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Free-rooted packings of arborescences

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Chapter 1

Introduction

In this thesis we prove new results about packings of arborescences. An r-arborescence is a directed tree in which each node has an in-degree of 1 except the root note r, which has an in-degree of 0. A packing of subgraphs in a graph means a collection of subgraphs, that are edge-disjoint. The most fundamental result of the study of packings of arborescences is the following result of Edmonds ([6]):

Theorem 1.0.1. *(Edmonds' Theorem (Weak form))[6]) Let $D = (V, A)$ a digraph and $s \in V$. There exists a packing of k spanning s-arborescences in G if and only if*

$$\varrho(X) \geq k \textit{ for all } \emptyset \neq X \subseteq V - s, \tag{1.1}$$

where $\varrho(X)$ denotes the in-degree of X.

This result has been generalized in multiple ways. Durand de Gevigney, Nguyen and Szigeti characterized the existence of matroid-based packings of arborescences in [5], Cs. Király, Szigeti, Tanigawa characterized the existence of matroid-based and matroid-restricted packings of arborescences in [23] and Bérczi and Frank characterized the existence of free-rooted packings of arborescences ([3]): packings, where the roots of the arborescences are not given (for further definitions see later sections).

This thesis generalizes results on free-rooted packings of arborescences. In Chapter 1 we give an overview of the previous results about packing arborescences, present results that we use in later chapters and describe some algorithms that we use in our algorithms as subroutines. In Chapter 2 we characterize the existence of free-rooted matroid-based and matroid-restricted packings of arborescences and prove some corollaries. In Chapter 3 we extend a result of Szigeti ([26]) about free-rooted packings of arborescences in mixed graphs. In Chapter 4 we look at the algorithmic aspects of the problems discussed in Chapter 2 and 3.

1.1 Definitions

1.1.1 Sets and set functions

Given a function $f : S \to \mathbb{R}$ and a finite set $Z \subset S$, let $\widetilde{f}(Z) := \sum_{s \in Z} f(s)$. Two subsets $X, Y \subseteq S$ are **intersecting**, if $X \cap Y \neq \emptyset$. A set function b on the ground set S is **subcardinal** if $b(X) \leq |X|$ for all $X \subseteq S$; **submodular** if

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \textit{ for all } X, Y \subseteq S; \tag{1.2}$$

and **supermodular** if

$$b(X) + b(Y) \le b(X \cap Y) + b(X \cup Y) \ for\ all\ X, Y \subseteq S. \tag{1.3}$$

A set function is **positively intersecting submodular** (**positively intersecting supermodular**) if (1.2) (respectively (1.3)) holds for intersecting subsets of S, for which $p(X) > 0$, $p(Y) > 0$.

An arc st with $s \in S$ and $t \in T$, where S and T are two non-empty subsets of a ground set V, is called an **st-arc**. A set function p on V is called **positively T-intersecting supermodular** if it satisfies (1.3) for subsets $X, Y \subseteq S \cup T$ for which $X \cap Y \cap T \neq \emptyset$ and $p_0(X) > 0$ and $p_0(Y) > 0$. The set function p is called **positively ST-crossing supermodular**, if (1.3) holds if $p(X) > 0$ and $p(Y) > 0$ and $X \subseteq V$ and $Y \subseteq V$ are **ST-crossing**, that is, $X \cap Y \cap T \neq \emptyset$ and $S - (X \cup Y) \neq \emptyset$. S and T are **ST-independent** if $X \cap Y \cap T = \emptyset$ and $S - (X \cup Y) = \emptyset$, that is, no ST-arc enters both sets.

1.1.2 Matroids

A **matroid** is a pair $M = (S, r)$ where S is called the **ground set** of M and $r : 2^S \to \mathbb{N}$ is called the **rank function** of M and satisfies the so called *rank axioms.*

(R_1) $r(\emptyset) = 0$.

(R_2) If $X \subseteq Y \subseteq S$ then $r(X) \le r(Y)$ (monotonity).

(R_3) $r(X) \le |X|$ for every $X \subseteq S$ (subcardinality).

(R_4) $r(X) + r(Y) \ge r(X \cap Y) + r(X \cup Y)$ for every $X, Y \subseteq S$ (submodularity).

A subset $X \subseteq S$ is said to be **independent** if $r(X) = |X|$. Matroids can also be defined with their independent sets. The pair $M = (S, \mathcal{I})$, where $\mathcal{I}$ is the set of independent sets, is a matroid if it satisfies the so called *independence axioms.*

(I_1) $\emptyset \in \mathcal{I}$.

(I_2) If $Y \in \mathcal{I}$ and $X \subseteq Y$ then $X \in \mathcal{I}$.

(I_3) The inclusion-wise maximal independent subsets of every $X \subseteq S$ have the same size.

If we define a matroid with its independent sets then the rank of a subset $X \subseteq S$ is defined as the size of the inclusion-wise maximal independent subsets of X. A **base** is an inclusion-wise maximal independent set. An element $s \in S$ is a **loop** if $r(\{s\}) = 1$. Two elements $x, y \in S$ are **parallel** if $r(\{x, y\}) = 1$.

The **direct sum** of two matroids $M_1 = (S_1, \mathcal{I}_1)$ and $M_2 = (S_2, \mathcal{I}_2)$ on disjoint ground sets is the matroid $M_1 \bigoplus M_2$ on the ground set $S_1 \cup S_2$ where a set $X \subseteq S_1 \cup S_2$ is independent if and only if $X \cap S_1 \in \mathcal{I}_1$ and $X \cap S_2 \in \mathcal{I}_2$. The **sum/union** of two matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ on the same ground set S is the matroid $M_1 + M_2$ on the ground set S where $X \subseteq S$ is independent if and only if there exists a set $Y \subseteq X$ such that $Y \in \mathcal{I}_1$ and $X - Y \in \mathcal{I}_2$.

The **k-uniform matroid** on the ground set S is a matroid whose independent sets are exactly the subsets of S with a size of at most k. Let $\mathcal{P} = \{P_1, \dots, P_k\}$ be a partition of S and $b_1, \dots, b_k$ positive integers. Let $\mathcal{I} := \{X \subseteq S : \ |X \cap P_i| \le b_i \text{ for all } i = 1, \dots, k\}$. The matroid $(S, \mathcal{I})$ is called a **partition matroid**. The **graphic matroid** of a graph $G = (V, E)$ is the matroid on E where an edge-set $F \subseteq E$ is independent if and only if F contains no cycles.

For a more thorough introduction to matroids, see [10, Chapters 5].

1.1.3 Directed graphs and packings of arborescences

Let $G = (S, T, E)$ be a bipartite graph and $p : T \to \mathbb{Z}$ a positively intersecting supermodular set function for which $p(\emptyset) = 0$ holds. Let $m_S \in \mathbb{N}^S$ be a degree prescription on S. We say that G **fits** m_S if the degree of every $s \in S$ is $m_S(s)$, that is, $d_G(s) = m_S(s)$. Let $\boldsymbol{\Gamma(Y)}$ be the set of neighbours of Y in G. We say that G **covers** p, if $\forall Y \subseteq T : p(Y) \leq |\Gamma(Y)|$. Given a matroid $M = (S, r)$, we say that G M**-covers** p, if $\forall Y \subseteq T : p(Y) \leq r(\Gamma(Y))$.

Let $D = (V + s, A)$ be a rooted digraph, where s is called the root. The in-degree of s is 0 and the outgoing arcs are called **root-edges**. We call an s-rooted arborescence an s**-arborescence**. For $X, Z \subseteq V + s$, $B \subseteq A$ let $\boldsymbol{\partial_Z(X)}$ denote the set of edges that go from $Z - X$ to X and let $\boldsymbol{\varrho_Z(X)}$$= |\partial_Z(X)|$. For $B \subseteq A$, let $\boldsymbol{\partial^B(X)}$ be the set of edges in B entering X. Let $\boldsymbol{\varrho^B(X)}$$= |\partial^B(X)|$. If $X = V$ or $B = A$ then $\partial(X)$ and $\varrho(X)$ may be used to denote the set and number of in-going arcs of X, respectively.

Let $M_1 = (\partial_s(V), r_1)$ be a matroid on the root-edges of D. We call a packing of s-arborescences $T_1, \ldots, T_k$ M_1**-based**, if every T_i contains exactly one root-edge (e_i) and, for all vertices $v \in V$, $\{e_i : v \in V(T_i)\}$ is a basis of M_1. Let $M_2 = (A, r_2)$ be a matroid on the edges of D. We call a packing of s-arborescences M_2**-restricted** if the union of the edge sets of the arborescences in the packing is independent in M_2.

1.1.4 Mixed graphs and packings of arborescences

Let $F = (V, E \cup A)$ be a mixed graph where E is the set of undirected edges and A is the set of arcs. **Orienting** an edge $uv \in E$ means we replace it with either the arc uv or vu.A **mixed** $\boldsymbol{r}$**-arborescence** is a mixed graph that can be oriented to be an r-arborescence. An arc $st \in A$ **enters** a set $X \subseteq V$ if $s \notin X$ and $t \in X$. An edge $uv \in E$ enters $x \subseteq V$ if $|X \cap \{u, v\}| = 1$. For $B \subseteq E \cup A$, let $\boldsymbol{\partial^B(X)}$ be the set of arcs and edges in B entering X. Let $\boldsymbol{\varrho^B(X)}$$= |\partial^B(X)|$.

For $\overrightarrow{Z} \subseteq A$, Z denotes the underlying undirected edges of $\overrightarrow{Z}$. For $Z \subseteq E$ and $X \subseteq V$, the set of vertices covered by Z is denoted by $\boldsymbol{V(Z)}$ and the set of edges in Z that are induced by X is denoted by $\boldsymbol{Z(X)}$. For a family of sets $\mathcal{P}$ on V and $B \subseteq A \cup E$ let $\boldsymbol{\partial^B(\mathcal{P})}$ be the set of edges and arcs in B, that enter a member of $\mathcal{P}$ and $\boldsymbol{\varrho^B(\mathcal{P})}$$:= |\partial^B(\mathcal{P})|$.

For $f, g : V \to \mathbb{Z}_+$ we call a packing of arborescences $\boldsymbol{(f,g)}$**-bounded** if, for each $v \in V$, the number of v-arborescences in the packing is between $f(v)$ and $g(v)$. For $k, l, l' \in \mathbb{Z}_+ - \{0\}$ a packing of arborescences is $\boldsymbol{(l,l')}$**-limited** if the number of arborescences in the packing is between l and l', and $\boldsymbol{k}$**-regular**, if each vertex is in exactly k arborescences in the packing. We call a packing of mixed arborescences (f, g)-bounded/(l, l')-limited/k-regular, if we can orient the undirected edges such that we get an (f, g)-bounded/(l, l')-limited/k-regular packing of arborescences.

For a graph $G = (V, E)$, let $\boldsymbol{M_G}$ be the graphic matroid of G, and let $\boldsymbol{M_G^k}$ be the k-graphic matroid of G, that is the k-sum of M_G, which is a matroid on V, where a set is independent if and only if it can be partitioned into k independent sets of M_G. Let $F = (V, E \cup A)$ be a mixed graph. For a subpartition $\mathcal{P}$ of V, let $\boldsymbol{A(\mathcal{P})}$ and $\boldsymbol{E(\mathcal{P})}$ be the set of arcs and edges entering a class of $\mathcal{P}$ and $\boldsymbol{V(\mathcal{P})}$ be the nodes covered by the subpartition $\mathcal{P}$. Let $\mathbf{G_F} = (V, E \cup E_A)$ be the underlying undirected graph of F, and $\mathbf{D_F} = (V, A_E \cup A)$ the directed extension of F, where $A_E = \bigcup_{e \in E} A_e$, and if $e = uv \in E$, $A_e = \{\overrightarrow{uv}, \overrightarrow{vu}\}$ ($\overrightarrow{uv}$ is an arc from u to v). The extended k-graphic matroid $\mathbf{M_F^k}$ of F is a matroid on $A \cup A_E$, which we get from $M_{G_F}^k$ by replacing each edge $e \in E$ with two parallel copies of itself, associating these edges to the corresponding edges in A_E, and associating the edges of E_A with the corresponding arcs of A. It is shown in [16], that the rank function of M_F^k is the following ($Z \subseteq A \cup A_E$):

$$r_{M_F^k}(Z) = \min\{|Z \cap A(\mathcal{P})| + |\{e \in E(\mathcal{P}) : \ Z \cap A_e \neq \emptyset\}| + k(|V| - |\mathcal{P}|) : \mathcal{P} \text{ is a partition of } V\} \quad (1.4)$$

1.1.5 Hypergraphs

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. We assume that all the hyperedges in $\mathcal{E}$ are of size at least 2. A directed hypergraph or **dypergraph** for short is a pair $\overrightarrow{\mathcal{H}} = (V, \mathcal{A})$ where $\boldsymbol{V}$ denotes the set of vertices and $\boldsymbol{\mathcal{A}}$ denotes the set of **dyperedges**. By dyperedge, we mean a pair (Z, z) where $z \in Z$ is the **head** of the dyperedge and the elements of $Z - z$ are the **tails** of the dyperdge. We say that the dyperedge (Z, z) **enters** a set $X \subseteq V$ if the head of Z is in X and at least one of the tails is not. The **in-degree** $\boldsymbol{\varrho_{\mathcal{A}}(X)}$ is defined as the number of dyperedges entering X in $\mathcal{A}$.

By **trimming** a dypergraph $\overrightarrow{\mathcal{H}}$ we mean replacing each dyperedge (Z, z) with an arc tz where t is one of the tails of the dyperedge (Z, z). We say that the node v **can be reached** from the node u if there exists a sequence of dyperedges that can be trimmed to a directed uv path. If $s \in V$ then an ***s*-hyperarborescence** is a subgraph $\overrightarrow{\mathcal{T}}$ of $\overrightarrow{\mathcal{H}}$ which can be trimmed to an s-arborescence. An s-hyperarborescence is **spanning** if every node can be reached from s.

1.1.6 Polymatroids

Let p and b be two set functions on S. For a vector $x \in \mathbb{R}^S$ and $Z \subseteq S$, let $\widetilde{x}(Z) := \sum_{s \in S} x_s$. The polyhedron $\boldsymbol{Q(p,b)}$= $\{x \in \mathbb{R}^S : p(Z) \leq \widetilde{x}(Z) \leq b(Z) \ \forall Z \subseteq S\}$ is called a *generalized-polymatroid* or *g-polymatroid* if p and b have the following properties: $p(\emptyset) = b(\emptyset)$, p is supermodular, b is submodular and $b(X) - p(Y) \geq b(X - Y) - p(Y - X)$ for all $X, Y \subseteq S$. The Minkowski sum of the n g-polymatroids $Q(p_i, b_i)$ is denoted by $\boldsymbol{\sum_{i=1}^n Q(p_i, b_i)}$. For $\alpha, \beta \in \mathbb{R}$, the polyhedron $\boldsymbol{K(\alpha, \beta)}$= $\{x \in \mathbb{R}^S : \alpha \leq \widetilde{x}(S) \leq \beta\}$ is called a plank. For more details on g-polymatroids see [10, Chapter 14]. We will use the following results on g-polymatroids:

Theorem 1.1.1 (Frank [10])**.**

1. *Let $Q(p, b)$ be a g-polymatroid, $K(\alpha, \beta)$ a plank and $M = Q(p, b) \cap K(\alpha, \beta)$.*

   (i) $M \neq \emptyset$ if and only if $p \leq b$, $\alpha \leq \beta$, $\beta \geq p(S)$ and $\alpha \leq b(S)$.

   (ii) M is a g-polimatroid.

   (iii) If $M \neq \emptyset$, then $M = Q(p^\alpha_\beta, q^\alpha_\beta)$ with

$$p^\alpha_\beta(Z) = \max\{p(Z), \alpha - b(S - Z)\}$$

$$b^\alpha_\beta(Z) = \min\{b(Z), \beta - p(S - Z)\}$$

2. *Let $Q(p_1, b_1)$ and $Q(p_2, b_2)$ be two non-empty g-polymatroids and $M = Q(p_1, b_1) \cap Q(p_2, b_2)$.*

   (i) $M \neq \emptyset$ if and only if $p_1 \leq b_2$ and $p_2 \leq b_1$.

   (ii) If p_1, b_1, p_2, b_2 are integral then M is an integral polyhedron.

3. *Let $Q(p_i, b_i)$ be n nonempty g-polimatroids. Then $\sum_1^n Q(p_i, b_i) = Q(\sum_1^n p_i, \sum_1^n b_i)$.*

1.2 Background results

1.2.1 Matroid-based and matroid-restricted packings

The following theorem is a stronger version of Theorem 1.0.1.

Theorem 1.2.1 (Edmonds Theorem (Strong form) [6])**.** *Let $D = (V + s, A)$ be a rooted digraph, and let $\{B_1, \ldots, B_k\}$ be a partition of its root-edges. There exists a packing of spanning s-arborescences $T_1, \ldots, T_k$, where the root-edges of T_i are in B_i for every $i = 1, \ldots, k$ if and only if $\varrho_V(X) \geq |\{i \in \{1, \ldots, k\} : B_i \cap \partial_s(X) = \emptyset\}|$ for all $\emptyset \neq X \subseteq V$.*

The following is the outline of Lovász's proof ([24]) of Theorem 1.2.1:

Proof (sketch). Let $R_i \subseteq V$ be the set of the endpoints of the edges in B_i. Let $p(X) := |\{i \in \{1, \ldots, k\} : B_i \cap \partial_s(X) = \emptyset\}|$. A subset $X \subseteq V$. We call X *tight*, if $p(X) = \varrho_V(X)$ and *dangerous*, if it intersects both $V - R_1$ and R_1 (we assume, that $R_1 \neq V$, otherwise the solution is trivial).

Step 0. Prove the necessity of the condition.

Step 1. Prove that p is supermodular.

Step 2. Prove that the intersection and union of intersecting tight sets are also tight. This implies, that the inclusion-wise maximal tight sets are pairwise disjoint.

Step 3. Show that, if $R_1 \neq V$, then there exists an uv edge that leaves R_1.

Step 4. Show that, if uv does not enter any dangerous sets, then the graph $D - uv$ and the following partition $B_1 + sv, B_2, \ldots, B_k$ of the root-edges satisfy the conditions of the theorem, where sv is a new root-edge.

Step 5. Let $M \subseteq V$ be an inclusion-wise minimal dangerous set. Show that there exists an edge uv in D such that $u \in M \cap R_1$ and $v \in M - R_1$.

Step 6. Show that this edge does not enter any dangerous sets.

□

Let M be a partition matroid on the root-edges with partition $\{B_1, \ldots, B_k\}$ and bounds 1 for all the partition classes. Then an M-based packing of s-arborescences corresponds to a packing satisfying the requirements of Theorem 1.2.1. Indeed, if we take the union of all the arborescences from the packing that have their root-edges in the same partition class then, by the definition of an M-based packing, we get an s-arborescence. Since there are k partition classes and the rank of M is k we get k spanning s-arborescences. Conversely, a spanning s-arborescence with all of its edges in the same class of the partition can be divided into sub-arborescences containing exactly 1 root-edge. In [5], Durand de Gevigney, Nguyen and Szigeti characterized the existence of matroid-based packings of arborescences. The theorem can be proved in a way that is similar to the outlined proof of Edmonds theorem (1.2.1).

Theorem 1.2.2. *(Durand de Gevigney, Nguyen, Szigeti [5]) We are given a graph $D = (V + s, A)$ and a matroid $M = (\partial_s(V), r)$. There is an M-based packing of s-arborescences in D if and only if*

$$\varrho_V(X) \geq r(M) - r(\partial_s(X)) := p(X) \tag{1.5}$$

Furthermore, if we want the S to be the root set of arborescences, then the following must also hold

$$\partial_s(v) \text{ is independent in } M \text{ for every } v \in V. \tag{1.6}$$

The following is the outline of the proof.

Proof (sketch). Let $X \subseteq V$. We say that X is *tight* if $\varrho_V(X) = p(X)$. We say that X *spans* $Y \subseteq V$, if $\varrho_s(Y) \subseteq span_M(\partial_s(X))$. An edge uv is called *bad*, if v spans u, otherwise it is called *good*. If $X \subseteq V$ is tight and there exists a good uv edge that enters X and X spans u, than X is called *dangerous*.

Step 0. Prove the necessity of the condition.

Step 1. Prove that p is supermodular.

Step 2. Prove that the intersection and union of tight sets are also tight.

Step 3. Let $X, Y \subseteq$ be tight sets. Prove that, if a node v is spanned by both X and Y, then $X \cap Y$ also spans v. Note that this statement follows from Statement 1.2.1.

Step 4. Prove that, if $X \subseteq V$ does not span any good edges, then every $v \in X$ spans X.

Step 5. Prove that, if there are no good edges, then we can construct an appropriate packing of arborescences using only the root-edges.

Step 6. Let uv be a good edge that does not enter any dangerous sets. Prove that we can choose a root-edge e entering u such that it is not spanned by the root-edges entering v and if we modify D and M the following way, 1.5 still holds: we delete uv form D and create a new root-edge entering v. Let the corresponding new matroid element be parallel with e.

Step 7. Let X be an inclusion-wise minimal dangerous set. Prove that X induces a good edge.

Step 8. Prove that this edge does not enter any dangerous sets.

□

The previous proof uses the following statement about matroids:

Statement 1.2.1. *Let $M = (S, r)$ be a matroid and $X, Y \subseteq S$ two subsets such that $r(X) + r(Y) = r(X \cup Y) + r(X \cap Y)$. Then if $s \in \text{span}(X) \cap \text{span}(Y)$ then $s \in \text{span}(X \cap Y)$.*

Proof. The span of a set X is defined as $span(X) := \{s \in S : r(X \cup \{s\}) = r(X)\}$. By the submodularity and subcardinality of r and the assumptions of the statement:

$$r(X \cap Y) + r(X \cup Y) = r(X) + r(Y) = r(X + s) + r(Y + s)$$

$$\geq r((X \cap Y) \cup \{s\}) + r(X \cup Y \cup \{s\}) \geq r(X \cap Y) + r(X \cup Y).$$

From this we have $r(X \cap Y) = r((X \cap Y) \cup \{s\})$, which means that $s \in span(X \cap Y)$. □

The packings characterized in the theorems above can also be characterized as common bases of two matroids. Edmonds gave a minmax theorem in [8] for the size of the largest common independent set of two matroids on the same ground set and a polynomial time algorithm to find it.

Theorem 1.2.3. *(Edmonds, [8]) Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be two matroids with rank functions r_1 and r_2. Then*

$$\max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min\{r_1(A) + r_2(S - A) : A \subseteq S\} \tag{1.7}$$

For example in the case of Theorem 1.0.1, let $M_1 = (A, \mathcal{I}_1)$ be the k-graphic matroid of the underlying undirected graph $G = (V, E)$ of D and $M_2 = (A, \mathcal{I}_2)$ be the direct sum of uniform matroids on $\partial(v)$ with rank k for $v \in V - s$ (we can assume that $\varrho(s) = 0$). Note that $r(M_1) = r(M_2) = k(|V| - 1)$. The following statement is common folklore.

Statement 1.2.2. *The subset $B \subseteq A$ is a common basis of M_1 and M_2 if and only if it is the edge-set of a packing of spanning s-arborescences.*

Proof. If B is the edge-set of a packing of spanning s-arborescences, then it is obviously a common basis of M_1 and M_2.

If there exists a common basis B then by Theorem 1.2.3, $r_1(F) + r_2(A - F) \geq k(|V| - 1)$ for all $F \subseteq A$. Our goal is to show that this implies condition (1.1). Since, by restricting M_1 and M_2 to B, we do not change the ranks of the matroids, this implies by Theorem 1.0.1 that B is the edge set of a packing of k s-arborescences.

Assume that a counter example to (1.1) exists, that is, there exists an $X \subseteq V - s$ such that $\varrho(X) < k$. Let F be the set of edges induced by X. Let Y be the set of nodes that have in-going edges in $A - F$. Then $r_2(E - F) = \sum_{v \in Y} \min\{\varrho(v), k\}$. Since $\varrho(X) < k$, $\sum_{v \in Y \cap X} \min\{\varrho(v), k\} < k$. Since $s \notin Y$, $\sum_{v \in Y - X} \min\{\varrho(v), k\} \leq k|Y - X| \leq k(|V| - |X| - 1)$. Therefore, $r_1(F) + r_2(A - F) < k(|X| - 1) + k(|V| - |X| - 1) = k(|V| - 1)$, contradicting (1.7). □

In the case of Theorem 1.2.1 let M_1 be the union of the graphic matroids of G_i ($i = 1, \ldots, k$) where we get G_i by contracting the endpoints of the edges in B_i (the edges induced by the endpoints of the edges in B_i become loops) and let M_2 be the direct sum of uniform matroids on $\partial(v)$ with rank $k - \partial_s(v)$ for $v \in V$. Then the common bases of M_1 and M_2 will exactly be the edge-sets of packings of arborescences satisfying the conditions of Theorem 1.2.1 as proved by Edmonds in [7].

The following theorem characterizes the existence of matroid-based and matroid-restricted packings of s-arborescences, which is a generalization of Theorem 1.2.2 and therefore, all previously mentioned theorems about packings. The proof of this theorem also relies on characterizing the edge-sets of the packings as the common bases of two matroids.

Theorem 1.2.4. *(Cs. Király, Szigeti, Tanigawa [23]) We are given a graph $D = (V + s, A)$, a matroid $M_1 = (\partial_s(V), r_1)$ with a rank function r_1, a matroid M_2 on A, which is the direct sum of the matroids $M_v = (\partial(v), r_v)$. There exist in D an M_1-based M_2-restricted packing of s-arborescences if and only if*

$$r_1(F) + r_2(\partial(X) - F) \geq r_1(\partial_s(V)) \tag{1.8}$$

holds for all $\emptyset \neq X \subseteq V$ and $F \subseteq \partial_s(X)$. If on the neighbouring edges of s $M_2 = M_2|_{\partial_s(V)} \oplus M_2|_{E(V)}$ and $M_2|_{\partial_s(V)}$ is the free matroid, then the condition is the following:

$$r_1(\partial_s(X)) + r_2(\partial(X) - \partial_s(X)) \geq r_1(\partial_s(V)) \tag{1.9}$$

for all $\emptyset \neq X \subseteq V$.

We can use any polynomial algorithm for finding a maximum size common independent set of two matroids (see [10, Chapter 1]) to find an M_1-based M_2-restricted packing.

1.2.2 Reachability packings

If, in Theorem 1.2.1, not every node is reachable from s through the root-edges in B_i for some $i \in \{1, \dots, k\}$, then there obviously does not exists a packing of spanning-arborescences satisfying the conditions of the Theorem. If we only require the arborescences to span the set of nodes that can be reached from the endpoints of their root-edges, we talk about **reachability packings**. The following theorem characterizes the existence of reachability packings in directed graphs.

Theorem 1.2.5. *(Kamiyama, Katoh es Takizawa [20]) Let $D = (V + s, A)$ be a rooted digraph, and let $\{B_1, \dots, B_k\}$ be a partition of its root-edges. There exists a packing of arborescences $T_1, \dots, T_k$, where the root-edges of T_i are in B_i for all $i = 1, \dots, k$ and T_i spans exactly the vertices reachable from s using root-edges only from B_i if and only if*

$$\varrho_V(X) \geq |\{i \in \{1, \dots, k\} : B_i \cap \partial_s(P(X)) \neq \emptyset\}| - |\{i \in \{1, \dots, k\} : B_i \cap \partial_s(X) \neq \emptyset\}|$$

for all $\emptyset \neq X \subseteq V$, where $\boldsymbol{P(X)}$ is the set of nodes in $\boldsymbol{V}$ from which $\boldsymbol{X}$ is reachable

We present the outline of the proof of Theorem 1.2.5. The main idea of the proof is to use induction on the number of strong components of the graph, where the **strong components** of a digraph are the maximal subgraphs that are strongly connected, that is, every node can be reached from every node on a directed path in the subgraph. The idea of the proof comes from Hörsch and Szigeti [17].

Proof (sketch). Let $R_i \subseteq V$ be the set of the endpoints of the edges in B_i.

Step 0. Prove the necessity of the condition.

Step 1. Let C be a strong component of D. Prove that in a packing required by the theorem, if T_i spans a node of C, then it spans C.

Step 2. Let C be a strong component of D such that no edge enters C. Let F be the set of edges in D that enter C. Let $T := \{t_{uv} : uv \in F\}$ and let $D' := (C \cup T, A(C) \cup \{t_{uv}v : uv \in F\} \cup k\{vt_{uv} : uv \in F\})$ a graph where $A(C)$ is the set of edges of D induced by C and for a set S, kS is the multiset we get by taking k copies of every element. Let $R'_i := (R_i \cap C) \cup \{t_{uv} \in T : u \textit{ is reachable from } R_i \textit{ on a directed path}\}$.

Prove that the intersection with C of a packing satisfying the conditions of the theorem corresponds to a packing of spanning arborescences in D'.

Step 3. Show that if D satisfies the conditions of Theorem 1.2.5, then D' satisfies the conditions of the Strong Edmonds Theorem (1.2.1).

Step 4. Prove the theorem by induction on the number of strong components.

□

We call a packing of arborescences $T_1, \dots, T_k$ an M**-reachability-based packing** if every T_i contains exactly one e_i root-edge and for all nodes $v \in V$, the set $\{e_i : v \in V(T_i)\}$ spans $\varrho_s(P(v))$ in M.

Theorem 1.2.6. *(Cs. Király [22]) The graph D contains an M-reachability-based packing of s-arborescences if and only if*

$$\varrho_V(X) \geq r(\partial_s(P(X))) - r(\partial_s(X)) := p(X)$$

for all $X \subseteq V$, where $P(X)$ is the set of nodes in V, from which X is reachable.

This theorem can also be proved with the idea of Hörsch and Szigeti, that is, by induction on the number of strong components.

Proof (sketch). For an edge e and an integer k, $k \cdot e$ means we take k parallel copies of e.

Step 0. Prove the necessity of the condition.

Step 1. Let C be a strong component of D. Prove that, in a packing required by the theorem, every node is in exactly $r(\partial_s(P(C)))$ arborescences.

Step 2. Let $D_1 := D - C$ and $M_1 := M|_{\partial_s(V-C)}$. Show that D_1 contains an M_1-reachability-based packing $T_1^1, \ldots, T_q^1$ with root-edges $e_1^1, \ldots, e_q^1$.

Step 3. Let $T := \{t_{uv} : uv \in A, u \in V - C, v \in C\}$. Let $D' := (C \cup T + s, A')$, where $A' = E(D[C+s]) \cup \{t_{uv}v, (r(\partial_s(P(C))) - r(\partial_s(P(u)))) \cdot vt_{uv}, r(\partial_s(P(u))) \cdot st_{uv} : t_{uv} \in T\}$.

Let M' be the following matroid on the root-edges of D': $M'|_{\partial_s(C)} := M|_{\partial_s(C)}$ and map the root-edges of the arborescences from the packing $T_1^1, \ldots, T_q^1$ that contain u to the new root-edges between s and t_{uv}. If we have two nodes t_{uv} and $t_{uv'}$, than let the root-edges that have the same element of M mapped to them be parallel in M'.

Prove that we can combine the packing $T_1^1, \ldots, T_q^1$ with an M'-based packing of arborescences in D' such that we get an M-reachability-based arborescence packing in D.

Step 4. Show that, if D satisfies the conditions of Theorem 1.2.6, then D' satisfies the conditions of Theorem 1.2.2.

Step 5. Prove the theorem by induction on the number of strong components.

□

This method can be used to derive the reachability versions of the theorems in the following sections.

1.2.3 Free-rooted packings

In [2], Bérczi and Frank characterized the existence of packings of spanning arborescences without specified root-sets, which we call free-rooted packings.

Theorem 1.2.7. *(Bérczi, Frank [2]) Let $D = (V, A)$ be a digraph with n nodes and let $\mu_1, \ldots, \mu_k$ positive integers. The following statements are equivalent:*

(A) *There exists in D a packing of k edge-disjoint spanning arborescences $B_1, \ldots, B_k$, for which $|B_i| = \mu_i$ for all $i = 1, \ldots, k$.*

(B1) *For every subpartition* $\{V_1, \ldots, V_q\}$ *of* V*:*

$$\sum_{j=1}^{k} \max\{0, q - (n - \mu_j)\} \leq \sum_{i=1}^{q} \varrho(V_i) \tag{1.10}$$

(B2) *Let* $[k] = \{1, 2, \ldots, k\}$*. For every subpartition* $\{V_1, \ldots, V_q\}$ *of* V *and for all* $X \subseteq [k]$*:*

$$q(k - |X|) - \sum_{j \in [k]-X} (n - \mu_j) \leq \sum_{i=1}^{q} \varrho(V_i) \tag{1.11}$$

Their proof relies on the following theorem.

Theorem 1.2.8. *(Bérczi, Frank [2]) Let* m_S *be a degree-specification on* S *for which* $\widetilde{m}_S(S) = \gamma$*. Let* p_T *be a positively intersecting supermodular function on* T *with* $p_T(\emptyset) = 0$*. Suppose that*

$$m_S(s) \leq |T| \ \forall s \in S. \tag{1.12}$$

The following statements are equivalent:

(A) *There exists a simple bipartite graph* $G = (S, T, E)$ *which covers* p_T *and fits the degree-specification* m_S*.*

(B1) *For every subpartition* $\{T_1, \ldots, T_q\}$ *of* T *and* $X \subseteq S$*,*

$$\widetilde{m}_S(X) + \sum_{i=1}^{q} p_T(T_i) - q|X| \leq \gamma. \tag{1.13}$$

(B2) *For every subpartition* $\{T_1, \ldots, T_q\}$ *of* T*,*

$$\sum_{i=1}^{q} p_T(T_i) \leq \sum_{s \in S} \min\{m_S(s), q\}. \tag{1.14}$$

In [3], a generalization of Theorem 1.2.8 is provided.

Theorem 1.2.9. *(Bérczi, Frank [3]) We are given a simple bigraph* $H_0 = (S, T, F_0)$*, a matroid* $M = (S, r)$*, a positively intersecting supermodular function* p_T *on* T *with* $p_T(\emptyset) = 0$ *and a degree-specification* m_S *on* S *for which* $\widetilde{m}_S(S) = \gamma$*. There is a simple bigraph* $G = (S, T, E)$ *fitting* m_S *for which* $G^+ = G + H_0$ *is simple and* M*-covers* p_T *if and only if*

$$m_S(s) + d_{H_0}(s) \leq |T| \text{ for every } s \in S, \tag{1.15}$$

and, for every subpartition $\{T_1, \ldots, T_q\}$ *of* T *and* $X \subseteq S$*:*

$$\widetilde{m}_S(X) + \sum_{i=1}^{q} [p_T(T_i) - r(X \cup \Gamma_{H_0}(T_i))] \leq \gamma. \tag{1.16}$$

For completeness, we shall present the outline of the proof given in [3]. The proof uses the following theorem.

Theorem 1.2.10. *(Supermodular arc-covering, set function version, Frank, Jordán [11]) We are given a ground set V, two subsets $S, T \subseteq V$ and a positively ST-crossing supermodular function p for which $p(V') \leq 0$ if $S \subseteq V'$ or $V' \cap T = \emptyset$. The function p can be covered by γ ST-arcs (that is, $\varrho(V') \geq p(V')$ for every $V' \subseteq V$ in the resulting graph) if and only if $\widetilde{p}(\mathcal{I}) \leq \gamma$ holds for every ST-independent family $\mathcal{I}$ of subsets of V.*

We now give the outline of the proof of Theorem 1.2.9.

Proof (sketch). The proof of necessity is straightforward hence we only prove sufficiency.

Let $V := S \cup T$, and for a bigraph $H = (S, T, F)$ let $\overrightarrow{H}$ be the directed graph we get by directing every edge of F from S to T. Let $\mathcal{H}_0 := \{V' \subseteq V : \text{ no arc of } \overrightarrow{H}_0 \text{ enters } V'\}$. Let us define the following set function on ground set V:

$$p_0(X \cup Y) = \begin{cases} p_T(Y) - r(X) & if\ X \cup Y \in \mathcal{H}_0,\ X \subseteq S,\ Y \subseteq T., \\ 0 & \text{otherwise.} \end{cases}$$

Claim 1.2.1. *p_0 is positively T-intersecting supermodular.*

For $s \in S$ let $V_s := \{v \in V - s : sv \notin F_0\}$. Note that $V_s \in \mathcal{H}_0$ for $s \in S$. Let the set function p_1 on V be defined as follows:

$$p_1(U) = \begin{cases} m_S(s) & U = V_s \text{ for some } s, \\ p_0(U) & \text{otherwise.} \end{cases}$$

Claim 1.2.2. *$p_1(V_s) \geq p_0(V_s)$ $\forall s \in S$.*

By Claim 1.2.2, we can prove the following statement.

Claim 1.2.3. *The set function p_1 is positively ST-crossing supermodular.*

Let ν denote the maximum total p_1-value of ST-independent sets, that is, $\nu := \max\{\widetilde{p}_1(\mathcal{I}) : \mathcal{I} \text{ is a family of } ST\text{-independent sets }\}$.

Proposition 1.2.1. *$\nu = \gamma$.*

Proof. Since $\mathcal{L} := \{V_s : s \in S\}$ is ST-independent, $\nu \geq \widetilde{p}_1(\mathcal{L}) = \widetilde{m}_S(S) = \gamma$. For the sake of a contradiction, assume that there exists an ST-independent family $\mathcal{I}$ for which $\widetilde{p}_1(\mathcal{I}) = \nu > \gamma$. We can assume that $|\mathcal{I}|$ is minimal, which means that $p_1(V') > 0$ for all $V' \in \mathcal{I}$.

Claim 1.2.4. *There are no two T-intersecting members V_1 and V_2 of $\mathcal{I}$ such that $p_1(V_i) = p_0(V_i)$ for $i = 1, 2$.*

Let $\mathcal{I}_1 := \{V' \in \mathcal{I} : p_1(V') = p_0(V')\} := \{V_1, \ldots, V_q\}$, and let $\mathcal{T} := \{T_1, \ldots, T_q\}$, where $T_i = V_i \cap T$ for $i = 1, \ldots, q$. Since $p_1(V') > 0$ for all $V' \in \mathcal{I}$, all members of $\mathcal{T}$ are non empty, and by Claim 1.2.4, $\mathcal{T}$ is a subpartition of T.

Let $\mathcal{I}_2 := \mathcal{I} - \mathcal{I}_1$. Every member I of $\mathcal{I}_2$ is of the form $I = V_s$ for some $s \in S$ such that $m_S(s) = p_1(I) > p_0(I)$. Let $X := \{s \in S : V_s \in \mathcal{I}_2\}$. It follows from the definitions that $\mathcal{I}_1$ and $\mathcal{I}_2$ form a partition of $\mathcal{I}$.

Claim 1.2.5. *$X \cup \Gamma_{H_0}(T_i) \subseteq V_i \cap S$ for $i = 1, \ldots, q$.*

By the previous claim and the fact that $\mathcal{T}$ is a partition of T we have

$$\gamma < \nu = \widetilde{p}_1(\mathcal{I}) = \widetilde{p}_1(\mathcal{I}_1) + \widetilde{p}_1(\mathcal{I}_2) =$$
$$\sum_{i=1}^{q}(p_T(T_i) - r(V_i \cap S)) + \sum_{s\in X} m_S(s) \le \sum_{i=1}^{q}(p_T(T_i) - r(X \cup \Gamma_{H_0}(T_i))) + \widetilde{m}_s(X),$$

which contradicts (1.16), completing the proof of the lemma. □ □

Claim 1.2.6. *If $S \subseteq U$ or $U \cap T = \emptyset$ for some $U \subseteq V$ then $p_1(U) \le 0$.*

Therefore, the conditions of Theorem 1.2.10 hold for p_1 and we can apply Theorem 1.2.10 to get a bipartite digraph $D = (S, T, A)$ which contains $\gamma = \nu$ ST-arcs and covers p_1. Let the underlying bipartite graph of D be $G = (S, T, E)$.

Claim 1.2.7. *$d_G(s) = m_S(s)$ for every $s \in S$.*

Claim 1.2.8. *$G^+ = (S, T, F_0 + E)$ is simple.*

Claim 1.2.9. *$r(\Gamma_{G^+}(Y)) \ge p_T(Y)$ for every $Y \subseteq T$.*

Proof. Let $X := \Gamma_{G^+}(Y)$ and $V' := X \cup Y$. Then $0 = \varrho(V') \ge p_1(V') \ge p_T(Y) - r(X)$, from which the claim follows. □

These mean that G satisfies the requirements of the theorem which concludes the proof.

□ □ □

1.2.4 Mixed graphs

Szigeti characterized the existence of free-rooted packings in mixed graphs in the following theorem.

Theorem 1.2.11. *(Szigeti [26]) Let $F = (V, E \cup A)$ be a mixed graph, $f, g : V \to \mathbb{Z}_+$ functions and $k, l, l' \in \mathbb{Z}_+ - \{0\}$. There exists an (f, g)-bounded k-regular (l, l')-limited packing of arborescences in F if and only if $g_k(v) \ge f(v)$ for every $v \in V$, $\min\{\widetilde{g}_k(V), l'\} \ge l$ and*

$$\varrho^{A\cup E}(\mathcal{P}) \ge k|\mathcal{P}| - \min\{l' - f(V - \cup\mathcal{P}), \widetilde{g}_k(\cup\mathcal{P})\} \text{ for every subpartition } \mathcal{P} \text{ of } V. \tag{1.17}$$

The proof relies on the following theorem, which give a polyhedral characterization for free-rooted packings in mixed graphs.

Theorem 1.2.12. *(Szigeti [26]) Let $F = (V, E \cup A)$ be a mixed graph, $f, g : V \to \mathbb{Z}_+$ functions and $k, l, l' \in \mathbb{Z}_+ - \{0\}$. Let $M_v := (\partial^{A\cup A_E}(v), r_v)$ be the free matroid for every $v \in V$. Let M_F^k be the extended k-graphic matroid of F on $A \cup A_E$. Let $T := Q(0, r_{M_F^k}) \cap (\sum_{v\in V}((Q(0, r_v)) \cap K(k - g_k(v), k - f(v))) \cap K(k|V| - l', k|V| - l)$.*

- ***(A)*** *The characteristic vectors of the edge sets of (f, g)-bounded k-regular (l, l')-limited packings of arborescences in orientations of F are exactly the integer points of T.*

(B) *$T \neq \emptyset$ if and only if $g_k(v) \geq f(v)$ for every $v \in V$, $\min\{\widetilde{g}_k(V), l'\} \geq l$ and for every $Z \subseteq A \cup A_E$,*

$$\sum_{v \in V} \max\{0, k - g_k(v) - \partial^Z(v)\} \leq r_{M_F^k}(A \cup A_E - Z) \tag{1.18}$$

$$k|V| - l' - \sum_{v \in V} \min\{\partial^Z(v), k - f(v)\} \leq r_{M_F^k}(A \cup A_E - Z) \tag{1.19}$$

(C) *(3.2) and (3.3) are equivalent to (3.1).*

1.2.5 Hypergraphs

We can state the results shown in the previous sections for dypergraphs. The theorems are very similar to the corresponding digraph versions and most of the time follow from them by constructing a directed graph based on the hypergraph (see [9]). To illustrate the method, we prove here the theorem about packing hyperarborescences corresponding to the Weak Edmonds Theorem.

Theorem 1.2.13. *(Frank, T. Király, Z. Király [12]) Let $\overrightarrow{\mathcal{H}} = (V, \mathcal{A})$ be a dypergraph and $s \in V$ a node. There exists a packing of k spanning s-hyperarborescences if and only if*

$$\varrho_{\mathcal{A}}(X) \geq k \text{ for all } \emptyset \neq X \subseteq V - s. \tag{1.20}$$

Proof. The proof of necessity is straightforward. To prove sufficency, we contruct a directed graph $D = (V \cup \mathcal{A}, A)$. Let $A_1 := \{(Z, z)z : (Z, z) \in \mathcal{A}\}$, $A_1 := \{t(Z, z) : (Z, z) \in A,\ t \in Z - z\}$ and let $A := A_1 \cup (k \cdot A_2)$ where $k \cdot A_2$ is the multiset consisting of k copies of every element in A_2.

First we show that if (1.20) holds for $\overrightarrow{\mathcal{H}}$ then (1.1) holds for D. Let $X \subseteq V \cup \mathcal{A} - s$. If there exists a dyperedge $(Z, z) \in \mathcal{A}$ and a tail node $t \in Z - z$ such that $t \notin X$ and $(Z, z) \in X$ then the k copies of the arc $t(Z, z)$ enter X. Otherwise, $\varrho_A(X) = \varrho_{\mathcal{A}}(X \cap V) \geq k$.

Therefore, by Theorem 1.0.1, D contains a packing of k spanning s-arborescences $\overrightarrow{T}_1, \ldots, \overrightarrow{T}_k$. Let $\overrightarrow{\mathcal{T}_i}$ $(i = 1, \ldots, k)$ be the subdypergraph of $\overrightarrow{\mathcal{H}}$ induced by the dyperedges (Z, z) that have an out-degree of exactly 1 in the packing. It is easy to see that each $\overrightarrow{\mathcal{T}_i}$ $(i = 1, \ldots, k)$ is a spanning s-hyperarborescence and $\overrightarrow{\mathcal{T}_1}, \ldots, \overrightarrow{\mathcal{T}_k}$ are pairwise dyperedge-disjoint. □

1.3 Truncation

The **lower truncation** of a set function b on the ground set S is defined as $b^\vee(X) = \min\{\widetilde{b}(\mathcal{P}) :$ $\mathcal{P}$ is a partition of X $\}$. Let b be finite and intersecting submodular. As proved in [13], the lower truncation of an intersecting submodular function is fully submodular. We describe the algorithm of Frank and Tardos [13] to compute the lower truncation of a finite-valued intersecting submodular function.

The input of the algorithm is a subset $X \subseteq S$ and the function b and the output is $b^\vee(X)$. Let $X = \{u_1, \ldots, u_k\}$. For $i = 1, \ldots, k$, compute $z(u_i) = \min\{b(B) - \widetilde{z}(B - u_i) :\ B \subseteq \{u_1, \ldots, u_i\},\ u_i \in B\}$ and save the set B_i that minimizes the right side. The output of the algorithm is $\widetilde{z}(X)$ and the partition $\mathcal{P}$ of X that we get by taking the connected components of the hypergraph $(X, \{B_i,\ i = 1, \ldots, k\})$.

To compute $\min\{b(B) - z(B - u_i) :\ B \subseteq \{u_1, \ldots, u_i\},\ u_i \in B\}$, we can use a submodular minimization algorithm (see Section 1.4), since $b(B) - \widetilde{z}(B - u_i)$ is submodular on the subsets of $\{u_1, \ldots, u_i\}$. Indeed, $\widetilde{z}$ is modular, b is intersecting modular and we minimize over a pairwise intersecting family of sets (every set contains u_i).

1.4 Submodular optimization

In this section we present the algorithm of Iwata [18] for minimizing a submodular function. The first polynomial algorithm for submodular function minimization was developed by Grötschel et al. in [14] and subsequently, a strongly polynomial version was also developed by Grötschel et al. in [15, Chapter 10]. These algorithms employ the ellipsoid method. The first strongly polynomial combinatorial algorithms were developed by Iwata et al. [19] and Schrijver [25]. All of these algorithms use multiplications and divisions, although the problem of submodular function minimization does not involve these operations. Schrijver [25] has asked if there exists a strongly polynomial fully combinatorial algorithm for submodular function minimization, that is, it uses only additions, subtractions, comparisons and function evaluations and does not use divisions. The algorithm of Iwata [18] presented in this section is the first such algorithm.

1.4.1 Definitions

Let $D = (V, F)$ be an acyclic digraph. A subset $Y \subseteq V$ is called an **ideal**, if no arc leaves Y in D. Let $\mathcal{D}$ denote the set of ideals of D.

Let $g : \mathcal{D} \to \mathbb{R}$ be a submodular function with $g(\emptyset) = 0$. The **base polyhedron** of g is

$$B(g) = \{y |\ y \in \mathbb{R}^V,\ \widetilde{y}(V) = g(V),\ \widetilde{y}(X) \leq g(X)\ \forall X \in \mathcal{D}\}$$

A vector $y \in B(g)$ is called a base, and an extreme point of $B(g)$ is called an **extreme base**. An extreme base can be computed by the greedy algorithm [10, Section 14.5].

Let $L = (v_1, \ldots, v_n)$ be a linear ordering of V. Let $L(v_i) := L_i := \{v_1, \ldots, v_i\}$. L is a **reverse topological ordering** of V, if $L_i \in \mathcal{D}$ for $i = 1, \ldots, n$. The greedy algorithm generates an extreme base y with respect to a reverse topological ordering L the following way:

$$y(v_i) = g(L_i) - g(L_{i-1})$$

where i goes from 1 to n. Moreover, every extreme base $y \in B(g)$ can be generated by the greedy algorithm with respect to a reverse topological ordering (see [10, Section 14.5]).

Let L be a reverse topological ordering, in which the node u immediately succeeds the node v. If $(u, v) \notin F$, then the linear ordering we get by swapping u and v is also a reverse topological ordering. We get the extreme base $y' \in B(g)$ by $y' = y + \beta(\chi_u - \chi_v)$, where

$$\beta = g(L(u) - v) - y(L(u) - v) \geq 0$$

and for a node $w \in V$, χ_w is the characteristic vector of the subset $\{w\} \subseteq V$.

1.4.2 The algorithm

We are given a submodular set function $f : 2^S \to \mathbb{R}$ on the ground set S. Our goal is to find a subset $Y \subseteq S$ such that $f(Y)$ is minimal.

In an iteration of the algorithm, we have a subset $Z \subseteq S$ which is included in every minimizer of f, a partition on $S - Z$ and an acyclic digraph $D = (V, F)$. The nodes of the graph correspond to the classes of the partition, if $v \in V$ then let $T(v)$ denote the corresponding class and for $X \subseteq V$ let $T(X) := \bigcup_{v \in X} T(v)$. The following property holds for the edges of D: if $uv \in F$ then every minimizer of f that contains $T(u)$ also contains $T(v)$. At the start of the algorithm $Z := \emptyset$, $V := S$ and $F := \emptyset$.

Let $\hat{f} : \mathcal{D} \to \mathbb{R}$ such that $\hat{f}(Y) := f(T(Y) \cup Z) - \min\{f(Z), f(S)\}$ if $Y \neq \emptyset$ or V and $\hat{f}(\emptyset) = \hat{f}(V) := 0$. It is easy to check that $\hat{f}$ is submodular. Note that, if W minimizes f, than it can be written as $W = T(Y) \cup Z$ where Y is an ideal of D. Furthermore:

Lemma 1.4.1 ([18])**.** *If $W \subseteq S$ is a minimizer of f, then it can be represented as $W = T(Y) \cup Z$ where Y is a minimizer of $\hat{f}$.*

For a node $v \in V$ let $R(v)$ be the set of nodes that are reachable from v in D. Then obviously $R(v)$ is an ideal of D. In an iteration, the algorithm first computes the following quantity:

$$\alpha := \max_{v \in V}\{\hat{f}(R(v)) - \hat{f}(R(v) - v)\} \tag{1.21}$$

Let $\sigma := |V|\alpha$.

Lemma 1.4.2 ([18])**.** *If $\alpha \leq 0$, then either S or Z is a minimizer of f.*

By Lemma 1.4.2 if $\alpha \leq 0$, the algorithm returns $\text{argmin}\{f(Z), f(S)\}$. If $\alpha > 0$, let $u \in V$ be a node such that $\alpha = \hat{f}(R(u)) - \hat{f}(R(u) - u)$. Then, either $2\hat{f}(R(u)) \geq \alpha$ or $2\hat{f}(R(u) - u) < -\alpha$.

Case 1. $2\hat{f}(R(u) - u) < -\alpha < 0$

In this case, the algorithm calls the subroutine $Fix(\hat{f}, \alpha, \sigma)$ (the subroutine is explained in 1.4.3), and finds a node $w \in V$ that is contained by every minimizer of $\hat{f}$. By Lemma 1.4.1, $T(w)$ is contained by every minimizer of f. The algorithm then adds $T(W)$ to Z and removes w from V (that is, $Z := Z \cup T(W)$ and $V := V - w$). Z and D continue to satisfy the required properties.

Case 2. $2\hat{f}(R(u)) \geq \alpha$

The algorithm calls $Fix(\hat{f}_u, \alpha, \sigma)$ and finds a node $w \in V - R(u)$, that is contained in every minimizer of $\hat{f}_u$ where $\hat{f}_u(X) := \hat{f}(X \cup R(u)) - \hat{f}(R(u))$ when $X \subseteq V - R(u)$ and $X \cup R(u)$ is an ideal of D. It is easy to see that $\hat{f}_u$ is submodular, and $2\hat{f}_u(V - R(u)) = 2(\hat{f}(V) - \hat{f}(R(u))) \leq -\alpha$. A set $X \subseteq V - R(u)$ minimizes $\hat{f}_u$ if and only if $X \cup R(u)$ minimizes $\hat{f}$ on $\{D \in \mathcal{D} | u \in D\}$. This implies that if $u \in D \in \mathcal{D}$ minimizes $\hat{f}$, than $w \in D$ too. Which in turn, by Lemma 1.4.1, implies that if $Y \subseteq S$ minimizes f and $T(u) \subseteq Y$, then $T(w) \subseteq Y$.

The algorithm adds (uw) to F. If this creates a directed cycle, we contract the nodes of the cycle into a new node, because a minimizer of f must include all or non of the elements of S represented by the nodes of the cycle. The resulting Z and D continue to satisfy the required properties.

In every iteration with $\alpha > 0$ either $|V|$ decreases or $|F|$ increase, which means that the algorithm stops after $O(n^2)$ iterations. The following lemma also holds, ensuring, that the procedure Fix can be called.

Lemma 1.4.3. *([18])*

- *In Case 1., every exchange capacity in $B(\hat{f})$ is at most σ.*
- *In Case 2., every exchange capacity in $B(\hat{f}_u)$ is at most σ.*

The following theorem gives the running time of the algorithm.

Theorem 1.4.1. *The algorithm finds a minimizer of f by $O(n^9 \log^2 n)$ oracle calls and fundamental operations.*

1.4.3 The Fixing Procedure

In this section, we describe the subroutine $Fix(g, \alpha, \sigma)$. The procedure finds a node $w \in V$ that is contained in every minimizer of g. We assume, that every exchange capacity in $B(g)$ is less than σ, there exists an ideal $Y \in \mathcal{D}$ such that $2g(Y) \leq -\alpha$ and $\sigma \leq n\alpha$. The subroutine also uses the graph $D = (V, F)$ used in the main algorithm.

The subroutine consists of scaling phases with a scale parameter $p \in \mathbb{Z}$ which we double at the end of a phase until it is sufficiently large. Initially $p := 1$. We have a set of linear orderings $\{L_i | i \in I\}$ of V, the corresponding extreme bases $\{y_i | i \in I\}$ which we get from the orderings by the greedy algorithm and non-negative integral coefficients $\{\lambda_i | i \in I\}$ such that $\sum_{i \in I} \lambda_i = p$. Let $x := \sum_{i \in I} \lambda_i y_i$. Initially $I = \{0\}$, L_0 is an arbitrary linear ordering and $\lambda_0 = 1$.

Furthermore, we also have a flow on the complete directed graph on the vertex set V, represented by a function $\varphi : V \times V \to \mathbb{R}$. The function satisfies $\varphi(u, v) + \varphi(v, u) = 0$ for all $u, v \in V$. The capacities of every arc in E/F are σ, that is $\varphi(u, v) \leq \sigma$, where E is the arc set of the complete directed graph. The boundary of φ is defined as $\partial\varphi(u) = \sum_{v \in V} \varphi(u, v)$, that is, $\partial\varphi(u)$ the size of the flow leaving the node $u \in V$.

Let $z := \partial\varphi + x$. The goal of a scaling phase is to increase $z^-(V)$. Let $G(\varphi) := (V, A(\varphi))$ with $A(\varphi) := F \cup \{(u, v) | u \neq v,\ \varphi(u, v) \leq 0\}$. Let $S := \{v | z(v) \leq -\sigma\}$ and Let $T := \{v | z(v) \geq \sigma\}$. A directed path from S to T in $G(\varphi)$ is called an augmenting path (increasing the flow among this path increases $Z^-(V)$).

Let W be the set of nodes reachable from S in $G(\varphi)$. W is an ideal of D, since D is a subgraph of $G(\varphi)$. Let $i \in I$, $u \in W$ and $v \in V - W$. The triple (i, u, v) is called an **active triple**, if u immediately succeeds v in L_i. The operation Double-Exchange works on an active triple (i, u, v), and modifies φ and x without changing z. The goal is to modify L_i by exchanging u and v and modify φ such that v becomes reachable from S in $G(\varphi)$. The procedure first computes the exchange capacity:

$$\beta = g(L_i(u) - v) - y(L_i(u) - v) \geq 0$$

Note, that $\beta \leq \sigma$ by our assumption. There are two cases based on the size of β:

Case 1. $\varphi(u, v) \geq \lambda_i \beta$ (Saturating case)

In this case we replace y_i by exchanging u and v: $y_i := y_i + \beta(\chi_u - \chi_v)$. We need to modify φ such that z does not change: $\varphi(u, v) := \varphi(u, v) - \lambda_i\beta$ and $\varphi(v, u) := \varphi(v, u) + \lambda_i\beta$. The resulting flow continues to satisfy the capacities (note, that the original $\varphi(u, v) > 0$, otherwise v be part of W) and z remains unchanged.

Case 2. $\varphi(u, v) \leq \lambda_i \beta$ (Non-saturating case)

In this case we have to be careful to keep φ feasible. First, we calculate $q := \lceil \frac{\varphi(u,v)}{\beta} \rceil$. This can be done without dividing (we want a fully combinatorial algorithm) by repeatedly subtracting β from $\varphi(u, v)$. Since $\varphi(u, v) \leq \lambda_i\beta$, the number of required subtractions is $\leq \lambda_i \leq p$. We add a new index k to I, and make $y_k := y_i$, $L_k := L_i$ and $\lambda_k := q$. Note, that $\varphi(u, v) \leq \lambda_i\beta = q\beta \leq \lambda_i\beta + \beta$.

Let $\lambda_i := q$ (this way, $\sum \lambda_j = p$ remains true), and now replace y_i and modify φ the same way as in the saturating case: $y_i := y_i + \beta(\chi_u - \chi_v)$, $\varphi(u, v) := \varphi(u, v) - \lambda_i\beta$ and $\varphi(v, u) := \varphi(v, u) + \lambda_i\beta$. The resulting φ satisfies $-\sigma \leq -\beta \leq \varphi(u, v) \leq 0$, thus it satisfies the capacities and v becomes reachable in $G(\varphi)$ from S.

We now describe the procedure Fix(g, α, σ):

Step 0 Let $p := 1$, $I := \{0\}$, $\lambda_0 := 1$, L_0 an arbitrary ordering, y_0 the extreme base computed from L_0 with the greedy algorithm and $\varphi(u, v) := 0$ for all $u, v \in V$.

Step 1 If there exists an augmenting path P in $G(\varphi)$:

$\varphi(u,v) := \varphi(u,v) + \sigma$ and $\varphi(v,u) := \varphi(v,u) - \sigma$. In this case $z^-(V)$ increases by σ.

Otherwise, apply Double-Exchange(i,u,v) to an active triple.

Step 2 If $\exists w \in V$ such that $x(w) < -n^2\sigma$, return w.

Step 3 $p := 2p$, $\lambda_i := 2\lambda_i$ for all $i \in I$. Go to Step 1.

The following lemmas ensure, that the algorithm returns a node, that is contained in every minimizer of g:

Lemma 1.4.4. *([18]) At the end of a scaling phase, $z^-(V) \geq pg(W) - n\sigma$ holds.*

Lemma 1.4.5. *([18]) If $x(w) < -n^2\sigma$ holds at the end of a scaling phase, then w is contained in every minimizer of g.*

This last lemma ensures that the procedure is finite.

Lemma 1.4.6. *([18]) The procedure consists of $O(\log n)$ scaling phases.*

Chapter 2

Free-rooted packings of arborescences with matroid constraints

In this chapter we characterize the existence of free-rooted matroid-based and matroid-restricted packings of arborescences, give two characterizations of the existence of free-rooted matroid-based packings of arborescences with an in-degree prescription and provide a new characterization for the existence of a free-rooted arborescence packing with an in-degree prescription.

2.1 Free-rooted matroid-based matroid-restricted packings

Using Theorem 1.2.4 and Theorem 1.2.9, we can characterize the existence of a free-rooted matroid-based and matroid restricted packing of arborescences.

Theorem 2.1.1. *Let $D=(V,A)$ be a digraph, let $M_1=(S,r_1)$ be a matroid with rank function r_1 and rank k and let M_2 be a matroid on A which is the direct sum of the matroids $M_v=(\partial(v),r_v)$. Let s be a node not in V. The following statements are equivalent:*

- ***(A)*** *We can add new possibly parallel arcs from s to some of the nodes of V and map the elements of S to the new edges such that there exists an M_1-based M_2'-restricted packing of s-arborescences, where M_2' is defined to be the direct sum of the free matroid on the new root-edges and M_2.*
- ***(B)*** *For every subpartition $\{V_1,\dots,V_q\}$ of V and $X\subseteq S$:*

$$(k-r_1(X))q-|S-X|\le\sum_{i=1}^{q}r_2(\partial(V_i)) \tag{2.1}$$

Proof. Necessity. Suppose that such a packing exists. Then at most $r_2(\partial(Y))$ and at least $k-\partial_s(Y)$ edges of the packing enter a set $Y\subset V$, thus

$$\sum_{i=1}^{q}(k-r_1(\partial_s(V_i)))\le\sum_{i=1}^{q}r_2(\partial(V_i))$$

Using the properties of the rank function we can show that, for every $X\subseteq S$

$$\sum_{i=1}^{q} r_1(\partial_s(V_i)) \leq \sum_{i=1}^{q} r_1(\partial_s(V_i) \cap X) + r_1(\partial_s(V_i) - X)) \leq q r_1(X) + |S - X|.$$

Hence

$$\sum_{i=1}^{q} r_2(\partial(V_i)) \geq \sum_{i=1}^{q} (k - r_1(\partial_s(V_i))) \geq qk - q r_1(X) - |S - X|.$$

Sufficiency. Let $m_S : S \to \mathbb{Z}_+$ be 1 for every element of S. Let $T := V$ and let us define the following set function on T,

$$p_T(Y) = \begin{cases} k - r_2(\partial(Y)) & \varnothing \subsetneq Y \subseteq T, \\ 0 & Y = \varnothing. \end{cases}$$

Since r_2 is submodular, p_T is intersecting supermodular. From the conditions of the theorem,

$$(k - r_1(X))q - \sum_{i=1}^{q} r_2(\partial(V_i)) \leq |S - X| = \widetilde{m}_S(S - X) = \widetilde{m}_S(S) - \widetilde{m}_S(X).$$

Thus,

$$-r_1(X)q + \sum_{i=1}^{q} (k - r_2(\partial(V_i))) \leq \widetilde{m}_S(S) - \widetilde{m}_S(X).$$

Hence,

$$\sum_{i=1}^{q} p_T(V_i) + \widetilde{m}_S(X) - r_1(X)q \leq \widetilde{m}_S(S).$$

This is the condition of Theorem 1.2.9 with $F_0 = \emptyset$, therefore there exists a simple bipartite graph $G = (S, V, E)$, which fits m_S and M_1-covers p_T, that is, $r_1(\Gamma(Y)) \geq k - \varrho(Y)\ \forall Y \subset V$. Orient the edges of G from S to T, add the edges of D in T and contract the nodes of S into a new node s. $\Gamma(Y) = \partial_s(Y)$ holds therefore, since G M_1-covers p_T, $r_1(\partial_s(Y)) \geq k - r_2(\partial(Y))$ holds, which is the condition of Theorem 1.2.4 with matroids M_1 and M_2'. Therefore, Theorem 1.2.4 implies that there exists an M_1-based M_2-restricted packing of s-arborescences. □

2.2 Free-rooted matroid-based packings with in-degree prescriptions

Using the previous theorem, we can characterize the existence of a free-rooted matroid-based packing of arborescences with an in-degree prescription.

Corollary 2.2.1. *Let $M = (S, r)$ be a matroid with rank function r, let $D = (V, A)$ be a digraph with n nodes and let $m_{in} : V \to \mathbb{Z}^+$ be an in-degree prescription for which $0 \leq m_{in}(v) \leq \varrho(v)$, $m_{in}(v) \leq r(M)$ for all $v \in V$ and $\widetilde{m}_{in}(V) = |V|r(M) - |S|$ holds. Let s be a node not in V. The following statements are equivalent:*

(A) *We can add new arcs from s to some of the nodes of V and map the elements of S to the new edges such that there exists an M-based packing of s-arborescences and if the edge set of the packing is F, then $\varrho_V^F(v) = m_{in}(v)$ holds for every $v \in V$.*

(B) *For all $X \subseteq S$ and subpartition $\{V_1, \ldots, V_q\}$ of V:*

$$(r(M) - r(X))q - |S - X| \leq \sum_{i=1}^{q} \sum_{v \in V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\} \tag{2.2}$$

Proof. Let $M_1 := M$ and $\forall v \in V$ let M_v be the uniform matroid on $\partial(v)$ with rank $m_{in}(v)$. Let M_2 be the direct sum of the matroids M_v. Then

$$r_2(\partial(V_i)) = \sum_{v \in V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\},$$

therefore

$$\sum_{i=1}^{q} r_2(\partial(V_i)) = \sum_{v \in \bigcup_{i=1} V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\}.$$

Hence (2.2) is the same as the condition of Theorem 2.1.1, thus there exists an M_1-based M_2-restricted packing of arborescences. This means that at most $m_{in}(v)$ arborescence enters every node v. Since $\widetilde{m}_{in}(V) = |V|r(M) - |S|$ and the right side is the number of edges in an M-based restriction, exactly $m_{in}(v)$ edge enters every node. □

Let R be a set of nodes. A graph is called an R-branching if all of its connected components are arborescences and the set of the roots of the components is R. The size of branching is the number of its edges. Using Corollary 2.2.1 we can prove a new characterization for the existence of a free-rooted packing of branchings with prescribed sizes and an in-degree prescription.

Corollary 2.2.2. *let $D = (V, A)$ be a digraph with n nodes and let $m_{in} : V \to \mathbb{Z}^+$ be an in-degree prescription for which $0 \leq m_{in}(v) \leq \varrho(v)$ and $m_{in}(V) \leq k$ for all $v \in V$. Let $\mu_1, \ldots, \mu_k$ be k positive integers, for which $\sum_{i=1}^{k} \mu_i = \widetilde{m}_{in}(V)$. The following statements are equivalent:*

(A) *There exist in D a packing of spanning branchings $B_1, \ldots, B_k$, for which $|B_i| = \mu_i$ and if $\bigcup_{i=1}^{k} B_i = F$, then $v \in V : \varrho^F(v) = m_{in}(v)$.*

(B) *For every subpartition $\{V_1, \ldots, V_q\}$ of V:*

$$\sum_{i=1}^{k} \max\{0, q - (n - \mu_i)\} \leq \sum_{v \in \bigcup_{i=1}^{q} V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\} \tag{2.3}$$

Proof. Let $n - \mu_j := m_j$. This is the number of roots for a spanning branching with μ_j edges.

Let M be a partition matroid with k classes, where the size of the ith class is m_i and the bound is 1 for every class. If there exists a packing satisfying the requirements of (A) then add a new node s and a root-edge from s to each of the roots of the branchings in the packing (if a node is the root of more than one branchings then we add a root-edge for each of the branchings). Then, in the resulting digraph, each of the branchings can be divided into s-arborescences containing exactly one root-edge and the packing of s-arborescences we get by dividing all of the branchings is an M-based packing. Conversely, a free-rooted M-based packing corresponds to a packing of branchings satisfying the requirements of (A).

Therefore, by Corollary 2.2.1, $(r(M) - r(X))q - |S - X| \leq \sum_{v \in \bigcup_{i=1}^{q} V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\}$ is equivalent to (A). We can assume that the set X contains either the entire partition class or it is disjoint from it. This is because if it intersects a class, then if we add the elements from the class that are not contained in X, then the left side increase and the right side stays the same. So if $I = \{1, \dots, k\}$, then

$$(A) \Leftrightarrow (k - |X|)q - \widetilde{m}(S - X) \leq \sum_{v \in \bigcup_{i=1}^{q} V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\} \forall X \subseteq I$$

The left side is maximized by $X = \{i \in I : m(i) > q\}$ and for this set $(k - |X|)q - \widetilde{m}(S - X) = \sum_{i=1}^{k} \max\{0, q - (n - \mu_i)\}$ holds. □

In [2], Bérczi and Frank provide a different characterization for the same problem with the following condition:

For all $Y \subseteq V$ and subpartition $\{V_1, \dots, V_q\}$ of $V - Y$:

$$\sum_{i=1}^{k} \max\{0, q + |Y| - (n - \mu_i)\} \leq \widetilde{m}_{in}(Y) + \sum_{i=1}^{q} \varrho(V_i) \tag{2.4}$$

This result follows from the following theorem, which gives a different characterization for the existence of a free-rooted matroid-based packing of arborescences with an in-degree prescription with a seemingly weaker condition. The proof is based on Theorems 1.2.2 and 1.2.9.

Theorem 2.2.1. *Let $M = (S, r)$ be a matroid with rank function r, let $D = (V, A)$ be a digraph with n nodes and let $m_{in} : V \to \mathbb{Z}^+$ be an in-degree prescription for which $0 \leq m_{in}(v) \leq \varrho(v)$, $m_{in}(v) \leq r(M)$ for all $v \in V$ and $\widetilde{m}_{in}(V) = |V|r(M) - |S|$ holds. Let s be a node not in V. The following statements are equivalent:*

- ***(A)*** *We can add new arcs from s to some of the nodes of V and map the elements of S to the new edges such that there exists an M-based s-arborescence packing and if the edge set of the packing is F, then $\varrho_V^F(v) = m_{in}(v)$ holds for every $v \in V$.*
- ***(B)*** *For all $Y \subseteq V$, subpartition $\{V_1, \dots, V_q\}$ of $V - Y$, and $X \subseteq S$,*

$$(|Y| + q)(r(M) - r(X)) - |S - X| \leq \widetilde{m}_{in}(Y) + \sum_{i=1}^{q} \varrho(V_i) \tag{2.5}$$

Furthermore, (2.2) implies (2.5).

Proof. First we will prove that (2.2) implies (2.5). By Theorem 2.2.1, this implies the necessity of (2.5).

Let us suppose that (2.2) holds and we are given a set $Y \in V$ and a subpartition $\mathcal{P} = \{V_1, \dots, V_q\}$ of $V - Y$. Let us define the following partition of V: $\mathcal{P}' = \mathcal{P} \cup \bigcup_{v \in Y} \{v\}$. Then $|\mathcal{P}'| = q + |Y|$, $m_{in}(Y) = \sum_{v \in Y} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\}$ and $\sum_{v \in \bigcup_{i=1}^{q} V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\} \leq \sum_{i=1}^{q} \varrho(V_i)$, so (2.5) holds.

To prove sufficiency let $m_S : S \to \mathbb{Z}_+$ be 1 for every element of S. Let $T := V$ and let us define the following set function on T:

$$p_T(Y) = \begin{cases} r(M) - \varrho(Y) & Y \subseteq T, |Y| \geq 2, \\ r(M) - m_{in}(v) & Y = \{v\}, v \in V \\ 0 & Y = \varnothing. \end{cases}$$

Since $m_{in}(v) \leq \varrho(v)$, $k - m_{in}(v) \geq k - \varrho(v)$ so p_T is intersecting supermodular.

Let $\mathcal{T} = \{V_1, \ldots, V_q, \ldots, V_{q'}\}$ be a subpartition of T, where the last $q' - q$ classes are singletons. Let $\mathcal{P} = \{V_1, \ldots, V_q\}$ and $Y = V_{q+1} \cup \cdots \cup V_{q'}$.

From the definition of p_T,

$$\sum_{i=1}^{q'} p_T(V_i) = \sum_{i=1}^{q} [r(M) - \varrho(V_i)] + \sum_{i=q+1}^{q'} [r(M) - \widetilde{m}_{in}(V_i)] = (|Y| + q)r(M) - \sum_{i=1}^{q} \varrho(V_i) - \widetilde{m}_{in}(Y).$$

If we apply the condition of Theorem 1.2.9 to $\mathcal{T}$, a set $X \subset S$ and $F_0 = \emptyset$ and we use the previous equation for $\sum_{i=1}^{q'} p_T(V_i)$, we get the following.

$$\widetilde{m}_S(X) + (|Y| + q)r(M) - \sum_{i=1}^{q} \varrho(V_i) - \widetilde{m}_{in}(Y) - q'r(X) \leq \widetilde{m}_S(S).$$

If we reorder the terms and use that $q' = |Y| + q$ and $\widetilde{m}_S(S) - \widetilde{m}_S(X) = |S - X|$ we get the condition in (2.5). Hence there exists a simple bipartite graph $G = (S, V, E)$ which covers p_T-t and satisfies the degree prescription. From this we get that for every $v \in V$-re $r(M) - m_{in}(v) \leq r(\Gamma_G(v)) \leq d_G(v)$ (where $\Gamma_G(v)$ is the set of neighbours of v in G), thus

$$\sum_{v \in V} [r(M) - m_{in}(v)] \leq \sum_{v \in V} d_G(v) = \sum_{s \in S} d_G(s) = |S|.$$

Since $\widetilde{m}_{in}(V) = |V|r(M) - |S|$, the left hand side of the previous equation is $|S|$, hence we have equality everywhere, thus we get $d_G(v) = r(\Gamma_G(v)) = r(M) - m_{in}(v)\ \forall v \in V$.

Since $\forall Y \subset V\ r(M) - \varrho(Y) \leq r(\Gamma_G(Y))$ also holds, if we contract S, orient its outgoing edges towards $T = V$ and add the edges of D on V, then the condition of Theorem 1.2.2 holds for the resulting digraph, thus there exists an M-based packing of s-arborescences. Since $r(\Gamma_G(v)) = r(M) - m_{in}(v)$, at least $m_{in}(v)$ arborescence enters v with a non-root-edge. We can assume that exactly $r(\Gamma_G(v))$ root-edge is in the packing (since otherwise we can exchange certain edges of the arborescences), so there exists a packing which enters v with exactly $m_{in}(v)$ edges.

□

Chapter 3

Free-rooted packings of arborescences in mixed graphs

In this Chapter, we characterize the existence of (f,g)-bounded k-regular (l,l')-limited M-restricted packing of arborescences in mixed graphs, which is a generalization of a result of Szigeti [26] (see section 1.2.4). As a corollary, we also characterize the existence of (f,g)-bounded k-regular (l,l')-limited packing of arborescences in mixed graphs with degree-constraints.

3.1 Matroid-restricted free-rooted packings of arborescences in mixed graphs

The main result of this chapter is the following theorem.

Theorem 3.1.1. *Let $F=(V,E\cup A)$ be a mixed graph, $f,g:V\to\mathbb{Z}_+$ functions and $k,l,l'\in\mathbb{Z}_+-\{0\}$. Let $M_v:=(\partial^{A\cup A_E}(v),r_v)$ be a matroid for every $v\in V$, and let $M:=\bigoplus_{v\in V}M_v$ with a rank function r. There exists an (f,g)-bounded k-regular (l,l')-limited M-restricted packing of arborescences in F if and only if $g_k(v)\geq f(v)$ for every $v\in V$, $\min\{\widetilde{g}_k(V),l'\}\geq l$ and*

$$R(\mathcal{P})\geq k|\mathcal{P}|-\min\{l'-f(V-\cup\mathcal{P}),\widetilde{g}_k(\cup\mathcal{P})\}\text{ for every subpartition }\mathcal{P}\text{ of }V \tag{3.1}$$

where $R(\mathcal{P})=\max\{r(\overrightarrow{\partial^{A\cup E}(\mathcal{P})})\}$, where $\overrightarrow{\partial^{A\cup E}(\mathcal{P})}$ is an orientation of $\partial^{A\cup E}(\mathcal{P})$.

The proof of this theorem relies on the following theorem.

Theorem 3.1.2. *Let $F=(V,E\cup A)$ be a mixed graph, $f,g:V\to\mathbb{Z}_+$ functions and $k,l,l'\in\mathbb{Z}_+-\{0\}$. Let $M_v:=(\partial^{A\cup A_E}(v),r_v)$ be a matroid for every $v\in V$, and let $M:=\bigoplus_{v\in V}M_v$ with a rank function r. Let M_F^k be the extended k-graphic matroid of F on $A\cup A_E$. Let $T:=Q(0,r_{M_F^k})\cap K(k|V|-l',k|V|-l)\cap\sum_{v\in V}[(Q(0,r_v))\cap K(k-g_k(v),k-f(v))]$.*

- ***(A)*** *The characteristic vectors of the edge sets of (f,g)-bounded k-regular (l,l')-limited M-restricted packings of arborescences in orientations of F are exactly the integer points of T.*
- ***(B)*** *$T\neq\emptyset$ if and only if $g_k(v)\geq f(v)$ for every $v\in V$, $\min\{\widetilde{g}_k(V),l'\}\geq l$ and for every $Z\subseteq A\cup A_E$,*

$$\sum_{v\in V}\max\{0,k-g_k(v)-r_v(\partial^Z(v))\}\leq r_{M_F^k}(A\cup A_E-Z) \tag{3.2}$$

$$k|V| - l' - \sum_{v\in V} \min\{r_v(\partial^Z(v)), k - f(v)\} \le r_{M_F^k}(A \cup A_E - Z) \tag{3.3}$$

(C) *(3.1) implies (3.2) and (3.3).*

Proof. **(A)**

By Theorem 1.2.12/(A), the integer points of T are characteristic vectors of the edge sets of (f, g)-bounded k-regular (l, l')-limited packings of arborescences in orientations of F. Since an integer point of T is also in $\sum_{v\in V} Q(0, r_v)$, the corresponding packing is also M-restricted.

By the other direction of Theorem 1.2.12/(A) and since the characteristic vector of an M-restricted packing must be in $\sum_{v\in V} Q(0, r_v)$, the integer points of T are exactly the characteristic vectors of the edge sets of the required packings.

(B)

By Theorem 1.1.1.1, $Q(0, r_v)) \cap K(k - g_k(v), k - f(v))$ is non empty if and only if $k - g_k(v) \le k - f(v)$ (which is equivalent to $g_k(v) \ge f(v)$), $k - f(v) \ge 0$ (which is true because $0 \le k - g_k(v) \le k - f(v)$) and $k - g_k(v) \le r_v(\partial^{A\cup A_E}(v))$ (which we will see later).

If $Q(0, r_v)) \cap K(k - g_k(v), k - f(v)) \ne \emptyset$ then it is equal to $Q(p_v, b_v)$, where by Theorem 1.1.1.1/(iii), for $Z \subseteq A \cup A_E$ and $Z_v = Z \cap \partial^{A\cup A_E}(v)$,

$$p_v(Z_v) = \max\{0, k - g_k(v) - r(\partial^{A\cup A_E - Z_v}(v))\}$$

$$b_v(Z_v) = \min\{r(\partial^{Z_v}(v), k - f(v)\}$$

By Theorem 1.1.1.3, $\sum_{v\in V} Q(p_v, b_v) = Q(p_\Sigma, b_\Sigma)$, where $p_\Sigma = \sum_{v\in V} p_v$, $b_\Sigma = \sum_{v\in V} b_v$.

By Theorem 1.1.1.1, $Q(0, r_{M_F^k}) \cap K(k|V| - l', k|V| - l) \ne \emptyset$ if and only if $k|V| - l' \le k|V| - l$ (which is equivalent to $l' \ge l$), $k|V| - l \ge 0$ (which follows from $k|V| - l \ge \widetilde{g}_k(V) - l \ge 0$) and $k|V| - l' \le r_{M_F^k}(A \cup A_e)$, which is (3.3) for $Z = A \cup A_E$. Thus, by Theorem 1.1.1.1/(iii), $Q(0, r_{M_F^k}) \cap K(k|V| - l', k|V| - l) = Q(p, b)$ where $p(Z) = \max\{0, k|V| - l' - r_{M_F^k}(A \cup A_E - Z)\}$ and $b(Z) = \min\{r_{M_F^k}(Z), k|V| - l\}$.

By Theorem 1.1.1.2, $Q(p, b) \cap Q(p_\Sigma, b_\Sigma) \ne \emptyset$ if and only if $p_\Sigma \le b$ and $p \le b_\Sigma$, that is

$$\sum_{v\in V} \max\{0, k - g_k(v) - r_v(\partial^{A\cup A_E - Z_v}(v))\} \le \min\{r_{M_F^k}(Z), k|V| - l\}$$

and

$$\max\{0, k|V| - l' - r_{M_F^k}(A \cup A_E - Z)\} \le \sum_{v\in V} \min\{r_v(\partial^{Z_v}(v)), k - f(v)\}$$

The first inequality is equivalent to (3.2) by the fact that $\max\{0, k|V| - l' - r_{M_F^k}(A \cup A_E - Z)\} \le \sum_{v\in V} k - g_k(v) \le k|V| - l$ (here we use that $\min\{\widetilde{g}_k(V), l'\} \ge l$). Since $k - f(v) \ge k - g_k(v) \ge 0$ and $r_v \ge 0$, $0 \le \sum_{v\in V} \min\{r_v(\partial^{Z_v}(v)), k - f(v)\}$, and $k|V| - l' - r_{M_F^k}(A \cup A_E - Z) \le \sum_{v\in V} \min\{r_v(\partial^{Z_v}(v)), k - f(v)\}$ is equivalent to (3.3).

Finally, $k - g_k(v) \le r_v(\partial^{A\cup A_E}(v))$ follows from $p_\Sigma(\emptyset) \le b(\emptyset)$ and the proof is complete.

(C)

Note, that (3.2) is equivalent to

$$k|V| - g_k(V) - \sum_{v\in V} \min\{r_v(\partial^Z(v)), k - g_k(v)\} \le r_{M_F^k}(A \cup A_E - Z). \tag{3.4}$$

Let $Z \subseteq A \cup A_E$. By (1.4), there exists a partition $\mathcal{P}$ of V such that for $K = \{e \in E(\mathcal{P}) : (A \cup A_E - Z) \cap A_e \neq \emptyset\}$:

$$r_{M_F^k}(A \cup A_E - Z) = |(A \cup A_E - Z) \cap A(\mathcal{P})| + |K| + k(|V| - |P|). \tag{3.5}$$

Let $\mathcal{P}_h := \{X \in \mathcal{P} : \ r_v(\partial^Z(v)) \leq k - h(v) \ (\forall v \in X)\}$, where $h \in \{f, g_k\}$. Then $\mathcal{P}_h$ is a subpartition of V and for every $X \in \mathcal{P} - \mathcal{P}_h$ there exists a $v_X \in X$ such that $r_v(\partial^Z(v)) > k - h(v)$.

By the definition of K, we have

$$A_{E(\mathcal{P}_h)-K} \subseteq Z \cap A_{E(\mathcal{P}_h)}. \tag{3.6}$$

Thus, by (3.5), the definition of $\mathcal{P}_h$ and v_X, $r_v(\partial^Z(v)) \geq 0$, $k - h \geq 0$, $h \geq 0$ and $r(X) \leq |X|$ (that is, the subcardinality of the rank function of a matroid), we have

$$\begin{aligned}
&r_{M_F^k}(A \cup A_E - Z) + \sum_{v \in V} \min\{r_v(\partial^Z(v)), k - h(V)\} \\
=&|(A \cup A_E - Z) \cap A(\mathcal{P})| + |K| + k(|V| - |\mathcal{P}|) + \sum_{v \in \cup\mathcal{P}} \min\{r_v(\partial^Z(v)), k - h(V)\} \\
&+ \sum_{v \in V - \cup\mathcal{P}} \min\{r_v(\partial^Z(v)), k - h(V)\} \\
\geq&|(A \cup A_E - Z) \cap A(\mathcal{P}_h)| + \sum_{v \in \cup\mathcal{P}} r_v(\partial^Z(v)) + \sum_{X \in \mathcal{P} - \mathcal{P}_h} \sum_{v \in X} \min\{r_v(\partial^Z(v)), k - h(V)\} \\
&+ |K| + k(|V| - |\mathcal{P}|) \\
\geq&|(A \cup A_E - Z) \cap A(\mathcal{P}_h)| + r((Z \cap A(\mathcal{P}_h)) \cup (Z \cap A_{E(\mathcal{P}_h)})) + \sum_{X \in \mathcal{P} - \mathcal{P}_h} (k - h(v_X)) \\
&+ |K| + k(|V| - |\mathcal{P}|) \\
\geq&r((A \cup A_E - Z) \cap A(\mathcal{P}_h)) + r((Z \cap A(\mathcal{P}_h)) \cup (Z \cap A_{E(\mathcal{P}_h)})) + \sum_{X \in \mathcal{P} - \mathcal{P}_h} (k - h(X)) \\
&+ |K| + k(|V| - |\mathcal{P}|) \\
\geq&r((A \cup A_E - Z) \cap A(\mathcal{P}_h)) + r((Z \cap A(\mathcal{P}_h)) \cup (Z \cap A_{E(\mathcal{P}_h)})) + k(|\mathcal{P}| - |\mathcal{P}_h|) - h(V - \cup\mathcal{P}_h) \\
&+ |K| + k(|V| - |\mathcal{P}|) \\
=&r((A \cup A_E - Z) \cap A(\mathcal{P}_h)) + r((Z \cap A(\mathcal{P}_h)) \cup (Z \cap A_{E(\mathcal{P}_h)})) - k|\mathcal{P}_h| - h(V - \cup\mathcal{P}_h) + |K| + k|V|.
\end{aligned}$$

By (3.6) and the submodularity of r, we get

$$\begin{aligned}
&r((A \cup A_E - Z) \cap A(\mathcal{P}_h)) + r((Z \cap A(\mathcal{P}_h)) \cup (Z \cap A_{E(\mathcal{P}_h)})) \\
\geq&r(A(\mathcal{P}_h) \cup (Z \cap A_{E(\mathcal{P}_h)})) + r(((A \cup A_E - Z) \cap A(\mathcal{P}_h)) \cap (Z \cap A_{E(\mathcal{P}_h)}))) \\
\geq&r(A(\mathcal{P}_h) \cup A_{E(\mathcal{P}_h)-K}) + r(\emptyset) \\
\geq&R(\mathcal{P}_h) - |K|.
\end{aligned}$$

In the last inequality we use $r(X - K) \geq r(X) - |K|$. By the previous two inequalities, we get

$$r_{M_F^k}(A \cup A_E - Z) + \sum_{v \in V} \min\{r_v(\partial^Z(v)), k - h(V)\} \geq R(\mathcal{P}_h) - k|\mathcal{P}_h| - h(V - \cup\mathcal{P}_h) + k|V|.$$

Using this inequality for $h = f$ and (3.1) we get (3.3), and if we apply it for $h = g_k$, we get (3.4). $\square$

Next, we prove Theorem 3.1.1.

Proof of Theorem 3.1.1. Necessity. The necessity of $g_k \geq f$ and $\min\{\widetilde{g}_k(V), l'\} \geq l$ is trivial. Let $\mathcal{P}$ be a subpartition of V and let B be the arc set of an (f,g)-bounded k-regular (l,l')-limited M-restricted packing of arborescences in an orientation $\overrightarrow{F}$ of F. For a node v, let the number of v-arborescences in the packing be $q(v)$. Let C be a class of $\mathcal{P}$. By k-regularity, there is at least k arborescences in the packing, which have arcs induced by C. If the root of an arborescence is not in C, then it enters it. Thus, the number of edges in B that enter C is at least $k - \sum_{v\in C} q(v)$. The number of edges in B entering a class of $\mathcal{P}$ is therefore at least $k|\mathcal{P}| - \sum_{C\in\mathcal{P}} \widetilde{q}(C) = k|\mathcal{P}| - \widetilde{q}(\cup\mathcal{P})$. Since the packing is (f,g)-bounded and (l,l')-limited, we have $\widetilde{q}(\cup\mathcal{P}) \leq \min\{l' - f(V - \cup\mathcal{P}), \widetilde{g}_k(\cup\mathcal{P})\}$, therefore the right side of (3.1) is a lower bound on the number of edges in B, that enter a member of $\mathcal{P}$. Since B is independent in M, we get (3.1).

Sufficiency. Let $(F = (V, E\cup A), f, g, k, l, l')$ be an instance of Theorem 3.1.1, that satisfies the necessary conditions. Since (3.1) holds, by Theorem 3.1.2/(C), (3.2) and (3.3) hold. Since $g_k \geq f$ and $\min\{\widetilde{g}_k(V), l'\} \geq l$ also hold, by Theorem 3.1.2/(B), T (as defined in Theorem 3.1.2) is nonempty, thus, by Theorem 1.1.1/2./(ii), it contains an integral element x. By Theorem 3.1.2/(A), x is the characteristic vector of the edge sets of an (f,g)-bounded k-regular (l,l')-limited M-restricted packing of arborescences in an orientation $\overrightarrow{F} = (V, \overrightarrow{E}\cup A)$ of F. Replacing the arcs in $\overrightarrow{E}$ with the edges in E, we get the required packing. $\square$

3.2 Free-rooted packings of arborescences in mixed graphs with degree-constraints

If we choose the matroid in Theorem 3.1.1 to be a partition matroid, we can prescribe bounds on the in-going arcs and edges in the packing, as follows.

Corollary 3.2.1. *Let $F = (V, E\cup A)$ be a mixed graph, $f, g, p, q : V \to \mathbb{Z}_+$ functions and $k, l, l' \in \mathbb{Z}_+ - \{0\}$. There exists an (f,g)-bounded k-regular (l,l')-limited packing of arborescences in F with $\varrho^{A\cap T}(v) \leq q(v)$ and $\varrho^{A_E\cap \overrightarrow{T}}(v) \leq p(v)$ for every $v \in V$ (where T is the edge set of the packing and $\overrightarrow{T}$ is an orientation of T where we orient each mixed-arborescence in the packing so that they become arborescences), if and only if $g_k(v) \geq f(v)$ for every $v \in V$, $\min\{\widetilde{g}_k(V), l'\} \geq l$ and for every subpartition $\mathcal{P}$ of V*

$$\min_{B\subseteq\partial_E(\mathcal{P})}\{|B| + \sum_{v\in V(\mathcal{P})} \min\{p(v), |\partial^E(v)\cap(\partial^E(\mathcal{P}) - B)|\}\} + \sum_{v\in V(\mathcal{P})} \min\{q(v), |\partial^A(v)\cap\partial^A(\mathcal{P})|\}$$
$$\geq k|\mathcal{P}| - \min\{l' - f(V - \cup\mathcal{P}), \widetilde{g}_k(\cup\mathcal{P})\} \tag{3.7}$$

Proof. For every $v \in V$, let M_v a partition matroid with partition classes $\partial^A(v)$ and $\partial^E(v)$, and bounds $q(v)$ and $p(v)$ and let $M := \bigoplus_{v\in V} M_v$ with a rank function r. Then, an M-restricted packing with edge set T satisfies $\varrho^{A\cap T}(v) \leq q(v)$ and $\varrho^{A_E\cap T}(v) \leq p(v)$.

Let R be as defined in Theorem 3.1.1. Since $M = M|\partial^A(\mathcal{P}) \oplus M|\partial^A(\mathcal{P})$, $R(\mathcal{P}) = R_E(\mathcal{P}) + r(\partial^A(\mathcal{P}))$ where $R_E(\mathcal{P}) = \max\{r(\overrightarrow{\partial^E(\mathcal{P})}) : \overrightarrow{\partial^E(\mathcal{P})}$ is an orientation of $\partial^E(\mathcal{P})\}$.

It is easy to see, that $r(\partial^A(\mathcal{P})) = \sum_{v\in V(\mathcal{P})} \min\{q(v), |\partial^A(v)\cap\partial^A(\mathcal{P})|\}$. To compute $R_E(\mathcal{P})$, we have to find an orientation $\overrightarrow{\partial^E(\mathcal{P})}$ that maximizes

$$\sum_{v \in V(\mathcal{P})} \min\{p(v), d(v)\} \tag{3.8}$$

where $d(v) := \varrho_{\overrightarrow{\partial^E(\mathcal{P})}}(v)$. We can assume, that in the orientation, every edge enters a class of the subpartition $\mathcal{P}$.

Let F be the set of arcs we get by orienting every edge in $uv \in \partial^E(\mathcal{P})$ that has both of its endpoints in $V(\mathcal{P})$ and oriented towards $V(\mathcal{P})$ if only one endpoint is in $V(\mathcal{P})$, that is $F = \{\overrightarrow{uv} : uv \in \partial^E(\mathcal{P}),\ v \in V(\mathcal{P})\}$. Let M_1 be a partition matroid on F with rank function r_1 where the classes are $\{\overrightarrow{vu}, \overrightarrow{uv}\}$ if $u, v \in V(\mathcal{P})$ and $\{\overrightarrow{uv}\}$ if only $v \in V(\mathcal{P})$, and the bounds are one for every class of the partition. Let M_2 also be a partition matroid on F with rank function r_2, where the classes of the partition are $\varrho_F(v)$ for every $v \in V(\mathcal{P})$ a with bounds $p(v)$.

Let $\overrightarrow{\partial^E(\mathcal{P})}$ be an orientation of $\partial^E(\mathcal{P})$. Delete edges until every in-degree is at most $p(v)$. The set of arcs we get is a common independent set of M_1 and M_2. The size of the constructed independent set will be 3.8.

Let I be a common independent set of M_1 and M_2. Since I is independent in M_1 it does not contain any parallel edges from F, so it defines an orientation of a subset of $\partial^E(\mathcal{P})$. Since I is independent in M_2, every in-degree in the orientation is at most $p(v)$. If I is a maximum size common independent set and we direct the rest of the edges arbitrarily, then we get an orientation, that maximizes 3.8.

By Theorem 1.2.3, we have

$$\max\{|I| : I \in M_1 \cap M_2\} = \min\{r_1(B) + r_2(F - B) : B \subseteq F\} \tag{3.9}$$

Let $B \subseteq F$ and assume that both $\overrightarrow{uv} \in F$ and $\overrightarrow{vu} \in F$. Then, $r_1(B - \overrightarrow{uv}) = r_1(B)$ and $r_2(F - B + \overrightarrow{uv}) \geq r_1(F - B)$ so we can assume that the minimizer of (3.9) contains either both or zero edges of a parallel pair. From this we get

$$R_E(\mathcal{P}) = \min\{|B| + \sum_{v \in V(\mathcal{P})} \min\{p(v), |\partial^E(v) \cap (\partial^E(\mathcal{P}) - B)|\} : B \subseteq \partial^E(\mathcal{P})\}$$

Thus, we get that for a subpartition $\mathcal{P}$ of V:

$$\begin{aligned} R(\mathcal{P}) = min_{B \subseteq \partial^E(\mathcal{P})}\{|B| + \sum_{v \in V(\mathcal{P})} \min\{p(v), |\partial^E(v) \cap (\partial^E(\mathcal{P}) - B)|\}\} \\ + \sum_{v \in V(\mathcal{P})} \min\{q(v), |\partial^A(v) \cap \partial^A(\mathcal{P})|\} \end{aligned} \tag{3.10}$$

Therefore (3.7) is equivalent to (3.1) with f, g, k, l, l' and the matroid M which proves the statement.

□

If $f(v) = g(v)$ for all $v \in V$, then we may reformulate Corollary 3.2.1 so that we have both lower and upper bounds on the in-going arcs in the packing:

Corollary 3.2.2. *Let $F = (V, E \cup A)$ be a mixed graph, $f, p, q : V \to \mathbb{Z}_+$ functions and $k, l, l' \in \mathbb{Z}_+ - \{0\}$. There exists an (f, f)-bounded k-regular (l, l')-limited packing of arborescences in F with $p(v) \leq \varrho^{A \cap T}(v) \leq q(v)$ for every $v \in V$ (where T is the edge set of the packing), if and only if $\min\{\widetilde{f}_k(V), l'\} \geq l$ and for every subpartition $\mathcal{P}$ of V*

$$
\begin{aligned}
&min_{B\subseteq\partial_E(\mathcal{P})}\{|B| + \sum_{v\in V(\mathcal{P})}\min\{k - f(v) - p(v), |\partial^E(v)\cap(\partial^E(\mathcal{P}) - B)|\}\} \\
&+ \sum_{v\in V(\mathcal{P})}\min\{q(v), |\partial^A(v)\cap\partial^A(\mathcal{P})|\} \geq k|\mathcal{P}| - \min\{l' - f(V - \cup\mathcal{P}), \widetilde{f}_k(\cup\mathcal{P})\}
\end{aligned} \tag{3.11}
$$

Proof. Apply Corollary 3.2.1 to $F, f, f, k, l, l', k - f - p$ and q. □

Chapter 4

Algorithms

In this chapter we will present algorithms to find arborescence packings which satisfy the properties of the theorems proved in Chapter 2 and 3.

4.1 Algorithm for finding a free-rooted matroid-based and matroid-restricted packing of arborescences

In this section we give an algorithm to find a free-rooted packing of arborescences satisfying the conditions of Theorem 2.1.1. That is, we are given a digraph $D = (V, A)$, a matroid $M_1 = (S, r_1)$ with rank function r_1 and rank k, and a matroid M_2 on A which is given as the direct sum of matroids $M_v(\partial_A(v), r_v)$. Let s be a node not in V (all matroids are given by its rank oracles). Our aim is to add new possibly parallel arcs from s to some of the nodes of V and map the elements of S to the new edges such that there exists an M_1-based M_2'-restricted packing of s-arborescences, where M_2' the direct sum of the free matroid on the new edges and M_2.

The algorithm is based on the proof of Theorem 2.1.1. Let m_S be a degree-prescription on S for which $m_S(s) = 1$ for every $s \in S$. Let the set function p_T on T be defined as in the proof of Theorem 2.1.1, that is,

$$p_T(Y) = \begin{cases} k - r_2(\partial(Y)) & \varnothing \subsetneq Y \subseteq T, \\ 0 & Y = \varnothing. \end{cases}$$

Now we find a bipartite graph $G = (S, V, E)$ fitting m_S that M_1-covers p_T using the algorithm described in Section 4.1.1. If the condition does not hold then the algorithm gives a set $X \subseteq S$ and a subpartition $\mathcal{T}$ of T that does not satisfy condition (1.16) and equivalently, condition (2.1). Otherwise, let $D' = (V + s, A')$ be the digraph we get by directing the edges of E from S to T, contracting S to get the node s and adding the arcs A on V. As can be seen in the proof of Theorem 2.1.1, the graph D' and matroids M_1 and M_2' satisfy the conditions of Theorem 1.2.4, that is, we can find a M_1-based M_2'-restricted packing of s-arborescences in D' using the algorithm in [23], ending our algorithm.

Since the algorithm described in Section 4.1.1 is polynomial and the matroid intersection problem is also polynomially solvable, our algorithm is polynomial.

4.1.1 Degree-specified matroidal augmentation

In this section we will present an algorithm from an unpublished manuscript of Bérczi and Frank ([1]) to find a bipartite graph which satisfies the properties in Theorem 1.2.9. They actually give an algorithm for a slightly different version of Theorem 1.2.9 in which we have a degree-prescription not only on S, but also on T. We only use the version of the theorem presented in the thesis, therefore the algorithm is slightly modified.

Let m_S be a vector on S. Bérczi and Frank gave a polyhedral description to the vectors that satisfy the conditions of Theorem 1.2.9 in [1]. The vector m_S has to satisfy $m_S(s) \leq |T| - d_{H_0}(s)$ for all $s \in S$ and condition (1.16):

$$\widetilde{m}_S(X) + \sum_{i=1}^{q} [p_T(T_i) - r(X \cup \Gamma_{H_0}(T_i))] \leq \gamma,$$

Where $X \subseteq S$ and $\mathcal{T} = \{T_1, \ldots, T_q\}$ is a subpartition of T. We define the following set function b_0 on S.

$$b_0(X) = \min\{\sum_{U \in \mathcal{T}}^{q} [r(X \cup \Gamma_{H_0}(U)) - p_T(U)] : \mathcal{T} \text{ is a subpartition of } T\}. \tag{4.1}$$

Then Condition (1.16) is equivalent to the following.

$$\widetilde{m}_S(X) \leq b_0(X) - \gamma. \tag{4.2}$$

Bérczi and Frank proved the following theorem.

Theorem 4.1.1. *(Bérczi, Frank [1])* b_0 *is fully submodular.*

The idea of the proof is to use the uncrossing procedure (replacing certain pairs of T-intersecting sets with their union and intersection) on a family of sets we construct from the subpartitions minimizing the right side of (4.1) for two subsets of S so that we can use the supermodular inequality on the resulting family which does not contain any properly T-intersecting pairs of sets.

The following corollary to Theorem 4.1.1 gives a polyhedral characterization to degree-vectors satisfying the conditions of Theorem 1.2.9.

Corollary 4.1.1. *(Bérczi, Frank [1]) Given a matroid* $M = (S, r)$ *and a positively intersecting supermodular set function* p_T *on* T*, an integral vector* $m_S \in \mathbb{Z}^S$ *is the degree-vector restricted to* S *of a simple bigraph* $G + (S, T, E)$ *which* M*-covers* p_T *if and only if it is in the polyhedron* $\{x \in \mathbb{R}^S : \widetilde{x} \leq b_0 - \gamma,\ \widetilde{x}(S) = \gamma,\ 0 \leq x(s) \leq |T|\ \forall s \in S\}$

Now we describe the algorithm to find a bigraph satisfying the conditions of Theorem 1.2.9 when p_T is intersecting supermodular. We are given a simple bigraph $H_0 = (S, T, F_0)$, a matroid $M = (S, r)$, an intersecting supermodular set function p_T on T and a degree-specification m_S on S for which $\widetilde{m}_S(S) = \gamma$ and $m_S(s) + d_{H_0}(s) \leq |T|$ for every $s \in S$.

It suffices to show that we can check the validity of (1.16) in polynomial time. Indeed, there is a solution to the original problem if and only if there is an edge $st \notin F_0$ $(s \in S,\ t \in T)$ for which the modified problem with $H_0' := H_0 + e$, $m_S'(s) := m_S(s) - 1$ has a solution. Therefore, we can find a solution by trying to add edges to H_0 on-by-one and checking the validity of (1.16) until the degree-prescription becomes 0 at each node.

By Corollary 4.1.1, (1.16) holds if and only if the vector m is in the polyhedron $\{x \in \mathbb{R}^S : \widetilde{x} \leq b_0 - \gamma,\ \widetilde{x}(S) = \gamma,\ 0 \leq x(s) \leq |T|\ \forall s \in S\}$. By Theorem 4.1.1, b_0 is submodular and since the set function $X \mapsto \sum_{s \in X} m_S(s)$ $(X \subseteq S)$ is modular, the set function $b_0 - \gamma - \widetilde{x}$ is submodular. Hence, checking $\widetilde{x} \leq b_0 - \gamma$ for $x = m_S$ can be done by checking if the minimum value of $b_0 - \gamma - \widetilde{x}$ is non-negative using a submodular minimization algorithm (see Section 1.4) if a subroutine for evaluating b_0 is at hand. That is, our aim is to show that

$$b_0(X) = \min\{\sum_{U \in \mathcal{T}}^{q} [r(X \cup \Gamma_{H_0}(U)) - p_T(U)] : \mathcal{T} \text{ is a subpartition of } T\}$$

can be determined for a fixed set $X \subseteq S$. Let us define a set function on T as follows:

$$b_X(U) := r(X \cup \Gamma_{H_0}(U)) - p_T(U)$$

.

Claim 4.1.1. *(Bérczi, Frank [1]) The function b_X is intersecting submodular.*

By definition,

$$b_0(X) = \min\{\sum_{U \in \mathcal{T}} b_X(U) : \ \mathcal{T} \text{ is a subpartition of } T\}.$$

Let the set function $\hat{b}_X$ be defined as

$$\hat{b}_X(U) = \begin{cases} b_X(U) & \varnothing \subsetneq U \subseteq T, \\ 0 & U = \varnothing. \end{cases}$$

The lower truncation of $\hat{b}_X$ is defined as

$$\hat{b}_X^{\vee}(W) := \min\{\sum_{U \in \mathcal{P}} b_X(U) : \ \mathcal{P} \text{ is a partition of } W\}. \quad (4.3)$$

Since by Claim 4.1.1 b_X is intersecting submodular, $\hat{b}_X$ is also intersecting submodular, therefore the minimum value of the right side of (4.3), together with a partition attaining the minimum, can be calculated by using the algorithm of Frank and Tardos ([13], described in Section 1.3).

By definition,

$$b_0(X) = \min\{\hat{b}_X^{\vee}(W) : \ W \subseteq T\}, \quad (4.4)$$

and since the lower truncation of an intersecting submodular function is fully submodular [13], $b_0(X)$ can be computed using a submodular minimization algorithm.

If (1.16) does not hold then the subset $X \subseteq S$ for which $\widetilde{m}(X) \not\leq b_0(X) - \gamma$ we get by minimizing the submodular function $b_0(X) - \widetilde{m}(X) - \gamma$ with the algorithm described in Section 1.4 and the subpartition of T that minimizes the right side of (4.1) for X provides a counterexample to (1.16).

Since the algorithm calls the submodular minimization algorithm at most $|E|$ times and evaluating b_0 can be done in polynomial time, the resulting algorithm is also polynomial.

4.2 Mixed graphs

In this section we show that finding a mixed-arborescence packing satisfying the conditions of Theorem 3.1.1 can be done in polynomial time. That is, we are given a mixed graph $F = (V, E \cup A)$, functions $f, g : V \to \mathbb{Z}_+$, positive integers k, l, l', and a matroid $M_v := (\partial^{A \cup A_E}(v), r_v)$ for every $v \in V$ (we assume M_v is given by its rank oracle). Let $M := \bigoplus_{v\in V} M_v$ with a rank function r. Our aim is to find an (f,g)-bounded k-regular (l,l')-limited M-restricted packing of arborescences.

In Theorem 3.1.2 we showed that the characteristic vectors of the edge sets of (f,g)-bounded k-regular (l,l')-limited M-restricted packings of arborescences in orientations of F are exactly the integer points of $T = Q(0, r_{M_F^k}) \cap K(k|V| - l', k|V| - l) \cap \sum_{v\in V}[(Q(0, r_v)) \cap K(k - g_k(v), k - f(v))]$. Therefore, our goal is to find an integer point in T. Finding an integer point in a polyhedron is NP-hard in general ([21]).

Claim 4.2.1. *$T = Q(0, r_{M_F^k}) \cap K(k|V| - l', k|V| - l) \cap \sum_{v\in V}[(Q(0, r_v)) \cap K(k - g_k(v), k - f(v))]$ is an integral polyhedron.*

Proof. By Theorem 1.1.1.1/(ii), both $Q(0, r_{M_F^k}) \cap K(k|V|-l', k|V|-l)$ and $(Q(0, r_v)) \cap K(k-g_k(v), k-f(v))$ are g-polymatroids. By Theorem 1.1.1.3, $\sum_{v\in V}[(Q(0, r_v)) \cap K(k - g_k(v), k - f(v))]$ is also a g-polymatroid. Therefore, T is the intersection of two g-polymatroids and since $f, q, k, l, l', r_{M_F^k}$ and r_v are all integral, by Theorem 1.1.1.2/(ii), T is an integral polyhedron. □

If we can decide for a vector $x \in \mathbb{R}^{A\cup A_E}$ if $x \in T$ then we can use any polynomial algorithm for optimizing over polyhedrons (see [4, Chapters 8 and 9]) to find an integer point in T. Deciding $x \in K(k|V|-l', k|V|-l) \cap \sum_{v\in V}[(Q(0, r_v)) \cap K(k-g_k(v), k-f(v))]$ is trivial. To decide if $x \in Q(0, r_{M_F^k})$ for a vector $x \geq 0$, we need to be able to compute its rank function (1.4) which can be done in polynomial time as shown in [16]. Then we can minimize the submodular function $r_{M_F^k} - \widetilde{x}$ and if the minimum is negative then $x \notin Q(0, r_{M_F^k})$, otherwise $x \in Q(0, r_{M_F^k})$.

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