antipode*

$$S = \sum_{n \geq 0} (-1)^n m^{n-1} \circ \pi^{\otimes n} \circ \Delta^{n-1},$$

where $\pi = Id - \eta \circ \varepsilon : H \to H$, and we employ the notational convention that $m^{-1} = \eta$, $\Delta^{-1} = \varepsilon$, and $m^0 = \Delta^0 = Id$.

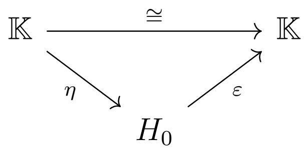
Notice that the above sum is *finite* for any $h \in H$, since $\Delta^{n-1}(H_m) = 0$ for $n > m$:

$$\Delta^{n-1}(H_m) \subseteq \bigoplus_{i_1 + \ldots + i_n = m} H_{i_1} \otimes \cdots \otimes H_{i_n}$$

implies that when $n > m$, in every possible partition $i_1, \ldots, i_n$ of m there is an index j such that $i_j = 0$. This means that $H_{i_j} = H_0$.

Lemma 1.3.11. $\pi(H_0) = 0$

Proof. We will make use of the fact that a graded bialgebra H is connected if and only if Id and $\eta \circ \varepsilon$ coincide on H_0, i.e. $Id|_{H_0} = \eta \circ \varepsilon|_{H_0}$. To see why this is the case, consider the following commutative diagram:



If H is not connected, then $\dim H_0 > 1$. Hence $\dim(Id(H_0)) > 1$ while $\dim(\eta(\varepsilon(H_0))) = 1$, meaning that the maps clearly cannot coincide on H_0. On the other hand, if $\dim H_0 = 1 = \dim \mathbb{K}$, then η is an isomorphism (its image must be subspace, hence the whole of H_0), and ε is its inverse. Therefore Id and $\eta \circ \varepsilon$ are indeed the same on $H_0 = \mathbb{K}$. Since we defined π to be $Id - \eta \circ \varepsilon$, this is exactly what we wanted. □

Now we know that π sends every "constant" (element of degree zero) to zero, and hence $(\pi^{\otimes n} \circ \Delta^{n-1})(h_m)$ (for $h_m \in H_m$) is also zero whenever $n > m$. Therefore the sum is finite, and the definition of S indeed makes sense.

Proof. (Takeuchi) To show that S defined as above is an antipode, we need to show that it is the (two-sided) inverse of Id for the convolution product. First notice that by the definition of the convolution given in Proposition 1.2.8, $m^{n-1} \circ \pi^{\otimes n} \circ \Delta^{n-1}$ is precisely π^{*n}, that is, π convolved with itself n times. Then

$$\begin{aligned} S * Id = \left(\sum_{n \geq 0} (-1)^n \pi^{*n} \right) * (\pi + \eta \circ \varepsilon) = \\ = \sum_{n \geq 0} (-1)^n \pi^{*(n+1)} + \sum_{n \geq 0} (-1)^n \pi^{*n} * (\eta \circ \varepsilon) = \\ = - \sum_{n \geq 0} (-1)^{n+1} \pi^{*(n+1)} + \sum_{n \geq 0} (-1)^n \pi^{*n} = \\ = \pi^{*0} = m^{-1} \circ \Delta^{-1} = \eta \circ \varepsilon \end{aligned}$$

which was what we wanted. $Id * S = \eta \circ \varepsilon$ follows similarly. □

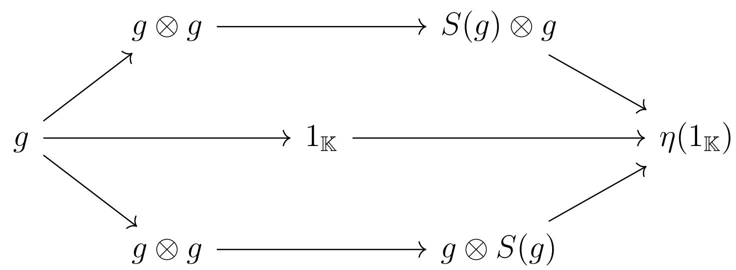
Takeuchi's theorem provides us with an explicit formula for the antipode, but in practice, this is usually inefficient to calculate. However, what matters most for our purposes is the fact that the antipode exists – that we are right to speak of a Hopf algebra whenever we have a graded and connected bialgebra.

Proposition 1.3.12. *Let H be a Hopf algebra. Then the set $\Gamma(H)$ of its group-like elements (see Definition 1.2.6) is a group.*

Proof. Let $g, h \in \Gamma(H)$. We need to show that $gh \in \Gamma(H)$, and that g has an inverse in $\Gamma(H)$. For simplicity, we will denote multiplication by juxtaposition, both in H and in $H \otimes H$. Then since Δ is an algebra morphism, and by the properties of group-like elements and the tensor product,

$$\Delta(gh) = \Delta(g)\Delta(h) = (g \otimes g)(h \otimes h) = gh \otimes gh.$$

Furthermore, since



commutes, $S(g)$ provides an inverse for g in H. To see that $S(g) \in \Gamma(H)$ also holds, we need the following lemma:

Lemma 1.3.13. *In a Hopf algebra H, $\Delta \circ S = (S \otimes S) \circ T \circ \Delta$.*

Proof. First, notice that the following identities follow directly from the definition of the convolution product and the fact that $\nu = \eta \circ \varepsilon$ is its identity element:

$$\nu(h) = (S * Id)(h) = (m \circ (S \otimes Id) \circ \Delta)(h) = m\left(S\left(h^{(1)}\right) \otimes h^{(2)}\right) =$$
$$= S\left(h^{(1)}\right) h^{(2)} = h^{(1)} S\left(h^{(2)}\right)$$
$$h = Id(h) = (\nu * Id)(h) = (m \circ (\nu \otimes Id) \circ \Delta)(h) = \nu\left(h^{(1)}\right) h^{(2)} = h^{(1)} \nu\left(h^{(2)}\right)$$
$$S(h) = (\nu * S)(h) = (m \circ (\nu \otimes S) \circ \Delta)(h) = \nu\left(h^{(1)}\right) S\left(h^{(2)}\right) = S\left(h^{(1)}\right) \nu\left(h^{(2)}\right)$$

Now, consider H with the coalgebra structure and $H \otimes H$ with the tensor product algebra structure. We know from Proposition 1.2.8 that $Hom(H, H \otimes H)$ is an algebra for the convolution product with unit $\eta_{H \otimes H} \circ \varepsilon_H$. For our convenience, let us define the maps $F, G : H \to H \otimes H$ as

$$F = \Delta \circ S \text{ and } G = (S \otimes S) \circ T \circ \Delta.$$

We will show that Δ is a left inverse for F and a right inverse for G w.r.t. convolution. Then, indeed, $G = G * (\Delta * F) = (G * \Delta) * F = F$ as promised.

Let us now perform the necessary calculations, making use of the above identities whenever needed. (A word of caution: we will use Sweedler notation heavily and unapologetically.) For any $h \in H$ we have, since Δ is an algebra morphism,

$$(\Delta * F)(h) = \Delta\left(h^{(1)}\right) F\left(h^{(2)}\right) = \Delta\left(h^{(1)}\right) \Delta\left(S\left(h^{(2)}\right)\right) = \Delta\left(h^{(1)} S\left(h^{(2)}\right)\right)$$
$$= \Delta(\nu(h)) = \Delta(\eta(\varepsilon(h))) = \Delta(\varepsilon(h)\eta(1_{\mathbb{K}})) = \varepsilon(h)\eta(1_{\mathbb{K}}) \otimes \eta(1_{\mathbb{K}}) =$$
$$= (\eta_{H \otimes H} \circ \varepsilon_H)(h)$$

and, by reindexing[1] the terms whenever convenient,

$$
\begin{aligned}
(G * \Delta)(h) &= G\left(h^{(1)}\right)\Delta\left(h^{(2)}\right) = \left(S\left(h^{(1)(2)}\right) \otimes S\left(h^{(1)(1)}\right)\right)\left(h^{(2)(1)} \otimes h^{(2)(2)}\right) = \\
&= \left(S\left(h^{(2)}\right) \otimes S\left(h^{(1)}\right)\right)\left(h^{(3)} \otimes h^{(4)}\right) = S\left(h^{(2)}\right)h^{(3)} \otimes S\left(h^{(1)}\right)h^{(4)} = \\
&= S\left(h^{(2)(1)}\right)h^{(2)(2)} \otimes S\left(h^{(1)}\right)h^{(3)} = \nu\left(h^{(2)}\right) \otimes S\left(h^{(1)}\right)h^{(3)} = \\
&= \varepsilon\left(h^{(2)}\right)\eta(1_{\mathbb{K}}) \otimes S\left(h^{(1)}\right)h^{(3)} = \eta(1_{\mathbb{K}}) \otimes S\left(h^{(1)}\right)\varepsilon\left(h^{(2)}\right)h^{(3)} = \\
&= \eta(1_{\mathbb{K}}) \otimes S\left(h^{(1)}\right)\varepsilon\left(h^{(2)(1)}\right)h^{(2)(2)} = \eta(1_{\mathbb{K}}) \otimes S\left(h^{(1)}\right)\nu\left(h^{(2)(1)}\right)h^{(2)(2)} = \\
&= \eta(1_{\mathbb{K}}) \otimes S\left(h^{(1)}\right)h^{(2)} = \eta(1_{\mathbb{K}}) \otimes \varepsilon(h)\eta(1_{\mathbb{K}}) = (\eta_{H\otimes H} \circ \varepsilon_H)(h).
\end{aligned}
$$

Therefore $(\Delta * F) = \eta_{H\otimes H} \circ \varepsilon_H = (G * \Delta)$ as we wanted.

□

Getting back now to the proof of the proposition, we have, by the previous lemma,

$$\Delta(S(g)) = ((S \otimes S) \circ T \circ \Delta)(g) = (S \otimes S)(g \otimes g) = S(g) \otimes S(g),$$

which was what we wanted to show. □

[1] As permitted by the coassociativity of Δ.

Chapter 2

Hopf algebras in linguistics

2.1 Motivation and linguistic background

We first strive to summarize the basics of modern (generative) syntax, the study of sentence structure, for the interested mathematician. However, it cannot be stressed enough that the following is only a bird's-eye view, and for a comprehensive treatment, the first few chapters of [2] should be consulted.

In generative syntax, the structure of a sentence is usually modeled as a binary rooted tree. This means that when analyzing sentence structure, we start with the whole sentence, and then repeatedly divide it in two parts (called *constituents*) until we arrive at the individual words. For instance, when analyzing the sentence "The boy ate the sandwich", we instinctively feel that it is comprised of the parts "the boy" and "ate the sandwich" (and not e.g. "the boy ate" and "the sandwich") – these words somehow seem to belong more strongly together. We can of course repeat this process to identify "ate the sandwich" as a composition of "ate" and "the sandwich" and "the boy" as "the" + "boy", soon arriving at the following hierarchical structure:

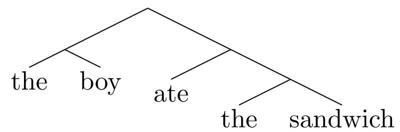


Figure 2.1: A very schematic syntactic tree

Here, the uppermost node (the root) represents the whole sentence, and the leaves are labeled with the words of the sentence, but what is at the other, intermediate nodes? We can roughly think of them as representing either *parts of speech* or *phrases*, as can be seen in Figure 2.2:

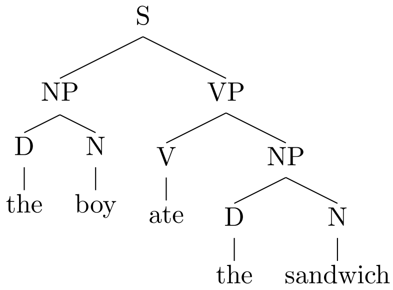


Figure 2.2: A little more sophisticated, but still quite schematic syntactic tree

We used S to denote the sentence, and the other single letters to denote parts of speech: N stands for *noun*, V for *verb*, and D for *determiner*. Determiners are essentially articles (the, a, an), demonstratives (this, that), possessives (my, their) or quantifiers (many, all). Further parts of speech that are often used are Adj for adjectives, Adv for adverbs, and P for prepositions. The other type of node that appears in our tree is of the form XP, which is short for *X phrase*: noun phrase (NP), verb phrase (VP) etc., there exists a corresponding phrase structure for every part of speech. The main idea is that in every constituent, there is one word that dominates: for example, most of the properties of the phrase "the boy" are determined by the properties of "boy" rather than those of "the" (more precisely, by the fact that "boy" is a noun (N)). This is called the *head* of the phrase, and the phrase inherits the head's part of speech, hence, in this case, it becomes an NP.[1]

We have previously promised that we would model sentence structure as rooted *binary* trees. Consider, however, the following structure:

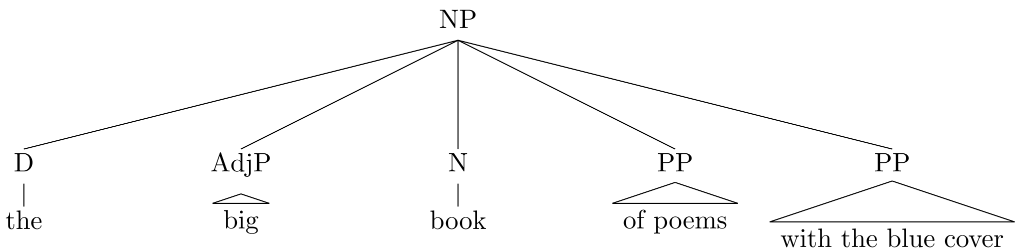


Figure 2.3: A seemingly non-binary syntactic tree

[1] To be more precise, "boy" is the *semantic* head of the phrase, as it clearly dominates the meaning. If one considered instead "the", the *syntactic* head, the phrase could be regarded as a DP.

This example is taken from [2]. A quick note on notation: a triangle in a syntactic tree just represents a part of the tree that contains some further structure which we do not want to analyze in detail.

The above tree is clearly not binary, what is more, we could append as many prepositional phrases as we want (e.g. *the big book of poems with the blue cover from Blackwell by Burns etc.*), arriving at an arbitrarily wide but flat structure. The linguists' response to this is that there is in fact a binary structure behind even these examples, and further nodes, further categories (beside those denoted X and XP) need to be introduced to model this structure. This is far from obvious at first sight, but the need for its existence is strongly motivated by so-called *constituency tests*. One such test is *one-replacement*, which essentially states that if a sequence of words can be replaced by the word *one* in the following sense, then it should form a constituent in the tree:

(1) I bought the big [book of poems with the blue cover], not the small [one].

Hence the phrase *book of poems with the blue cover* should be its own constituent in whatever tree structure we assume, that is, there should be a node in the tree such that its descendant leaves are exactly the words of the above phrase. The same argument can be applied again for *book of poems* as in

(2) I bought the big [book of poems] with the blue cover, not the small [one] with the red cover.

This implies rather the following deep structure for the whole phrase:

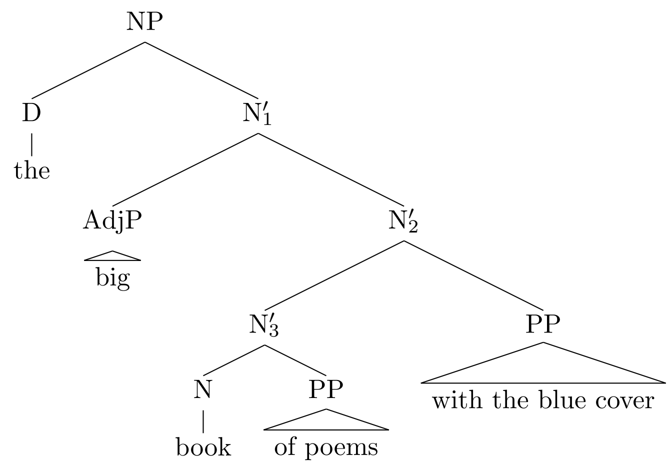


Figure 2.4: A tree according to X-bar theory

The assumption of this structure is called *X-bar theory*. Here we denote the intermediate categories by X′, but the name comes from the fact that they are often denoted $\overline{X}$. In the figure above, N'_3 corresponds to the constituent *book of poems*, N'_2 to *book of poems with the blue cover*, and so on.

In this example, we introduced intermediate categories for smaller constituents within an NP, but we observe the same phenomenon within VPs, AdjPs, PPs, etc. Of course, there is a difference in which type of children a certain node can have (i.e. which type of phrases are allowed to modify it), but we see the same need for X-bar categories arise. Abstracting away from the specifics, we can formulate rules within the framework of X-bar theory to describe the hierarchical structure.[2] These rules are meant to be interpreted as a description of what kind of children a certain type of node is allowed to have in the tree.

Now we have a relatively good theory that represents sentences as binary rooted trees, as promised. However, there are some phenomena that even X-bar theory cannot account for. Such are the inverted word order in English questions (cf. *You saw John.* and *Have you seen John?*), passives (cf. *The boy ate the bread.* and *The bread was eaten.*), and topicalization (emphasizing a certain phrase by having it appear at the front of the sentence, cf. *I like bagels.* and *Bagels, I like.*), the placement of negation and of adverbs in some languages like French, and VSO[3] languages like Irish overall. Transformational grammarians explain these phenomena by *transformations* of the trees called *movement*. Movement is a process in which certain subsections of a syntactic tree are extracted and placed onto a different part of the tree. Note that – for reasons internal to a particular understanding of transformational theory – movement is only possible upwards in the tree.

Figure 2.5: Illustration of movement in a tree

In order to be able to describe movement, it is clear that we need a mathematical structure capable of representing such transformations, i.e. allowing the repositioning of subtrees within some (linguistically motivated) constraints. In recent papers [7, 5], Marcolli, Chomsky, and Berwick propose that Hopf algebras provide a suitable

[2] Though the number of bar-levels and the exact format of the rules are debated.

[3] VSO is short for Verb–Subject–Object, and refers to the word order of a language. For instance, English is an SVO language, as typical word order is subject–verb–object as in *The girl read the book.* SVO and SOV are the most common among the languages of the world, but a non-negligible 9% of languages follow a VSO word order.

framework. Intuitively, the coproduct will allow us to cut up our tree and extract the corresponding subtree in order to insert it into another spot in the tree.

2.2 Mathematical formulation

Let us now introduce the precise formulation of how Hopf algebras may be used to handle and manipulate syntactic trees. This short introduction closely follows [7] and [4], a recent talk on the topic given by Marcolli specifically for mathematicians.

So far, we have been considering syntactic trees as the result of *analyzing* a sentence in a top-down manner: we start with a sentence, divide it into two constituents, and so on, repeating this process until we arrive at the single words. As a matter of fact, sentences are actually thought to be *generated* through this process, hence the name *generative* syntax. However, we might as well interpret the process in the other direction, starting out with the individual words, combining (*merging*) them via simple binary set formation, then repeatedly combining the resulting objects until we arrive at the whole sentence. This latter, bottom-up view will provide a useful framework for us in the following.

Mathematically, this is to say that we start with a set $\mathcal{SO}_0$ of *lexical items* (like *the, dog, eat, elephant, pretty, etc.*) and *syntactic features* (like English past tense marker -*ed*), and we denote the commutative, nonassociative operation that is binary set formation by $\mathfrak{M}$. In other words, clearly

$$\{\alpha, \beta\} = \mathfrak{M}(\alpha, \beta) = \mathfrak{M}(\beta, \alpha) = \{\beta, \alpha\},$$

but

$$\{\gamma, \{\alpha, \beta\}\} = \mathfrak{M}(\gamma, \mathfrak{M}(\alpha, \beta)) \neq \mathfrak{M}(\mathfrak{M}(\gamma, \alpha), \beta) = \{\{\gamma, \alpha\}, \beta\}.$$

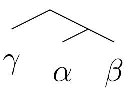
Here α, β and γ may be elements of $\mathcal{SO}_0$, or they may be more complex objects that we obtained via $\mathfrak{M}$. The set of *syntactic objects* $\mathcal{SO}$ is the set of all elements that can be obtained from $\mathcal{SO}_0$ by repeated application of $\mathfrak{M}$:

Definition 2.2.1. A *magma* is a set S equipped with a binary operation $*$ such that S is closed under $*$. (No further properties required.)

Definition 2.2.2. The set of syntactic objects $\mathcal{SO}$ is the free nonassociative commutative magma generated by $\mathcal{SO}_0$.

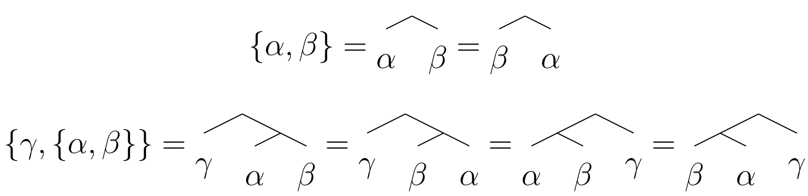
In other words, a syntactic object is simply a result of repeated binary set formation like {{the, students}, {are, exhausted}} or {{the, dogs}, {{eat, everything}, {in, {the, house}}}}.

The free nonassociative commutative magma on a set S is canonically isomorphic to the set $\mathfrak{T}_S$ of abstract binary rooted trees with leaves decorated by elements of S. The identification is clear – for instance, the syntactic object $\{\gamma, \{\alpha, \beta\}\}$ corresponds to the tree



The word *abstract* means that there is no planar embedding assigned to the trees, namely, that the leaves are not ordered. This is at first sight counterintuitive, since we usually perceive language linearly, as a sequence of strings or words. However, it proves useful to assume a nonplanar deep structure in the background, and regard the apparent (ordered) surface-level structure only as a projection of the true deep structure. Notice that the commutativity of $\mathfrak{M}$ guarantees precisely that the leaves are not fully[4] ordered.

Example 2.2.3 (Syntactic trees are abstract)**.** Multiple planar embeddings of an abstract tree can correspond to the same syntactic object:



Let us now introduce the concept of *workspaces*. We think of the workspace as the space where all syntactic operations happen, where sentences are formed and transformed. It contains all available computational resources (i.e. all syntactic objects, even multiple copies if needed), and our central operation, which we will call *Merge*, will transform workspaces into new workspaces.

Definition 2.2.4. A *workspace* F is a binary forest whose connected components are (finitely many) abstract binary rooted trees:

$$F = T_1 \sqcup \ldots \sqcup T_n \quad \text{with } T_i \in \mathfrak{T}_{\mathcal{SO}_0}$$

[4]Note that this interpretation does not allow all possible permutations of the leaves. For instance, in Example 2.2.3, the remaining two permutations, $\beta\gamma\alpha$ and $\alpha\gamma\beta$, are not permitted, even though they do appear in some languages. Consider, for example, the phrase *call someone up* in English. Such cases are usually handled by so-called *discontinuous constituents*.

In linguistics, the T_i's are called the *members* of the workspace. Denote the set of all workspaces $\mathcal{WS}$. Then $\mathcal{WS} \cong \mathfrak{F}_{\mathcal{SO}_0}$, where the latter denotes the set of all finite binary non-planar forests with leaves decorated by elements of $\mathcal{SO}_0$.

As mentioned previously, certain substructures of syntactic objects sometimes need to be extracted in order to be repositioned within the tree. We will call these subtrees *accessible terms*.

Definition 2.2.5. *Accessible terms of a syntactic object* T are subtrees $T_v \subsetneq T$, where v is a non-root vertex of T, and T_v is the subtree under v.

Accessible terms of a workspace F are the accessible terms of its members *and* the members themselves.

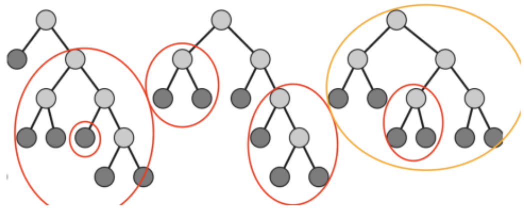


Figure 2.6: Some accessible terms of a workspace F. Source: [4]

Definition 2.2.6 (Admissible cuts)**.** Given a tree $T \in \mathfrak{T}_{\mathcal{SO}_0}$, consider forests $F_{\underline{v}} \subsetneq T$ of the form

$$F_{\underline{v}} = T_{v_1} \sqcup \ldots \sqcup T_{v_n}$$

where $T_{v_i} \subsetneq T$ are disjoint accessible terms. Every such $F_{\underline{v}}$ corresponds to an *admissible cut* C of T with forest $\pi_C(T) = F_{\underline{v}}$ and remaining tree $\rho_C(T)$ attached to the root. In a similar fashion, an admissible cut $F_{\underline{v}}$ of a forest F corresponds to the union of some disjoint accessible terms of F.

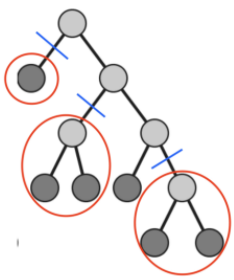


Figure 2.7: A subforest of accessible terms and the corresponding admissible cut. Source: [4]

Now we are almost ready to define a Hopf algebra structure on $\mathcal{WS}$. We still need one last definition:

Definition 2.2.7. Let T be a tree, and let T_v be a subtree of T consisiting of vertex v and its descendants (note: v can also be the root vertex). Then the *quotient* T/T_v is defined as the rooted binary tree obtained by removing the entire tree T_v from T, and contracting the remaining edges if necessary.

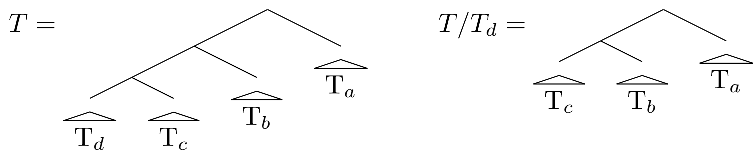


Figure 2.8: Quotient trees and the contraction of edges

Remark 2.2.8. Note that the above definition does not exactly coincide with how quotients in the Hopf algebra of rooted trees are usually defined in mathematics and theoretical physics. This is, in part, to ensure that the quotient T/T is the empty tree, which will also happen to be the unit in our Hopf algebra structure. Further explanation and context is provided in [7, p. 6].

Lemma 2.2.9. *Let* $T \in \mathfrak{T}_{\mathcal{SO}_0}$ *be a rooted binary tree, and let* $F_{\underline{v}}$ *be a subforest of* T *such that* $F_{\underline{v}} = \bigsqcup_i T_{v_i}$ *is the union of disjoint subtrees* $T_{v_i} \subseteq T$*,* $\underline{v} = (v_1, \ldots, v_k)$ *– that is,* $F_{\underline{v}}$ *is either an admissible cut of* T *or, in the trivial case, the whole tree* T*. Then the quotient* $T/F_{\underline{v}}$ *given by*

$$T/F_{\underline{v}} = (\cdots (T/T_{v_1})/T_{v_2} \cdots)/T_{v_k}$$

is well-defined and independent of the order of $v_1, \ldots, v_k$*. This extends to quotients of forests* $F/F_{\underline{v}}$*, in particular,* $F_{\underline{v},\underline{w}}/F_{\underline{v}} = F_{\underline{w}}$*.*

Proof. Fairly straightforward to check, see [7, Lemma 2.6]. □

There are many admissible cuts of a given tree or forest, and we want to keep track of all of them to be able to access all accessible terms simultaneously. Hence instead of defining the Hopf algebra structure on $\mathfrak{F}_{\mathcal{SO}_0}$ itself, we will consider the space of formal finite linear combinations of elements of $\mathfrak{F}_{\mathcal{SO}_0}$ over $\mathbb{Q}$, denoted $\mathcal{V}(\mathfrak{F}_{\mathcal{SO}_0})$.

Definition 2.2.10 (The Hopf algebra of forests)**.** Consider the vector space $\mathcal{V}(\mathfrak{F}_{\mathcal{SO}_0})$.

Let the product operation on $\mathcal{V}(\mathfrak{F}_{\mathcal{SO}_0})$ be simply given by $\sqcup$: that is, let the product of two workspaces be their disjoint union, and extend linearly. This is a commutative, associative linear map. Hence $(\mathcal{V}(\mathfrak{F}_{\mathcal{SO}_0}), \sqcup, \eta)$ forms an algebra, the unit η being the constant mapping to the empty forest.

Let the coproduct operation $\Delta : \mathcal{V}(\mathfrak{F}_{\mathcal{SO}_0}) \to \mathcal{V}(\mathfrak{F}_{\mathcal{SO}_0}) \otimes \mathcal{V}(\mathfrak{F}_{\mathcal{SO}_0})$ be defined as follows:

On trees $T \in \mathfrak{T}_{\mathcal{SO}_0}$, define

$$\Delta(T) = \sum_{\underline{v}} F_{\underline{v}} \otimes T/F_{\underline{v}}$$

with $F_{\underline{v}}$ as per Lemma 2.2.9. Extend to forests $F = \bigsqcup_a T_a$ by letting $\Delta(F) = \bigsqcup_a \Delta(T_a)$, which we write as

$$\Delta(F) = \sum_{\underline{v}} F_{\underline{v}} \otimes F/F_{\underline{v}},$$

then extend linearly to $\mathcal{V}(\mathfrak{F}_{\mathcal{SO}_0})$. Let ε be the linear mapping that maps each formal linear combination in $\mathcal{V}(\mathfrak{F}_{\mathcal{SO}_0})$ to the empty forest's coefficient in it. These maps fulfill the conditions of Definition 1.2.1, and thus $(\mathcal{V}(\mathfrak{F}_{\mathcal{SO}_0}), \Delta, \varepsilon)$ is a coalgebra.

Proposition 2.2.11. $(\mathcal{V}(\mathfrak{F}_{\mathcal{SO}_0}), \sqcup, \eta, \Delta, \varepsilon)$ *is a Hopf algebra.*

Proof. One needs to check coassociativity, the counitary property, and the compatibility axioms of Definition 1.3.1. For the full proof, we refer the reader to [7, Lemma 2.7]. Note that $\mathcal{V}(\mathfrak{F}_{\mathcal{SO}_0})$ is graded by the number of leaves, and connected, thus the existence of the antipode is automatic by Takeuchi's theorem (1.3.10). □

Now that we have all the necessary apparatus at hand, let us describe how the transformation of workspaces happens. Let us call the operation that transforms workspaces into new workspaces (the exact mechanism to be specified below) *Merge*. More precisely, Merge is a family of operations of the form $\mathfrak{M}_{S,S'}$ with syntactic objects $S, S' \in \mathcal{SO}$. Given a workspace F, $\mathfrak{M}_{S,S'}$ searches among all accessible terms of F for a matching pair $T_v \simeq S$ and $T_w \simeq S'$. If a match is found, these two terms are merged into

$$\mathfrak{M}(T_v, T_w) = \overset{\frown}{\mathrm{T}_v \quad \mathrm{T}_w},$$

T_v and T_w are deleted from their original positions, and the above new tree is appended to F. Let T_v and T_w originally be contained in the components T_i and T_j respectively. Then all the other components $T_a \subset F$, $a \neq i, j$ are left unchanged by the action of $\mathfrak{M}(S, S')$. The above can be summarized by the formula

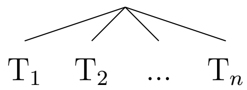
$$\mathfrak{M}_{S,S'} : F \mapsto F' = \mathfrak{M}(T_v, T_w) \sqcup T_i/T_v \sqcup T_j/T_w \sqcup \bigsqcup_{a \neq i,j} T_a.$$

Utilizing the Hopf algebra structure on $\mathcal{V}(\mathfrak{F}_{\mathcal{SO}_0})$, the action of Merge can be expressed as

$$\mathfrak{M}_{S,S'} = \sqcup \circ (\mathcal{B} \otimes 1) \circ \delta_{S,S'} \circ \Delta,$$

that is to say: The coproduct Δ first extracts all the accessible terms. Then $\delta_{S,S'}$ finds a matching pair of accessible terms, which are then *grafted* – joined via adding a new root – by the grafting operator $\mathcal{B}$

$$\mathcal{B} : T_1 \sqcup \ldots \sqcup T_n \mapsto$$



Finally, the product $\sqcup$ reassembles the new workspace.

To conclude, a few brief remarks are due on all that we could not cover in the scope of this thesis. The Merge operation as defined above in fact allows more types of transformations than linguistically motivated, cases that violate the aforementioned principle that movement is only possible upwards in a tree. Hence a slight modification, a mechanism called Minimal Search is needed to eliminate unwanted types of Merge. Furthermore, we have only been able to summarize the very basics of how Hopf algebras can be used in the study of syntax – however, in an even more recent paper [6], Marcolli, Chomsky, and Berwick also extend this Hopf algebraic formulation to the interaction of syntax and semantics. To grasp the full picture of Hopf algebras in linguistics, a thorough read of all three papers [7, 5, 6] would be necessary.

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NYILATKOZAT

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NEPTUN azonosító: XX3GQU

Szakdolgozat címe:
Hopf algebras and their applications in linguistics

A **szakdolgozat** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

Budapest, 2024.06.03.

Kövér Blanka

a hallgató aláírása