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Quantitative Helly-type theorems for axis-parallel boxes

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Chapter 1

Introduction

Helly's theorem states that *if the intersection of any* $d+1$ *members of a finite family of convex sets in* $\mathbb{R}^d$ *have a non-empty intersection, then all members of the family have a non-empty intersection.* In other words, if the intersection of a finite family of convex sets is empty, then already the intersection of a $(d + 1)$-tuple from the family is empty. A significant property of this statement is that the size of the tuples does not depend on the size of the family.

Several similar theorems have been proven regarding a plethora of different objects ranging from different kinds of geometric sets, to trees or hypergraphs [1] [2] [3]. The property of the families of objects that are implied by these theorems can also vary greatly. Bárány, Katchalski and Pach [4] showed, that *if there is a lower bound on the volume of the intersection of any* $2d$*-tuple from a family of convex sets of dimension* d*, then there is a lower bound on the volume of the intersection of the whole family which does not depend on the size of the family.* This result is the so-called quantitative volume theorem.

Another interesting property of a family of sets is called pierceability by tuples of points. A family of sets is n-pierceable, if there are n points such that any member of the family contains at least one of the points. Danzer and Grünbaum [5] explored whether there are similar Helly-type theorems about the pierceability of axis-parallel boxes in $\mathbb{R}^d$. Among other things, they showed that

(1): n*-pierceability of all* $(n + 1)$*-tuples from a finite family of intervals implies* n*-pierceability of the whole family,*

(2): 2*-pierceability of all* $(3d - 1)$*-tuples, respectively* $3d$*-tuples from a finite family of axis-parallel boxes in* $\mathbb{R}^d$ *implies the 2-pierceability of the whole family for odd, respectively even* d *and* $d \geq 2$.

Furthermore, they also proved, that for smaller tuple-sizes, the above statements are not true. This is also the case for Helly's theorem and the stated quantitative volume theorem.

The smallest tuple-size for which a certain Helly-type statement holds will be called its *Helly-number*.

The goal of this thesis is to obtain quantitative variants of Danzer and Grünbaum's results about pierceability of axis-parallel boxes and to find the Helly-numbers. Section 1.1 provides an overview of Helly-type theorems in general and Sections 1.1.1 and 1.1.2 present some of the most notable and relevant ones to our results.

In Section 1.2, we introduce the notion of *punching* - a new framework for quantitative variants of statements about piercing - and analyze some of its basic properties. A family of sets is n-punchable, if there are n sets of volume (at least) 1 such that any member of the family contains at least one of the sets. In Section 1.3, we show that certain statements about punching convex sets imply statements about piercing convex sets. Therefore, Danzer and Grünbaum's results preclude the existence of some Helly-type theorems about punching. We also show that certain statements about punching by sets of translates are reducible to statements about piercing.

Our main results are presented in Chapter 2. In Section 2.1 we prove **Theorem** 2.1.1 and **Theorem** 2.1.2, two Helly-type theorems about n-punching of families of intervals. The former states that *if every $(n+1)$-tuple is n-punchable from a finite family of intervals, then the whole family is n-punchable*, while the latter is a so-called colorful version of this theorem. It is also shown that $n+1$ is the Helly-number for these statements. Section 2.2 offers some insights into the properties of 2-punching of axis-parallel boxes in $\mathbb{R}^d$ and the possible Helly-type theorems regarding 2-punching. While it remains to be seen whether there are Helly-type theorems about 2-punching axis-parallel boxes, it can be shown that for certain tuple-sizes, a given Helly-type statement is certainly not true, i.e. lower bounds for the Helly-number can be shown. In Section 2.3 several lower bounds for h are given for the following statement: *2-punchability of all h-tuples from a finite family of axis-parallel boxes in $\mathbb{R}^d$ implies 2-punchability of the whole family*. In **Theorem** 2.3.1 we show a lower bound of $4d-1$ for any dimension d, by a constructive proof. We examine some principles for giving a stronger lower bound in Section 2.3.2, while the principles are implememted to obtain a stronger lower bound for 3-dimensional boxes in **Theorem** 2.3.4.1, which is 18. In Section 2.3.4, a rather different constructive proof of **Theorem** 2.3.8 shows that in the plane, 10 is a lower bound for the Helly-number.

Finally, some open questions are adressed in Chapter 3.

1.1 Helly-type theorems

Classical Helly-type theorems state that if a condition A holds for any subfamily $\mathcal{H}$ of a given finite size h from a family of sets $\mathcal{F}$, then some condition B holds for the whole family $\mathcal{F}$ which is of arbitrary finite size N. An equivalent and often useful formulation provided by negations is that if $\mathcal{F}$ does not satisfy condition B, then some subfamily $\mathcal{H}$ of size h does not satisfy condition A, i.e. showing a given $\mathcal{H}$ is sufficient to disprove condition A for $\mathcal{F}$. The minimal number h for which a given Helly-type statement holds will be referred to as the **Helly-number**.

Helly's original statement is about the emptyness of the intersection of a family of convex sets in Euclidean space.

Theorem 1.1.1 (Helly)**.** *For a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$, if any $(d+1)$-tuple of sets in $\mathcal{F}$ has a non-empty intersection, then all sets in $\mathcal{F}$ have a non-empty intersection.*

The terms (n-)tuple and (sub)family (of size n) will be used interchangibly, and will usually refer to a proper subset of the family of objects $\mathcal{F}$. Any family discussed in this thesis will be assumed to be finite. Note that in Helly's theorem, condition A and B are the same. Not only is it true, but $d+1$ is the lowest tuple-size for which a statement of the same form holds, so $d+1$ is the Helly-number in this case. **Lovász** and later **Bárány** introduced a variant of the classic Helly-theorem where there is not only one, but a multitude of families of objects and the chosen tuples are colorful selections, that is, they are systems of distinct representatives of the different families. The resulting theorem, the so called **Colorful Helly Theorem**, is a stronger result, as the original Helly theorem is the subcase of this statement in which all families are the same.

Theorem 1.1.2 (Colorful Helly Theorem. Lovász, Bárány [6])**.** *For finite families (color classes) $\mathcal{F}_1, ..., \mathcal{F}_{d+1}$ of convex sets in $\mathbb{R}^d$, if any colorful selection $C_1 \in \mathcal{F}_1, ..., C_{d+1} \in \mathcal{F}_{d+1}$ has a non-empty intersection, then there is a family $\mathcal{F}_i$ such that all sets in $\mathcal{F}_i$ have a non-empty intersection.*

Here, clearly $d+1$ is the Helly-number, as the tuple-size cannot be lower because the orginal Helly theorem is a trivial consequence. **Bárány**, **Katschalski** and **Pach** showed a Helly-type theorem about a stronger condition B on the family of convex sets. Their **Quantitative Volume Theorem** provides a condition not only for the emptyness of the intersection, but also gives a lower bound for the volume of intersection of sets that only depends on the dimension d.

Theorem 1.1.3 (Quantitative Volume Theorem. Bárány, Katschalski, Pach [4]). *For a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$, if any $2d$-tuple has an intersection of volume at least 1, then all sets in $\mathcal{F}$ have an intersection of volume at least $c_d = d^{-2d^2}$*

Note that here in terms of the framework presented in the introduction, condition B is weaker than A, although both are lower bounds on the volume of the intersection. This is often the case with quantitative volume theorems. Note also, that the Helly number is larger than in the original Helly theorem. The constant c_d was later reduced to d^{-2d} by **Naszódi** [7] and even $d^{-3/2d}$ by **Brazitikos** [8].
At this point, it is visible that when condition A is stronger, more strict, the Helly-numbers can increase (by moving from emptyness to volume of the intersection, the Helly-numbers increased from $d + 1$ to $2d$ in this case.)

1.1.1 Colorful Volume Theorems

The results of **Damásdi**, **Földvári** and **Naszódi** combine the conditions of the colorful version and the quantitative version of Helly's theorem giving a lower bound on the volume of the intersection of a family of convex sets if there is a common lower bound on the intersection of every colorful selection.

Theorem 1.1.4 (Damásdi, Földvári, Naszódi [9]). *Let $\mathcal{C}_1, \ldots, \mathcal{C}_{3d}$ be finite families of convex bodies in $\mathbb{R}^d$. Assume that for any colorful selection of $2d$ sets, $C_{i_k} \in \mathcal{C}_{i_k}$ for each $1 \leq k \leq 2d$ with $1 \leq i_1 < \ldots < i_{2d} \leq 3d$, the intersection $\bigcap_{k=1}^{2d} C_{i_k}$ contains an ellipsoid of volume at least 1. Then, there exists an $1 \leq i \leq 3d$ such that $\mathrm{Vol}\left(\bigcap_{C \in \mathcal{C}_i} C\right) \geq c^{d^2} d^{-7d^2/2}$ with an absolute constant $c \geq 0$.*

This is a generalization of the original quantitative volume theorem in the same way as the result of Lovász and Bárány generalized the original Helly theorem. A notable difference is that here the size of the colorful selections is smaller than the number of color classes (a statement with smaller tuple-size implies the same statement with larger tuple-size, as the latter is a stronger condition.)
The same authors, concurring with Sarkar, Xue and Soberón [10], show an improved lower bound on the volume with the same condition A, while the number of color classes increases.

Theorem 1.1.5 (Quantitative volume theorem - many color classes [9]). *Let $\mathcal{C}_1, \ldots, \mathcal{C}_{d(d+3)/2}$ be finite families of convex bodies in $\mathbb{R}^d$. Assume that for any colorful selection of $2d$ sets, $C_{i_k} \in \mathcal{C}_{i_k}$*

for each $1 \leq k \leq 2d$ *with* $1 \leq i_1 < \ldots < i_{2d} \leq d(d+3)/2$*, the intersection* $\bigcap_{k=1}^{2d} C_{i_k}$ *is of volume at least 1. Then, there exists an* $1 \leq i \leq d(d+3)/2$ *such that Vol* $\left(\bigcap_{C \in C_i} C\right) \geq d^{-O(d)}$.

Here, again, a stronger condition (more color classes) allows for an improved lower bound.

Note that even though the number of color classes differed, all of the listed quantitative volume theorems had tuple-sizes of $2d$. This is the Helly-number for these statements, since $2d$ is indeed minimal, as will be shown here as well by examining boxes in $\mathbb{R}^d$.

1.1.2 Piercing boxes

Another possible variant of Helly's theorem generalizes the notion of intersection with the notion of piercing.

Definition: A set P **pierces** a family of sets $\mathcal{F}$, if for any set $S \in \mathcal{F}$, there is an element $p \in P$ such that $p \in S$. If there exists a P such that $|P| = n$, then $\mathcal{F}$ is n**-pierceable**.

Note that an intersection of sets is non-emty if and only if it is 1-pierceable. Note also that in some other contexts this notion might be called covering by points.

All previously discussed Helly-type statements were about general families of convex sets. However, for $n > 1$ the folowing statement is not true: *"If all subfamilies $\mathcal{H}$ of finite size h from a family of convex sets is n-pierceable, then $\mathcal{F}$ is n-pierceable."* Not all types of convex sets allow statements of this form about n-piercing. For example **Chakraborty**, **Rameshwar**, **Sasanka** and **Shubhangi** showed [11] that *for any constant $h > 0$ there exists a family of circles in the plane such that any subfamily of size h is 2-pierceable but the whole family is not 2-pierceable.*

On the other hand, some types of convex sets are viable objects for Helly-type statements about n-piercing for $n > 1$, at least in some cases.

Definition: An **axis-parallel box** B in $\mathbb{R}^d$ is a set of the form $\prod_{i=1}^{d} [a_i, b_i]$, where $a_i < b_i \in \mathbb{R}$ if $1 \leq i \leq d$ and $[a_i, b_i]$ denotes the closed interval $\{x \in \mathbb{R} : a_i \leq x \leq b_i\}$. Note that boxes are closed sets.

Throughout this thesis, the term box will be used to refer to axis-parallel boxes and "axis-parallel" will be usually omitted for brevity. Even though they are relatively simple objects, axis-parallel boxes have some rather interesting properties when it comes to piercing and Helly-type theorems, some of which are known.

Danzer and **Grünbaum** showed the **Helly-number** for all possible classical Helly-type theorems about n**-piercing** families of axis-parallel boxes in Euclidean space where both condition A and B are n-piercing.

Theorem 1.1.6 (Danzer, Grünbaum [5]). *If $h = h(d,n)$ is the smallest positive integer such that for any finite family $\mathcal{F}$ of axis-parallel boxes in $\mathbb{R}^d$ every h-tuple from $\mathcal{F}$ is n-pierceable implies that $\mathcal{F}$ is n-pierceable then following are the values of h:*

$$h(d,1) = 2 \ \forall d \in \mathbb{N}$$

$$h(1,n) = n+1 \ \forall n \in \mathbb{N}$$

$$h(d,2) = \begin{cases} 3d & : \quad d \equiv 0 \mod 2 \\ 3d-1: & \quad d \equiv 1 \mod 2 \end{cases}$$

$$h(2,3) = 16$$

$$h(d,n) = \aleph_0 \quad n \geq 3, (d,n) \neq (2,3)$$

It is interesting to note that here there are some Helly-numbers that are independent of the dimension of the boxes and also, $3d$ is the largest (finite) Helly-number, that was discussed here thus far. In a way, this reflects the relative simplicity of boxes and also the complexity of piercing.
Chakraborty, **Ghosh** and **Nandi** generalized previous statements and showed a colorful Helly-type theorem for n-piercing intervals and 2-piercing axis-parallel boxes.

Theorem 1.1.7 (Chakraborty, Ghosh, Nandi [12]). *If $h = h(d,n)$ is the smallest positive integer such that for any finite families $\mathcal{F}_1, ..., \mathcal{F}_h$ of axis-parallel boxes in $\mathbb{R}^d$ every colorful h-tuple from $\mathcal{F}$ is n-pierceable implies that for some $1 \leq i \leq h$ the family $\mathcal{F}_i$ is n-pierceable then following are the values of h:*

$$h(1,n) = n+1 \ \forall n \in \mathbb{N}$$

$$h(d,2) = 3d \ \forall d \in \mathbb{N}$$

Note that for odd d, the colorful Helly-number is larger than in the classic case. While the colorful Helly-number is an upper limit for any respective classical Helly-number, it is not necessarily the same. A colorful version implies a classical Helly-type statement, but the tuple-size is not necessarily minimal for the classical version.
Note that the cases $n = 3, d = 2$ are unadressed. Whether there is a finite colorful Helly-number for these parameters is an open question as of yet.

In addition to the conclusions of both **Theorem** 1.1.2 and **Theorem** 1.1.7, it can be added that the color class for which condition B holds can be extended with a colorful tuple so that the condition A still holds. [12] [13]

1.2 Punching holes into boxes

This section presents an overview of the possible quantitative variants of the box-piercing theorems. Thus, it introduces a framework which allows for statements about volume that generalize box-piercing. This is achieved by the notion of punching holes into boxes. One of the most obvious ways to generalize piercing by points is punching by sets (of points).

Definition: A family of sets $\mathcal{H}$ **punches** another family of sets $\mathcal{F}$ if

$$\forall S \in \mathcal{F} \quad \exists H \in \mathcal{H} \quad H \subset S \tag{1.1}$$

One can also say that $\mathcal{H}$ is punching. Note that if every element of $\mathcal{H}$ has only one element, then punching is essentially the same as piercing (there is a natural bijection between piercing tuples and such punching tuples.) Furthermore, if $\mathcal{F}$ is punchable by an n-tuple, then naturally it is also n-pierceable. The elements of $\mathcal{H}$ can be referred to as **holes**. By slight abuse of terminology, if $\mathcal{H} = \{H\}$ is a 1-tuple that punches $\mathcal{F}$, then one can also say that H punches $\mathcal{F}$. $\mathcal{F}_H \subset \mathcal{F}$ denotes the subfamily of boxes that contain a hole $H \in \mathcal{H}$. One can say that the elements of $\mathcal{F}_H$ are **punched together**. Note that $\{\mathcal{F}_H : H \in \mathcal{H}\}$ is not necessarily a partition of $\mathcal{F}$ and that $\mathcal{H}$ punches $\mathcal{F}$ if and only if $\bigcup\limits_{H \in \mathcal{H}} \mathcal{F}_H = \mathcal{F}$.

As the goal is to obtain quantitative volume theorems, the sets in question have to have a meaningful definition of volume, thus they should be measurable. It might be an interesting question to examine how the choice of measure can affect such statements, but as this thesis deals with boxes, this question is of limited relevance here and beyond scope. The Lebesgue-measure can be used, for example, but a direct definition of volume for boxes will also be given later, which is sufficient for all purposes.

Now follows a framework, which gives meaning to punching in terms of volume. The first notion prescribes which volumes the holes in a punching n-tuple should have with what multiplicity.

Definition: For a set of possible volumes $\mathcal{V} \subset \mathbb{R}_{>0}$ (volume set) and numeration $\nu : \mathcal{V} \to \mathbb{Z}_{>0}$ a family of (measurable) sets $\mathcal{F}$ is **$(\mathcal{V}, \nu)$-punchable** if there is a family (tuple) of (measurable) sets $\mathcal{H}$ such that it punches $\mathcal{F}$, i.e. (1.1) holds and

$$\sum_{v \in \mathcal{V}} \nu(v) = |\mathcal{H}| \tag{1.2}$$

$$\forall v \in \mathcal{V} \quad \nu(v) = |\{H \in \mathcal{H} : \mathrm{Vol}(H) = v\}| \tag{1.3}$$

are also satisfied. For such a tuple $\mathcal{H}$ one can say that it $(\mathcal{V}, \nu)$-punches $\mathcal{F}$.

This rather complicated condition will not be discussed in this thesis, only a much simpler subcase where the elements of the n-tuple share the same prescribed volume. This condition is equivalent to prescribing only the volume of the smallest hole, or having a common lower bound on the volume of all the holes.

Definition: If the volume set has 1 element $\mathcal{V} = \{v\}$ and $\nu(v) = n$ and there is a family $\mathcal{H}$ for which (1.1), (1.2), (1.3) hold, then $\mathcal{F}$ is (v, n)**-punchable**. In the case $v = 1$ $\mathcal{F}$ is simply n**-punchable**.

So, if a family is $(\mathcal{V}, \nu)$-punchable, then it is also $(\min \mathcal{V}, \sum_{v \in \mathcal{V}} \nu(v))$-punchable as any set of larger volume contains a set of any smaller volume. Also note that 1-punchability is equivalent to having an intersection of volume at least 1.

It is also useful to introduce a total order on the punching tuples to facilitate their description.

In the ordering, for any two tuples the sets with smaller volume have higher priority and the smallest j for which the jth smallest boxes have different size determines the direction of the relation among the tuples. (This ordering is not necessarily restricted to boxes but is meaningful for any measurable sets with proper defintion of volume.) In other words, let the elements of all tuples be ordered by volume and then, the tuples themselves are ordered by lexicographic order. Here follows a rigourous definition.

Definition: Let $\mathcal{N} \subset \mathcal{P}(\mathbb{R}^d)^n$ be the set of n-tuples of (measurable) sets in $\mathbb{R}^d$ for a given $d \in \mathbb{N}$ and $n \in \mathbb{N}$. For any $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{N}$ the sets M_{ij} and $\mathcal{M}_{ij}$ are the following. For $i \in \{1, 2\}$ and $\mathcal{M}_{i0} = \mathcal{H}_i$ the set $M_{i0} = \arg\min_{S \in \mathcal{M}_{i0}} \mathrm{Vol}(S)$ and recursively $M_{ij} = \arg\min_{S \in \mathcal{M}_{ij}} \mathrm{Vol}(S)$ where $\mathcal{M}_{ij} = \mathcal{M}_{i(j-1)} \setminus \{M_{i(j-1)}\}$ for $1 \leq j \leq n-1$. By the **ordering** $(\mathcal{N}, \leq)$ for any two tuples $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{N}$, the relation $\mathcal{H}_1 \leq \mathcal{H}_2$ holds if and only if $Vol(M_{1j}) \leq Vol(M_{2j})$ for every $0 \leq j \leq n-1$. If j is the smallest index for which $\mathrm{Vol}(M_{1j}) \neq \mathrm{Vol}(M_{2j})$ then $\mathcal{H}_1 < \mathcal{H}_2$ if and only if $\mathrm{Vol}(M_{1j}) < \mathrm{Vol}(M_{2j})$.

Tuples will be referred to as **smaller** or **larger** according to the above ordering. The **volume** of a tuple $\mathcal{H}$ will be the volume of the smallest set, i.e. $\mathrm{Vol}(M_0) = \min_{S \in \mathcal{H}} \mathrm{Vol}(S)$. Note that by these definitions, a tuple can be smaller or larger than another with the same volume and also that if a tuple has smaller volume, then it is also smaller (according to the ordering). Of course in general, for sets in $\mathbb{R}^d$ the terms larger and smaller will refer to ordering by volume.

The unique containment-wise maximal hole that punches a family $\mathcal{F}$ is the intersection $\bigcap_{S \in \mathcal{F}}$ and is also a largest such hole. For subfamilies $\mathcal{F}_{H_1}, ..., \mathcal{F}_{H_n}$ defined by a punching n-tuple $\mathcal{H} = \{H_i : 1 \leq i \leq n\}$, the containment-wise maximal n-tuple that punches the elements of $\mathcal{F}_{H_i}$ together for any $1 \leq i \leq n$ is composed of the intersections of all members within these subfamilies. So, for any $\mathcal{F}$ that is punchable by an n-tuple there is a largest n-tuple that is the intersections from an n-partition of $\mathcal{F}$ and is also containment-wise maximal.

1.3 Preliminary results

We make some preliminary observations about the introduced notions of punching and their relation to boxes.

First, some useful definitions to describe boxes.

Definition: The ith (canonical) unit vector in $\mathbb{R}^d$ is $\underline{e_i} = \left(\begin{cases} 1 : j = i \\ 0 : j \neq i \end{cases} \right)_{1 \leq j \leq d}$.

Also, the notation $\underline{1} = (1)_{1 \leq i \leq d} \in \mathbb{R}^d$ will be used throughout.

Definition: For $1 \leq i \leq d$ the ith axis in $\mathbb{R}^d$ is $\{\lambda \underline{e_i} : \lambda \in \mathbb{R}\}$.

Definition: The projections to the ith axis $\pi_i : \mathbb{R}^d \to \mathbb{R}$ for $1 \leq i \leq d$ map $(x_1, ..., x_d) \mapsto x_i$.

For a set $S \subset \mathbb{R}^d$, by extension of notation, let $\pi_i(S) = \{\pi_i(\underline{x}) : \underline{x} \in S\}$.

For a box B, the numbers $a_{i,B} = \min \pi_i(B)$ and $b_{i,B} = \max \pi_i(B)$ are the left and right endpoints of the projection to the ith axis.

The notion of punching was defined for general sets in $\mathbb{R}^d$, however it will be shown here, that without loss of generality one can assume that any n-tuple $\mathcal{H}$ that punches a family of boxes $\mathcal{F}$ is an n-tuple of boxes.

For any hole $H \in \mathcal{H}$ of a given tuple let B_H be a containment-wise minimal box that contains H.

Observation: *B_H is unique and is equal to the box* $\prod_{i=1}^{d} [\min \pi_i(H), \max \pi_i(H)]$.

Observation: *H punches a box B if and only if B_H punches B.*

These observations are true because a box B contains a point P if and only if $\pi_i(P) \in \pi_i(B)$

for every axis i.
Therefore, a tuple $\mathcal{H}$ punches a family of boxes $\mathcal{F}$ if and only if $\mathcal{H}_\square = \{B_H : H \in \mathcal{H}\}$ punches $\mathcal{F}$.
From now on, any punching n-tuple in the context of boxes will be assumed to be a tuple of boxes and the ordering of tuples will be restricted to such tuples as well.
Observation: *The intersection of a family of boxes $\mathcal{F}$ in $\mathbb{R}^d$ is also a (possibly empty) box* $B = \prod_{i=1}^{d} [\max_{B \in \mathcal{F}} a_{i,B}, \min_{B \in \mathcal{F}} b_{i,B}]$.
Therefore, the largest punching hole for a family is unique and is the box defined by the intersection.

The next part presents some basic considerations regarding Helly-type theorems about punching boxes and their relation to piercing. Since punching was introduced as a sort of generalization of piercing, it is evident to examine in what way it is different to piercing and whether there are some similarities and pertinent properties.
A first notable difference is that n-pierceability is only determined by the structure of the intersections, while n-punchability is not.

Statement 1. *Given a bijection between two families $f : \mathcal{F} \longrightarrow \mathcal{F}'$ such that for any subfamily $\mathcal{H} \subset \mathcal{F}$, having non-empty intersection implies $f(\mathcal{H})$ has non-empty intersection, then if $\mathcal{F}'$ is n-pierceable, $\mathcal{F}$ is n-pierceable as well.*

Proof. A family $\mathcal{F}$ is n-pierceable if and only if there are subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_n$ with non-empty intersection such that $\bigcup_{i=1}^{n} \mathcal{F}_i = \mathcal{F}$. □

In other words, for any family $\mathcal{F} \subset \mathcal{P}(S)$, any injective map of S preserves any pierceability of $\mathcal{F}$.
Before we state our main theorem regarding the connection of statements about punching and piercing, we recall some definitions to help describe closed and convex sets in $\mathbb{R}^d$.
Definition: In $\mathbb{R}^d$ a **hyperplane** is a set P for which there is an isometry $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that f maps P to $\{(x_1, \ldots, x_d) : x_1 = 0\}$.
Definition: For $c \in \mathbb{R}$ and a set $S \subset \mathbb{R}^d$ for $d \in \mathbb{N}$ let $cS = \{c \cdot \underline{v} : \underline{v} \in S\}$. Note that the notation adheres to the vector space structure of $\mathbb{R}^d$ and scalar multiplication is the homothety of scale c with center $\underline{0} \in \mathbb{R}^d$. A **homothety** in general is the map $\underline{x} \mapsto c(\underline{x} - \underline{c})$ for some center $\underline{c} \in \mathbb{R}^d$. In addition for a family of sets $\mathcal{F}$, by further

extension $c\mathcal{F} = \{cS : S \in \mathcal{F}\}$.

Definition: The **open ball** of center $\underline{x}$ and radius $\varepsilon > 0$ is $B_\varepsilon(x) = \{\underline{y} \in \mathbb{R}^d : |\underline{x} - \underline{y}| < \varepsilon\}$.

Definition: The **boundary** of a set $S \subset \mathbb{R}^d$ is $\partial S = \{\underline{x} \in \mathbb{R}^d : \forall \varepsilon > 0\ B_\varepsilon(\underline{x}) \cap S \neq \emptyset, B_\varepsilon(\underline{x}) \cap \left(\mathbb{R}^d \setminus S\right) \neq \emptyset\}$.

For two points $a, b \in \mathbb{R}^d$ let $d(a, b)$ denote their Eucledian distance.

Defintion: For sets $A, B \subset \mathbb{R}^d$ their **distance** is $\operatorname{dist}(A, B) = \inf_{a \in A, b \in B} d(a, b)$.

Theorem 1.3.1 (Punching implies piercing)**.** *Let $\mathscr{C}$ be a set of closed convex sets in $\mathbb{R}^d$ closed under homothety. Given some volumes $v_1 \geq v_2$ and tuple-sizes $n_1 \leq n_2$, statement (1) implies (2).*

(1) For a finite family $\mathcal{F} \subset \mathscr{C}$ of closed convex sets in $\mathbb{R}^d$ if any h-tuple of sets $\mathcal{H}$ in $\mathcal{F}$ is (v_1, n_1)-punchable, then $\mathcal{F}$ is (v_2, n_2)-punchable.

(2) For a finite family $\mathcal{F} \subset \mathscr{C}$ of closed convex sets in $\mathbb{R}^d$ if any h-tuple of sets $\mathcal{H}$ in $\mathcal{F}$ is n_1-pierceable, then $\mathcal{F}$ is n_2-pierceable.

Remark. It is left as an exercise to the reader to show that statements of the form of (1) are not possible if $v_2 > v_1$ and neither type is possible for $n_2 < n_1$. Also, by a homothety of ratio $\frac{1}{v_1}$ or $\frac{1}{v_2}$, statements of type (1) are equivalent to statements were the h-tuples are n_1-punchable and $\mathcal{F}$ is $(\frac{v_2}{v_1}, n_2)$-punchable or h-tuples are $(\frac{v_1}{v_2}, n_1)$-punchable and $\mathcal{F}$ is n_2-punchable respectively.

Proof. Assume $\mathcal{F}$ is a finite family of closed convex sets in $\mathbb{R}^d$ such that any subfamily $\mathcal{H}$ of size h is n_1-pierceable.

Suppose every h-tuple is also (v, n_1)-punchable, for some $v > 0$, then for $c > \left(\frac{v_1}{v}\right)^{1/d}$ by a homothety of ratio c the resulting family $\mathcal{F}' = c\mathcal{F}$ is isomorphic with respect to intersections. In addition, every h-tuple of $\mathcal{F}'$ is (v_1, n)-punchable due to the homothety. Then, by (1) the whole family is (v_2, n_2)-punchable. This immediately yields that $\mathcal{F}$ is n_2-pierceable. Since $\mathcal{F}'$ is isomorphic, $\mathcal{F}$ is also n_2-pierceable.

If the previous condition does not hold, then it is possible that some of the sets in $\mathcal{F}$ have volume zero. First, we will assume that they do not.

If there are h-tuples which are not (v, n)-punchable for any positive v, then there will be a separate homothety with ratios sufficiently small for every set in $\mathcal{F}$ instead of one for the whole family that creates a proper $\mathcal{F}'$.

In this case, for some subfamily $\mathcal{H} \subset \mathcal{F}$ the non-empty intersection I is of volume zero. Since the sets are closed, I must be the intersection of boundaries. For every set $S \in \mathcal{H}$, there will be a homothety H_S such that I is in the inside of $H_S(S)$, while the modified family of sets where every $S \in \mathcal{H}$ is replaced with $H_S(S)$ is isomorphic to $\mathcal{F}$ with respect to intersections.

For every $S \in \mathcal{F}$ let P_S be an inner point of S (this exists because S has positive volume.) For some $S' \in \mathcal{F}'$ that does not intersect S for any point $P' \in S'$ the half-line with endpoint P_S that contains P' is denoted by $\overline{P_S P'}^+$, while the point $P_{P'} = \overline{P_S P'}^+ \cap \partial S$ is the point where this half-line crosses the boundary of S. As S is closed and convex, $P_{P'}$ is unique and is the furthest point of $\overline{P_S P'}^+ \cap S$ from P_S. For any $P' \in S'$ the ratio $\frac{d(P_S, P')}{d(P_S, P_{P'})}$ is the ratio of the homothety with center P_S that maps $P_{P'}$ to P'. Since S and S' are closed and convex, $c_{S'} = \min\limits_{P' \in S'} \frac{d(P_S, P')}{d(P_S, P_{P'})}$ exists and is greater than 1 if S and S' are disjoint. This means that by definition, for any homothety H of center P_S and ratio $c \in (1, c_{S'})$ the image $H(S)$ is disjoint from S'. At the same time $\partial S \subset H(S)$ as S is convex and the center is in S. Let $c_S = \min\limits_{S' \in \mathcal{F}: S \cap S' = \emptyset} c_{S'}$ and then let H_S be the homothety with ratio $c \in (1, c_S)$ and center P_S.

Let $\mathcal{H}_1, \ldots, \mathcal{H}_{N'}$ be the subfamilies with non-empty intersection of volumes zero and let $\mathcal{H} = \bigcup\limits_{i=1}^{N'} = \{S_i : 1 \leq i \leq N\}$. Let $\mathcal{F}_0 = \mathcal{F}$ and for $1 \leq i \leq N$ let H_{S_i} be the homothety for $S_i \in \mathcal{F}_{i-1}$ constructed based on the elements of $\mathcal{F}_{i-1}$ according to the above considerations and $\mathcal{F}_i = \mathcal{F}_{i-1} \setminus \{S_i\} \cup \{H_{S_i}(S_i)\}$. Finally, $\mathcal{F}_N$ is isomorphic to $\mathcal{F}$ and every non-empty intersection is in the inside or outside of any set in $\mathcal{F}_N$, so the intersections of $\mathcal{F}$ of positive volume remain positive in $\mathcal{F}_N$, while the previously zero-sized non-empty intersections become positive.

Therefore, $\mathcal{F}_N$ satisfies the previous case, thus it is possible to construct an isomorphic $\mathcal{F}'$ that proves that $\mathcal{F}$ is n_2-pierceable.

If some sets in $\mathcal{F}$ have volume zero, then they will be replaced by thin cylinders, in a similar procedure as in the previous cases.

If a convex set $S \in \mathcal{F}$ has volume zero, then it is contained in a hyperplane. Let $\underline{n}$ be the unit normal of this hyperplane and $d = \min\limits_{S' \in \mathcal{S}: S \cap S' = \emptyset} \operatorname{dist}(S, S')$. Then, the cylinder $C_S = \{\lambda \underline{n} + \underline{x} : \underline{x} \in S, \lambda \in [-c, c]\}$ for some $0 < c < d$ does not intersect any S' disjoint from S and keeps all previous intersections.

Therefore, by the same type of iterative process, one can obtain an intersection-isomorphic family with no elements of volume zero.

□

Remark. Proving this statement solely for boxes is somewhat simpler and can be achieved by using projections to the coordinate axes and using a similar sort of cylindrical expansion as in the last part of this proof.

This statement is significant as **Theorem** 1.1.6 is a collection of statements of type (2). Since **Danzer** and **Grünbaum** showed that there are no Helly-type statements for certain types of piercing of boxes, there cannot be any Helly-type statements for certain types of punching of boxes.

Corollary 1.3.1.1 (Negative Helly-type theorems for punching)**.** *If $n \geq 4$ and $d \geq 2$ or $n = 3$ and $d \geq 3$ and v_1, v_2 are any volumes, then there is no finite h for which assuming every h-tuple from a family of boxes $\mathcal{F}$ in $\mathbb{R}^d$ is (v_1, n)-punchable implies $\mathcal{F}$ is (v_2, n)-punchable.*

By **Theorem** 1.1.6, the possible cases are therefore any punching on the line (in dimension one), 2-punching in any dimension, and 3-punching on the plane. The main results of this thesis are about the characterization of these cases, presented in the next chapter.
Of course, 1-punching Helly-theorems are also possible and are about the volume of intersection, just as **Theorem** 1.1.3, the original quantitative volume theorem about convex sets. The special case of boxes will be discussed here briefly.
First, we present some definitions regarding boxes.
Definition: The **volume** of a box $B = \prod_{i=1}^{d}[a_i, b_i]$ is $\mathrm{Vol}(B) = \prod_{i=1}^{d}(b_i - a_i)$. If $\mathrm{Vol}(B_1) < \mathrm{Vol}(B_2)$, then B_1 is the smaller, B_2 is the larger box. The terms volume, and size will be used interchangeably for boxes. In contrast, the tuple-size of an n-tuple is n, i.e. the number of elements.
Definition: A **facet** of an axis-parallel box B is $F_{B,i,\mathrm{sgn}(b_j - b_{j+1 \mod 2})} = \partial B \cap \{(x_1, ..., x_d) : x_i = b_j\}$ for some $1 \leq i \leq d$ and $b_j \in \partial\pi_i(B) = \{b_0, b_1\}$. This means that $F_{B,i,+}$ and $F_{B,i,-}$ denote the "upper" and "lower" facets a with normal $\underline{e_i}$, perpendicular to the ith axis.
Definition: A box B **borders** another box A if a facet F_B of B contains a facet F_A of A and their relative interiors are not disjoint. Similarly, a **hyperplane borders** a box A if it contains a facet of A.
Definition: A **vertex** of a box is a point $\underline{v} \in B$ that is contained in d facets of B. These facets are the (neighboring) facets of $\underline{v}$. Two vertices with no facets in common are called **diagonally opposite**.
Observation: *A pair of points $\underline{v} = (v_i)_{1 \leq i \leq d}$, $\underline{w} = (w_j)_{1 \leq j \leq d}$ defines a unique box $B = \prod_{i=1}^{d}[\min\{v_i, w_i\}, \max\{v_i, w_i\}]$. The points $\underline{v}$ and $\underline{w}$ are diagonally opposite vertices of B.*

The box defined by a pair of diagonally opposite vertices $\underline{v}, \underline{w}$ will be denoted $\underline{v} \square \underline{w}$. Note that $\underline{v} \square \underline{w} = \underline{w} \square \underline{v}$.

Proposition 1 (Quantitative volume theorem for boxes - 1-punching)**.** *For a finite family $\mathcal{F}$ of boxes in $\mathbb{R}^d$ if any $2d$-tuple has an intersection of volume at least 1, i.e. is 1-punchable, then all sets in $\mathcal{F}$ have an intersection of volume at least 1, i.e. are 1-punchable.*

Proof. As mentioned previously, this statement is equivalent to showing that if $\mathcal{F}$ is not 1-punchable, then there is a $2d$-tuple that is not 1-punchable either. The largest box B that punches $\mathcal{F}$ is the intersection. Assuming B has positive volume, it has $2d$ faces, each of which are bordered by some box of $\mathcal{F}$. Therefore, there are at most $2d$ boxes of $\mathcal{F}$ that border B and since they are bordering, there is no larger box that punches them. Therefore, if $\mathcal{F}$ is not 1-punchable, then a $2d$-tuple containing a bordering box for each facet of the intersection is not 1-punchable either.
If the intersection B has volume zero, then there are two boxes in $\mathcal{F}$ with disjoint relative interiors. Since $d \leq 1$, any $2d$-tuple containing them will have intersection of volume zero as well. □

This is again a case where the condition of this statement (being restricted to boxes) is stronger, and the lower bound on the volume is greater than in the case of **Theorem** 1.1.3. It also demostrates a case where boxes behave in a simpler way than general convex sets. In addition, $2d$ is also the Helly-number, so as mentioned before it is also the Helly-number for convex sets as boxes are convex.

Proposition 2 (Lower bound on Helly-number for 1-punching)**.** *For every dimension $d \in \mathbb{Z}_{\geq 0}$ and any volumes $v_1 > v_2$ there is a family of boxes $\mathcal{F}$ in $\mathbb{R}^d$ that is not $(v_2, 1)$-punchable, but every subfamily of size $2d - 1$ is $(v_1, 1)$-punchable.*

Proof. For $1 \leq i \leq d$ and $0 < \varepsilon_1 < v_1^{1/d}$ let $\varepsilon_2 > \frac{\varepsilon_1^{d-1}}{v_2}$ so $(\varepsilon_1/2 + \varepsilon_2)\varepsilon_1^{d-1} > v_2$. Then $B_{i+} = \prod_{j=1}^{i-1}[-\varepsilon_2, \varepsilon_2] \times [-\varepsilon_1/2, \varepsilon_2] \times \prod_{j=i+1}^{d}[-\varepsilon_2, \varepsilon_2]$ and $B_{i-} = \prod_{j=1}^{i-1}[-\varepsilon_2, \varepsilon_2] \times [-\varepsilon_2, \varepsilon_1/2] \times \prod_{j=i+1}^{d}[-\varepsilon_2, \varepsilon_2]$.
The family $\mathcal{F} = \{B_{i+} : 1 \leq i \leq d\} \cup \{B_{i-} : 1 \leq i \leq d\}$ is maximally punched by its intersection $B = -\frac{\varepsilon_1}{2}\underline{1} \square \frac{\varepsilon_1}{2}\underline{1}$. Clearly, $\mathrm{Vol}(B) = \varepsilon^d < 1$. By removing B_{i+} from $\mathcal{F}$, the box $B'_{i+} = \prod_{j=1}^{i-1}[-\varepsilon/2, \varepsilon/2] \times [-\varepsilon/2, \varepsilon_2] \times \prod_{j=i+1}^{d}[-\varepsilon/2, \varepsilon/2]$ of volume $(\varepsilon_2 + \varepsilon/2)\varepsilon^d > v_2$. □

Corollary 1.3.1.2 (Helly-number for quantitative volume theorems on convex sets)**.** *The* *smallest h for which assuming every h-tuple from a family of convex sets in $\mathcal{F}$ in $\mathbb{R}^d$ is* 1*-punchable implies $\mathcal{F}$ is $(c_d, 1)$-punchable for some constant $c_d > 0$ is at least $2d$.*

To recapitulate, it was shown that for the introduced definition of n-punching, there are no Helly-type statements in a lot of cases. It begs the question then, whether for some other definition of punching, or by introducing some additional restriction on the punching tuples, it is possible to obtain viable Helly-type theorems. Other definitions will be adressed in chapter 3, while a simple restriction on the punching tuples will be examined here, as it is also relevant to the punching theorems in dimension 1.

An arguably natural type of restriction on the punching tuples is to introduce some sort of similarity between the holes. For boxes it has been shown that it can be assumed that when punching boxes, the tuples also contain boxes. Since this is not strict enough for some cases, the tuples will now be restricted to contain only holes that are translates of each other. Note that this idea is also adressed and used by Damásdi, Fölvári and Naszódi in the proof of **Theorem** 1.1.4 [9].

For this question the following concepts will be useful.

Definition: The **Minkowski sum** of sets $A \subset \mathbb{R}^d$ and $B \subset \mathbb{R}^d$ is $A + B = \{\underline{a} + \underline{b} : \underline{a} \in A, \underline{b} \in B\}$.

Note that $\{\underline{v}\} + A$ is a translate of A.

Definition: The **Minkowski difference** of two sets $A, B \subset \mathbb{R}^d$ is $A \sim B = \bigcap\limits_{\underline{b} \in B} A + \{-\underline{b}\}$.

Note that $A \sim B = \{\underline{v} \in \mathbb{R}^d : \{\underline{v}\} + B \subset A\}$, i.e. the Minkowski-difference defines the translations of the subtrahend which move it into the minuend. (Note also, that sometimes the term Minkowski difference is defined differently to mean a sort of inverse of Minkowski sum.)

Observation: *If B is a box, then $B \sim A$ is a (possibly empty) box for any A.*

This is due to the fact that the intersection of boxes is a box and Minkowski difference was defined as the intersection of some translates of B. Therefore, Minkowski difference preserves convexity as well in general.

Observation: *A hole H punches a family $\mathcal{F}$ of boxes if and only if the intersection of boxes $\bigcap\limits_{B \in \mathcal{F}} B \sim H$ is not empty.*

Therefore, if the holes are translates of each other, there is a direct correspondence between punching and piercing.

Proposition 3 (Punching by translates equals piercing)**.** *Given some tuple-sizes $n_1 \geq n_2$ and a hole $H \subset \mathbb{R}^d$, statements (1) and (2) are equivalent.*

(1) For a finite family of boxes $\mathcal{F}$ in $\mathbb{R}^d$ if any h-tuple $\mathcal{H}$ in $\mathcal{F}$ is punchable by an n_1-tuple of translates of H, then $\mathcal{F}$ is punchable by an n_2-tuple of translates of H.

(2) For a finite family of boxes $\mathcal{F}$ in $\mathbb{R}^d$ if any h-tuple $\mathcal{H}$ in $\mathcal{F}$ is n_1-pierceable, then $\mathcal{F}$ is n_2-pierceable.

Proof. Given a family of boxes $\mathcal{F}$ let $F_{\sim H} = \{B \sim H : B \in \mathcal{F}\}$ also be a family of boxes and let the bijection that maps $B \mapsto B \sim H$ be f. Then, any subfamily $\mathcal{H} \subset \mathcal{F}$ is punchable by an n-tuple of translates $\{H + \{\underline{v_i}\} : 1 \leq i \leq n\}$ of H if and only if the n-tuple $(v_i)_{1 \leq i \leq n}$ pierces $f(\mathcal{H})$. □

This means that these types of statements are equivalent to the Helly-type statements about piercing such as **Theorem** 1.1.6 and **Theorem** 1.1.7.

Corollary 1.3.1.3. *For a finite family $\mathcal{F}$ of boxes in $\mathbb{R}^d$, if assuming any h-tuple of sets $\mathcal{H}$ in $\mathcal{F}$ is (v_1, n_1)-punchable implies $\mathcal{F}$ is (v_2, n_2)-punchable, then $h \geq 3d - 1$ for odd d and $h \geq 3d$ for even d.*

The results presented in the next chapter will make it clear that if only the volume of the punching tuples is restricted, then for dimension 1, punching theorems do not become more complicated. However, in case of multiple dimensions, the problem becomes much more complicated indeed.

Chapter 2

Main results

This chapter presents the main results of this thesis, Helly-type theorems about the introduced notion of n-punching of boxes in dimension d for cases $d = 1$ and $n \in \mathbb{N}$, and for $d \geq 2$ and $n = 2$.

2.1 Helly-type theorems about punching intervals

As mentioned previously, even when the punching tuples are less restricted, and the notion of n-punching is concerned, the Helly-type theorems for n-punching intervals in dimension 1 are equivalent to the Helly-type theorems about punching with tuples of translates, which were discussed in the previous chapter.
This is due to the following simple but crucial observation.
Observation 2.1: *All intervals of the same volume are translates of each other.*
Therefore, a family of intervals is n-punchable if and only if it is punchable by an n-tuple of translates of the unit interval $[0, 1]$. This yields the following theorems.

Theorem 2.1.1 (Classical Helly theorem about n-punching intervals)**.** *For a finite family of intervals $\mathcal{F} = \{I_i = [a_i, b_i] \subset \mathbb{R} : i \in \mathcal{I}\}$, if any subfamily of $h = n + 1$ elements is n-punchable, then $\mathcal{F}$ is n-punchable. Furthermore, this is the smallest h for which this holds, so $n + 1$ is the Helly-number for every n.*

Remark. Even though this theorem is a direct consequence of **Theorem** 1.1.6 and **Proposition** 3 by the above observation, a direct proof will also be given. This proof has essentially the same structure as the part of **Danzer** and **Grünbaum's** proof about piercing of intervals.

Proof. This proof explicitly shows n holes that punch $\mathcal{F}$ assuming the condition holds. Note that if every $(n+1)$-tuple is n-punchable, then every interval has to have length at least 1.

For $n = 1$, the interval $L = \arg\max_{I\in\mathcal{F}} \min I$ has the rightmost left endpoint in $\mathcal{F}$ and $R = \arg\min_{I\in\mathcal{F}} \max I$ has the leftmost right endpoint (calling $-\infty$ "to the left" and ∞ "to the right"). Then $L \cap R \subset I$ for every interval in $\mathcal{F}$. Therefore $L \cap R = \bigcap_{I\in\mathcal{F}} I$. Since $\{L, R\}$ is 1-punchable therefore $L \cap R$ is at least 1, so $\mathcal{F}$ is also 1-punchable.

The proof will proceed by induction on n. Assume the statement holds for $n-1$. Let the set of n-tuples that are not $(n-1)$-punchable be $\mathcal{D} \subset \mathcal{P}(\mathcal{F})$. If $\mathcal{D} = \emptyset$ then by the induction hypothesis, $\mathcal{F}$ is $(n-1)$-punchable, so it is also n-punchable. If there is an n-tuple $\mathcal{G}$ that is not $(n-1)$-punchable, then for any $\mathcal{G} \in \mathcal{D}$ let $I_{i,\mathcal{G}}$ be the interval with the ith left end from the right in $\mathcal{G}$. Then let $L_1 = \arg\max_{\mathcal{G}\in\mathcal{D}} \min I_{1,\mathcal{G}}$ and recursively for $1 < i \leq n$ let $L_i = \arg\max_{\mathcal{G}\in\mathcal{D}:\ L_j\in\mathcal{G},\ 1\leq j<i} \min I_{i,\mathcal{G}}$, i.e. L_1 maximizes the first left endpoint and for larger indices among those tuples with maximal previous endpoints the ith left endpoint is maximized. Then let $R_i = \arg\min_{I\in\mathcal{F}:\ \min L_i+1\leq\max I} \max I$, i.e the intervals with the next right endpoint at least at distance 1 after the left endpoint of L_i.

Observation: *The intervals $H_i = L_i \cap R_i$ have length at least 1.*

Claim: *The intervals $\mathcal{H} = \{H_i : 1 \leq i \leq n\}$ punch $\mathcal{F}$.*

Proof: Let $\mathcal{L} = \{L_i : 1 \leq i \leq n\}$. For any $I \in \mathcal{F}$ by the Helly-condition, $\mathcal{L} \cup \{I\}$ is n-punchable.

If the right endpoint of I is an inner point of some $H_i \in \mathcal{H}$ then I has to be punched together with L_{i-1}, since $I \cap L_i$ has to be smaller than 1 by the definition of R_i. Furthermore, in this case the left endpoint of I cannot be an inner point of L_{i-1} either, since then $\mathcal{L} \setminus \{L_{i-1}\}$ would not be $(n-1)$-punchable, but this contradicts the definition of L_{i-1} because it could be replaced with I. Therefore $L_{i-1} \cap R_{i-1} \subset I$. Note that the right endpoint of I cannot be an inner point of H_1, because then either R_1 could be replaced with I or $\{I\} \cup \mathcal{L}$ would not be n-punchable.

Assume then, that the right endpoint of I is not an inner point of any H_i and I and L_i are punched together in an n-punching of $\{I\} \cup \mathcal{L}$ (there is such an i because $\mathcal{L}$ is not $(n-1)$-punchable). If the left endpoint of I is not an inner point of L_i, then $H_i \subset I$, while if the left endpoint is an inner point, yet $H_{i+1} \not\subset I$, then $\mathcal{L} \setminus \{L_i\} \cup \{I\}$ would not be $(n-1)$-punchable, which contradicts the definition of L_i as L_i could be replaced by I.

So in each case, I is punched by one of the holes in $\mathcal{H}$. □

Remark. The only essential difference of this proof and the proof about n-piercing is the addition of the parts involving R_i.

Similarly, the following colorful version of the previous result is a direct consequence of **Theorem** 1.1.7 and **Proposition** 3.

Theorem 2.1.2 (Colorful Helly theorem about n-punching intervals)**.** *For any families of intervals $\mathcal{C}_1, ..., \mathcal{C}_h$ (color classses), if any colorful selection $I_1 \in \mathcal{C}_1, ..., I_h \in \mathcal{C}_h$ of $h = n + 1$ elements is n-punchable, then there is a color class $\mathcal{C}_i$ that is n-punchable with $1 \leq i \leq h$. Furthermore, this is the smallest h for which this holds, so $n + 1$ is the Helly-number for every n.*

2.2 Helly-type theorems about 2-punching boxes in dimension $d > 1$

This section provides some insight into the possible Helly-type theorems about 2-punching boxes. However, whether there is such a statement (a finite Helly-number) remains an open question.

Since contrary to intervals, not all boxes of the same volume are translates of each other for higher dimensions, 2-punching is not directly reducible to 2-piercing in this case. Indeed, it will be shown in the next section, that for the tuple-sizes ($3d$ and $3d - 1$) given in **Theorem** 1.1.6, similar statements about 2-punching do not hold.

As previously observed, by homothety, statements about (v, n)-punching are equivalent to statements about n-punching. Also, a classic Helly-type theorem states that if condition B does not hold for the whole family $\mathcal{F}$, then condition A does not hold for some h-tuple from $\mathcal{F}$. Therefore, to prove a classic Helly-theorem about n-punching boxes in the form of statement (1) in **Theorem** 1.3.1, it has to be shown that if the largest pair of boxes that punches $\mathcal{F}$ has smaller volume than v_2, then some h-tuple can only be punched by a pair of volume smaller than v_1. First, the case $v_1 = v_2$ shall be examined. In this case, the goal is to show that the largest punching pair for $\mathcal{F}$ is also the largest pair for some h-tuple (the largest holes for $\mathcal{F}$ have to punch any subfamily as well obviously.) This tuple will be referred to as the **proving tuple**.

It was shown previously that for higher dimensions, the case of 1-punching is fairly simple. The largest hole is bordered by at most $2d$ boxes, which cannot be punched by any larger box.

For 2-punching, the situation becomes more complicated and this consideration fails due to an inherent complication that arises with n-punching for $n \geq 2$.

Let $\{H_1, H_2\}$ be the largest pair of holes that punch a family of boxes $\mathcal{F}$. While it is true that H_1 and H_2 are the largest holes in $\mathcal{F}_{H_1}$ and $\mathcal{F}_{H_2}$ respectively, and that both are bordered by at most $2d$ boxes each, which cannot be punched by any larger hole when punched together, these bordering boxes might be punched by a pair of larger volume if they are *regrouped*. So, in the subfamily of bordering boxes of the holes H_1 and H_2 of size at most $4d$, the largest punching pair does not necessarily punch all elements of $\mathcal{F}_{H_1}$ and $\mathcal{F}_{H_2}$ together respectively. Examples for this will be shown in Section 2.3.4.

A pertient question then is, whether the bordering boxes of the largest hole need to be in the proving tuple. Let $\mathcal{B}_{H_1}$ and $\mathcal{B}_{H_2}$ be all the boxes that border H_1 and H_2 respectively while $\mathcal{P}$ is the proving tuple. If some facet $F_1 = F_{H_1,i,+}$ of H_1 and $F_2 = F_{H_2,j,+}$ of H_2 are not bordered by a box in $\mathcal{P}$, then for some sufficiently small $\varepsilon > 0$, the holes H_1 and H_2 can be extended along the axes i and j. This means, that the extended boxes $H_1' = \prod_{k=1}^{i-1} \pi_k(H_1) \times [a_{i,H_1}, b_{i,H_1} + \varepsilon] \times \prod_{k=i+1}^{d} \pi_k(H_1)$ and $H_2' = \prod_{l=1}^{j-1} \pi_l(H_2) \times [a_{l,H_2}, b_{j,H_2} + \varepsilon] \times \prod_{l=j+1}^{d} \pi_l(H_2)$ will also punch $\mathcal{P}$ and have obviously larger volume than the original pair as $H_1 \subsetneq H_1'$ and $H_2 \subsetneq H_2'$. ε has to be chosen such that $\varepsilon < \min\{\min_{B \in \mathcal{P} \cap \mathcal{F}_{H_1}} b_{i,B} - b_{i,H_1}, \min_{B \in \mathcal{P} \cap \mathcal{F}_{H_2}} b_{i,B} - b_{i,H_2}\}$. Similarly, if a lower facet $F_{H_1,i,-}$ is unbordered, then the hole can be extended across the lower limit a_{i,H_1}.

This implies that all of the facets of the smaller hole in the largest punching pair have to be bordered in $\mathcal{P}$.

The above argument shows that at least some of the bordering boxes of the largest punching pair have to be in the proving tuple. It is therefore relevant to examine how to describe them in terms of the other boxes of $\mathcal{F}$. It turns out, that some of them are simple to describe while others, not so apparently. This means that some of them can be defined for $\mathcal{F}$ without any reference to punching.

Clearly, the largest pair of boxes that 2-punch $\mathcal{F}$ are $\{H_1, H_2\} = \arg\max_{\mathcal{F}_1 \sqcup \mathcal{F}_2 = \mathcal{F}} \left(\bigcap_{B_1' \in \mathcal{F}_1} B_1', \bigcap_{B_2' \in \mathcal{F}_2} B_2' \right)$ i.e. the largest pair of intersections from the possible partitions of $\mathcal{F}$. Note that here max was taken with respect to the introduced ordering on the tuples. It will now be shown that some of the borders of H_1 and H_2 are in some way extremal in the projections to the axes. This extremality will be described first.

For a family of boxes $\mathcal{F}$, for all axes $1 \leq i \leq d$ the numbers $a_{i,\mathcal{F}} = \max_{B \in \mathcal{F}} a_{i,B}$ and $b_{i,\mathcal{F}}$ will be referred to as the **outward limit** values and the hyperplanes $\alpha_{i,\mathcal{F}} = \{(x_j)_{1 \leq j \leq d} : x_1 = a_{i,\mathcal{F}}\}$

and $\beta_{i,\mathcal{F}} = \{(x_j)_{1\leq j\leq d} : x_1 = b_{i,\mathcal{F}}\}$ as outward limits. This means, that along every axis the "leftmost right endpoint" and "rightmost left endpoint" of the projection of $\mathcal{F}$ is the corresponding outer limit. The outward limits $\alpha_{i,\mathcal{F}}$ and $\beta_{i,\mathcal{F}}$ are the borders of the half-spaces $\alpha^+_{i,\mathcal{F}} = \{(x_j)_{1\leq j\leq d} : x_i \geq a_{i,\mathcal{F}}\}$ and $\beta^+_{i,\mathcal{F}} = \{(x_j)_{1\leq j\leq d} : x_i \leq b_{i,\mathcal{F}}\}$ respectively for $1 \leq i \leq d$, which will be called **outward limit halves**.

The boxes contained in an outward limit half are called **outward limit boxes**. Outward limit boxes are bordered by an outward limit $\gamma_{i,\mathcal{F}} \in \{\alpha_{i,\mathcal{F}}, \beta_{i,\mathcal{F}}\}$ and the relative interior of the box B is in the corresponding outward limit half $\gamma^+_{i,\mathcal{F}}$. For $\alpha_{i,\mathcal{F}}$, the set of corresponding outward limit boxes is $\mathcal{L}_i = \{B \in \mathcal{F} : B \subset \alpha^+_{i,\mathcal{F}}\}$, while it is $\mathcal{R}_i = \{B \in \mathcal{F} : B \subset \beta^+_{i,\mathcal{F}}\}$ for $\beta_{i,\mathcal{F}}$. $\bigcup_{i=1}^{d} \mathcal{L}_i \cup \mathcal{R}_i = \mathcal{O}$ are all the outward limit boxes.

The term outward limit refers to the following property.

Observation: *For any outward limit $\gamma_{i,\mathcal{F}} \in \{\alpha_{i,\mathcal{F}}, \beta_{i,\mathcal{F}}\}$, no box $B \in \mathcal{F}$ is contained in the inside of the outward limit half $\gamma^+_{i,\mathcal{F}}$ for any $1 \leq i \leq d$.*

So, in a sense, outward limit boxes are most the most "outward facing" or "outward starting" boxes along each axis.

We defined outward limit boxes without any reference to punching, yet they all border the largest punching pair of $\mathcal{F}$.

Proposition 4 (Outward limits border largest punching pair)**.** *If $\{H_1, H_2\}$ are the largest pair of boxes that 2-punch $\mathcal{F}$, then all of the outward limit boxes border some element of $\{H_1, H_2\}$*

Proof. Without loss of generality, for any $1 \leq i \leq d$, assume that the outward limit box $O \in \mathcal{L}_i$ is punched by H_1. Since H_1 punches O, we know $a_{i,O} \leq a_{i,H_1}$. As H_1 is member of a largest punching pair, it is the intersection of the members of $\mathcal{H}_\infty$. This means that $a_{i,\mathcal{F}_{H_1}} = a_{i,H_1}$, where $a_{i,\mathcal{F}_{H_1}} = \max_{B\in\mathcal{F}_{H_1}} a_{i,B}$. By definition, $a_{i,\mathcal{F}_{H_1}} \leq a_{i,\mathcal{F}}$ as $\mathcal{F}_{H_1} \subset \mathcal{F}$. However, by definition of outward limits, $a_{i,O} = a_{i,\mathcal{F}}$. Therefore, we have $a_{i,O} \leq a_{i,H_1} = a_{i,\mathcal{F}_{H_1}} \leq a_{i,\mathcal{F}} \leq a_{i,O}$, which implies equality between all of the numbers. This means that since $H_1 \subset O$, O has to border H_1.

The same argument applies for $\mathcal{R}_i$ by considering the other extrema, e.g. $b_{i,\mathcal{F}}$. □

The outward limits are not only related to the largest punching pair, but provide a structure for all punching pairs. This is due to the fact that an outward limit box is punched only if its bordering outward limit halves are punched. Therefore, every outward limit half has to be punched by a pair that punches $\mathcal{F}$. Thus, the punching pairs can be characterized according to which outward limit halves are punched together.

The structure of punching varies according to which outward limit halves are punchable together. If $\alpha^+_{i,\mathcal{F}}$ and $\beta^+_{i,\mathcal{F}}$ (the outward limit halves for the ith axis) intersect, then they can be punched together by a hole H_1, while H_2 has to punch neither, while if the intersection is empty, then each of the holes punches exactly one of the outward limit halves, and each is punched by exactly one hole.

If the outward limit halves have disjoint relative interiors, then the punching pairs have a structure which is fairly simple to describe.

Observation: $\bigcap\limits_{i=1}^{d} \gamma^+_i$ *is an orthant where* $\gamma^+_i \in \{\alpha^+_i, \beta^+_i\}$ *for* $1 \leq i \leq d$.

These orthants defined by the outer limits halves shall be called **corners**. The point $\bigcap\limits_{i=1}^{d} \gamma_{i,\mathcal{F}}$ is the vertex of the corner.

If $C = \bigcap\limits_{i=1}^{d} \gamma^+_i$ and $C' = \bigcap\limits_{i=1}^{d} \gamma^{+'}_i$ are two corners such that $\gamma^+_i \in \{\alpha^+_i, \beta^+_i\} \setminus \{\gamma^{+'}_i\}$ for $1 \leq i \leq d$ then they are called **diagonally opposite corners**. This means that if a box has exactly one vertex in each corner, then diagonally opposite vertices are in diagonally opposite corners.

Statement 2. *If for every* $1 \leq i \leq d$*, the outward limit halves* $\alpha^+_{i,\mathcal{F}}$ *and* $\beta^+_{i,\mathcal{F}}$ *have disjoint relative interiors, then any pair of boxes punching* $\mathcal{F}$ *punches a pair of diagonally opposite corners.*

Proof. Let H_1 and H_2 be a pair of boxes punching $\mathcal{F}$. As mentioned earlier, for each axis $1 \leq i \leq d$, there is a bijection $p_i : \{H_1, H_2\} \to \{\alpha^+_{i,\mathcal{F}}, \beta^+_{i,\mathcal{F}}\}$ between the holes and the outward limit halves such that H_j uniquely punches $p_i(H_j)$ for $j \in \{1, 2\}$. This means, that for the ith axis $p_i(H_1) \in \{\alpha^+_{i,\mathcal{F}}, \beta^+_{i,\mathcal{F}}\} \setminus \{p_i(H_{(j+1 \mod 2)+1})\}$ for any $j \in \{1, 2\}$. Therefore, clearly $\bigcap\limits_{i=1}^{d} p_i(H_1)$ and $\bigcap\limits_{i=1}^{d} p_i(H_2)$ are diagonally opposite corners. □

Therefore, the holes of the largest punching pair share a vertex each with opposite members of the punched diagonally opposite pair of corners in this axis-wise disjoint case. The neighboring facets of this vertex are thus bordered by outward limits boxes.

Suppose that in this axis-wise disjoint case, punching were restricted to a pair of diagonally opposite corners, i.e. only considering punching pairs that punch both corners. If it were possible to construct a restricted proving tuple of fixed size (not depending on the size of $\mathcal{F}$) for a pair of diagonally disjoint corners, then taking the union of these restricted proving tuples for all pairs of diagonally opposite corners would give a proving tuple for the whole of $\mathcal{F}$ as the largest punching pair cannot increase by adding boxes to a (sub)family. If apart from the outward limit boxes a restricted proving tuple were to contain h_r boxes, then there would be a Helly-type theorem for the axis-disjoint case with tuple-size $h = 2d + h_r \cdot 2^{d-1}$ as there are 2^{d-1} pairs of diagonally opposite corners and 2d outward limits.

Unfortunately, it is not known whether there is a proving tuple even for this restricted case. For example, as mentioned previously, $2d$ boxes that border the the largest punching pair for a given pair of diagonally opposite corners have to be in the proving tuple for the smaller hole, so it is an immediate idea to check whether these form a proving tuple in the restricted case. However, these bordering boxes can also be regrouped even within the restricted area resulting in larger holes than for the whole family. An example of this is shown in section 2.3.4.

If the outward limit halves can be intersecting, then the situation might be even more complicated, as then, not all punching pairs have to punch diagonally opposite corners, and this is true even for the largest punching pair. For example, if the 1-punching construction from **Proposition** 2 is sufficiently extended in some direction along a given axis, then the largest punching pair will not be contained in corners.

Similarly as before, for some $0 < \varepsilon$ let $B_{1-} = [-1, \varepsilon/2] \times \prod_{j=2}^{d}[-1, 1]$. Now, for some sufficiently large $c \geq \frac{2^{d-1}(1+\varepsilon/2)}{\varepsilon^{d-1}}$, let $B_{1+} = [-\varepsilon/2, c] \times \prod_{j=2}^{d}[-1, 1]$ and for $2 \leq i \leq d$ let $B_{i+} = [-1, c] \times \prod_{j=2}^{i-1}[-1, 1] \times [-\varepsilon/2, 1] \times \prod_{j=i+1}^{d}[-1, 1]$ and $B_{i-} = [-1, c] \times \prod_{j=2}^{i-1}[-1, 1] \times [-1, \varepsilon/2] \times \prod_{j=i+1}^{d}[-1, 1]$. This means, that in the direction $\underline{e_1}$, each box has been sufficiently extended, except, the lower box B_{1-}. The largest pair that punches family $\mathcal{F} = \{B_{i+} : 1 \leq i \leq d\} \cup \{B_{i-} : 1 \leq i \leq d\}$ is $\{B_{1-}, \bigcap_{B \in \mathcal{F} \setminus \{B_{1-}\}} B\}$ since no punching tuple can have larger volume than the smallest box in $\mathcal{F}$. Clearly, the smallest box is the unextended B_{1-} of volume $2^{d-1}(1 + \varepsilon/2)$ which is punched alone by the first hole. By the choice of c, the intersection $\bigcap_{B \in \mathcal{F} \setminus \{B_{1-}\}} B\}$ has volume $\varepsilon^{d-1}(c + \varepsilon/2) > 2^{d-1}(1 + \varepsilon/2) = \mathrm{Vol}(B_{i-})$, so there is no punching tuple with larger volume. Yet B_{1-} is not contained in any corner.

2.3 Lower bounds on the Helly-numbers for 2-punching boxes

Even though it is unclear whether Helly-type statements about 2-punching are possible for a finite tuple-size h, lower bounds for the required tuple-size can be given. In other words, lower bounds can be given on the Helly-number (which may turn out to be infinite.) This is achieved by showing constructions of families $\mathcal{F}$ violating the possible Helly-statement, that is, even though $\mathcal{F}$ cannot be punched by a pair of large enough volume, every h-tuple can be punched by a sufficiently large pair of holes.

First, a lower bound will be shown for the general statement of type (1) in **Theorem** 1.3.1 for boxes, i.e. applying to any volumes v_1 and v_2.

Of course, a statement about 2-punching implies the corresponding statement about 1-punching with the same tuple-size, so this means, that $2d$ is an immediate lower bound for 2-punching Helly-numbers.

Proposition 2 about 1-punching boxes contained a construction with $2d$ elements, where removing any box created a large hole for the subfamily. It might be an immediate idea to consider adding a disjoint translated copy of the 1-punching construction to create a 2-punching construction.

For $1 \geq c_d > 0$, $\varepsilon \in (0, \sqrt[d]{c_d})$ and $I = [-1, 1]$, consider the family $\mathcal{F}_1 = \{B_{i+} : 1 \leq i \leq d\} \cup \{B_{i-} : 1 \leq i \leq d\}$ where $B_{i+} = \prod_{j=1}^{i-1} I \times [-\varepsilon/2, 1] \times \prod_{j=i+1}^{d} I$ and $B_{i-} = \prod_{j=1}^{i-1} I \times [-1, \varepsilon/2] \times \prod_{j=i+1}^{d} I$. Then, add a translated copy $\mathcal{F}_2$ such that all members are disjoint from all members of $\mathcal{F}_1$. Thus, for some $c > 2$ let $\mathcal{F}_2 = \{B + c\underline{1} : B \in \mathcal{F}_1\}$ and the whole family is $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. The two copies of the original define two obvious classes $\mathcal{F}_1 = \mathcal{F}_{H_1}$ and $\mathcal{F}_2 = \mathcal{F}_{H_2}$ and no other partition has classes whose boxes can be punched together as the two copies are disjoint. However, each box only borders exactly one of the holes H_1 and H_2 of equal volume smaller than c_d, so any subfamily that contains all of the boxes from one of the copies of the 1-punching construction is not punchable by a pair of larger volume than the whole family. Therefore, up to $2d-1$ boxes can be removed without increasing the volume of the pair of holes of the $4d$ boxes in total. If, however, $2d$ boxes are removed, either there are boxes missing from both copies, which increases the intersection in both classes, or, exactly one of the classes remains. However, if a family has non-empty intersection, then the intersection is not a member of the largest punching pair. So any subfamily of size $2d$ is punched by a larger pair than $\mathcal{F}$.

Although this simple idea proved not so effective, it does increase the lower bound by one, so the Helly-number for 2-punching theorems is at least $2d + 1$.

The bounds $2d$ and $2d+1$ are of course immaterial due to **Corollary** 1.3.1.3, which implies a lower bound of $3d - 1$ and $3d$ for odd and even d respectively.

However, these lower bounds can be improved significantly, by modifying the previous simple construction.

2.3.1 Lower bound $4d - 1$ of Helly-number for 2-punching boxes in any dimension d

Theorem 2.3.1. *For any dimension d and $1 \geq \varepsilon > 0$, there is a family of boxes $\mathcal{F}$ in $\mathbb{R}^d$ such that any $(4d-2)$-tuple is 2-punchable, and $\mathcal{F}$ is $(\varepsilon, 2)$-punchable, but is not $(\varepsilon', 2)$-punchable for any $\varepsilon' > \varepsilon$.*

Proof. For dimension d let $B'_{ij} = \prod_{k=1}^{d} I_k$ for $1 \leq i \leq d$, $1 \leq j \leq 4$, where $I_k = [-2, 2] = I$ for $k \neq i$ and $I_i = \begin{cases} [-2+\varepsilon/2, -1+\varepsilon/2] : j = 1 \\ [-1-\varepsilon/2, -\varepsilon/2] : j = 2 \\ [\varepsilon/2, 1+\varepsilon/2] : j = 3 \\ [1-\varepsilon/2, 2-\varepsilon/2] : j = 4 \end{cases}$

Then for all i and j let $B_{ij} = cB'_{ij}$ where $c = \varepsilon^{-\frac{d-1}{d}}$. Finally, let the family of boxes be $\mathcal{F} = \{B_{ij} : 1 \leq i \leq d, 1 \leq j \leq 4\}$.

We will refer to the index i for box B_{ij} as the box's **narrow** dimension and call boxes $B_{ij}, j \in \{1,2,3,4\}$ the i-**narrow** boxes.

Observation: *or every dimension i there are 4 i-narrow boxes in $\mathcal{F}$: two pairs, B_{i1},B_{i2} and B_{i3}, B_{i4}, which are intersecting respectively. However, no member of the first pair intersects any member of the second pair.*

Lemma 2.3.2. *$\mathcal{F}$ is $(\varepsilon, 2)$-punchable, but not $(\varepsilon', 2)$-punchable for any $\varepsilon' > \varepsilon$.*

Proof. Consider the box $A = [-c, c]^d$ with vertices $V(A) = \{\underline{v} \in \{-c, c\}^d\}$. The family of axis-parallel hypercubes $\mathcal{A} = \{B_{\underline{v}} : \underline{v} \in V(A)\}$ is defined as $B_{\underline{v}} = \prod_{i=1}^{d} [v_1 - c\varepsilon/2, v_i + c\varepsilon/2]$ with edge-lenghts $c\varepsilon$ having $\underline{v} \in V(A)$ at their center. These boxes are bordered by the boxes in $\mathcal{F}$.

For any axis $1 \leq i \leq d$, the disjoint i-narrow boxes B_{i1} and B_{i4} are outward limit boxes. Therefore, $\mathcal{F}$ is only punchable by a pair punching a pair of diagonally opposite corners. However, for any pair of diagonally opposite corners, each box in $\mathcal{F}$ only intersects exactly one of the corners. So, for any diagonally opposite pair of vertices $\underline{v}, -\underline{v}$ the pair of boxes $B_{\underline{v}}, B_{-\underline{v}}$ are a largest punching pair in the family $\mathcal{F}$. Clearly, $\mathrm{Vol}(B_v) = (c\varepsilon)^d = \varepsilon$ for any $\underline{v} \in V(A)$. □

Lemma 2.3.3. *Any subfamily $\mathcal{H} \subset \mathcal{F}$ of size $4d-2$ is 2-punchable.*

Proof. Since $|\mathcal{F}| = 4d$ and $|\mathcal{H}| = 4d - 2$ there is either *A)*: an index $1 \geq i \geq d$ for which there are only two i-narrow boxes in $\mathcal{H}$ or *B)*: there are two indices $1 \geq i_1, i_2 \geq d, i_1 \neq i_2$, for which there is an i_1-narrow and i_2-narrow box missing from $\mathcal{H}$.

In case *A)* there is a pair of hypercubes from $\mathcal{A}$ that punches $\mathcal{H}$ and is bordered by boxes of $\mathcal{H}$ for all facets. Let $B_{\underline{v}}$ and $B_{\underline{w}}$ be such a punching pair and let i be the dimension missing boxes belonging to it in $\mathcal{H}$, while $B_{ij_1} \supset B_{\underline{v}}$ and $B_{ij_2} \supset B_{\underline{w}}$ are the boxes in $\mathcal{H}$ with narrow dimension i. If $B_{\underline{v}} = \prod_{k=1}^{d} [v_k - c\varepsilon/2, v_k + c\varepsilon/2]$ and $B_{\underline{w}} = \prod_{k=1}^{d} [w_k - c\varepsilon/2, w_k + c\varepsilon/2]$, then Boxes $B_1 = \left(\prod_{k=1}^{i-1} [v_k - c\varepsilon/2, v_k + c\varepsilon/2]\right) \times \pi_i \left(B_{ij_1}\right) \times \left(\prod_{k=i+1}^{d} [v_k - c\varepsilon/2, v_k + c\varepsilon/2]\right)$ and $B_2 = \left(\prod_{k=1}^{i-1} [w_k - c\varepsilon/2, w_k + c\varepsilon/2]\right) \times \pi_i \left(B_{ij_2}\right) \times \left(\prod_{k=i+1}^{d} [w_k - c\varepsilon/2, w_k + c\varepsilon/2]\right)$ is a punching pair in $\mathcal{H}$. Note that $B_{\underline{v}} \subset B_1$ and $B_{\underline{w}} \subset B_2$ and one can think of B_1 and B_2 as a sort of extension of $B_{\underline{v}}$ and $B_{\underline{w}}$ along the dimension i such that they fill B_{ij_1} and B_{ij_2} along the narrow dimension. Since $\pi_i (B_1), \pi_i (B_2) \subset \pi_1 (B_{kl})$ for any $B_{kl} \in \mathcal{H} \setminus \{B_{ij_1}, B_{ij_2}\}$ we have that $B_{\underline{v}} \subset B_{kl}$ implies $B_1 \subset B_{kl}$ and $B_{\underline{w}} \subset B_{kl}$ implies $B_2 \subset B_{kl}$, so B_1 and B_2 truly punches $\mathcal{H}$.

In case *(B)* let i_1 and i_2 be the dimensions with missing boxes in $\mathcal{H}$. Since 3 narrow boxes belong to both dimensions i_1 and i_2 in $\mathcal{H}$ respectively, there is a pair $B_{\underline{v}}, B_{-\underline{v}}$ which is contained in two i_1-narrow and i_2-narrow boxes respectively. Let j_1 and j_2 be the indices of the i_1-narrow and i_2-narrow boxes in $\mathcal{H}$ which do not intersect any other narrow boxes of their respective dimensions. We can extend $B_{\underline{v}}$ and $B_{-\underline{v}}$ similarly as in case *A)* so that we get a punching pair of boxes with bigger volume: $B_1 = \left(\prod_{k=1}^{i_1-1} [v_k - c\varepsilon/2, v_k + c\varepsilon/2]\right) \times \pi_{i_1} \left(B_{i_1 j_1}\right) \times \left(\prod_{k=i_1+1}^{d} [v_k - c\varepsilon/2, v_k + c\varepsilon/2]\right)$ and $B_2 = \left(\prod_{k=1}^{i_2-1} [-v_k - c\varepsilon/2, -v_k + c\varepsilon/2]\right) \times \pi_{i_2} \left(B_{i_2 j_2}\right) \times \left(\prod_{k=i_2+1}^{d} [-v_k - c\varepsilon/2, -v_k + c\varepsilon/2]\right)$. By the same argument as in case *A)*, $\pi_{i_i} \left(B_{i_1 j_1}\right) \subset \pi_{i_1} (B_{kl})$ and $\pi_{i_2} \left(B_{i_2 j_2}\right) \subset \pi_{i_2} (B_{kl})$ for any $B_{kl} \in \mathcal{H} \setminus \{B_{i_1 j_1}, B_{i_2 j_2}\}$, so (B_1, B_2) punches $\mathcal{H}$ indeed.

In both cases, we can see that the volume of B_1 and B_2 is $(c\varepsilon)^{d-1} \cdot c = c^d \varepsilon^{d-1} = \left(\varepsilon^{-\frac{d-1}{d}}\right)^d \varepsilon^{d-1} = \varepsilon^{-(d-1)} \varepsilon^{d-1} = 1$. □

Thus, we have shown a punching pair of volume 1 for any subfamily $\mathcal{H} \subset \mathcal{F}$ of size $|\mathcal{H}| = 4d - 2$, while the whole family $\mathcal{F}$ is only $(\varepsilon, 2)$-punchable. □

Corollary 2.3.3.1. *For a finite family $\mathcal{F}$ of boxes in $\mathbb{R}^d$, if assuming any h-tuple of sets $\mathcal{H}$ in $\mathcal{F}$ is 2-punchable implies $\mathcal{F}$ is $(c_d, 2)$-punchable for some $c_d > 0$, then $h \geq 4d - 1$.*

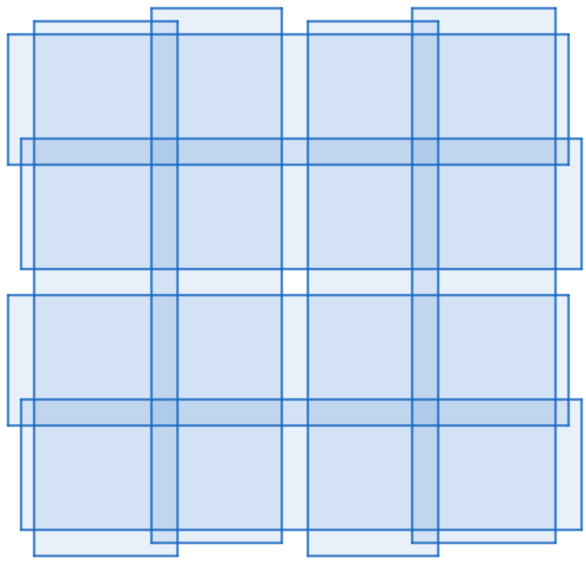

Figure 2.1: Construction for dimension 2 which is only $(\varepsilon, 2)$-punchable, but any $(4d-2)$-tuple is 2-punchable. For visual purposes, the boxes are slightly moved in the picture related to the written construction in the proof.

So, it is demonstrated that even for this broader class of quantitative volume theorem about 2-punching, the Helly-number has to be at least $4d-1$. This result also shows that even 2-punching is in a sense more complicated than the previously discussed conditions in Helly-type theorems, e.g. piercing.

2.3.2 Framework of construction for stronger lower bound

For the restricted case $v_1 = v_2$ of the general 2-punching theorem, there are some other interesting approaches to constructions. In this case, the largest punching pairs for the given subfamilies do not have to be arbitrarily large as in the previous case, just larger than the largest pair that punches the whole family $\mathcal{F}$. Several considerations will be taken into account to find an appropriate construction. This will be achieved by imposing constructing principles so that correctness of the construction depends on a discrete structure as simple as possible.

In the previous construction, the holes in the largest punching pair were of equal size. This means that a bordering box from both holes has to be removed to increase the volume of the largest punching pair. If every largest punching pair has a larger and smaller hole, then only a border from the smaller hole has to be removed. Since considering tuples with only 1 box missing might be simpler, the goal will be to ensure this property, which will be called the **unequal punching condition**.

If the goal is for the largest punching pair to increase in volume by the removal of any box, then any box has to border the smaller hole from a largest punching pair of $\mathcal{F}$ such that no other box boundary contains the same facet of this hole. This condition will be called the **unique border condition**. So, it is clear for example, that adding any box to the previous construction violates this condition, so it cannot be extended in a way that increases the lower bound. Also, an obvious consequence is that every box has to be unique in $\mathcal{F}$, so if $B_1 = B_2 \subset \mathbb{R}^d$, then they cannot be both in $\mathcal{F}$, as every facet in the boundary of B_1 is in the boundary of B_2 as well.

For simplicity, it is reasonable to first consider the case where the outer limit halves have disjoint insides, as then, diagonally opposite corners are punched in any case. Satisfying the unique border condition can be achieved by making the other bordering boxes (apart from the outward limits) only slightly larger in the direction of the non-bordering axes and making them sufficiently small in the diagonally opposite corner. Here, the general properties of such a construction approach will be described, and a concrete example constructed.

Given an integer $d > 2$ let $V \subset \mathbb{R}^d$ be the vertices of the d-dimensional box $B_0 = [-1;1]^d$ and for a vertex $\underline{v} \in V$ the diagonally opposite vertex pair is $-\underline{v}$. Let $\varepsilon > 0$ be sufficiently small. The dilated box $T = (1+\varepsilon)B_0$, called the total box, will contain the whole family $\mathcal{F}$. Let $\mathcal{F}$ be composed of two disjoint parts $\mathcal{O}$, the outward limit boxes and $\mathcal{I}$, the bordering boxes of the largest punching pairs apart from $\mathcal{O}$, which will be called **inward bordering boxes**. The outward limit halves are simply the different (positive and negative) halves of T along each dimension, that is $T \cap \{(x_i)_{1\leq i \leq d} : (-1)^{k_j} x_j \geq 0\}$ for $1 \leq j \leq d$ and $k_j \in \{0,1\}$. If $k_j = 0$, the outward limit box will be denoted O_{j+} and O_{j-} otherwise. By the above definition, note that $|\mathcal{O}| = 2d$. Since these will be outward limit boxes the outward limit values will be $a_i = b_i = 0$.

The corners in this construction will be therefore the orthants of the canonical coordinate system. Since the insides of the outward limit halves are disjoint for every axis, diagonally opposite corners have to be punched in any case. Since there are 2^{d-1} pairs of diagonally opposite corners, ensuring that there is a punching pair of largest volume for as many such pairs as possible is promising in finding a construction with more than $4d$ boxes.

For further simplicity, the goal will be to ensure that the smaller holes in the pairs of largest volume will be the boxes $\underline{0}\square\underline{v}$ or $\underline{0}\square-\underline{v}$ for $\underline{v} \in V$ of volume 1, which will be called **unit boxes**. The hole in the opposite corner has to be larger by the unequal punching condition.

Since little is understood about the regrouping of tuples, it can be fruitful to look at constructions, where the boxes bordering the largest punching pair cannot be regrouped when punching is restricted to the punched pair of diagonally opposite of corners. Therefore, any inward bordering box should have a small intersection with the corner opposite from where it is bordering. This property will be called **unique grouping condition**.

The hyperplanes $\sigma_{i+} = \{(x_j)_{1\le j\le d} : x_i = 1\}$ and $\sigma_{i-} = \{(x_j)_{1\le j\le d} : x_i = -1\}$ for $1 \le i \le d$ will be called **small borders** and shall thus contain the facets of the smaller holes of the largest punching pairs and define the half-spaces $\sigma^+_{i+} = \{(x_j)_{1\le j\le d} : x_i \le 1\}$ and $\sigma^+_{i-} = \{(x_j)_{1\le j\le d} : x_i \ge -1\}$ that intersect with the total box T on a large box of volume $(2+2\varepsilon)^{d-1}(2+\varepsilon)$ and $T \setminus \sigma^+_{i+}$ has very small volume $(2+2\varepsilon)^{d-1}\varepsilon$. Additionally, the hyperplanes $\tau_{i+} = \{(x_j)_{1\le j\le d} : x_i = \varepsilon\}$ and $\tau_{i-} = \{(x_j)_{1\le j\le d} : x_i = -\varepsilon\}$ will be called **tiny borders** and define half-spaces $\tau^+_{i+} = \{(x_j)_{1\le j\le d} : x_i \le \varepsilon\}$ and $\tau^+_{i-} = \{(x_j)_{1\le j\le d} : x_i \ge -\varepsilon\}$ such that $T \setminus \tau^+_{i+}$ has much larger volume $(2+2\varepsilon)^{d-1}(1+2\varepsilon)$. In other words, the σ^+ half-spaces cut off a small box (or a "slice" of width ε) from T, while the τ^+ half-spaces cut off a large box (or a "slice" of width 1) from T.

The boxes of $\mathcal{I}$ will be the intersection of the total box T and some of the half-spaces σ^+ and τ^+. The intersection(s) with σ^+(s) will ensure that an inward bordering box indeed borders a unit hole $\underline{0}\square\underline{v}$ for $\underline{v} \in V$ while the intersection(s) with τ^+(s) ensure that the boxes have a small intersection with the opposite corners (a small intersection of positive in stead of 0 volume ensures the unique bordering of outward limit boxes). This is achieved by a sufficiently small $\varepsilon > 0$ such that $\varepsilon(1+\varepsilon)^{d-1} < 1$. Such an ε exists, because $\lim\limits_{x\to 0} x(1+x)^{d-1} = 0$ and $f(x) = x(1+x)^{d-1} > 0$ for $x > 0$ and f is continuous.

A construction composed of such boxes is required to satisfy the conditions that were introduced at the beginning of this part while ensuring a number of boxes as large as possible.

Observation: *If an inward bordering box $B \in \mathcal{I}$ is bordered by τ_{i+} or τ_{i-}, then B cannot be punched by a hole of size at least 1 in the corners which intersect with τ_{i+} or τ_{i-}. Also, B cannot border a unit box in a corner which intersects τ_{i+}, or τ_{i-}.*

Therefore, if τ_{i+} borders a box in $\mathcal{I}$, then τ_{i-} cannot border it. This also means, that if an inward bordering box has $1 \leq t \leq d$ tiny borders, then then it is 1-punchable in $2^{d-t}$of the possible 2^d corners. Furthermore, since any tiny border intersects with exactly one element of every pair of diagonally opposite corners, if each box is bordered by a tiny border, then the unique grouping condition is satisfied for $\mathcal{F}$.

Because of the unique border condition, every bordering box has to have at least one small border.

Observation: *If an inward bordering box $B \in \mathcal{I}$ has a small border σ_{i+} or σ_{i-} and for $0 \leq k \leq d$ is also bordered by tiny borders $\tau_{i_1}, \ldots, \tau_{i_k}$ where $1 \leq i_1 < \ldots < i_k \leq d$ and $i_j \neq i$ for $j \in [k]$ and $\tau_{i_j} \in \{\tau_{i_j+}, \tau_{i_j-}\}$ then B borders the unit boxes of 2^{d-1-k} corners that are also bordered by $\sigma i+$ or σ_{i-}.*

Thus, if an inward bordering box B has $1 \leq t \leq d$ tiny borders and $1 \leq s \leq d-t$ small borders, $\max\{0, s+t-d\} \leq o \leq \min\{s, d-1\}$ of which are opposite a tiny border, then the boundary of B contains $(s-o)2^{d-1-t} + o2^{d-1-t-1}$ facets of unit boxes in different corners. So, the more tiny borders a box has, the less pairs of diagonally opposite corners are available as a location for punching, thus also restricting the number of possible boxes in $\mathcal{I}$. On the other hand, the less tiny borders an inward bordering box has, the less available unit box facets there are for other inward bordering boxes to border uniquely. Furthermore, the number of small borders of a box also limits the unit box facets for other boxes to border uniquely.

Since there can be only one tiny border for a box along each axis, the number of boxes with $k \leq d$ tiny borders and $l \leq d-k$ small borders cannot be more than $\binom{d}{k}\binom{2d-k}{l}$ as for each box, only one tiny border can be chosen for each of the k chosen axes.

Therefore, it is not trivial what types of borders should be chosen to obtain a proper construction of maximal size, since the number of small borders restricts the other boxes because of the unique border condition, while it is directly related to the possible number of unique boxes with the same number of small borders (up to d, but the allowed number of small boxes is directly related in any range of course.) Simultaneously, the allowed number of tiny borders is also directly related to number of possible unique boxes, while the number of tiny borders limits the number of boxes due to 2-punchability but also loosens the restriction imposed by small borders because of the unique border condition. However, it is worth noting that creating a proper construction with the above properties is now only dependent on a purely discrete structure of the small and tiny borders of the inward bordering boxes and the corners. Furthermore, it is indeed possible to give such a construction that is larger than $4d$.

2.3.3 Improved lower bound of Helly-number for 2-punching boxes in dimension 3

For $d = 3$, we will show an explicit construction along the previously layed out considerations which yields the following results.

Theorem 2.3.4 (Proper construction for lower bound, $d = 3$)**.** *There is a family of boxes $\mathcal{F}$ in $\mathbb{R}^3$ that is 2-punchable, but not $(v, 2)$-punchable for any $v > 1$. Furthermore, any proper subfamily is $(1 + \varepsilon, 2)$-punchable for $\varepsilon > 0$ and $\varepsilon(1 + \varepsilon)^2 < 1$.*

Corollary 2.3.4.1 (Lower bound on Helly-number for 2-punching boxes in $\mathbb{R}^3$)**.** *For a family of boxes $\mathcal{F}$ in $\mathbb{R}^3$ and tuple-size h, if assuming any subfamily $\mathcal{H} \subset \mathcal{F}$ of size h is 2-punchable implies $\mathcal{F}$ is 2-punchable, then $h \geq 18$.*

Proof of Theorem 2.3.4. Let V be the vertices of the previously defined box $B_0 = [-1, 1]^d$ in $\mathbb{R}^3$ and the total box is $T = (1 + \varepsilon)B_0$ for sufficiently small $\varepsilon > 0$.
The outward limit boxes $\mathcal{O}$ are defined as previously and the inward bordering boxes $\mathcal{I}$ will all be congruent. Each will have exactly one tiny border and one small border not opposite to the tiny border. So, each member of $\mathcal{I}$ will each be 1-punchable in half of the corners and will each border 2 unit boxes. Figure 2.2 shows the construction by highlighting the facets of the unit boxes that are bordered (which uniquely identifies each inward bordering box).
In each pair of diagonally opposite corners where $\mathcal{F}$ can be 2-punched, one the holes will be the unit box and in the opposite corner, one of the facets of the unit box has to be unbordered by all inward bordering boxes by the unequal punching condition. Since there are 4 pairs of diagonally opposite corners, there are 4 possible largest punching pairs, each of which contains the unit box from one of the corners of a diagonally opposite pair. These 4 boxes have 12 facets that are not bordered by outward limit boxes (these shall be the inward facets). Therefore, since each inward bordering box has to uniquely border a facet of the smaller hole of a largest punching pairs, there can be at most 12 inward bordering boxes. Together with $\mathcal{O}$ this amounts to at most 18 boxes in total in $\mathcal{F}$. It will be shown here, that this limit is sharp and it is possible to give a construction of size $18 > 12 = 4d$ for $d = 3$, which therefore yields a stronger result than the previous construction.
Consider the graph of $G(V, E)$ where $(v_1, v_2) \in E$ if and only if they are connected by an

edge of B_0 for any $v_1, v_2 \in V$. G is bipartite (for any d) and each color class covers all pairs of diagonally opposite vertex pairs (for any odd d). Let C be one of the color classes (these vertices are the vertices of a tetrahedron in the box B_0).
The smaller holes of each largest punching pair will be unit boxes $\underline{0}\Box\underline{c}$ for $\underline{c} \in C$. For $j \neq i$ and $\epsilon_1, \epsilon_2 \in \{+,-\}$, if $\sigma_{i\epsilon_1}$ and $\tau_{j\epsilon_2}$ borders an inward bordering box, then it will be denoted $B_{i\epsilon_1,j\epsilon_2}$ (so the first index reflects the small, the second the tiny border). This also uniquely identifies each box.
Let

$$\mathcal{I}_1 = \{B_{1+,2-}, B_{2+,3-}, B_{3+,1-}\}$$
$$\mathcal{I}_2 = \{B_{1+,3+}, B_{2-,1-}, B_{3-,1-}\}$$
$$\mathcal{I}_3 = \{B_{2+,1+}, B_{3-,2-}, B_{1-,2-}\}$$
$$\mathcal{I}_4 = \{B_{3+,2+}, B_{1-,3-}, B_{2-,3-}\}$$

Be the inward bordering boxes.

Let $\mathcal{F} = \mathcal{O} \cup \bigcup_{i=1}^{d} \mathcal{I}_i$. A box $B_{i\pm,j\pm}$ is 1-punchable in the same corners as outward limit box $O_{j\mp}$.

Lemma 2.3.5. *$\mathcal{F}$ is 2-punchable.*

Proof. The pair of holes $H_1 = \underline{0}\Box\underline{1}$, $H_2 = \underline{0}\Box - (1+\varepsilon)\underline{1}$ punches $\mathcal{F}$. Clearly, $\mathrm{Vol}(H_1) = 1$ and $\mathrm{Vol}(H_2) = (1+\varepsilon)^3$, so the pair has volume 1. The two classes $\mathcal{F}_{H_1}$ and $\mathcal{F}_{H_2}$ will be shown to cover $\mathcal{F}$. $\mathcal{F}_{H_1} = \{O_{1+}, O_{2+}, O_{3+}\} \cup \mathcal{I}_1 \cup \{B_{2-,1-}, B_{3-,2-}, B_{1-,3-}, B_{3-,1-}, B_{1-,2-}, B_{2-,3-}\}$ because the listed outward limit boxes and the boxes of $\mathcal{I}_1$ border H_1 and the other six boxes have only tiny borders τ_{1-}, τ_{2-} and τ_{3-}, so they can be punched in all corners of O_{1+},O_{2+} and O_{3+} respectively, so H_1 also punches them. $\mathcal{F}_{H_2} = \{O_{1-}, O_{2-}, O_{3-}\} \cup \{B_{1+,3+}, B_{2+,1+}, B_{3+,2+}\}$ because all of the listed boxes border H_2 and the tiny borders τ_{1+}, τ_{2+} and τ_{3+} and small borders σ_{1+}, σ_{2+} and σ_{3+} do not intersect H_2. $\Box$

The punching pair in the previous proof however, is not the only punching pair of volume 1. The unit boxes $\underline{0}\Box(1,-1,-1)$, $\underline{0}\Box(-1,1,-1)$ and $\underline{0}\Box(-1,-1,1)$ are bordered by the boxes of $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$ respectively. Note that $\underline{c_2} = (1,-1,-1)$, $\underline{c_3} = (-1,1,-1)$ and $\underline{c_4} = (-1,-1,1)$ are vertices of the tetrahedron C (while $\underline{c_1} = \underline{1}$).
Note, that $\mathcal{F}$ has rotational symmetry of 120 degrees around axis $\underline{c_1}$ passing through the

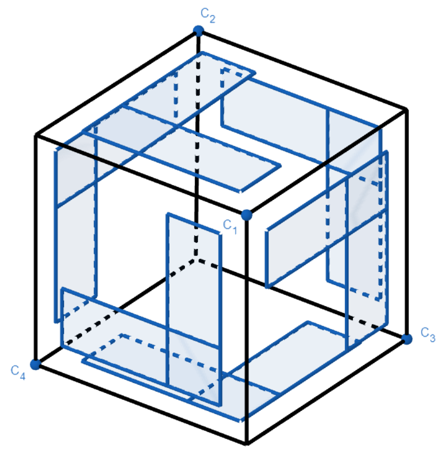

Figure 2.2: The inward bordering boxes of the construction $\mathcal{F}$. The edges of B_0 are in black, while each blue rectangle represents the facets of each inward bordering box, which contain the facets of a unit box, uniquely identifying each inward bordering box. The blue rectangles are not the actual facets, but are smaller and only highlight which halves of each facet of B_0 is bordered bo a box. The points C_i mark the corners where a unit box is the smaller hole of a largest punching pair of $\mathcal{F}$. These unit boxes are bordered along every inward facet. The unit box in every other corner has a facet that is not bordered by any box.

origin. Therefore, by examining the case $\mathcal{I}_2$ is sufficient.

Let $H_1 = \underline{0}\square\underline{c_2}$ and $H_2 = \underline{0}\square(-1,1,1+\varepsilon)$ be holes. $\mathcal{F}_{H_1} = \{O_{1+}, O_{2-}, O_{3-}\} \cup \mathcal{I}_2 \cup \{B_{3+,1-}, B_{3+,2+}\}$ since $B_{3+,1-}$ and $B_{3+,2+}$ have tiny borders τ_{1-} and τ_{2+} which do not intersect H_1. $\mathcal{F}_{H_2} = \{O_{1-}, O_{2+}, O_{3+}\} \cup \mathcal{I}_3 \cup \{B_{1+,2-}, B_{2+,3-}, B_{1-,3-}, B_{2-,3-}\}$ because none of the listed inward bordering boxes have tiny borders τ_{1-}, τ_{2+} or τ_{3+} and small border σ_{3+} which would intersect H_2. Since $\mathrm{Vol}(H_1) = 1$ and $\mathrm{Vol}(H_2) = 1+\varepsilon$ the punching pair has volume 1.

At the same time, there are no larger punching pairs in $\mathcal{F}$. This is due to the unique grouping property of $\mathcal{F}$.

Lemma 2.3.6. *$\mathcal{F}$ is not $(v, 2)$-punchable for any $v > 1$.*

Proof. Assume for some $\underline{c} \in C$, the punching pair H_1 and H_2 punches diagonally opposite corners containing $\underline{c}$ and $-\underline{c}$. Then, as seen previously, the boxes from exactly one of the subfamilies $\mathcal{I}_i$ from $i \in \{1,2,3,4\}$ borders $\underline{0}\square\underline{c}$. Since the boundaries of all three inward bordering boxes contain an inward facet of $\underline{0}\square\underline{c}$ each, if $\mathcal{I}_i$ is punched in this corner, then $\underline{0}\square\underline{c}$ is the largest hole. Therefore, in the corner containing $\underline{c}$ a larger hole can only punch $\mathcal{F}$ if at least one of the boxes of $\mathcal{I}_i$ is punched in the opposite corner containing $-\underline{c}$. However, an inward bordering box has a tiny border in the opposite corner, so their intersection is of volume $\varepsilon(1+\varepsilon)^2 < 1$ by choice of ε. Therefore, any such regrouped punching pair has volume smaller than 1. □

Lemma 2.3.7. *Any proper subfamily is* $(1+\varepsilon, 2)$*-punchable.*

Proof. Since each outward limit box contains two vertices of the tetrahedron C, the smaller holes of two largest punching pairs punch it. Therefore, if any outward limit box is removed, then the two smaller holes lose their borders, since outward limit boxes uniquely border some facets of the holes of largest punching pairs. Therefore, in any subfamily $\mathcal{H} \subseteq \mathcal{F} \setminus \{O_{i\pm}\}$ either the outward limit value $a_{i,\mathcal{F}}$ or $b_{i,\mathcal{F}}$ changes to $a_{i,\mathcal{H}} \geq -\varepsilon$ or $b_{i,\mathcal{H}} \leq \varepsilon$. Therefore, if H_1 is a smaller hole in O_{i+} or O_{i-} from a largest punching pair $\{H_1, H_2\}$, then for the hole $H_1' = \prod_{j=1}^{i-1} \pi_j(H_1) \times I \times \prod_{j=i+1}^{d} \pi_j(H_1)$ the pair $\{H_1', H_2\}$ punches $\mathcal{H}$ for $I = [-\varepsilon, 1]$ or $I = [-1, \varepsilon]$ respectively. Clearly, $\mathrm{Vol}(H_1') = 1+\varepsilon$, while $\mathrm{Vol}(H_2) \geq 1+\varepsilon$ so the pair is of volume $1+\varepsilon$.

Since the inward limit boxes all uniquely border a given facet of the smaller box from the largest punching pairs, removing any of them clearly also increases the largest punching pair. So, if $B_{i\pm,j+}$ or $B_{i\pm,j-}$ is removed, which uniquely bordered smaller hole H_1 from largest punching pair H_1, H_2, then for $I = [0, 1+\varepsilon]$ or $I = [-\varepsilon, 0]$ respectively H_1 can be replaced by $H_1' = \prod_{j=1}^{i-1} \pi_j(H_1) \times I \times \prod_{j=i+1}^{d} \pi_j(H_1)$ such that H_1', H_2 punches any $\mathcal{H} \subseteq \mathcal{F} \setminus \{B_{i\pm,j+}\}$ or $\mathcal{H} \subseteq \mathcal{F} \setminus \{B_{i\pm,j-}\}$. Again, the pair $\{H_1', H_2\}$ is clearly of volume $1+\varepsilon$. □

This completes the proof of **Theorem** 2.3.4. □

2.3.4 Improved lower bound of Helly-number for 2-punching boxes in the plane

Another approach for a construction of a lower bound also assumes that outer limit halves are disjoint along each axis, but while the previous approach had punching pairs for different (all) diagonally opposite corners this approach only focuses on one pair. Such a construction of size $10 > 4d - 1$ will be presented for $d = 2$, which yields the following results.

Theorem 2.3.8. *The family of boxes $\mathcal{F} = \mathcal{S} \cup \mathcal{O}_\mathcal{S} \cup \mathcal{B} \cup \mathcal{O}_\mathcal{B} \cup \mathcal{R}$ in $\mathbb{R}^2$ is 2-punchable, is not $(v, 2)$-punchable for any $v > 1$. Furthermore, any proper subfamily of $\mathcal{F}$ is $(v, 2)$-punchable for some $v > 1$.*

Corollary 2.3.8.1. *For a family of boxes $\mathcal{F}$ in the plane and tuple-size h, if assuming all subfamilies $\mathcal{H} \subset \mathcal{F}$ of size h are 2-punchable implies $\mathcal{F}$ is 2-punchable, then $h \geq 10$.*

Proof of Theorem 2.3.8. Consider the family $\mathcal{F} = \mathcal{S} \cup \mathcal{O}_\mathcal{S} \cup \mathcal{B} \cup \mathcal{O}_\mathcal{B} \cup \mathcal{R}$ consisting of five types of boxes. The boxes in $\mathcal{S}$ will be referred to as the small boxes, $\mathcal{B}$ contains the big boxes, $\mathcal{O}_\mathcal{S}$ and $\mathcal{O}_\mathcal{B}$ are the outer limits boxes for the small and big boxes and $\mathcal{R}$ are the restricting boxes.

All boxes of $\mathcal{F}$ will be contained in the total box $T = [-4, 4]^2$. Let $\mathcal{O}_\mathcal{S}$ and $\mathcal{O}_\mathcal{B}$ be the positive and negative halves of T. For sufficiently small $1 > \varepsilon > 0$ and $a > 0$ such that $2.1 \cdot a < 1$ and $a^2 + 1.9 \cdot a > 1$ (for example, $a = 0.45$ is an appropriate choice) the other boxes will be the following.

$$\mathcal{S} = \{(3, 1)\square(-0.5, -4), (1, 3)\square(-4, -0.5)\}$$
$$\mathcal{B} = \{(-1 - \varepsilon, -3)\square(0.5, 4), (-3, -1 - \varepsilon)\square(4, 0.5)\}$$
$$\mathcal{R} = \{(-a, -a)\square(2.1, 1.9), (-a, -a)\square(1.9, 2.1)\}$$

Note that the members of a given pair are axial reflections of each other for axis $\{(x, y) : x = y\}$.

Lemma 2.3.9. *$\mathcal{F}$ is 2-punchable.*

Proof. Clearly, the family is punched by $H_1 = \underline{0}\square\underline{1}$ and $H_2 = \underline{0}\square-(1+\varepsilon)\underline{1}$ as $\mathcal{F}_{H_1} = \mathcal{O}_\mathcal{S} \cup \mathcal{S} \cup \mathcal{R}$ and $\mathcal{F}_{H_2} = \mathcal{O}\mathcal{B} \cup \mathcal{B}$. Clearly, $\mathrm{Vol}(H_1) = 1$ and $\mathrm{Vol}(H_2) = (1 + \varepsilon)^2$, so the punching pair has volume 1. □

Lemma 2.3.10. *$\mathcal{F}$ is not (v, n)-punchable for any $v > 1$.*

Proof. For any intersection point outside the For any point P in the set $S = \{(x, y) : xy < 1\} \cap \{(x, y) : xy < -1\}$ the box $\underline{0}\square P$ has volume smaller than 1, while if $P \in \mathbb{R}^2 \setminus S$, then $\mathrm{Vol}(\underline{0}\square P) \geq 1$. Figure 2.3 shows the relationship of $\mathcal{F}$ to the set S.

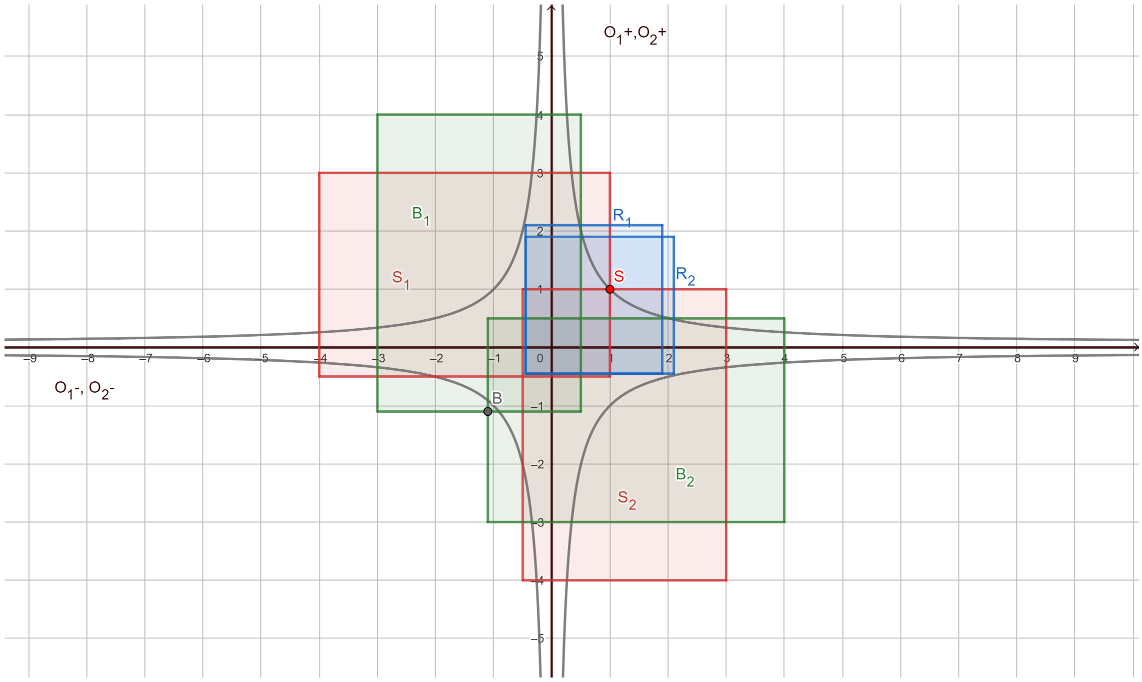

Figure 2.3: A construction which is 2-punchable, not $(v, 2)$-punchable for any $v > 1$, but for which any subfamily is $(v, 2)$-punchable for some $v > 1$. The pair of boxes $\{\underline{0}\square S, \underline{0}\square B\}$ is the largest punching pair. The hyperboles are $\{(x, y) : xy = 1\}$ and $\{(x, y) : xy = -1\}$, so a box $\underline{0}\square P$ is larger than 1 if and only if P is "outside" of the hyperboles.

First, punching will be considered in the diagonally opposite corners $C_+ = O_{1+} \cap O_{2+}$ and $C_- = O_{1-} \cap O_{2-}$ (by slight extension of the term, since they are not orthants). If a restricting box is not punched in C_+, then $\underline{0}\square - a\underline{1}$ of volume $a^2 < 1$ is the largest hole for it in C_-. Both restricting boxes and a big box cannot be punched together in C_+, because their intersections are $\underline{0}\square(1.9, 0.5)$ or $\underline{0}\square(0.5, 1.9)$ of volume 0.55. If $\mathcal{R}$ and only one small box is punched together, then the big boxes and the other small box are punched in C_-, where $\underline{0}\square(-0.5, -1 - \varepsilon)$ or $\underline{0}\square(-1 - \varepsilon, -0.5)$ are the intersections of volume $\frac{1+\varepsilon}{2} < 1$.
If a restricting box is punched in the other pair of corners, then the intersections are of volume $2.1 \cdot a < 1$. □

Lemma 2.3.11. *Any proper subfamily of $\mathcal{F}$ is $(v, 2)$-punchable for some $v > 1$.*

Proof. By removing a small box, a unique border for the smaller box of the largest punching pair for $\mathcal{F}$ is removed, so a subfamily missing a small box has a punching pair of volume larger than 1. The holes $\{\underline{0}\square(1, 1.9), \underline{0}\square - (1 + \varepsilon)\underline{1}\}$ or $\{\underline{0}\square(1.9, 1), \underline{0}\square - (1 + \varepsilon)\underline{1}\}$ are punching if a small box is missing.

By removing a big box, the restricting boxes and only one small box can be punched together, while only one small and big box are punched together in C_- by a hole larger than 1. $\underline{0}\square(1.9,1)$ or $\underline{0}\square(1,1.9)$ punch $\mathcal{R} \cup \{S_1\}$ or $\mathcal{R} \cup \{S_2\}$ while $\underline{0}\square(-0.5,-4)$ or $\underline{0}\square(-4,-0.5)$ punch $\{S_2, B_2\}$ or $\{S_1, B_1\}$. Clearly, both punching pairs are of volume 1.9.

If a restricting box is removed, then for $i \in \{1,2\}$ the subfamilies $\{S_i, B_i, R_i\}$ and $\{S_{(i+1 \mod 2)+1}, B_{(i+1 \mod 2)+1}\}$ are punched together respectively by punching pairs $\{\underline{0}\square(0.5,2.1), \underline{0}\square(-0.5,-4)\}$ or $\{\underline{0}\square(2.1,0.5), \underline{0}\square(-4,-0.5)\}$ of volume 1.05.

By removing an outward limit for a small box, the unique border for the smaller box from the largest punching pair of $\mathcal{F}$ is removed so the largest punching pair is of larger volume in the subfamily. The pairs $\{(-a,0)\square\underline{1}, \underline{0}\square(-1-\varepsilon)\underline{1}\}$ or $\{(0,-a)\square\underline{1}, \underline{0}\square(-1-\varepsilon)\underline{1}\}$ of volume $(1+\varepsilon)^2$ punches $\mathcal{F} \setminus \{O_{i+}\}$ for $i \in \{1,2\}$.

By removing an outward limit for a big box, one of the restricting boxes becomes punchable in the other outward limit box for a big box with the largest hole crossing corners. The pairs $\{-a\underline{1}\square(1.9,0), \underline{0}\square(0.5,2.1)\}$ or $\{-a\underline{1}\square(0,1.9), \underline{0}\square(2.1,0.5)\}$ of volume $a^2 + 1.9 \cdot a > 1$ punch $\{O_{i-}, S_i, B_i, R_{(i+1 \mod 2)+1}\}$ and $\{O_{1+}, O_{2+}, S_{(i+1 \mod 2)+1}, B_{(i+1 \mod 2)+1}, R_i\}$ together respectively for $i = 1$ or $i = 2$. □

This completes the proof of **Theorem** 2.3.8. □

Chapter 3

Open problems

It remains to be seen whether there is a Helly-type theorem about 2-punching of the following form: *There is a finite tuple-size h_d for which 2-punchability of all h_d-tuples from a finite family of boxes in $\mathbb{R}^d$ implies 2-punchability of the whole family.* An open question is whether the construction in Section 2.3.4 can be generalized to any dimension d, which would yield a lower bound of $5d$ for h_d. It is also unkown whether it is possible to construct to an improved lower bound using the principles set in Section 2.3.2 for dimensions other than 3.

It is also unanwsered for which cases $n_1 < n_2$ a not n_1-punchable tuple of fixed size can disprove the n_2-punchability of a family from which the tuple is taken, i.e. for which cases *n_1-punchability of all h-tuples from a family of boxes implies n_2-punchability*. In this case, condition B of the whole family is weakened. This also an open question for the same type of statement about piercing, as Danzer and Grünbaum only examined the cases $n_1 = n_2$.

A further possible line of inquiry is to examine the more general notion of $(\mathcal{V}, \nu)$-punching, which was introduced in Section 1.2. It is unknown whether prescribing different sizes for the holes in the punching tuples yields different Helly-type statements than for n-punching. A similar approach is to prescribe not a lower bound on the volume of holes, but a lower bound on the sum of the volume of the holes. This might yield the following definition.

Definition: A family of d-dimensional boxes is **n-sum-s-punchable** if there is a family of

d-dimensional boxes $\mathcal{H}$ that **punches** $\mathcal{F}$ such that

$$\sum_{H \in \mathcal{H}} \mathrm{Vol}(H) = s \tag{3.1}$$

$$|\mathcal{H}| = n \tag{3.2}$$

It can be immediately seen for example, that *2-sum-1-punchability of any 3-tuple from a family of intervals does not imply the 2-sum-1-punchability of the whole family.* This is shown for example, by the family of intervals $\{[0,1],[1,2],[2,3],[3,4]\}$. It is also interesting to explore whether there is a Helly-type statement of this sort and what the consequences of further restrictions on the family or the punching tuples might be.

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