Bachelor Thesis

Singularities of Plane Algebraic Curves

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Abstract

This thesis is devoted to the study of plane algebraic curves, with a particular emphasis on their singular points. Singularities are not only natural objects arising in geometry, but also understanding them reveals deep connections between algebra, geometry, and topology.

In the first chapter, we introduce the necessary background without delving into detailed proofs. We then focus on intersection multiplicities and present two fundamental results: Bézout's theorem and Noether's fundamental theorem. Next, we explore the local structure of curves around singular points via Puiseux parametrization, a powerful tool allowing for precise local descriptions. We also investigate Newton diagrams as a combinatorial method for understanding singularities. Finally, we discuss resolution of singularities and investigate different topological aspects.

Our goal is to gain insight into the rich and complex world of singularities and plane algebraic curves by approaching them from multiple perspectives. The content and structure of the thesis are inspired by the lectures of Professor András Némethi.

Contents

1 Preliminaries

In this section, we review the basic notions of algebraic curves and present the necessary definitions, propositions, and theorems from various fields of mathematics. Proofs are omitted, as these results serve only as tools.

1.1 Complex projective plane

In this paper, we will work mostly in the complex projective plane, usually denoted by $\mathbb{CP}^2$. We give a formal definition.

Definition 1.1.1. (Complex projective line $\mathbb{CP}^1$). Consider the 2-dimensional space $\mathbb{C}^2 = \{(z_1, z_2) \mid z_1, z_2 \in \mathbb{C}\}$. The 1-dimensional projective space, or projective line, is defined to be the quotient of $\mathbb{C}^2 \setminus \{\mathbf{0}\}$ by the equivalence relation

$$(z_1, z_2) \sim (z_1', z_2') \iff \exists\, \lambda \in \mathbb{C} \setminus \{\mathbf{0}\} \text{ such that } z_1' = \lambda z_1,\ z_2' = \lambda z_2.$$

The set of equivalence classes is denoted by $\mathbb{CP}^1$. A point in $\mathbb{CP}^1$is written as $[z_1 : z_2]$, where $(z_1, z_2) \in \mathbb{C}^2 \setminus \{\mathbf{0}\}$.

Proposition 1.1.1. [Kir92] The projective line is compact in the standard topology.

Definition 1.1.2. (Complex projective plane $\mathbb{CP}^2$). Consider the 3-dimensional space $\mathbb{C}^3 = \{(z_1, z_2, z_3) \mid z_1, z_2, z_3 \in \mathbb{C}\}$. The 2-dimensional projective space, or projective plane is defined to be the quotient of $\mathbb{C}^3 \setminus \{0\}$ by the equivalence relation

$$(z_1, z_2, z_3) \sim (z_1', z_2', z_3') \iff \exists \lambda \in \mathbb{C} \setminus \{\mathbf{0}\} \text{ s.t. } z_1' = \lambda z_1, z_2' = \lambda z_2, z_3' = \lambda z_3.$$

The set of equivalence classes is denoted by $\mathbb{CP}^2$. A point in $\mathbb{CP}^2$ is written as $[z_1 : z_2 : z_3]$, where $(z_1, z_2, z_3) \in \mathbb{C}^3 \setminus \{\mathbf{0}\}$.

Proposition 1.1.2. [Kir92] $\mathbb{CP}^2$ is compact and Hausdorff in the standard topology.

Definition 1.1.3. (Projective transformation). A projective transformation of the complex projective plane $\mathbb{CP}^2 = \mathbb{CP}^2$ is a bijection

$$f : \mathbb{CP}^2 \to \mathbb{CP}^2$$

that is induced by a linear isomorphism $\alpha : \mathbb{C}^3 \to \mathbb{C}^3$, i.e., for every point $[x : y : z] \in \mathbb{CP}^2$, we have

$$f([x : y : z]) = [\alpha(x, y, z)].$$

Remark 1.1.3. Since two linear maps differing by a nonzero scalar define the same projective map, the group of all such transformations is the projective general linear group $\mathrm{PGL}(3, \mathbb{C})$.

Remark 1.1.4. A projective transformation can also be interpreted as a coordinate change. If $T \in \mathrm{PGL}(3, \mathbb{C})$, then the new coordinates of a point or geometric object $X \subset \mathbb{CP}^2$ under the coordinate change T are denoted by X^T, which represents the image of X under the transformation induced by T.

Proposition 1.1.5. Projective transformations send lines to lines and preserve cross ratios.

Proposition 1.1.6. [Kir92] Given three distinct points in $\mathbb{CP}^1$: P_1, P_2 and P_3. There exists a unique projective transformation sending P_1 to $[1 : 0]$, P_2 to $[0 : 1]$ and P_3 to $[1 : 1]$.

Proposition 1.1.7. [Kir92] Given four distinct points in $\mathbb{CP}^2$: P_1, P_2, P_3, and P_4, no three of which are collinear. There exists a unique projective transformation sending P_1 to $[1 : 0 : 0]$, P_2 to $[0 : 1 : 0]$, P_3 to $[0 : 0 : 1]$ and P_4 to $[1 : 1 : 1]$.

1.2 Algebraic curves

In this subsection, and frequently throughout the thesis, we will denote by P the projective curve defined by the polynomial p.

Definition 1.2.1. (Affine algebraic plane curve). A set P of points in $\mathbb{C}^2$ is called an affine algebraic curve if there exists a polynomial $p \in \mathbb{C}[x,y]$ such that P is precisely the zero set of p.

Definition 1.2.2. (Projective algebraic plane curve). A set P of points in $\mathbb{CP}^2$ is called a projective algebraic curve if there exists a homogeneous polynomial $p \in \mathbb{C}[x,y,z]$ such that P is precisely the zero set of p.

Definition 1.2.3. (Homogenisation and dehomogenisation). Given a polynomial $p \in \mathbb{C}[x,y]$, its homogenisation is the unique homogeneous polynomial $p^* \in \mathbb{C}[x,y,z]$ such that $p^*(x,y,1) \equiv p(x,y)$ and $\deg p^*$ is minimal, that is $p^*(x,y,z) \equiv z^d p(x/z, y/z)$ where $d = \deg p^*$. Similarly, if given a homogeneous polynomial $p \in \mathbb{C}[x,y,z]$, its dehomogenisation is the unique polynomial $p_* \in \mathbb{C}[x,y]$ such that $p_*(x,y) \equiv p(x,y,1)$.

Proposition 1.2.1. [Ful08]

a) For $p, q \in \mathbb{C}[x,y]$ we have $(pq)^* = p^* q^*$.

b) For homogeneous $f, g \in \mathbb{C}[x,y,z]$ we have $(fg)_* = f_* g_*$.

c) For $p, q \in \mathbb{C}[x,y]$ we have $z^{\deg p + \deg q - \deg(p+q)}(p+q)^* = z^{\deg q} p^* + z^{\deg p} q^*$.

d) For homogeneous $p, q \in \mathbb{C}[x,y,z]$ we have $(p+q)_* = p_* + q_*$.

e) For $p \in \mathbb{C}[x,y]$ we have $(p^*)_* = p$.

f) For homogeneous $p \in \mathbb{C}[x,y,z]$ we have $z^r (p_*)^* = p$, where z^r is the greatest power of z dividing p.

Proposition 1.2.2. [Kir92] Every non-zero homogeneous polynomial in $\mathbb{C}[x,y]$ factors as product of linear polynomials. The number of these linear polynomials equals to the degree of the homogeneous polynomial.

Theorem 1.2.3. (Euler's Relation). [Kir92] Let $p(x,y,z) \in \mathbb{C}[x,y,z]$ be a homogeneous polynomial of degree d. Then

$$x\frac{\partial p}{\partial x}(x,y,z) + y\frac{\partial p}{\partial y}(x,y,z) + z\frac{\partial p}{\partial z}(x,y,z) = d \cdot p(x,y,z).$$

Definition 1.2.4. (Curves without common component). We say that algebraic curves have no common component if the polynomials defining them do not have a common component.

Definition 1.2.5. (Reducible, irreducible and minimal polynomials). A polynomial is called reducible if it can be written as the product of two polynomials (in the same ring) of positive degrees. A non-reducible polynomial is called irreducible or minimal.

Definition 1.2.6. (Smooth/regular and singular points). Consider an algebraic curve P defined by $p \in \mathbb{C}[x,y]$. Then P is smooth (or regular) at $X \in P$ if

$$\operatorname{grad}_p(X) = \left(\frac{\partial p}{\partial x}(X), \frac{\partial p}{\partial y}(X)\right) \neq (0,0).$$

Similarly, for homogeneous $p \in \mathbb{C}[x,y,z]$, $X \in P$ is a smooth (or regular) point if

$$\operatorname{grad}_p(X) = \left(\frac{\partial p}{\partial x}(X), \frac{\partial p}{\partial y}(X), \frac{\partial p}{\partial z}(X)\right) \neq (0,0,0).$$

If P is not smooth at X, then it is singular. The set of singularities will be denoted by $\mathrm{Sing}(P)$ or $\mathrm{Sing}(p)$.

Suppose that $p \in \mathbb{C}[x, y]$ is written as $p = a_0 + b_0x + b_1y + c_0x^2 + c_1xy + c_2y^2 \ldots$. We may assume that $a_0 = 0$, that is the curve contains the origin. Note that $\dfrac{\partial p}{\partial x}(0) = b_0$ and $\dfrac{\partial p}{\partial y}(0) = b_1$, so the origin is a singularity if and only if $b_0 = b_1 = 0$.

Definition 1.2.7. (Double point and multiple point). If at least one of c_0, c_1, c_2 is non-zero, then the origin is called a double point. More precisely:

- if $4c_0c_2 = c_1^2$ and the degree three terms are generic then the origin is called an ordinary cusp (see Figure 2 below);
- if $4c_0c_2 \neq c_1^2$ then the origin is called a node.

In general, if the first non-zero term in the expression of f has degree d then the origin is called a multiple point of order d.

Definition 1.2.8. (Tangent lines of algebraic curves). Given a plane curve P. Let $X \in P$ be a point on the curve. Translate the curve by the vector $\overrightarrow{XO}$, where O denotes the origin. Denote the new curve's defining polynomial by p. Let $p = p_{k_1} + p_{k_2} + \ldots + p_{k_l}$ where p_{k_i} is a homogeneous polynomial in $\mathbb{C}[x, y]$ of degree k_i, and $k_1 < k_2 < \ldots < k_l$. Write $p_{k_i} = l_1^{r_1} \cdot \ldots \cdot l_k^{r_k}$ where each l_i is a linear polynomial. The l_i polynomials define lines which are called the tangents to P at X. The multiplicity of the tangent line l_i is r_i.

Definition 1.2.9. (Simple tangent). In the previous definitions, if $r_i = 1$, then l_i is said to be a simple tangent.

Definition 1.2.10. (Tangent line at smooth point). The tangent line to P at a smooth point X is the line

$$T_XP := \left\{ (x, y) \in \mathbb{C}^2 \mid \frac{\partial p}{\partial x}(X)x + \frac{\partial p}{\partial y}(X)y = c \right\}$$

where c is a constant chosen such that X lies on the tangent line.

Definition 1.2.11. (Ordinary point). If a curve P has m different simple tangents at a point X, then we say that X is an ordinary multiple point on P.

Example 1.2.1. Here are a few examples of algebraic curves, visualised in the real affine plane. The figures were created using Desmos [Des24].

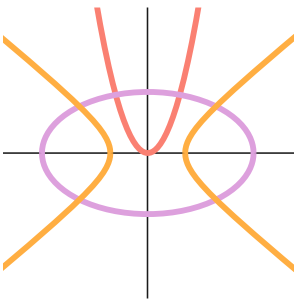

Figure 1
conic sections
(ellipse, parabola,
hyperbola)
no singularities

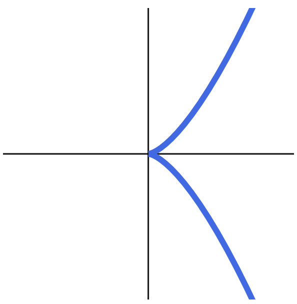

Figure 2
semicubical parabola
$x^2 - y^3$
cusp singularity

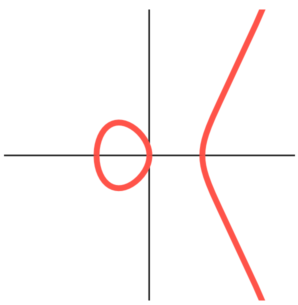

Figure 3
$x^3 - x - y^2$
no singularities

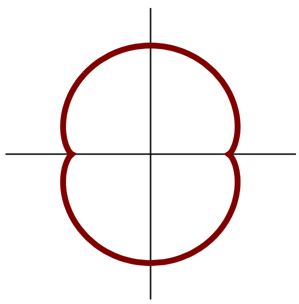

Figure 4
nephroid
$(x^2 + y^2 - 4)^3 - 108y^2$
two cusp singularities

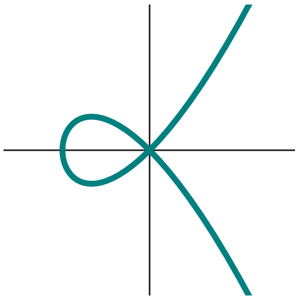

Figure 5
alpha curve
$x^3 + x^2 - y^2$
node singularity

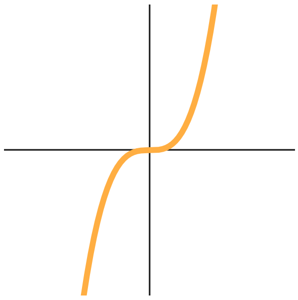

Figure 6
$x^3 - y$
singularity at
infinity

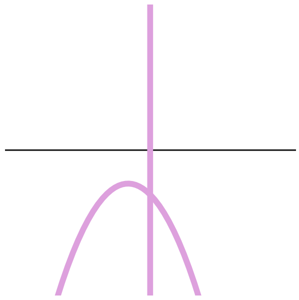

Figure 7
trident curve
$xy + x^3 + x^2 + x$
reducible curve

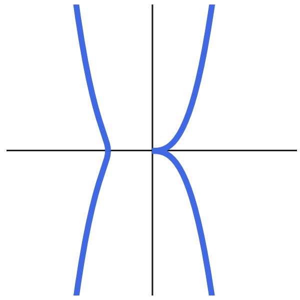

Figure 8
$x^5 + x^6 - y^2$
non-ordinary cusp
singularity

1.3 Inflection points

Definition 1.3.1. (Hessian matrix). Let $p(x, y, z)$ be a homogeneous polynomial of degree d. The Hessian $\mathcal{H}_p$ of p is the polynomial defined by

$$\mathcal{H}_p(x, y, z) = \det \begin{pmatrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{pmatrix}.$$

Remark 1.3.1. We used and we will use the common notations $p_u = \dfrac{\partial p}{\partial u}$ and $p_{uv} = \dfrac{\partial^2 u}{\partial u \partial v}$ for all $u, v \in \{x, y, z\}$.

Remark 1.3.2. Note that the second partial derivatives of p are homogeneous of degree $d-2$ in x, y, z, so $\mathcal{H}_p$ is a homogeneous polynomial of degree $3(d-2)$ in x, y, z.

Definition 1.3.2. (Point of inflection). A nonsingular point $[a : b : c]$ of the projective curve P in $\mathbb{CP}^2$ defined by $p(x, y, z)$ is called a point of inflection (or flex) of P if

$$\mathcal{H}_p(a, b, c) = 0.$$

Proposition 1.3.3. [Kir92] If $p(x, y, z)$ is a homogeneous polynomial of degree $d > 1$ then

$$z^2 \mathcal{H}_p(x, y, z) = (d-1)^2 \det \begin{pmatrix} p_{xx} & p_{xy} & p_x \\ p_{yx} & p_{yy} & p_y \\ p_x & p_y & \dfrac{d \cdot p}{d-1} \end{pmatrix}.$$

Proposition 1.3.4. [Kir92] If P defined by p is an irreducible projective curve of degree d, then every point of P is a point of inflection if and only if $d = 1$.

1.4 Algebraic properties

The following definitions can be generalized to the projective plane.

Definition 1.4.1. (V notation). For a set of polynomials $\mathcal{P}$ in $\mathbb{C}[x, y]$, let $V(\mathcal{P})$ denote the set of points (in $\mathbb{C}^2$) which are the common zeroes of $\mathcal{P}$, that is,

$$V(\mathcal{P}) = \{X \in \mathbb{C}^2 \mid p(X) = 0 \text{ for all } p \in \mathcal{P}\}.$$

Proposition 1.4.1. [Ful08] If $\mathcal{P} \subset \mathcal{Q}$ for some sets of polynomials in $\mathbb{C}[x, y]$, then $V(\mathcal{P}) \supset V(\mathcal{Q})$.

Definition 1.4.2. (Algebraic set). A set of points $\mathcal{S} \subset \mathbb{C}^2$ is called an algebraic set if there exists a set of polynomials $\mathcal{P} \subset \mathbb{C}[x, y]$ for which $V(\mathcal{P}) = \mathcal{S}$.

Proposition 1.4.2. [Ful08] Here are a few basic properties of algebraic sets:

a) Every algebraic set is equal to $V(I)$ for some ideal $I \subset \mathbb{C}[x, y]$.

b) The intersection of any collection of algebraic sets, and the union of any finite collection of algebraic sets is also an algebraic set.

c) Any finite subset of $\mathbb{C}^2$ is an algebraic set.

Theorem 1.4.3. (Hilbert Basis Theorem). Every ideal in $\mathbb{C}[x, y]$ is finitely generated. Therefore, every algebraic set can be defined by a finite set of polynomials.

Definition 1.4.3. (Reducibility of algebraic sets). An algebraic set $\mathcal{S}$ is reducible if there exist algebraic sets $\mathcal{S}_1, \mathcal{S}_2 \subsetneq \mathcal{S}$ such that $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$. A non-reducible algebraic set is called irreducible.

Definition 1.4.4. (Ideal of a set). For a subset $\mathcal{S} \subset \mathbb{C}^2$, we define the ideal of $\mathcal{S}$ to be

$$I(\mathcal{S}) = \{p \in \mathbb{C}[x, y] \mid p(s) = 0 \text{ for all } s \in \mathcal{S}\}.$$

Proposition 1.4.4. [Ful08]

a) $I(\emptyset) = \mathbb{C}[x, y]$.

b) $I(\{(a, b)\}) = (x - a, y - b)$.

Proposition 1.4.5. [Ful08] Let $\mathcal{P}$ be a set of polynomials in $\mathbb{C}[x, y]$, and let $\mathcal{S}$ be a set of points in $\mathbb{C}^2$.

a) $I(V(\mathcal{P})) \supset \mathcal{P}$ and $V(I(\mathcal{S})) \supset \mathcal{S}$.

b) $V(I(V(\mathcal{P}))) = V(\mathcal{P})$ and $I(V(I(\mathcal{S}))) = I(\mathcal{S})$.

Proposition 1.4.6. [Ful08] An algebraic set $\mathcal{S}$ is irreducible if and only if $I(\mathcal{S})$ is a prime ideal.

Proposition 1.4.7. [Ful08] Let $\mathcal{S}$ be an algebraic set. Then there exist unique irreducible algebraic sets $\mathcal{S}_1, \ldots, \mathcal{S}_k$ such that

$$\mathcal{S} = \bigcup_{i=1}^{k} \mathcal{S}_i \quad \text{and} \quad \mathcal{S}_i \not\subset \mathcal{S}_j \text{ for } i \neq j.$$

Definition 1.4.5. (Irreducible components). Using the notation in the previous proposition, the algebraic sets $\mathcal{S}_1, \ldots, \mathcal{S}_k$ are called the irreducible components of $\mathcal{S}$.

Theorem 1.4.8. (Hilbert's Nullstellensatz). Let $p, q \in \mathbb{C}[x, y]$. Then $V(p) = V(q)$ if and only if p and q have the same set of irreducible factors (up to multiplicity). This is equivalent to the existence of $k, \ell \in \mathbb{N}$ such that $p \mid q^k$ and $q \mid p^\ell$.

Corollary 1.4.9. Suppose $p, q \in \mathbb{C}[x, y]$ are irreducible. Then $V(p) = V(q)$ if and only if there exists a scalar $\lambda \in \mathbb{C} \setminus \{0\}$ such that $p = \lambda q$.

Corollary 1.4.10. When studying algebraic curves, we may assume that the defining polynomial has no repeated factors.

Definition 1.4.6. (Zariski topology). The Zariski topology on $\mathbb{C}^2$ or $\mathbb{CP}^2$ is the topology in which the closed sets are the algebraic sets.

1.5 Varieties

In this subsection, P will denote a point, not a curve.

Definition 1.5.1. (Variety). An irreducible algebraic set is called a variety.

Definition 1.5.2. (Coordinate ring). For a variety $V \subset \mathbb{C}^n$ we denote the ring $\mathbb{C}[x_1, x_2, \ldots, x_n]/I(V)$ by $\Gamma(V)$ and call it the coordinate ring of V.

Definition 1.5.3. (Residue). Let $\overline{M}$ denote the residue of $M \in \mathbb{C}[x_1, \ldots, x_n]$ in the coordinate ring.

Proposition 1.5.1. [Ful08] $\Gamma(V)$ is an integral domain, and thus has no zero divisors.

Definition 1.5.4. (Field of rational functions). The quotient field of the coordinate ring $\Gamma(V)$ is denoted by $\mathbb{C}(V)$ and called the field of rational functions on V.

Definition 1.5.5. (Local ring). Given a variety V, the local ring of V at $P \in V$ is the set of rational functions on V that are defined at P. It is denoted by $\mathcal{O}_P(V)$.

Proposition 1.5.2. [Ful08] $\Gamma(V) \subset \mathcal{O}_P(V) \subset \mathbb{C}(V)$.

Definition 1.5.6. (Maximal ideal). The maximal ideal of $\mathcal{O}_P(V)$ is denoted by $\mathfrak{m}_P(V)$.

Proposition 1.5.3. [Ful08] $\mathfrak{m}_P(V)$ is the ideal of all the rational functions that vanish at P.

Proposition 1.5.4. [Ful08] $\mathfrak{m}_P(V)$ is the set of non-units in $\mathcal{O}_P(V)$.

Proposition 1.5.5. [Ful08] Given an ideal I of $\mathbb{C}[x_1, \ldots, x_n]$. Suppose that $V(I)$ is the finite set consisting of $P_1, \ldots, P_k$. Then there exists a natural isomorphism

$$\mathbb{C}[x_1, \ldots, x_n]/I \cong \prod_{i=1}^{k} \mathcal{O}_{P_i}(\mathbb{C}^n)/I\mathcal{O}_{P_i}(\mathbb{C}^n)$$

where $I\mathcal{O}_{P_i}(\mathbb{C}^n)$ denotes the ideal of $\mathcal{O}_{P_i}(\mathbb{C}^n)$ generated by I.

Proposition 1.5.6. [Ful08] Let V be variety and P one of its points. Suppose that J is an ideal of $\mathbb{C}[x, y]$ containing $I(V)$. Let J' denote the image of J in $\Gamma(V)$. Then $\mathcal{O}_P(\mathbb{C}^2)/J\mathcal{O}_P(\mathbb{C}^2)$ is isomorphic to $\mathcal{O}_P(V)/J'\mathcal{O}_P(V)$.

Definition 1.5.7. (Regular functions). Let Y be an open subset of an (affine or projective) variety and $f : Y \to \mathbb{C}$ a function. For affine varieties, f is said to be regular at a point $P \in Y$ if there is an open neighbourhood U with $P \in U \subset Y$ and polynomials $g, h \in \mathbb{C}[x, y]$ such that h is nowhere zero on U and $f = \frac{g}{h}$ on U. For projective varieties, f is said to be regular at a point $P \in Y$ if there is an open neighbourhood U with $P \in U \subset Y$ and homogeneous polynomials $g, h \in \mathbb{C}[x, y, z]$ of same degree such that h is nowhere zero on U and $f = \frac{g}{h}$ on U. We say f is regular on Y if it is regular at every point of Y.

Definition 1.5.8. (Morphisms). If X, Y are two varieties, a morphism $\varphi : X \to Y$ is a continuous map such that for every open set $V \subset Y$ and for every regular function $f : V \to \mathbb{C}$ the function $f \circ \varphi : \varphi^{-1}(V) \to \mathbb{C}$ is regular. An isomorphism $\varphi : X \to Y$ of two varieties is a morphism which admits and inverse morphism $\psi : Y \to X$ with $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$.

Definition 1.5.9. (Rational Maps). Let X, Y be varieties. A rational map $\varphi : X \dashrightarrow Y$ is an equivalence class of pairs $\langle U, \varphi_U \rangle$ where U is a non-empty open subset of X, φ_U is a morphism of U to Y, and where $\langle U, \varphi_U \rangle$ and $\langle V, \varphi_V \rangle$ are equivalent if $\varphi_U = \varphi_V$ on $U \cap V$. The rational map φ is dominant if for some pair $\langle U, \varphi_U \rangle$, the image of φ_U is dense in Y.

Definition 1.5.10. (Birational map). Let X, Y be two varieties. A birational map is a rational map $\varphi : X \dashrightarrow Y$ which admits an inverse $\psi : Y \dashrightarrow X$. If we have such maps, we say that X and Y are birationally equivalent or simply birational.

1.6 Intersection number

In this subsection, F and G denotes two algebraic curves. P denotes a point, not a curve.

Definition 1.6.1. (Proper intersection). Given two curves F, G. They intersect at P properly if they have no common component passing through P.

Definition 1.6.2. (Transverse intersection). Given two curves F, G. They intersect at P transversally if P is a regular point on both F and G, and the tangent lines at P are different on the two curves.

Definition 1.6.3. (Multiplicity at the origin). Given a plane curve F defined by the polynomial $f \in \mathbb{C}[x, y]$. Let $f = f_{k_1} + f_{k_2} + \ldots + f_{k_l}$ where f_{k_i} is a homogeneous polynomial in $\mathbb{C}[x, y]$ with degree k_i, and $k_1 < k_2 < \ldots < k_l$. We define the multiplicity of F at the origin $(0, 0)$ by k_1, and denote it by $m_{(0,0)}(F)$.

Remark 1.6.1. The origin lies on a curve F if and only if the multiplicity is positive. The origin is a smooth point of a curve F if and only if its multiplicity is 1.

Definition 1.6.4. (Multiplicity in general). Given a point P and a curve F. Let τ denote the translation by the vector $\overrightarrow{PO}$ where O denotes the origin. Then we define the multiplicity of F at P by $m_O(F \circ \tau)$ and denote it by $m_P(F)$.

Definition 1.6.5. (Intersection number). For curves F, G let the intersection number at a point P be

$$I[F, G](P) := \dim_{\mathbb{C}}(\mathcal{O}_P(\mathbb{C}^2)/(F, G)).$$

Proposition 1.6.2. [Ful08] The intersection number defined above is the unique $I^*[F, G](P)$ number which satisfies the following properties:

a) If F and G intersect properly at P, then $I^*[F, G](P)$ is a non-negative integer, while if they don't intersect properly then $I^*[F, G](P) = \infty$.

b) $I^*[F, G](P) = 0$ if and only if $P \notin F \cap G$.

c) $I^*[F, G](P)$ only depends on the components of F and G that pass through P.

d) For a coordinate change T which sends Q to P, we have $I^*[F, G](P) = I^*[F \circ T, G \circ T](Q)$.

e) $I^*[F, G](P) = I^*[G, F](P)$.

f) $I^*[F, G](P) \geq m_P(F)m_P(G)$, where the equality occurs if and only if there is no common tangent at P.

g) $F = F_1^{a_1} \cdot F_2^{a_2} \cdot \ldots \cdot F_k^{a_k}$ and $G = G_1^{b_1} \cdot G_2^{b_2} \cdot \ldots \cdot G_l^{b_l}$ implies $I^*[F, G](P) = \sum_{1 \leq i \leq k, 1 \leq j \leq l} a_i b_j I^*[F_i, G_j](P)$.

h) For any $H \in \mathbb{C}[x, y]$ we have $I^*[F, G](P) = I^*[F, G + HF](P)$.

1.7 Inverse function theorem

Theorem 1.7.1. (Inverse function theorem). [Kir92]

a) Let $f : U \to V$ be a holomorphic bijection between open subsets U and V of $\mathbb{C}$. Then

$$f'(z) \neq 0 \quad \text{for all } z \in U$$

and the inverse

$$f^{-1} : V \to U$$

of f is holomorphic.

b) Let $f : U \to \mathbb{C}$ be a holomorphic function, defined on a neighbourhood U of a in $\mathbb{C}$, such that

$$f'(a) \neq 0.$$

Then the restriction of f to any sufficiently small open neighbourhood of a in U is a holomorphic bijection onto an open neighbourhood of $f(a)$ in $\mathbb{C}$.

Proposition 1.7.2. [Kir92] Suppose that U is an open subset of $\mathbb{C}$ and that $f : U \to \mathbb{C}$ is continuous. If the restriction of f to

$$U \setminus \{a_1, \dots, a_m\}$$

is holomorphic for some $a_1, \dots, a_m \in U$, then f is holomorphic.

1.8 Implicit function theorem

Theorem 1.8.1. (Implicit function theorem). [Hör66] Let $f_j(w, z)$, $j = 1, \dots, m$ be analytic functions of $(w, z) = (w_1, \dots, w_m, z_1, \dots, z_n)$ in a neighbourhood of w_0, z_0 in $\mathbb{C}^m \times \mathbb{C}^n$ and assume that

$$f_j(w_0, z_0) = 0, \quad j = 1, \dots, m$$

and that

$$\det \left(\frac{\partial f_j}{\partial w_k} \right)_{j,k=1}^{m} \neq 0$$

at (w_0, z_0). Then the equations $f_j(w, z) = 0$, $j = 1, \dots, m$, have a uniquely determined analytic solution $w(z)$ in a neighbourhood of z_0, such that $w(z_0) = w_0$.

Corollary 1.8.2. [Kir92] Let $a(z, w)$ be a polynomial with complex coefficients in two variables z and w. Suppose that

$$a(z_0, w_0) = 0 \neq \frac{\partial a}{\partial w}(z_0, w_0).$$

Then there is a holomorphic function $f : U \to V$ where U and V are open neighbourhoods of z_0 and w_0 in $\mathbb{C}$ such that

$$f(z_0) = w_0$$

and if $z \in U$ and $w \in V$ then

$$f(z) = w$$

implies

$$a(z, w) = 0.$$

Moreover

$$a(z, w) = (w - f(z)) \cdot b(z, w)$$

where $b(z, w)$ is a polynomial in w whose coefficients are holomorphic functions of z.

Corollary 1.8.3. [Kir92] Let $a(z, w)$ be a polynomial with complex coefficients in two variables z and w such that

$$a(z_0, w_0) = 0$$

and the polynomial $a(z_0, w)$ in w has a zero of order m at w_0. Then given any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|z - z_0| < \delta$ then the polynomial $a(z, w)$ in w has at least m zeros (counting multiplicities) in the disc

$$\{w \in \mathbb{C} : |w - w_0| < \varepsilon\}.$$

Remark 1.8.4. [Kir92] The corollary above holds for all continuous functions $a(z, w)$, for which $\frac{\partial a}{\partial w}(z, w)$ is continuous and for fixed w both $a(z, w)$ and $\frac{\partial a}{\partial w}(z, w)$ are holomorphic in z, while for fixed z they are both holomorphic in w.

1.9 Resultant

Definition 1.9.1. (Resultant). Let $f(x) = a_0x^n+a_1x^{n-1}+\cdots+a_n$ and $g(x) = b_0x^m+b_1x^{m-1}+\cdots+b_m$ be two complex polynomials. The resultant $\mathrm{Res}(f, g)$ is defined as the determinant of the $(n+m)\times(n+m)$ Sylvester matrix of f and g, that is,

$$\mathrm{Res}(f, g) = \det(S(f, g)),$$

where $S(f, g)$ is the matrix

$$S(f,g) = \begin{pmatrix} a_0 & a_1 & \cdots & a_n & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_n & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_m & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & b_m & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_0 & b_1 & \cdots & b_m \end{pmatrix},$$

where there are m rows coming from the coefficients of f and n rows from the coefficients of g.

Definition 1.9.2. (Resultant for polynomials in three variables). Let $f(x, y, z)$ and $g(x, y, z)$ be two complex polynomials. Assume that we regard f and g as polynomials in the variable x. Then the resultant $\mathrm{Res}_x(f, g)$ is defined as the resultant of f and g considered as polynomials in x: that is,

$$\mathrm{Res}_x(f, g) = \det(S(f, g)),$$

where $S(f, g)$ is the Sylvester matrix formed from the coefficients of f and g viewed as polynomials in x.

Proposition 1.9.1. [Kir92] Let $f, g \in \mathbb{C}[x]$. Then f and g have a nonconstant common factor if and only if $\mathrm{Res}(f, g) = 0$.

Proposition 1.9.2. [Kir92] Let $f, g \in \mathbb{C}[x, y, z]$ be nonconstant homogeneous polynomials with $f(1, 0, 0) \neq 0$ and $g(1, 0, 0) \neq 0$. Then f and g have a nonconstant homogeneous common factor if and only if $\mathrm{Res}_x(f, g) \equiv 0$.

Proposition 1.9.3. [Kir92] The degree of $\mathrm{Res}_x(f, g)$ is $\deg(f) \cdot \deg(g)$.

Proposition 1.9.4. [Kir92] Let $f(x) = (x-\alpha_1)\cdot\ldots\cdot(x-\alpha_m)$ and $g(x) = (x-\beta_1)\cdot\ldots\cdot(x-\beta_n)$. Then

$$\mathrm{Res}(f, g) = \prod_{1\le i\le m, 1\le j\le n} (\beta_j - \alpha_i).$$

Corollary 1.9.5. Let $f, g, h \in \mathbb{C}[x]$. Then $\mathrm{Res}(f, gh) = \mathrm{Res}(f, g) \cdot \mathrm{Res}(f, h)$.

Proposition 1.9.6. [Kir92] Let $f, g \in \mathbb{C}[x, y, z]$ and suppose that $f(x, y, z) = (x-a_1)\cdot\ldots\cdot(x-a_m)$ and $g(x) = (x-b_1)\cdot\ldots\cdot(x-b_n)$ where $a_1, \ldots, a_m, b_1, \ldots, b_n$ are polynomials in y, z. Then

$$\mathrm{Res}_x(f, g) = \prod_{1\le i\le m, 1\le j\le n} (b_j - a_i).$$

Corollary 1.9.7. Let $f, g, h \in \mathbb{C}[x, y, z]$. Then $\mathrm{Res}_x(f, gh) = \mathrm{Res}_x(f, g) \cdot \mathrm{Res}_x(f, h)$.

1.10 Formal power series

Definition 1.10.1. (Ring of formal power series). Let

$$\mathbb{C}[[x,y]] := \left\{ \sum_{(\mu,\nu)\in\mathbb{N}^2} a_{\mu,\nu} x^\mu y^\nu \mid a_{\mu,\nu} \in \mathbb{C} \right\}$$

denote the ring of (complex) formal power series (of two variables).

Remark 1.10.1. We similarly define the well-known ring of formal power series in one variable:

$$\mathbb{C}[[x]] = \left\{ \sum_{\nu\in\mathbb{N}} a_\nu x^\nu \mid a_\mu \in \mathbb{C} \right\}.$$

Definition 1.10.2. (homogeneous part). For any non-negative integer d, the homogeneous part of the power series $f = \sum_{(\mu,\nu)\in\mathbb{N}^2} a_{\mu,\nu} x^\mu y^\nu$ of degree d is

$$f_{(d)} := \sum_{\mu+\nu=d} a_{\mu,\nu} x^\mu y^\nu.$$

Definition 1.10.3. (Polynomial part). For any non-negative integer d, the polynomial part of the power series $f = \sum_{(\mu,\nu)\in\mathbb{N}^2} a_{\mu,\nu} x^\mu y^\nu$ of degree d is

$$f^{(d)} := \sum_{k=0}^{d} f_{(k)}.$$

Remark 1.10.2. We have a ring extension $\mathbb{C}[x,y] \subset \mathbb{C}[[x,y]]$ if we set $f + g = \sum_{k=0}^{\infty} (f_{(k)} + g_{(k)})$ and $fg = \sum_{k=0}^{\infty} \left(\sum_{p+q=k} (f_{(p)} g_{(q)} \right)$ for any $f, g \in \mathbb{C}[[x,y]]$.

Definition 1.10.4. (Order). For all $f \in \mathbb{C}[[x,y]]$ we define its order by

$$\operatorname{ord} f := \begin{cases} \min\{k : f_{(k)} \neq 0\} & f \neq 0, \\ \infty & f = 0. \end{cases}$$

Proposition 1.10.3. [Fis01] For all $f, g \in \mathbb{C}[[x,y]]$ we have $\operatorname{ord}(f+g) \geq \min\{\operatorname{ord} f, \operatorname{ord} g\}$ and $\operatorname{ord}(fg) = \operatorname{ord} f + \operatorname{ord} g$.

Corollary 1.10.4. $\mathbb{C}[[x,y]]$ is an integral domain.

Definition 1.10.5. ($\mathfrak{m}^{(k)}$). Let $\mathfrak{m} := \{f \in \mathbb{C}[[x,y]] \mid \operatorname{ord} f \geq 1\}$ and for $k \geq 2$ let $\mathfrak{m}^{(k)} := \{f \in \mathbb{C}[[x,y]] \mid \operatorname{ord} f \geq k\}$.

Proposition 1.10.5. [Fis01] $\mathfrak{m}$ is the unique maximal ideal of $\mathbb{C}[[x,y]]$.

Proposition 1.10.6. [Fis01] For $k \geq 2$, $\mathfrak{m}^{(k)} = \mathfrak{m}^k$.

Proposition 1.10.7. [Fis01] The elements of $\mathbb{C}[[x,y]]$ of order 0 are precisely the units.

Definition 1.10.6. (Formal convergent sequence). A sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{C}[[x,y]]$ is called formally convergent to $f \in \mathbb{C}[[x,y]]$ if for every positive integer k there exists an $N \in \mathbb{N}$ such that $f - f_n \in \mathfrak{m}^k$ for all $n \geq N$.

Definition 1.10.7. (Cauchy sequence). A sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{C}[[x,y]]$ is called a (formal) Cauchy sequence if for every positive integer k there exists an $N \in \mathbb{N}$ such that $f_m - f_n \in \mathfrak{m}^k$ for all $m, n \geq N$.

Proposition 1.10.8. [Fis01] Every Cauchy-sequence in $\mathbb{C}[[x,y]]$ is formally convergent.

Proposition 1.10.9. [Fis01] For a formal power series $f(x,y) = \sum_{(\mu,\nu)\in\mathbb{N}^2} a_{\mu,\nu}x^\mu y^\nu$ the following are equivalent:

a) There exists non-zero complex numbers x_0, y_0 such that $f(x_0, y_0)$ converges.

b) There exists positive real numbers r, s such that $f(r, s)$ converges.

c) There exists positive real numbers r, s such that $\sum_{(\mu,\nu)\in\mathbb{N}^2} |a_{\mu,\nu}|r^\mu s^\nu$ converges.

Definition 1.10.8. (Convergent power series). A formal power series is called convergent if it satisfies one of the (equivalent) conditions in the previous proposition. The set of convergent power series is denoted by $\mathbb{C}\langle x,y\rangle$.

Definition 1.10.9. (Generality). Let $f \in \mathbb{C}[[x,y]]$ and let $\overline{f}$ denote $f(x,0)$. Then f is general in x if $\overline{f} \neq 0$. If $\operatorname{ord}\overline{f} = d$ then we say that f is general in x of order d. We similarly define the general property in y.

1.11 Some basic topology

Proposition 1.11.1. [Ken11] Any algebraic curve in $\mathbb{CP}^2$ is pathwise connected.

Corollary 1.11.2. Any algebraic curve in $\mathbb{CP}^2$ is connected.

Proposition 1.11.3. [Ken12] Any smooth irreducible algebraic curve in $\mathbb{CP}^2$ is orientable.

Definition 1.11.1. (Triangulation). Let S be a finite CW-complex of dimension 2. A triangulation of S is given by the following data:

a) a finite nonempty set V of points of X called vertices,

b) a finite nonempty set E of continuous maps $e : [0,1] \to X$ called edges,

c) a finite nonempty set F of continuous maps $f : \Delta \to X$ called faces, where

$$\Delta = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y \leq 1\}$$

satisfying

i) $V = \{e(0) : e \in E\} \cup \{e(1) : e \in E\}$;

ii) if $e \in E$ then $e(t) \in V$ if and only if $t \in \{0,1\}$, and the restriction of e to $(0,1)$ is a homeomorphism onto its image in X;

iii) if $\tilde{e}, \bar{e}$ are distinct edges then $\tilde{e}(t) \neq \bar{e}(s)$ for all $s, t \in (0,1)$;

iv) if $f : \Delta \to X$ is a face then the restriction of f to

$$\Delta^0 = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0, x + y < 1\}$$

is a homeomorphism of Δ^0 onto a connected component K_f of

$$X - \bigcup_{e\in E} e([0,1]),$$

and if $\sigma_i : [0,1] \to \Delta$ for $1 \le i \le 3$ and $r : [0,1] \to [0,1]$ are defined by

$$\sigma_1(t) = (t,0) \quad \sigma_2(t) = (1-t,t) \quad \sigma_3(t) = (0,1-t), \quad r(t) = 1-t$$

then either $f \circ \sigma_i$ or $f \circ \sigma_i \circ r$ is an edge $e_i^f \in E$ for $1 \le i \le 3$;

v) the mapping $f \mapsto K_f$ from F to the set of connected components of $X - \bigcup_{e \in E} e([0,1])$ is a bijection;

vi) for every $e \in E$ there is exactly one face $f_+^e \in F$ such that $e = f_+^e \circ \sigma_i$ for some $i \in \{1,2,3\}$ and exactly one face $f_-^e \in F$ such that $e = f_-^e \circ \sigma_i \circ r$ for some $i \in \{1,2,3\}$.

$\#V, \#E, \#F$ denotes the number of elements of sets V, E, F, respectively.

Definition 1.11.2. (Euler number). The Euler number χ of a triangulation is defined by $\chi := \#V - \#E + \#F$.

Proposition 1.11.4. [Kir92] The Euler number is independent of triangulations.

Proposition 1.11.5. [Kir92] $\chi(\mathbb{CP}^1) = 2$.

Definition 1.11.3. (Topological genus). Let S be a compact, connected, orientable smooth surface. The topological genus $g(S)$ is defined as

$$g(S) = 1 - \frac{1}{2}\chi(S).$$

where $\chi(S)$ is the Euler characteristic of the surface.

Proposition 1.11.6. [Hat02] Let S be a compact, connected, orientable smooth surface. The topological genus $g(S)$ is the number of handles on S, that is, the number of tori appearing in a connected sum decomposition

$$S \cong \underbrace{T^2 \# T^2 \# \cdots \# T^2}_{(g \text{ times})} \quad \text{if } g \ge 1, \qquad S \cong \mathbb{CP}^1 \quad \text{if } g = 0.$$

Definition 1.11.4. (Covering projection). A continuous map $\pi : Y \to X$ between topological spaces X and Y is called a covering projection if each $x \in X$ has an open neighbourhood U in X such that $\pi^{-1}(U)$ is a disjoint union of open subsets of Y, each of which is mapped homeomorphically onto U by π.

Proposition 1.11.7. [Kir92] Let $\pi : Y \to X$ be a covering projection and let $f : A \to X$ be a continuous map. Suppose that A is simply connected, path connected and locally path connected (i.e. every $a \in A$ has arbitrarily small path connected open neighbourhoods in A). Then given any $a \in A$ and $y \in Y$ such that $f(a) = \pi(y)$ there is a unique continuous map $F : A \to Y$ such that $F(a) = y$ and

$$\pi \circ F = f.$$

If moreover f is a homeomorphism onto its image $f(A)$, then F is a homeomorphism onto a connected component of $\pi^{-1}(f(A))$.

Proposition 1.11.8. [Kir92] Let $\pi : Y \to X$ be a continuous map and suppose that every $x \in X$ has an open neighbourhood U in X such that each connected component of $\pi^{-1}(U)$ contains at most one point of $\pi^{-1}(x)$. Suppose that Y is compact and that V is an open subset of X such that $\pi : \pi^{-1}(V) \to V$ is a covering projection. If $f : [0,1] \to X$ is continuous and $f^{-1}(V)$ contains the open interval $(0,1)$ then given $\tau \in (0,1)$ and $y \in Y$ such that $\pi(y) = f(\tau)$ there is a unique continuous map $F : [0,1] \to Y$ such that $F(\tau) = y$ and $\pi \circ F = f$.

Remark 1.11.9. It is easy to modify this argument to apply when $[0,1]$ is replaced by Δ and $(0,1)$ is replaced by

$$\Delta - \{(0,0), (1,0), (0,1)\}.$$

1.12 Riemann surfaces

Definition 1.12.1. (Locally homeomorphic spaces). Let X and Y be topological spaces. We say X is locally homeomorphic to Y if for all $x \in X$ there exist open sets $U \subset X$ and $V \subset Y$ such that $x \in U$ and U is homeomorphic to V.

Definition 1.12.2. (n-manifold). An n-manifold is a Hausdorff topological space with a countable basis that is locally homeomorphic to $\mathbb{R}^n$.

Definition 1.12.3. (Biholomorphic Function). Let $U \subset \mathbb{C}$ be open. A function $f : U \to V \subset \mathbb{C}$ is called biholomorphic if it is a holomorphic bijection with a holomorphic inverse.

Definition 1.12.4. (Complex Atlas). Let X be a 2-manifold. A complex atlas on X consists of an open cover $\{U_i\}_{i\in I}$ and a collection of associated homeomorphisms $\{\phi_i : U_i \to V_i \subset \mathbb{C}\}_{i\in I}$ with the following property:

$$\phi_i \circ \phi_j^{-1} \text{ is biholomorphic on } \phi_j(U_i \cap U_j) \quad \forall i, j \in I.$$

The homeomorphisms belonging to a complex atlas are called charts. Two charts are called compatible if they satisfy the property above. Two complex atlases are considered equivalent if their union is itself an atlas.

Definition 1.12.5. (Riemann Surface). A Riemann surface X is a connected 2-manifold with a complex structure given by an equivalence class of atlases on X.

1.13 Meromorphic functions

Definition 1.13.1. (Isolated singularity). Let f be a holomorphic function on an open set $D \setminus \{c\}$, where $D \subset \mathbb{C}$ is a domain and $c \in D$. Then the point c is called an isolated singularity of f if there exists a punctured neighbourhood $U = \{z \in \mathbb{C} : 0 < |z - c| < \varepsilon\} \subset D$ such that f is holomorphic on U, but not defined or not holomorphic at c.

Proposition 1.13.1. [Kós24] Let f be a holomorphic function on a punctured neighbourhood of a point $c \in \mathbb{C}$. The following statements are equivalent:

1) The function f has a holomorphic extension to a full neighbourhood of c, i.e., there exists a function $\tilde{f}$ holomorphic on $|z - c| < \varepsilon$ such that $\tilde{f}(z) = f(z)$ for all $z \neq c$.

2) The singularity of f at c is removable in the sense that the Laurent series of f around c contains no terms of negative degree.

3) $\lim_{z\to c} f(z)$ exists and is finite.

4) f is bounded on a punctured neighbourhood of c, i.e., there exists $\varepsilon > 0$ and $M > 0$ such that $|f(z)| < M$ for all $0 < |z - c| < \varepsilon$.

5) $\lim_{z\to c}(z - c)f(z) = 0$.

Definition 1.13.2. (Pole). Let U be an open subset of $\mathbb{C}$. A function $f : U \to \mathbb{C}^2$ said to have a pole at a of order m if f is holomorphic on a punctured neighbourhood $\dot{B}(a, r)$ of a, but is not defined at a, and if

$$f(z) = \frac{g(z)}{(z - a)^m},$$

where m is a positive integer, $g(z)$ is holomorphic at a, and $g(a) \neq 0$.

Definition 1.13.3. (Meromorphic function). A meromorphic function on the open set U is a function
$$f : U \to \mathbb{C} \cup \{\infty\}$$
such that $f : W \setminus f^{-1}(\{\infty\}) \to \mathbb{C}$ is holomorphic and f has a pole at each $a \in f^{-1}(\{\infty\})$.

Definition 1.13.4. (Pair of meromorphic functions). An ordered pair (f, g) of meromorphic functions defined on an open neighbourhood of 0 in $\mathbb{C}$ simply a pair if f is not constant on any neighbourhood of 0 and the mapping defined by
$$t \mapsto (f(t), g(t))$$
is one-to-one near 0.

Definition 1.13.5. (Parameter change). A parameter change is a holomorphic function ρ defined on an open neighbourhood of 0 in $\mathbb{C}$ such that $\rho(0) = 0$ and $\rho'(0) \neq 0$.

Definition 1.13.6. (Equivalent pairs). We say that two pairs (f, g) and $(\widetilde{f}, \widetilde{g})$ are equivalent and write
$$(f, g) \sim (\widetilde{f}, \widetilde{g})$$
if there is a parameter change ρ such that $\widetilde{f} = f \circ \rho$ and $\widetilde{g} = g \circ \rho$ in some neighbourhood of 0. By Theorem 1.7.1, this is an equivalence relation on the set of pairs. The equivalence class of a pair (f, g) is called a meromorphic element. We use the notation $\langle f(t), g(t)\rangle$.

Definition 1.13.7. (Set of meromorphic elements). We denote the set of meromorphic elements by $\mathcal{M}$.

Definition 1.13.8. (Open sets $U(f, g, r)$). Let (f, g) be a pair and let $r > 0$ be sufficiently small that f and g are both defined and meromorphic on the open disc $D(0, r)$ of centre 0 and radius r, and the map
$$t \mapsto (f(t), g(t))$$
is one-to-one on $D(0, r)$. Then
$$U(f, g, r) := \{\langle f(t_0 + t), g(t_0 + t)\rangle : t_0 \in D(0, r)\} \subset \mathcal{M}.$$

Remark 1.13.2. This can be defined because $(f(t_0 + t), g(t_0 + t))$ is a pair for each $t_0 \in D(0, r)$.

Remark 1.13.3. $\langle f, g\rangle \in U(f, g, r)$.

Proposition 1.13.4. [Kir92] There is a topology on $\mathcal{M}$ such that a subset of $\mathcal{M}$ is open if and only if it is a union of subsets of the form $U(f, g, r)$ just defined.

Proposition 1.13.5. [Kir92] $\mathcal{M}$ is a Riemann surface with the holomorphic atlas
$$\Phi = \{\phi_\alpha : U_\alpha \to V_\alpha : \alpha \in \mathcal{A}\},$$
where

- $\mathcal{A}$ is the set of all ordered triples (f, g, r) for which (f, g) is a pair, $r > 0$ and f, g are defined meromorphic functions on $D(0, r)$ such that $t \mapsto (f(t), g(t))$ is a bijection on $D(0, r)$;
- $U_\alpha = U(f, g, r)$ for all $\alpha = (f, g, r) \in \mathcal{A}$;
- $V_\alpha = D(0, r)$ for all $\alpha = (f, g, r) \in \mathcal{A}$;
- $\phi_\alpha : U_\alpha \to V_\alpha$ is the inverse of the homeomorphism $D(0, r) \to U(f, g, r)$: $t_0 \mapsto \langle f(t_0 + t), g(t_0 + t)\rangle$.

Proposition 1.13.6. [Kir92] The maps $\langle f, g\rangle \mapsto f(0)$ and $\langle f, g\rangle \mapsto g(0)$ are meromorphic functions $\mathcal{M} \to \mathbb{C} \cup \{\infty\}$.

2 Investigation of intersections

In this section, we will focus on two vital results of classical algebraic geometry: Bézout's Theorem, which gives a precise count of the intersection points of plane curves, and Max Noether's Fundamental Theorem, which provides an even deeper insight into the ideal-theoretic structure of these intersections. Along the way, we introduce several important lemmas and propositions that we shall use later on.

2.1 Bézout's theorem

In this subsection, we will mostly rely on the notations, definitions and propositions of subsection 1.6 and subsection 1.9.

Proposition 2.1.1. Any two projective curves in $\mathbb{CP}^2$ intersect in at least one point.

Proof. [Kir92] Suppose that the homogeneous polynomials $f, g \in \mathbb{C}[x, y, z]$ define the curves F and G. By Proposition 1.9.3, the resultant $R_x(f, g)$ is a homogeneous polynomial of degree $\deg f \cdot \deg g$ in y and z. By Proposition 1.2.2, the resultant is either zero or equal to the product of $\deg f \cdot \deg g$ linear factors of form $bz - cy$ with $b, c \in \mathbb{C}$, not both being zero. In both cases, there exists $(b, c) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ such that $R_x(f, g)$ is zero at (b, c), that is, the polynomials $f(x, b, c)$ and $g(x, b, c)$ have a common root by Proposition 1.9.1. If we denote this root by a then

$$f(a, b, c) = g(a, b, c) = 0,$$

so $[a : b : c] \in F \cap G$. □

Proposition 2.1.2. If two projective curves F and G in $\mathbb{CP}^2$ defined by the homogeneous polynomials $f, g \in \mathbb{C}[x, y, z]$ have no common component, then they intersect in at most $\deg f \cdot \deg g$ points.

Proof. [Kir92] Indirectly assume that F and G intersect in at least $\deg f \cdot \deg g + 1$ distinct points. Let $\mathcal{I}$ denote the set of intersection points. Choose a point in $\mathbb{CP}^2$ not lying on F or G, or on any line determined by two points of $\mathcal{I}$. By Proposition 1.1.7 we can assume that this point is $[1 : 0 : 0]$. Then we have

$$f(1, 0, 0) \neq 0 \neq g(1, 0, 0),$$

so $[1 : 0 : 0] \notin F \cup G$. By Proposition 1.9.3, the resultant $R_x(f, g)$ is a homogeneous polynomial of degree $\deg f \cdot \deg g$ in y and z. If $R_x(f, g)$ is not identically zero, then it factors into $\deg f \cdot \deg g$ linear forms $bz - cy$ for $(b, c) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. Furthermore, such a factor $bz - cy$ divides $R_x(f, g)$ if and only if $f(x, b, c)$ and $g(x, b, c)$ have a common root a. If $[a : b : c] \in \mathcal{I}$, then the above vanishing holds. Since $[1 : 0 : 0] \notin \mathcal{I}$, each such $[b : c]$ gives a distinct linear factor $bz - cy$ of $R_x(f, g)$. If two such points in $\mathcal{I}$ gave proportional vectors (b, c) and (β, γ), then $[a : b : c]$, $[\alpha : \beta : \gamma]$, and $[1 : 0 : 0]$ would lie on the same line $bz = cy$, contradicting the choice of $[1 : 0 : 0]$.

Hence $R_x(f, g)$ has at least $\deg f \cdot \deg g + 1$ distinct linear factors, which is impossible unless it is identically zero. By Proposition 1.9.2, this implies that F and G share a common component, contradicting the assumption. □

Proposition 2.1.3. A nonsingular projective curve C in $\mathbb{CP}^2$ is irreducible.

Proof. [Kir92] Suppose, for contradiction, that $H = \{[x : y : z] \in \mathbb{CP}^2 : f(x, y, z)g(x, y, z) = 0\}$. By Proposition 2.1.1, there exists a point $[a : b : c] \in \mathbb{CP}^2$ such that

$$f(a, b, c) = g(a, b, c) = 0,$$

which implies that $[a : b : c]$ is a singular point of H (since at $[a : b : c]$ fg has no linear part). Contradiction. $\square$

Proposition 2.1.4. An irreducible projective curve F in $\mathbb{CP}^2$ has at most finitely many singular points.

Proof. [Kir92] Let F be defined by a homogeneous polynomial $f(x, y, z)$, and assume that $[1 : 0 : 0] \notin F$ (by Proposition 1.1.7). Then the coefficient of $x^{\deg f}$ in $f(x, y, z)$ is non-zero, implying that the partial derivative

$$g(x, y, z) = \left(\frac{\partial f}{\partial x}\right)(x, y, z)$$

is a non-zero homogeneous polynomial of degree $\deg f - 1$, hence it defines a curve G in $\mathbb{CP}^2$.

Since $\deg g < \deg f$ and F is irreducible, F and G have no common component. By Proposition 2.1.2, they intersect in at most $\deg f(\deg f - 1)$ points. Each singularity of F lies in $F \cap G$, so F clearly has only finitely many singularities. $\square$

Corollary 2.1.5. Any reduced algebraic curve has only finitely many singular points.

Proof. Let f be the polynomial defining the curve, and suppose that $f = f_1 \cdot f_2 \cdot \ldots \cdot f_k$, where each f_i is irreducible. Since the curve is reduced, each f_i is different. If F_i is the curve defined by f_i, then for any $1 \leq i < j \leq k$, F_i and F_j intersect in at most finitely many points due to Proposition 2.1.2 because they cannot have common component. Thus, by Proposition 2.1.4, we can only have finitely many singularities altogether. $\square$

Remark 2.1.6. For non-reduced curves, it is possible to have infinitely many singularities. As an example, for each $\lambda \in \mathbb{C}$, the point $[0 : \lambda : 1]$ is a singular point of the curve $x^2 = 0$.

Theorem 2.1.7. (Bézout's theorem). Let F and G be projective curves in $\mathbb{CP}^2$ with no common component. Then the total number of intersection points counted with multiplicities is $\deg f \cdot \deg g$, that is,

$$\sum_{P \in F \cap G} I[F, G](P) = \deg f \cdot \deg g.$$

Proof. [Kir92] By Proposition 1.1.7, there exists a projective coordinate system in which the following conditions hold:

- $[1 : 0 : 0] \notin F \cup G$,
- $[1 : 0 : 0]$ does not lie on any line containing two distinct points of $F \cap G$,
- $[1 : 0 : 0]$ does not lie on the tangent line to F or G at any point of $F \cap G$.

Suppose that F and G are defined by homogeneous polynomials $f(x, y, z)$ and $g(x, y, z)$ in this coordinate system. By Proposition 1.9.2, and Proposition 1.9.3, the resultant $R_x(f, g)$ is a homogeneous polynomial of degree $\deg f \cdot \deg g$ in y and z, and is not identically zero. Hence, by Proposition 1.2.2, we can write

$$R_x(f, g) = \prod_{i=1}^{k} (c_i z - b_i y)^{r_i},$$

where each r_i is a positive integer and

$$\sum_{i=1}^{k} r_i = \deg f \cdot \deg g.$$

The pairs (b_i, c_i) are distinct projective directions, meaning that $(b_i : c_i) \neq (b_j : c_j)$ for $i \neq j$. As we have seen in the proof of Proposition 2.1.2, for each factor there exists a unique $a_i \in \mathbb{C}$ such that $P_i := [a_i : b_i : c_i] \in C \cap D$, and $I[F, G](P_i) = r_i$. Hence, summing over all such intersection points P_i, we obtain the desired formula. $\square$

Example 2.1.1. Consider the algebraic plane curves $f = x + y$, $g = 5(x + y)^2 - x + y$ and $h = x^3 - y^2$. After homogenizing, we have $f^* = x + y$, $g^* = 5(x+y)^2 - xz + yz$ and $h^* = x^3 - y^2 z$. Note that $\deg f^* = 1$, $\deg g^* = 2$ and $\deg h^* = 3$. So what does Bézout's theorem (Theorem 2.1.7) tell us?

- f^* and g^* should intersect at $1 \cdot 2 = 2$ points, as they do not have a common component. In Figure 9, however, they only intersect at the origin. This is because one of the intersection points of the curves is not visible in the real plane, it is in fact $[1 : -1 : 0]$.
- f^* and h^* should intersect at $1 \cdot 3 = 3$ points, as they do not have a common component. In Figure 9, however, there are only two intersection points. This does not contradict Bézout's theorem, as here the origin has a multiplicity of 2.
- g^* and h^* should intersect at $2 \cdot 3 = 6$ points, as they do not have a common component. In Figure 9, only three of the intersections are visible. This is because the origin has a multiplicity of 2 and two other intersections are complex, thus lie outside of the real plane.

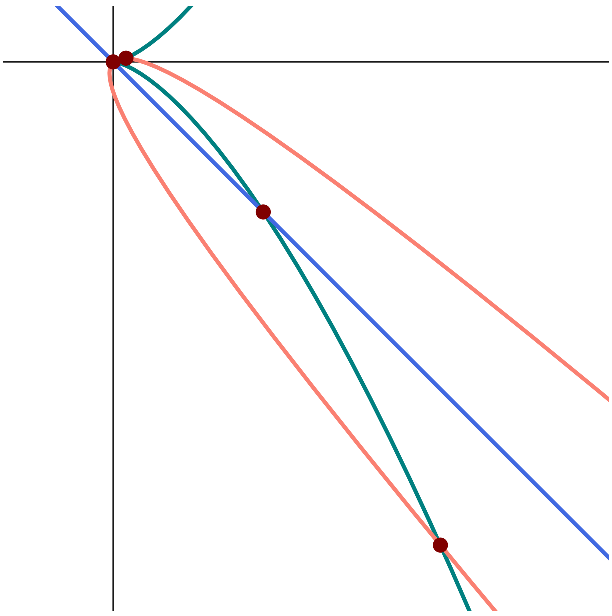

Figure 9
$f = x + y$, $g = 5(x + y)^2 - x + y$, $h = x^3 - y^2$ near the origin in the real plane.
The figure was created by [Des24].

Corollary 2.1.8. For projective curves F, G without common component defined by f and g if every point of $F \cap G$ is nonsingular on both curves and the tangents to F and G at these points are different lines, then $F \cap G$ consists of exactly $\deg f \cdot \deg g$ points.

Proof. It follows from Bézout's theorem and from Proposition 1.6.2 part f). $\square$

2.2 Noether's fundamental theorem

In this subsection, we will mostly rely on the notations, definitions and propositions of subsection 1.3, subsection 1.4, subsection 1.5 and subsection 1.6.

Definition 2.2.1. (Zero cycle). For points $P_1, P_2, \ldots, P_k \in \mathbb{CP}^2$ and non-zero integers $n_{P_1}, n_{P_2}, \ldots, n_{P_k}$ we define the zero cycle as the sum $\sum_{i=1}^{k} n_{P_i} P_i$.

Remark 2.2.1. For each point in $\mathbb{CP}^2$ not appearing in a zero cycle, we may consider it as appearing in the cycle with coefficient zero.

Remark 2.2.2. Zero cycles define a group (a free commutative group with base $\mathbb{CP}^2$).

Definition 2.2.2. (Degree). The degree of the zero cycle $\sum n_{P_i} P_i$ is $\sum n_{P_i}$.

Definition 2.2.3. (Positive zero cycle). A zero cycle is positive if for all P_i, $n_{P_i} \geq 0$.

Definition 2.2.4. (Relations). The zero cycle $\sum n_{P_i} P_i$ is said to be greater than or equal to $\sum m_{P_i} P_i$ (denoted by $\geq$), if $n_{P_i} \geq m_{P_i}$ for every P_i.

Definition 2.2.5. (Intersection cycle). The intersection cycle of projective curves F, G without common components

$$F \bullet G = \sum_{P \in F \cap G} I[F, G](P) \cdot P.$$

Proposition 2.2.3. The intersection cycle of curves F, G is a zero cycle of degree $\deg f \cdot \deg g$, where f and g are the polynomials defining F and G.

Proof. Follows immediately from Bézout's theorem Theorem 2.1.7). □

Proposition 2.2.4. Given algebraic curves F, G, H defined by the homogeneous polynomials f, g, h, respectively. Suppose no two of them have a common component.

a) Then $F \bullet GH = F \bullet G + F \bullet H$.

b) If $\deg h = \deg g - \deg f$ then $F \bullet (G + HF) = F \bullet G$.

Proof. Note that F and GH cannot have a component, since then either F and G or F and H would have to have a common component. Similarly, F and $G + HF$ cannot have a common component, as it would imply that F and G have one as well. Now, part **a)** follows from part g) of Proposition 1.6.2, while part **b)** is a direct consequence of part h) of Proposition 1.6.2. □

Definition 2.2.6. (Noether's condition). Suppose that the homogeneous polynomials f, g, h define the curves F, G, H. Let F and G be two projective curves passing through a point P, without a common component. We say that H satisfies Noether's condition at P if $\overline{h_*} \in (\overline{f_*}, \overline{g_*}) \subset \mathcal{O}_P(\mathbb{CP}^2)$, that is, there exist $a, b \in \mathcal{O}_P(\mathbb{CP}^2)$ such that $\overline{h_*} = a\overline{f_*} + b\overline{g_*}$.

Theorem 2.2.5. (Max Noether's Fundamental Theorem). Let F, G, H be projective curves such that F and G have no common component. Suppose they are defined by the homogeneous polynomials f, g, h, respectively. Then there exist homogeneous polynomials a, b (respectively of degrees $\deg h - \deg f$ and $\deg h - \deg g$), such that $h = af + bg$ if and only if Noether's condition is satisfied at every point $P \in F \cap G$.

Proof. [Ful08] Start with the simpler direction: if such a, b polynomials exist, then Noether's condition is satisfied everywhere. By dehomogenizing the equation $h = af + bg$, we see that $\overline{h_*} = a\overline{f_*} + b\overline{g_*}$, thus indeed Noether's condition is met everywhere.

Now let us prove the other direction. Take a line that does not pass through any intersection point of F and G (there are finitely many points to avoid due to Bézout's theorem Theorem 2.1.7). Transform this line into the ideal line using a projective transformation (which is possible by Proposition 1.1.7). By Noether's condition, the image of h_* in $\mathcal{O}_P(\mathbb{CP}^2)/(\overline{f_*}, \overline{g_*})$ is zero (as it is in the kernel of the natural projection). Now, due to Proposition 1.5.5, the image of h_* is also zero in $\mathbb{C}[x,y]/(f_*, g_*)$, that is, $h_* = af_* + bg_*$ for some $a, b \in \mathbb{C}[x,y]$. Then, using Proposition 1.4.7, for appropriate r,

$$z^r h = z^r (h_*)^* = z^r (af_* + bg_*)^* = af + bg, \tag{$\clubsuit$}$$

where a, b are homogeneous polynomials.

Now prove that multiplication by z in $\mathbb{C}[x,y,z]/(f,g)$ is injective. We need to see that if zh is in the kernel, that is, $zh = af + bg$, then h is also in it. Note that then $a(x,y,0)f(x,y,0) = -b(x,y,0)g(x,y,0)$. Since f and g have no common components, $f(x,y,0), g(x,y,0)$ are relatively prime, this is only possible if $b(x,y,0) = cf(x,y,0)$ and $a(x,y,0) = -cg(x,y,0)$ for some $c \in \mathbb{C}[x,y]$. Then, the polynomials $a + cg, b - cf$ both are multiples of z. Thus, for appropriate a^*, b^*, $zh = (a+cg)f + (b-cf)g = z(a^*f + b^*g)$, from which it follows that $h = a^*f + b^*g$.

Returning back to the theorem, from Equation $\clubsuit$ the image of h in $\mathbb{C}[x,y,z]/(f,g)$ is zero, so there are $a', b' \in \mathbb{C}[x,y,z]$ with $h = a'f + b'g$. If we write a' and b' as sums of homogeneous polynomials as $a' = \sum a'_i$ and $b' = \sum b'_i$ (where a'_i, b'_i are homogeneous polynomials of degree i), then we see that

$$h = a'_{\deg h - \deg f} f + b'_{\deg h - \deg g} g.$$

$\square$

Proposition 2.2.6. Let F, G, H be projective curves such that no two of them have no common component, and furthermore $F \bullet H \geq F \bullet G$. Then if Noether's condition is satisfied at every point $P \in F \cap G$, there exists such a curve G' such that $F \bullet G' = F \bullet H - F \bullet G$.

Proof. [Ful08] Due to Noether's theorem (Theorem 2.2.5), there are curves A, B such that $H = AF + BG$. Then, according to Proposition 2.2.4,

$$F \bullet H = F \bullet (AF + BG) = F \bullet BG = F \bullet B + F \bullet G,$$

thus $F \bullet B = F \bullet H - F \bullet G$. $\square$

Proposition 2.2.7. Let F, G, H be projective curves such that no two of them have no common component. If every point of $F \cap G$ is smooth on F and $F \bullet H \geq F \bullet G$, then Noether's condition is satisfied at every point $F \cap G$.

Proof. [Ful08] Let $P \in F \cap G$. Then $I[F,H](P) \geq I[F,G](P)$, thus $\overline{h_*} \in (\overline{g_*}) \subset \mathcal{O}_P(F)$. By Proposition 1.5.6 we have that $\mathcal{O}_P(F)/(\overline{g_*})$ is isomorphic to $\mathcal{O}_P(\mathbb{C}^2)/(f_*, g_*)$, thus the residue of h_* is also zero. Hence, Noether's condition is met. $\square$

Theorem 2.2.8. (Cayley-Bacharach Theorem). Let F, G, H be cubic curves, with F irreducible. If $F \bullet G = \sum_{i=1}^{9} P_i$, where the points P_i are smooth (but not necessarily distinct) and $F \bullet H = \sum_{i=1}^{8} P_i + Q$, then $Q = P_9$.

Proof. We prove by contradiction. Take a line L that passes through P_9 but not through Q. Then $L \bullet F = R + S + P_9$. Now

$$LH \bullet F = L \bullet F + H \bullet F = (R + S + P_9) + (P_1 + \cdots + P_8 + Q) = F \bullet G + Q + R + S.$$

Then $(LH/G) \bullet F = Q + R + S$. Since LH/G is quadratic, Q, R, S are collinear, and the line L passes through points R and S, but then it must also pass through Q, a contradiction. $\square$

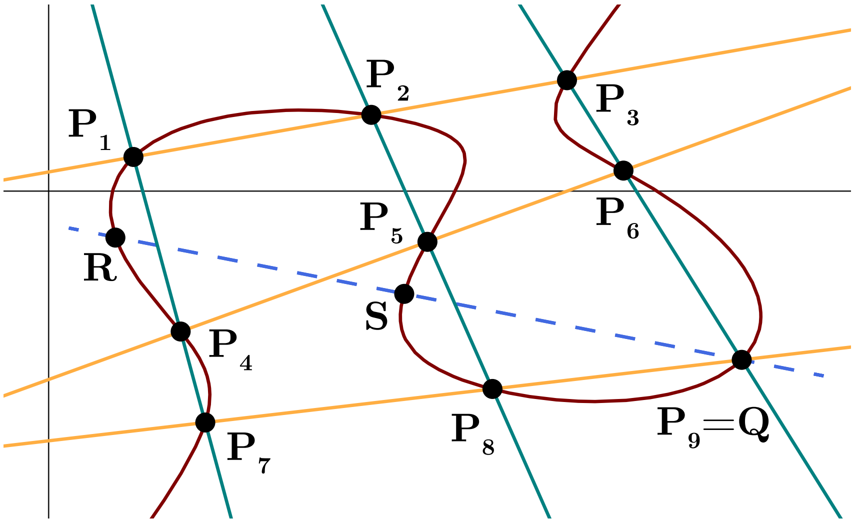

Figure 10
The Cayley-Bacharach theorem demonstrated with
$F = -8177009737992995840x^3 - 6802848028953690112x^2y + 2580569274240338000000x^2$
$-24182716091392440000xy^2 + 357001752705254700000xy - 2186389994062293200000x+$
$29405882122725134000y^3 + 40403905539519814000y^2 + 30127328537621300000y$
$+32585266942726656000$ [Nag25],
$G = (-5.1x + 1.38y + 14.08)(3.71x + 1.64y - 38.67)(2.64x + 1.65y - 45.35)$,
$H = (-1.23x + 6.95y - 3.85)(-2.62x + 7.21y + 39.73)(-0.98x + 8.39y + 61.22)$.
The figure was created by [Des24].

Lemma 2.2.9. A nonsingular complex cubic projective curve has exactly nine points of inflections (with counting multiplicity).

Proof. Let F be the the curve. By Remark 1.3.2, its Hessian is a homogenous polynomial of degree 3. We denote the curve defined by this polynomial by G. As F is nonsingular, it is irreducible due to Proposition 2.1.3. Thus, F and G can only have a common component if every point on F is a point of inflection. Luckily, Proposition 1.3.4 guarantees that this is not the case. Hence, we may apply Bézout's theorem (Theorem 2.1.7), and the result follows. $\square$

Remark 2.2.10. It can be shown that the Hessian matrix's polynomial intersects F transversally at each intersection, thus F has exactly nine different points of inflection. However, we will only use the existence of a point of inflection.

Definition 2.2.7. ($\oplus$ operation). Let F be a nonsingular cubic projective curve defined by a homogeneous polynomial of degree three. Fix a point of inflection O on the curve (which is possible by the lemma and remark above). Now for every point $P \in F \setminus \{O\}$, let $\overline{F}$ be the third intersection point of the line through O and A with F (counting multiplicities, well-defined by Bézout's theorem Theorem 2.1.7). Furthermore, for points $P, Q \in F$, let $P \oplus Q = \overline{R}$, where R is the third intersection point of the line through P and Q with F (with counting multiplicities).

Proposition 2.2.11. The operation $\oplus$ defined above on the nonsingular cubic projective curve F with $O \in F$ as the identity element determines a commutative group structure on F.

Proof. [Men11] It is easy to see that O indeed acts as the identity element. Furthermore, the inverse is well-defined, as for each point P, it will be the third intersection point of the line through O and P. The commutativity is trivial. So we only need to prove the associativity, that is $(A \oplus B) \oplus C = A \oplus (B \oplus C)$. Let S be the third intersection point of the line through $A \oplus B$ and C with F, and let T be the third intersection point of the line through $B \oplus C$ and A with

F. We want to prove that $\overline{S} = \overline{T}$, which is equivalent to $S = T$. Define two cubic curves: let M be the product of the lines through $AB, O(B \oplus C), CS$ and let N be the product of the lines through $BC, O(A \oplus B), AT$. Then

$$F \bullet N = A + B + C + O + A \oplus B + \overline{A \oplus B} + B \oplus C + \overline{B \oplus C} + S$$

and

$$F \bullet M = A + B + C + O + A \oplus B + \overline{A \oplus B} + B \oplus C + \overline{B \oplus C} + T.$$

Therefore, by Theorem 2.2.8, $S = T$. □

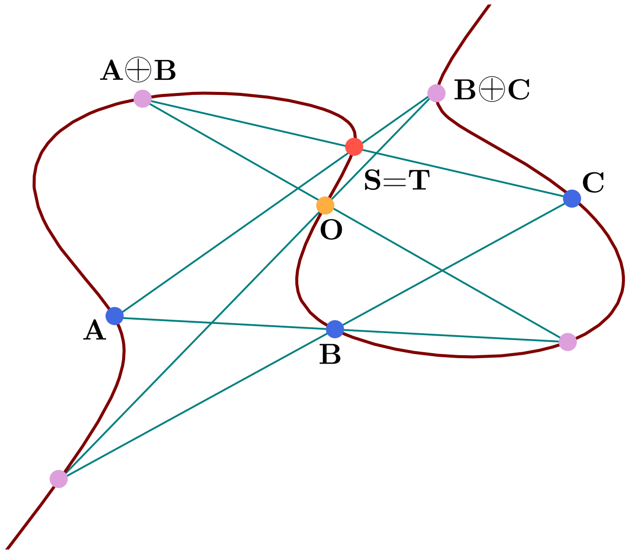

Figure 11
Proof of associativity of the $\oplus$ group structure on curve F from Figure 10.

Proposition 2.2.12. For cubic curves F, G, let $F \bullet G = \sum_{i=1}^{9} P_i$ and C be a conic such that $F \bullet C = \sum_{i=1}^{6} P_i$, where $P_1, \ldots, P_6$ are smooth points on F. Then P_7, P_8, and P_9 are collinear.

Proof. According to the condition $F \bullet G \geq F \bullet C$ and the points $F \cap C$ are smooth, thus by Proposition 2.2.7, Noether's condition is met everywhere. Proposition 2.2.6 states that then $P_7 + P_8 + P_9 = F \bullet B$, where $\deg B = 3 - 2 = 1$, thus the three points are indeed collinear. □

Theorem 2.2.13. (Pascal's Theorem). The intersection points of the opposite sides of a hexagon inscribed in a conic are collinear.

Proof. Let the hexagon be $P_1P_2P_3P_4P_5P_6$, C the conic, $F = (P_1P_2)(P_3P_4)(P_5P_6)$ and $G = (P_2P_3)(P_4P_5)(P_6P_1)$. Then $F \bullet G = P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + P_1P_2 \cap P_4P_5 + P_2P_3 \cap P_5P_6 + P_3P_4 \cap P_6P_1$. The previous proposition states that the last three points are on a line. □

Proposition 2.2.14. Let C be a nonsingular cubic curve. If the degrees of curves F, G are m and $F \bullet C = P_1 + \cdots + P_{3m}$, and $G \bullet C = P_1 + \cdots + P_{3m-1} + Q$. Then $Q = P_{3m}$.

Proof. We prove by contradiction. Let L be a line passing through P_{3m} and Q. Then L intersects C at a third point, R. Draw a tangent at Q, call it E. Notice that then

$$LG \bullet C = L \bullet C + G \bullet C = P_{3m} + Q + R + P_1 + \ldots P_{3m-1} + Q = F \bullet C + 2Q + R.$$

Thus $LG \bullet C \geq F \bullet C$, thus $2Q + R = L' \bullet C$ for some line L'. But then L' is the tangent at Q, thus $L' = E$. However, L' passes through Q, R, thus $L' = L$. Summing up, $L = E$. This is only possible if $Q = P_{3m}$. Contradiction. □

3 Resolutions

One of the fundamental goals in algebraic geometry is to resolve singularities, that is, to construct a well-behaved object that retains the essential geometric information of the original curve but is entirely nonsingular.

Definition 3.0.1. (Resolution of singularities). Given an open subset Y of a projective variety, a resolution of singularities of Y is a nonsingular variety X together with a birational map $\pi : X \to Y$ where π is an isomorphism above the nonsingular locus of Y.

As a first step, we examine the process of normalisation, which offers a global method for improving the structure of a curve. The normalisation of a curve replaces it with a new one that is normal — meaning it contains no singularities arising from algebraically inseparable or "glued" branches. The normalisation is canonical, that is, it always exists and is unique. For curves, the normalisation is, in fact, the resolution.

In what follows, we will study a key tool for constructing resolutions: the blow-up. Blow-ups serve as local modifications that gradually improve the geometry of a space. While a single blow-up may not always suffice to eliminate all singularities, iterated blow-ups can eventually yield a fully desingularized (smooth) model of the curve.

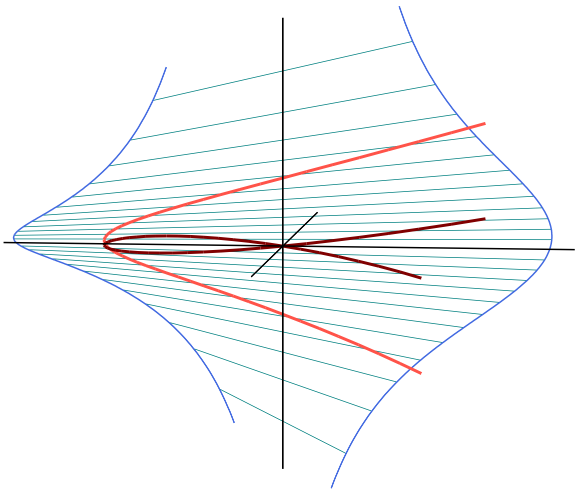

Figure 12
Desingularisation of the alpha curve (Figure 5).
Figure adapted from [Ken11], modified by the author.

3.1 Normalisation

In this subsection, we will mostly rely on the notations, definitions and propositions of subsection 1.13.

In this subsection, we will work with an irreducible plane curve P defined by the homogeneous polynomial $p \in \mathbb{C}[x, y, z]$ of degree d.

Definition 3.1.1. (Normalisation). Let P be an irreducible algebraic plane curve. A normalisation of P is a pair $(\widetilde{P}, \pi)$, where $\widetilde{P}$ is a smooth, connected, compact Riemann surface, and $\pi : \widetilde{P} \to P$ is a continuous and surjective map such that:

- π is holomorphic,
- the restriction $\pi : \widetilde{P} \setminus \pi^{-1}(\mathrm{Sing}(P)) \to C \setminus \mathrm{Sing}(P)$ is a biholomorphism,

- π is universal among all such maps from nonsingular Riemann surfaces to P, i.e., for any such map $\phi : S \to P$ where S is smooth and ϕ is birational, there exists a unique holomorphic map $f : S \to \widetilde{P}$ such that $\phi = \pi \circ f$.

Definition 3.1.2. (Riemann surface of irreducible curve). Let $p(x, y, z)$ be a nonconstant irreducible homogeneous polynomial of degree d. The Riemann surface S_p of $p(x, y, z)$ is the open subset of $\mathcal{M}$ consisting of all those elements $\langle f, g\rangle$ of $\mathcal{M}$ satisfying

$$p(f(t), g(t), t) = 0$$

for all t in some neighbourhood of 0. If P is the projective curve defined by p, then we write $\widetilde{P}$ for S_p and define $\pi : \widetilde{P} \to P$ by

$$\pi(\langle f, g\rangle) = \begin{cases} [f(0), g(0), 1] & \text{if } f \text{ and } g \text{ are both holomorphic near } 0, \\ [f^{(m)}(0), g^{(m)}(0), 0] & \text{otherwise} \end{cases}$$

where $f^{(m)}(t) = t^m f(t)$ and $g^{(m)}(t) = t^m g(t)$, and m is greater value amongst the multiplicity of the pole at 0 of f or g.

Proposition 3.1.1. The map $\pi : \widetilde{P} \to P$ is continuous, and its restriction to $\widetilde{P} \setminus \pi^{-1}(\mathrm{Sing}(P))$ is holomorphic.

Proof. [Kir92] Let $\langle f_0, g_0\rangle \in \widetilde{P}$ be such that both f_0 and g_0 are holomorphic in a neighbourhood of 0. Then π is continuous at $\langle f_0, g_0\rangle$ because it can be expressed locally as the composition of the continuous map $\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{CP}^2$, given by $\pi(x, y, z) = [x : y : z]$, and the map $\widetilde{P} \to \mathbb{C}^3 \setminus \{0\}$ defined by

$$\langle f, g\rangle \mapsto (f(0), g(0), 1)$$

which is holomorphic in each coordinate by Proposition 1.13.6. If f_0 has a pole at 0 of order at least equal to that of g_0, then near $\langle f_0, g_0\rangle$, the map π can be written as the composition of $\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{CP}^2$ with the map

$$\langle f, g\rangle \mapsto \left(1, \frac{g(0)}{f(0)}, \frac{1}{f(0)}\right),$$

which the coordinates are holomorphic near $\langle f_0, g_0\rangle$. Therefore, π is continuous at such points. The same reasoning applies if g_0 has a higher-order pole than f_0, since then π remains continuous at $\langle f_0, g_0\rangle$.

To establish that the restriction of π to $\widetilde{P} \setminus \pi^{-1}(\mathrm{Sing}(P))$ is holomorphic, choose an element $\alpha = \langle f, g, r\rangle \in \mathcal{A}$. Then the holomorphic chart $\varphi_\alpha : U_\alpha \to V_\alpha$ has inverse

$$t_0 \mapsto \langle f(t_0 + t), g(t_0 + t)\rangle$$

defined on $V_\alpha = D(0, r)$. A holomorphic chart ψ_β on $P \setminus \mathrm{Sing}(P)$ is constructed from one of the maps taking $[x : y : z]$ to one of the ratios $\frac{x}{z}$, $\frac{y}{z}$, $\frac{z}{x}$, $\frac{z}{y}$, $\frac{x}{y}$, or $\frac{y}{x}$.

The composition $\psi_\beta \circ \pi \circ \varphi_\alpha^{-1}$ is then defined wherever the expression corresponds to one of the functions f, g, g/f, f/g, $1/g$, or $1/f$. Each of these is meromorphic, meaning they extend to holomorphic functions on subsets of $\mathbb{C} \cup \{\infty\}$.

Since charts on P map into $\mathbb{C}$, the composition $\psi_\beta \circ \pi \circ \varphi_\alpha^{-1}$ is holomorphic on its domain. Consequently, the restriction of π to $\widetilde{P} \setminus \pi^{-1}(\mathrm{Sing}(P))$ is holomorphic. ∎

Proposition 3.1.2. The restriction of $\pi : \widetilde{P} \to P$ to $\widetilde{P} \setminus \pi^{-1}(\mathrm{Sing}(P))$ defines a bijection onto $P \setminus \mathrm{Sing}(P)$.

Proof. [Kir92] Let us assume that $[a : b : c]$ is a smooth point on P where $c \neq 0$. Without loss of generality, we may suppose that $c = 1$. In that case, either $\frac{\partial p}{\partial x}$ or $\frac{\partial p}{\partial y}$ must be non-zero at $(a, b, 1)$, as if both partial derivatives were zero, then by Theorem 1.2.3 the derivative with respect to z, namely $\frac{\partial p}{\partial z}$, would also vanish, implying that $[a : b : 1]$ is singular, contrary to our assumption. Hence, we can assume without loss of generality that

$$\frac{\partial p}{\partial y}(a, b, 1) \neq 0.$$

We aim to prove that there is a unique $\langle f, g\rangle \in \widetilde{P}$ such that $\pi(\langle f, g\rangle) = [a : b : 1]$. Note that the implicit function theorem (Corollary 1.8.2) guarantees the existence of open neighbourhoods U and V of a and b in $\mathbb{C}$, respectively, and a holomorphic function $h : U \to V$ such that

$$p(x, y, 1) = 0 \quad \text{if and only if} \quad y = h(x),$$

for all $x \in U$ and $y \in V$. Suppose now that f and g are holomorphic functions near 0 satisfying $f(0) = a$ and $g(0) = b$ and that

$$p(f(t), g(t), 1) = 0$$

for all t sufficiently close to 0. Then it must be that

$$g(t) = h(f(t))$$

for all t near 0. Therefore, if $\langle f, g\rangle$ is an element of $\widetilde{P}$, we conclude that f is injective near 0, and we may define a change of parameter $\rho(t) = f(t) - a$. This gives

$$\langle f, g\rangle = \langle a + \rho(t), h(a + \rho(t))\rangle = \langle a + t, h(a + t)\rangle,$$

and this is the unique element of $\widetilde{P}$ such that $\pi(\langle f, g\rangle) = [a : b : 1]$.

In the case where $c = 0$, a similar argument applies. One instead considers the polynomial $p(x, 1, z)$ or $p(1, y, z)$ in place of $p(x, y, 1)$, and uses the implicit function theorem with the functions $\frac{1}{g}$, $\frac{1}{f}$, $\frac{g}{f}$, or $\frac{f}{g}$ in place of f and g. ☐

Proposition 3.1.3. The map $\pi : \widetilde{P} \to P$ is surjective.

Proof. [Kir92] Let $[a : b : c] \in P$ be arbitrary. Our goal is to show that the preimage $\pi^{-1}([a : b : c])$ is non-empty. By choosing coordinates appropriately, we may suppose that $[0 : 1 : 0] \notin P$. Then $\frac{\partial p}{\partial y}$ is not identically zero. Since P is irreducible, Proposition 2.1.4 implies that there are only finitely many points in P where $\frac{\partial p}{\partial y} = 0$.

First, consider the case $c \neq 0$. Without loss of generality, we take $c = 1$. Then there exists $\varepsilon > 0$ such that for all $x \in \mathbb{C}$ with $0 < |a - x| \leq \varepsilon$, there is no $y \in \mathbb{C}$ such that $[x : y : 1] \in P$ and

$$\frac{\partial p}{\partial y}(x, y, 1) = 0.$$

Define $D_\pm(a, \varepsilon)$ as the open disc $D(a, \varepsilon)$ in $\mathbb{C}$ with the straight segment from a to $a \pm \varepsilon$ removed. The sets $D_\pm(a, \varepsilon)$ are simply connected. By Proposition 1.11.7, the holomorphic map $\phi : P \to \mathbb{CP}^1$ defined by $\phi([x : y : z]) = x$ is a homeomorphism when restricted to each connected component of $\phi^{-1}(D_\pm(a, \varepsilon))$. This restriction maps onto $D_\pm(a, \varepsilon)$, and there are d connected components in total. The inverse function theorem (Theorem 1.7.1) ensures that the inverse of

ϕ on the jth component of $\phi^{-1}(D_\pm(a,\varepsilon))$ is holomorphic and must be of the form

$$x \mapsto [x : f_j^\pm(x) : 1],$$

where

$$p(x, f_j^\pm(x), 1) = 0$$

for all $x \in D_\pm(a,\varepsilon)$. Furthermore, if $i \neq j$, then $f_i^\pm(x) \neq f_j^\pm(x)$ for $x \in D_\pm(a,\varepsilon)$, since $[x : f_i^\pm(x) : 1]$ and $[x : f_j^\pm(x) : 1]$ belong to different connected components of $\phi^{-1}(D_\pm(a,\varepsilon))$.

As P is compact, each function $f_j^\pm$ is bounded on $D_\pm(a,\varepsilon)$. Hence, the only possible limits as $x \to a$ are the finite set of values $y \in \mathbb{C}$ for which $[a : y : 1] \in P$. Among these, b is one such value, so $[a : b : 1]$ lies in the closure of $\phi^{-1}(D_\pm(a,\varepsilon))$, implying that b is the limit of $f_j^\pm(x)$ for some j as $x \to a$.

Since $D_\pm(a,\varepsilon)$ are disjoint open subsets of $D(a,\varepsilon)$ and each has d holomorphic branches $f_1^\pm, \ldots, f_d^\pm$, we may relabel them so that f_j^+ agrees with $f_{\sigma(j)}^-$ on the common boundary, where σ is a permutation of $\{1, \ldots, d\}$. We may assume this permutation satisfies

$$\sigma(i) = i + 1.$$

If $1 \leq i < m$, then $\sigma(i) = i+1$ and $\sigma(m) = 1$ for some $m \leq d$. This setup defines a holomorphic function

$$g : \left\{ t \in \mathbb{C} : 0 < |t| < \varepsilon^{1/m} \right\} \to \mathbb{C}$$

as follows:

$$g(t) = f_j^+(a + t^m) \quad \text{if} \quad \frac{(2j-2)\pi}{m} < \arg(t) < \frac{2j\pi}{m},$$

$$g(t) = f_j^-(a + t^m) \quad \text{if} \quad \frac{(2j-1)\pi}{m} < \arg(t) < \frac{(2j+1)\pi}{m}.$$

The function g is bounded and converges to b as $t \to 0$, so it extends holomorphically to the disc $D(0, \varepsilon^{1/m})$, and $g(0) = b$ by Proposition 1.13.1. Moreover,

$$p(a + t^m, g(t), 1) = 0 \quad \text{for all } t \in D(0, \varepsilon^{1/m}).$$

Hence, the mapping

$$t \mapsto (a + t^m, g(t))$$

is injective on $D(0, \varepsilon^{1/m})$ because $f_i^\pm(x) \neq f_j^\pm(x)$ whenever $i \neq j$. Thus, the element $\langle a + t^m, g(t) \rangle$ lies in $\widetilde{P}$ and maps via π to

$$\pi(\langle a + t^m, g(t) \rangle) = [a : b : 1].$$

Now, consider the case when $c = 0$. Then $a \neq 0$ since $[0 : 1 : 0] \notin P$, and we may take $a = 1$ without loss of generality. Then there exists $\varepsilon > 0$ such that for all $x \in \mathbb{C}$ with $|x| > 1/\varepsilon$, there is no y such that $[x : y : 1] \in P$ and

$$\frac{\partial p}{\partial y}[x : y : 1] = 0.$$

Define

$$D_\pm(\infty, \varepsilon) = \left\{ x \in \mathbb{C} : |x| > \frac{1}{\varepsilon},\ x \notin \mathbb{R}_\pm \right\},$$

where $\mathbb{R}_+$ and $\mathbb{R}_-$ denote the positive and negative real axes in $\mathbb{C}$, respectively. The argument used earlier can be repeated in this case, with a replaced by ∞, and $a + t^m$ replaced by t^{-m}.

This shows that there is an element of $\widetilde{P}$ of the form $\langle t^{-m}, g(t)\rangle$ such that $\pi(\langle t^{-m}, g(t)\rangle) = [1 : b : 0] = [a : b : c]$. This completes the proof. $\square$

Corollary 3.1.4. For any $a \in \mathbb{C} \cup \{\infty\}$, the meromorphic function $\varphi \circ \pi$ given by $\langle f, g\rangle \mapsto f(0)$ (see Proposition 1.13.6) on $\widetilde{C}$ takes the value a precisely d times (counted with multiplicities).

Proof. [Kir92] Recall from Proposition 1.13.6 that the map $\widetilde{C} \to \mathbb{CP}^1$, defined by $\langle f, g\rangle \mapsto f(0)$, is a meromorphic function on $\widetilde{P}$. Consider the permutation σ introduced during the proof of the previous proposition, and decompose it into disjoint cycles:

$$\sigma = \sigma_1 \cdots \sigma_l,$$

where each σ_i is a cycle of length $m_i \geq 1$, and the sum $m_1 + \cdots + m_l$ equals d.

Then the argument in the proof implies that for any $a \in \mathbb{C}$, there exist holomorphic functions $g_1, \ldots, g_l$ defined near $t = 0$ and corresponding elements

$$\langle a + t^{m_j}, g_j(t)\rangle \in \widetilde{P}$$

such that

$$(\varphi \circ \pi)^{-1}(a) = \{\langle a + t^{m_j}, g_j(t)\rangle : 1 \leq j \leq l\}.$$

Moreover, each point $\langle a + t^{m_j}, g_j(t)\rangle$ maps to a under $\varphi \circ \pi$ with multiplicity m_j, meaning that the order of vanishing of the function $\varphi \circ \pi - a$ at $\langle a + t^{m_j}, g_j(t)\rangle$ is exactly m_j. As a result, for all $a \in \mathbb{C}$, the meromorphic function $\varphi \circ \pi$ on $\widetilde{P}$ takes the value a precisely d times (counted with multiplicities). The same conclusion holds in the case where $a = \infty$. $\square$

Lemma 3.1.5. Let $\Phi : S \to \mathbb{CP}^1$ be a meromorphic function on a Riemann surface S, which assumes each value $a \in \mathbb{CP}^1$ exactly d times, counting multiplicities. Then S is compact.

Proof. [Kir92] Since $\mathbb{CP}^1$ is compact by Proposition 1.1.1, it suffices to prove that for every point $a \in \mathbb{CP}^1$, there exists an open neighbourhood $W_a \subset \mathbb{CP}^1$ such that the preimage $\Phi^{-1}(W_a)$ is contained within a compact subset $S_a \subset S$.

Let $\{W_a : a \in \mathbb{CP}\}$ be such an open cover. Since $\mathbb{CP}^1$ is compact , this cover admits a finite subcover $W_{a_1}, \ldots, W_{a_k}$. Then we can write:

$$S = \Phi^{-1}(W_{a_1}) \cup \cdots \cup \Phi^{-1}(W_{a_k}) = S_{a_1} \cup \cdots \cup S_{a_k},$$

which is a finite union of compact sets, hence compact.

Now, fix any $a \in \mathbb{C} \cup \{\infty\}$. Let $\Phi^{-1}(a) = \{s_1, \ldots, s_l\}$ be the set of points where Φ assumes the value a, and let m_j denote the multiplicity at s_j, so that $m_1 + \cdots + m_l = d$. For clarity, assume $a \neq \infty$; if $a = \infty$, the same reasoning applies to the function $1/\Phi$.

Choose local holomorphic charts $\varphi_j : U_j \to V_j$ near each s_j such that $\varphi_j(s_j) = 0$, and such that the neighbourhoods $U_1, \ldots, U_l$ are pairwise disjoint. Now define for each j the holomorphic function

$$f_j(z) := \Phi(\varphi_j^{-1}(z)) - a,$$

which vanishes at $z = 0$ with multiplicity m_j. Choose $\varepsilon_j > 0$ such that the closed disc $D(0, \varepsilon_j) := \{z \in \mathbb{C} : |z| \leq \varepsilon_j\} \subset V_j$. Then by Remark 1.8.4 there exists $\delta_j > 0$ such that for every $a' \in \mathbb{C}$ with $|a' - a| < \delta_j$, the function

$$f_j(z) - a' + a = \Phi(\varphi_j^{-1}(z)) - a'$$

has at least m_j zeros (counting multiplicities) in the open disc $D(0, \varepsilon_j)$. Now define

$$W_a := \{z \in \mathbb{C} : |z - a| < \min(\delta_1, \dots, \delta_l)\}\,.$$

and define

$$S_a := \bigcup_{1 \le j \le l} \varphi_j^{-1}(D(0, \varepsilon_j)).$$

Since each φ_j is a homeomorphism and each $D(0, \varepsilon_j)$ is compact, it follows that S_a is compact as well. Furthermore, if $a' \in W_a$, then the function Φ assumes the value a' at least m_j times (with multiplicities) in each open set $\varphi_j^{-1}(D(0, \varepsilon_j)) \subset U_j$. Since the sets $U_1, \dots, U_l$ are disjoint and $m_1 + \cdots + m_l = d$, we conclude that Φ assumes the value a' at least d times in total within S_a. Therefore,

$$\Phi^{-1}(a') \subset S_a.$$

This holds for every $a' \in W_a$, and thus

$$\Phi^{-1}(W_a) \subset S_a,$$

which completes the proof. □

Proposition 3.1.6. $\widetilde{P}$ is compact (in the standard topology).

Proof. It follows from the lemma above and Corollary 3.1.4. □

Proposition 3.1.7. $\widetilde{P}$ is connected.

Proof. [Kir92] Let A be a connected component of $\widetilde{P}$. Our aim is to prove that $A = \widetilde{P}$. As in the proof of Proposition 3.1.3, we choose coordinates such that $[0 : 1 : 0] \notin P$ and define the map

$$\phi([x : y : z]) = [x : z].$$

Let $a \in \mathbb{CP}^1$ and define $D_\pm(a, \varepsilon)$ as before. Then $\phi^{-1}(D_\pm(a, \varepsilon))$ consists of d components, and the restriction of ϕ to each component is a homeomorphism with inverse

$$x \mapsto [x : f_j^\pm(x) : 1],$$

where $f_j^\pm$ are holomorphic on $D_\pm(a, \varepsilon)$. We may index them so that f_j^+ and $f_{\sigma(j)}^-$ coincide on the overlap, for a permutation σ of $\{1, \dots, d\}$. Each component is either entirely contained in A or disjoint from it. Suppose $f_j^\pm(x) \in A$ for $1 \le j \le e$ and that σ restricts to a permutation on $\{1, \dots, e\}$. Then, by Corollary 3.1.4, we can write the permutation σ as a product of disjoint cycles:

$$\sigma = \sigma_1 \cdots \sigma_l$$

with lengths $m_1, \dots, m_l$ satisfying $m_1 + \cdots + m_l = d$.

If $a \neq \infty$, then there exist holomorphic functions $g_1, \dots, g_l$ near 0 and a non-zero constant K such that

$$p(x, y, 1) = K \prod_{1 \le j \le l} \prod_{1 \le s \le m_j} \left(y - g_j(e^{2\pi i s/m_j}(x - a)^{1/m_j})\right)$$

for x and y near a. (For $a = \infty$, we may interchange x and z.)

Assume that the restriction of σ to $\{1, \dots, e\}$ corresponds to $\sigma_1 \cdots \sigma_k$ with $m_1 + \cdots + m_k = e$. Define the function

$$q(x, y, 1) = K \prod_{1 \le j \le k} \prod_{1 \le s \le m_j} \left(y - g_j(e^{2\pi i s/m_j}(x - a)^{1/m_j})\right),$$

which is a degree e polynomial in y, with coefficients that are power series in the expressions $e^{2\pi is/m_j}(x-a)^{1/m_j}$, which converge for x near a. Because these expressions are invariant under permutation, the coefficients of $q(x,y,1)$ are symmetric polynomials in them, and hence can be written using the elementary symmetric functions. Therefore, these coefficients are power series in $x-a$. Thus, $q(x,y)$ is a polynomial in y whose coefficients are holomorphic functions of x near a. Moreover, for $x \in D_{\pm}(a,\varepsilon)$ we have

$$A \cap \phi^{-1}(x) = \left\{[x, g_j(e^{2\pi is/m_j}(x-a)^{1/m_j}), 1] : 1 \leq j \leq k, 1 \leq s \leq m_j\right\},$$

which implies that $q(x,y)$ is independent of the choice of $a \in P$, because P is connected (Corollary 1.11.2). In particular, $q(x,y)$ is a polynomial in y whose coefficients are holomorphic in x for all $x \in P$.

Now consider the case $a = \infty$. The same reasoning shows that the coefficients of $q(x,y)$ extend to meromorphic functions on $\mathbb{CP}^1$ with poles only at ∞. But such functions are polynomials. Hence, $q(x,y)$ is a polynomial in both x and y. This same argument applied to the complementary product

$$\prod_{k<j\leq l}\prod_{1\leq s\leq m_j}\left(y - g_j(e^{2\pi is/m_j}(x-a)^{1/m_j})\right)$$

also yields a polynomial in x and y. Therefore, $q(x,y)$ divides $p(x,y,1)$.

Let $\tilde{p}(x,y,z)$ be the homogenization q^*. Then $\tilde{p}$ divides $p(x,y,z)$. However, since $q(x,y)$ is nonconstant and p is irreducible, it must be that q and $p(x,y,1)$ are scalar multiples. Thus $k = l$ and $e = d$, and so $A = \widetilde{P}$. □

Finally, with all these propositions we have showed the following:

Theorem 3.1.8. (Normalisation). $\widetilde{P}$ is a compact connected Riemann surface. The map $\pi : \widetilde{P} \to P$ is continuous and surjective. If P is nonsingular then π is a holomorphic bijection, and in general $\pi^{-1}(\mathrm{Sing}(P))$ is finite and

$$\pi : \widetilde{P} \setminus \pi^{-1}(\mathrm{Sing}(P)) \longrightarrow P \setminus \mathrm{Sing}(P)$$

is a holomorphic bijection.

Remark 3.1.9. [BK86] Generally speaking, each resolution of singularities is a normalisation. For curves the two concept coincide, however, in higher dimensions it is much easier to find a normalisation than to find a resolution.

3.2 Blow-ups via quadratic transformations

In this subsection, we will mostly rely on the notations, definitions and propositions of subsection 1.2, subsection 1.4 and subsection 1.6.

Definition 3.2.1. (Fundamental points). Points $W_1 = [1:0:0]$, $W_2 = [0:1:0]$, $W_3 = [0:0:1] \in \mathbb{CP}^2$ are so-called the fundamental points.

Definition 3.2.2. (Exceptional lines). The lines $V(x)$, $V(y)$ and $V(z)$ are so-called the exceptional lines.

Definition 3.2.3. (Standard quadratic transformation). The standard quadratic transformation is defined by:

$$q : \mathbb{CP}^2 \setminus \{W_1, W_2, W_3\} \longrightarrow \mathbb{CP}^2, \quad [x:y:z] \mapsto [yz:xz:xy]$$

Example 3.2.1. Consider the algebraic projective curve

$$x^2y^2 + x^2z^2 + y^2z^2.$$

Each fundamental point is a singularity. Apply the standard quadratic transformation: the new curve is

$$x^2z^2x^2y^2 + y^2z^2x^2y^2 + y^2z^2x^2z^2 = x^2y^2z^2(x^2 + y^2 + z^2).$$

This is the union of three lines and a conic. It is essentially a nonsingular curve, since the lines are more or less insignificant. Unfortunately, we are hardly ever this lucky.

Remark 3.2.1. Let $U = \mathbb{CP}^2 \setminus V(xyz)$. Then q defines a morphism $q : \mathbb{CP}^2 \setminus \{W_1, W_2, W_3\} \to U \cup \{W_1, W_2, W_3\}$. For $[x : y : z] \in U$, we have

$$q(q(x, y, z)) = q(yz, xz, xy) = (xyz, yxyz, zxyz) = (x(xyz), y(xyz), z(xyz)) = (x, y, z),$$

so q is an isomorphism on U, and thus a birational map of $\mathbb{CP}^2$ with itself.

Definition 3.2.4. (The set U). From now on, we shall denote $\mathbb{CP}^2 \setminus V(xyz)$ by U.

Remark 3.2.2. Consider the *triangle* $W_1W_2W_3$. Lines through the vertices are determined by the ratios $y : z, x : z, y : x$. As the ratios go to their inverses under q, lines through the vertices are sent to their reflections in the corresponding angle bisector. Thus, the image of a point is its isogonal conjugate in the triangle. It follows that conics are sent to conics or lines.

Definition 3.2.5. (Algebraic transform). Let a homogeneous $p \in \mathbb{C}[x, y, z]$ define an irreducible curve P disjoint from the exceptional lines. Define the transform

$$p^q = p(yz, xz, xy) = p(q(x, y, z)),$$

a homogeneous form of degree $2 \cdot \deg p$. This is called the algebraic transform of p.

From now on, we will work with a homogeneous irreducible polynomial $p \in \mathbb{C}[x, y, z]$ with degree d. The curve defined by p will be denoted by P.

Proposition 3.2.3. Let r be the multiplicity of P at W_3, then z^r is the largest power of z dividing p^q.

Proof. [Rüd15] Using the standard identification of the affine plane $\mathbb{C}^2$ with the open subset of $\mathbb{CP}^2$ where $z \neq 0$, a point $[x : y : 1] \in \mathbb{CP}^2$ corresponds to $(x, y) \in \mathbb{C}^2$. In particular, the point $W_3 = [0 : 0 : 1]$ maps to the origin $(0, 0) \in \mathbb{C}^2$.

Let $p(x, y, 1) = p_r(x, y) + \cdots + p_d(x, y)$, where each p_i is a homogeneous polynomial of degree i. Homogenizing this expression gives

$$p(x, y, z) = p_r(x, y)z^{d-r} + p_{r+1}(x, y)z^{d-r-1} + \cdots + p_d(x, y).$$

Applying the transformation q, we compute:

$$\begin{aligned} p^q(x, y, z) &= p_r(yz, xz)(xy)^{d-r} + p_{r+1}(yz, xz)(xy)^{d-r-1} + \cdots + p_d(yz, xz) \\ &= z^r p_r(y, x)(xy)^{d-r} + z^{r+1}p_{r+1}(y, x)(xy)^{d-r-1} + \cdots + z^d p_d(y, x) \\ &= z^r \left(p_r(y, x)(xy)^{d-r} + zp_{r+1}(y, x)(xy)^{d-r-1} + \cdots + z^{d-r}p_d(y, x)\right), \end{aligned}$$

which completes the proof. □

Using this proposition for the other fundamental points too, we can define the following:

Definition 3.2.6. (Proper transform). Let $r_1 = m_{W_1}(P)$, $r_2 = m_{W_2}(P)$, $r_3 = m_{W_3}(P)$. Then $p^q = x^{r_1}y^{r_2}z^{r_3}p'$, where x, y, z do not divide p'. The polynomial p' is called the proper transform of p with degree $2d - (r_1 + r_2 + r_3)$. We denote by P' the curve defined by p'.

Remark 3.2.4. [Rüd15] As U is open, $U \cap P$ is open too in P and closed in U. Further, q and q^{-1} are well-defined on $U \cap P$. Moreover, $q^{-1}(U \cap P) = q(U \cap P)$ is a closed curve in U, since q is an involution. For $P^* = q(U \cap P)$ - by construction - $P \cap U$ and $P^* \cap U$ are isomorphic, so P and P^* are birational.

Proposition 3.2.5. We have $(p')^q = p$, $m_{W_1}(P) = d - r_2 - r_3$, $m_{W_2} = d - r_1 - r_3$, $m_{W_3} = d - r_1 - r_2$. Also, p' is irreducible and $P^* = P'$.

Proof. [Rüd15] We first prove the last assertion. Since $(p^q)^q = (xyz)^d p$, we compute:

$$\begin{aligned}(p^q)^q &= (x^{r_1}y^{r_2}z^{r_3}p')^q \\ &= (yz)^{r_3}(xz)^{r_2}(xy)^{r_1}(p')^q \\ &= (xyz)^d p = x^{r_2+r_3}y^{r_1+r_3}z^{r_1+r_2}(p')^q,\end{aligned}$$

thus

$$(p')^q = x^{d-r_2-r_3}y^{d-r_1-r_3}z^{d-r_1-r_2}p.$$

Since x, y, z do not divide p, and p is irreducible and not divisible by any coordinate function (being distinct from the tangent lines), we deduce that

$$(p')^q = p, \; m_{W_1}(P) = d - r_2 - r_3, \; m_{W_2}(P) = d - r_1 - r_3, \; m_{W_3}(P) = d - r_1 - r_2.$$

Also, p' must be irreducible — otherwise, by the equation above, p wouldn't be irreducible. Moreover, $q^{-1}(U \cap P) \subset V(p')$, so by irreducibility, we must have $P^* = q(U \cap P) = V(p')$. □

Definition 3.2.7. (Good position). We say that a curve P is in good position if none of the exceptional lines is tangent to P at any of the fundamental points.

Definition 3.2.8. (Excellent position). We say that P is in excellent position if it is in good position and:

- $V(x)$ intersects P transversally at d distinct non-fundamental points;
- both $V(y)$ and $V(z)$ intersect P transversally at $d - m_{W_1}(P)$ distinct non-fundamental points.

Proposition 3.2.6. Let P be irreducible and suppose $X \in \mathbb{CP}^2$ with $m_X(P) = r \geq 0$. Then there are infinitely many lines through X such that each intersects P in $d - r_1$ distinct points other than X.

Proof. [Rüd15] Suppose $X = W_2$, and for all $\lambda \in \mathbb{C}$, define $l_\lambda = \{[\lambda : t : 1] : t \in \mathbb{C}\} \cup \{X\} = V(x - \lambda z)$. For

$$p = a_{r_1}(x, z)y^{d-r_1} + \cdots + a_d(x, z), \quad a_{r_1} \neq 0 \tag{♥}$$

define $g_\lambda(t) = p(\lambda, t, 1)$. Then

$$g_\lambda(t) = a_{r_1}(\lambda, 1)t^{d-r_1} + \cdots + a_d(\lambda, 1),$$

so g_λ is a degree $d - r_1$ polynomial. If $a_{r_1}(\lambda, 1) \neq 0$, then g_λ has $d - r_1$ roots. Now $a_{r_1}(\lambda, 1)$ is a polynomial in λ, and it only vanishes for finitely many λ, so for all but finitely many λ, we have $a_{r_1}(\lambda, 1) \neq 0$, and hence g_λ has $d - r_1$ distinct roots. Since p is irreducible, g_λ must also be irreducible or separable (i.e., it has no multiple roots), so its roots are distinct complex numbers. Thus, for all but finitely many λ, the line l_λ intersects P transversally at $d - r_1$ distinct points other than X, as claimed. □

Corollary 3.2.7. Let $X_1 \in P$ be a point on P. Then there exists a projective change of coordinates T such that P^T is in excellent position and $T(0,0,1) = X_1$.

Proof. [Rüd15] Use the previous proposition with $X_1 \in P$ to construct two distinct lines l_1, l_2 through X_1, each intersecting P in $d - r_1$ distinct points. Now choose a point $X_2 \in l_1 \setminus P$, such that $X_2 \notin P$. Since $m_{X_2}(P) = 0$, we may apply the proposition again at X_2, yielding infinitely many lines intersecting P in d distinct points. Pick one, call it l_3, and let $X_3 = l_2 \cap l_3$.

Throughout this construction, at each step we had infinitely many lines to choose from, so we can ensure that none of the selected lines is tangent to P at any of the points X_1, X_2, X_3, and all intersections are transverse. Since tangency corresponds to a finite set of excluded directions, avoiding them is always possible.

Now define a projective coordinate change T such that: $T(X_1) = W_1,\ T(X_2) = W_2,\ T(X_3) = W_3$. Then the three lines l_1, l_2, l_3 will be sent to the exceptional lines $V(z), V(y), V(x)$, respectively. By construction, P^T is in excellent position. □

From now on, we shall continue to use the notation $m_{W_1}(P) = r$.

Proposition 3.2.8. If P is in good position, then:

a) P' is also in good position;

b) if $X_1, \dots, X_s$ are the non-fundamental points on $P' \cap V(x)$, then

$$m_{X_i}(P') \leq I[P', V(x)](X_i)$$

and

$$\sum_{i=1}^{s} I[P', V(x)](X_i) = r_1.$$

Proof. [Rüd15] By the definition of tangency and properties of intersection multiplicity (Proposition 1.6.2), $V(x)$ is tangent to P' at W_1 if and only if $I[P', V(x)](W_1) > m_{W_1}(P')$, that is using the notations from Equation ♥

$$I[V(a_{r_1}(y,x)x^{d-r_1-r_3}), V(x)](W_1) > d - r_1 - r_3.$$

This is equivalent to checking whether $a_{r_1}(y,x)$ is zero at $(1,0)$. However, since $V(x)$ is not tangent to P at W_1, we have $a_{r_1}(1,0) \neq 0$, so the inequality does not hold. By symmetry, the same holds at W_2 and W_3, so P' is in good position.

For part **b)**, using Bézout's Theorem (Theorem 2.1.7) and the same reasoning:

$$\sum_{i=1}^{s} I[P', V(x)](X_i) = \sum_{i=1}^{s} I[V(a_{r_1}(y,x)x^{d-r_1-r_3}), V(x)](X_i) = r_1,$$

since the degree of the intersection is d, and $a_{r_1}(y,x)$ has no multiple root in this configuration. □

Example 3.2.2. [Ful08] Consider the curve P defined by

$$p = 8x^3y + 8x^3z + 4x^2yz - 10xy^3 - 10xy^2z - 3y^3z.$$

Since

$$\frac{\partial p}{\partial x} = 24x^2y + 24x^2z + 8xyz - 10y^3 - 10y^2z,$$
$$\frac{\partial p}{\partial y} = 8x^3 + 4x^2z - 30xy^2 - 20xyz - 9y^2z,$$
$$\frac{\partial p}{\partial z} = 8x^3 + 4x^2y - 10xy^2 - 3y^3.$$

the point W_3 is a singular point of the curve (and a multiple point of order 2 to be precise). Also, the derivatives don't vanish in the x direction, so the curve is in good position. Now

$$p(x, y, 1) = 8x^3y + 8x^3 + 4x^2y - 10xy^3 - 10xy^2 - 3y^3,$$

so $r_1 = 2$. Similarly,

$$p(x, 1, z) = 8x^3 + 8x^3z + 4x^2z - 10x - 10xz - 3z,$$

so $r_1 = 1$. Finally,

$$p(1, y, z) = 8y + 8z + 4yz - 10y^3 - 10y^2z - 3y^3z,$$

implying $r_2 = 1$. As a result, $p' = \frac{p}{x^2yz}$. To compute the proper transform we blow up the point W_3. The blow-up map is given by $[x : y : z] \mapsto [x : xy : xz]$. Hence, we divide $p(x, xy, xz)$ by $x^2 \cdot xy \cdot xz = x^4yz$, and obtain

$$p'(x, y, z) = \frac{p(x, xy, xz)}{x^4yz}.$$

Substituting into p gives:

$$p(x, xy, xz) = 8x^4y + 8x^4z + 4x^4yz - 10x^4y^3 - 10x^4y^2z - 3x^3y^3z,$$

so we simplify:

$$p'(x, y, z) = 8y + 8z + 4yz - 10y^3 - 10y^2z - 3\frac{y^3z}{x}.$$

Now we homogenize this expression to obtain:

$$p'(x, y, z) = 8y^2z + 8y^3 + 4xy^2 - 10x^2z - 10x^2y - 3x^3.$$

Observe that the singularity of W_3 transformed into an ordinary multiple point.

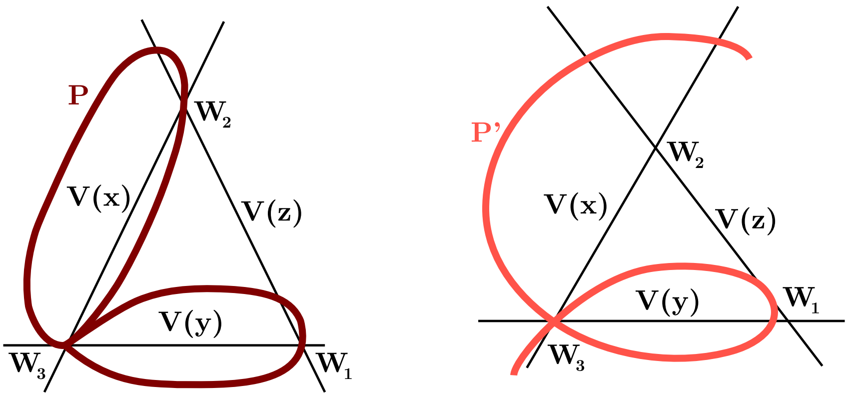

Figure 13
$p = 8x^3y + 8x^3z + 4x^2yz - 10xy^3 - 10xy^2z - 3y^3z$ and
$p' = 8y^2z + 8y^3 + 4xy^2 - 10x^2z - 10x^2y - 3x^3$
Figure adapted from [Ful08], modified by the author.

Proposition 3.2.9. If P is in excellent position, then:

a) Multiple points on $P' \cap U$ correspond to multiple points on $P \cap U$, preserving both multiplicity and whether the points are ordinary multiple points;

b) W_1, W_2, W_3 are ordinary multiple points on P', with multiplicities $d, d - r_1, d - r_1$;

c) There are no non-fundamental points on $P' \cap V(y)$ or $P' \cap V(z)$. If $X_1, \dots, X_s$ are the non-fundamental points on $P' \cap V(x)$, then:

$$m_{X_i}(P') \leq I[P', V(x)](X_i)$$

and

$$\sum_{i=1}^{s} I[P', V(x)](X_i) = r_1.$$

Proof. [Rüd15] Part **a)** follows from the fact that $P' \cap U$ and $P \cap U$ are isomorphic (Remark 3.2.4). Part **b)** follows from Proposition 3.2.8, since P' is in good position, and under excellent position, all non-fundamental intersections are transverse. In particular, since P is in excellent position, if $X_1, \dots, X_d$ are the non-fundamental points of $P \cap V(x)$, then:

$$\sum_{i=1}^{d} I[P', V(x)](X_i) = d,$$

and since the intersection numbers are all 1, we have $m_{W_1}(P') = d$. Likewise, for P'', we find $m_{W_2}(P') = m_{W_3}(P') = d - r_1$ by symmetry and the same argument.

Now, **c)** comes from part b) of Proposition 3.2.8, with W_1 we have the exact same result, and use it also with W_2 and W_3 but now the sum of the intersection numbers equals the multiplicity of W_2 or W_3, which are 0 because we are in excellent position. Hence, there are no such non-fundamental points. $\square$

Definition 3.2.9. (Pseudo arithmetic genus). Define

$$g^*(P) = \frac{(d-1)(d-2)}{2} - \sum_{M \text{ is a multiple point of } P} \frac{m_M(P)(m_M(P)-1)}{2}.$$

We will use the notation $r_M := m_M(P)$ from now on.

Remark 3.2.10. By definition, ordinary multiple points do not contribute to $g^*(P)$.

Proposition 3.2.11. $g^*(P)$ is a non-negative integer.

Proof. [Rüd15] It is clear that $g^*(P) \in \mathbb{Z}$. We prove that it is non-negative. Let $p' = \frac{\partial p}{\partial z}$, and consider the curve $P' = V(p')$. Let $X_1, \dots, X_s$ be the multiple points of P. Then, if $P' \neq 0$, the multiplicity of X_i on P' is at least $r_{X_i} - 1$, since differentiation reduces multiplicity by at most 1. Since P and P' are both irreducible and share no common component, Bézout's theorem (Theorem 2.1.7) gives:

$$\sum_{i=1}^{s} r_{X_i}(r_{X_i} - 1) \leq \sum_{i=1}^{s} I[V(p), V(p')](X_i) = d(d-1).$$

This inequality and the definition of $g^*(P)$ yield the desired result. Now suppose $p' = 0$. We find another irreducible curve $G = V(g)$ of degree $d - 1$ with no common component with P, such that $m_{X_i}(G) \geq r_{X_i} - 1$. To construct g, we need $\binom{d}{2}$ coefficients. Each multiplicity condition

$m_{X_i}(G) \geq r_{X_i} - 1$ imposes at most $\frac{r_{X_i}(r_{X_i}-1)}{2}$ linear constraints. The number of remaining degrees of freedom is:

$$D = \binom{d}{2} - 1 - \sum_{i=1}^{s} \frac{r_{X_i}(r_{X_i}-1)}{2}.$$

Use these to impose that G passes through D further points on P, ensuring that G and P are still distinct and have no common factor. Then again by Bézout's theorem:

$$\sum_{i=1}^{s} I[V(p), V(g)](X_i) = d(d-1) \geq \sum_{i=1}^{s} r_{X_i}(r_{X_i}-1) + D,$$

hence:

$$g^*(P) = \frac{(d-1)(d-2)}{2} - \sum_{i=1}^{s} \frac{r_{X_i}(r_{X_i}-1)}{2} \geq 0.$$

□

Proposition 3.2.12. If P is in excellent position and $X_1, \ldots, X_s$ are the non-fundamental points on $P' \cap V(x)$, then

$$g^*(P') = g^*(P) - \sum_{i=1}^{s} \frac{r_{X_i}(r_{X_i}-1)}{2}.$$

Proof. [Rüd15] This is a direct computation. We know that $\deg(p') = 2d - r_1$, so we can compute $g^*(P')$ using its definition. Also, since $P \cap U$ and $P' \cap U$ are in bijection (from the excellent position assumption), all components and multiplicities matching on these parts cancel out. The only remaining contributions to $g^*(P')$ are:

- On P, the only multiple point on the exceptional lines is X,
- On P', we get three ordinary multiple points — but they do not contribute to g^* by Remark 3.2.10.

Thus, the remaining contribution to the difference between $g^*(P)$ and $g^*(P')$ comes from the other multiple points on $V(x)$, giving the stated formula. □

Definition 3.2.10. (Quadratic transformation). Let T be any projective change of coordinates. Then we say that (P^T) is a quadratic transformation of P. If P^T is in excellent position and $T[0:0:1] = Q$, we say that the quadratic transformation is centred at Q.

Proposition 3.2.13. By a finite sequence of quadratic transformations, any projective plane curve may be transformed into a curve with only ordinary multiple points as singularities.

Proof. [Rüd15] When applying a quadratic transformation centred at a non-ordinary multiple point, we do not create new multiple points outside the exceptional lines. The point Q is sent to an ordinary multiple point, and two more ordinary multiple points appear on the exceptional lines. To see if any new multiple points appear on $V(x)$, recall that:

$$g^*(P) \in \mathbb{N}, \text{ and } g^*(P) = g^*(P') \iff \text{there is no non-ordinary multiple point on } P.$$

Otherwise,

$$g^*(P') = g^*(P) - \sum_{i=1}^{s} \frac{r_{X_i}(r_{X_i}-1)}{2} < g^*(P).$$

Therefore, the sequence

$$g^*(P) \geq g^*(P') \geq g^*(P'') \geq \ldots$$

is a non-increasing sequence of non-negative integers, and it must stabilize after at most $g^*(P)$ steps. So, by repeatedly applying the quadratic transformation centred at non-ordinary multiple points, we ensure that no new non-ordinary multiple points appear, and we eventually obtain a curve with only ordinary multiple points. □

Example 3.2.3. [Ken11] We present an easy example how a quadratic transformation transforms a non-ordinary singularity to an ordinary one. Consider the curve

$$x^4 - y^4 - y^2.$$

The lowest-degree part is y^2, and both directional derivatives are zero, so it has a non-ordinary singularity in the origin. Its homogenisation is $x^4 - y^2z^2 - y^4$. Applying the standard quadratic transformation we get

$$y^4z^4 - x^2z^2x^2y^2 - x^4y^4 = z^2(y^4 - z^2 - x^4y^2 - x^4z^2).$$

After setting $z = 1$ we get

$$y^4 - x^4y^2 - x^4.$$

It also has a singularity in the origin, but this time it is ordinary, since the lowest-degree part is $y^4 - x^4$, which factors as $(y+x)(y-x)(y+ix)(y-ix)$, from which it is clear that there are four tangents with slopes $1, -1, i, -1$.

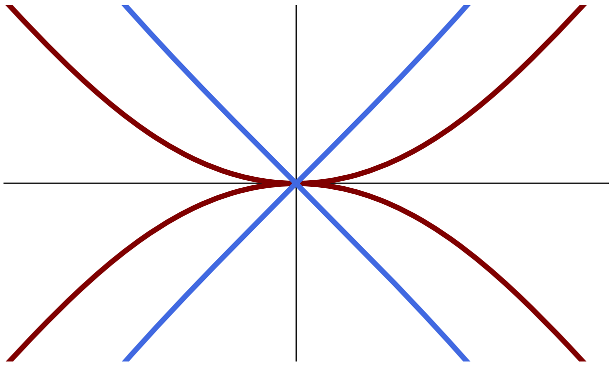

Figure 14
$x^4 - y^2 - y^2$ and $y^4 - x^4y^2 - x^4$ in the real plane near the origin.

Thus, we managed to prove the following:

Corollary 3.2.14. Any projective plane curve is birational to a projective plane curve with only ordinary multiple points.

It is also worth mentioning that:

Proposition 3.2.15. [BK86] Each birational transformation of the plane is a product of quadratic transformations.

Now we define the general blow-up.

Definition 3.2.11. (Blowing up at finitely many points). Let $X_1, \ldots, X_t \in \mathbb{CP}^2$ be distinct points. We define the blowing up of $\mathbb{CP}^2$ at these points as follows. Let $U = \mathbb{CP}^2 \setminus \{X_1, \ldots, X_t\}$, and choose coordinates such that $X_i = [a_{i0} : a_{i1} : 1] \in U \subset \mathbb{CP}^2$ for all $i = 1, \ldots, t$. Define functions $f_i : U \to \mathbb{CP}^1$ by

$$f_i(x_0, x_1, x_2) = (x_0 - a_{i0}x_2,\, x_1 - a_{i1}x_2).$$

Then define a map

$$f : U \to (\mathbb{CP}^1)^t, \quad f(P) = (f_1(P), \ldots, f_t(P)),$$

and let $G \subset \mathbb{CP}^2 \times (\mathbb{CP}^1)^t$ be the graph of f, that is $G = \{(P, f(P)) \mid P \in U\}$. Introduce homogeneous coordinates (y_{i0}, y_{i1}) on the ith copy of $\mathbb{CP}^1$, and define the blow-up $B \subset \mathbb{CP}^2 \times (\mathbb{CP}^1)^t$ by:

$$B = V\left(y_{i1}(x_1 - a_{i1}x_2) - y_{i0}(x_0 - a_{i0}x_2) : i = 1, \ldots, t\right).$$

Definition 3.2.12. (π and the exceptional divisor). Let $\pi : B \to \mathbb{CP}^2$ be the projection onto the first factor. Then π is a birational morphism, and the exceptional divisor over X_i is defined as $E_i := \pi^{-1}(X_i)$.

Proposition 3.2.16. $E_i = \{X_i\} \times \{f_1(X_i)\} \times \cdots \times \{f_{i-1}(X_i)\} \times \mathbb{CP}^1 \times \{f_{i+1}(X_i)\} \times \cdots \times \{f_t(X_i)\}$,

Proof. [Rüd15] Observe that the point $\{P_i, (y_{i0}, y_{i1}), \ldots, (y_{i0}, y + t1)\}$ lies in B whenever y_{i0}, y_{i1} are arbitrary and for each $j \neq i$, $(y_{j0}, y_{j1}) = f_j(X_i)$. These points are well-defined because the points X_i are distinct, and each f_j is defined everywhere on $\mathbb{CP}^2$ except at X_j.

Now let $j \in \{1, \ldots, t\}$ and suppose (y_{j0}, y_{j1}) lies in the j-th copy of $\mathbb{CP}^1$ in B. By definition, we have:

$$y_{j1}(a_1 - a_{j1}) - y_{j0}(a_0 - a_{j0}) = 0.$$

If $i = j$, then this equation imposes no condition, so y_{i0} and y_{i1} can be arbitrary. If $i \neq j$, then (up to scalar multiplication) the equation determines a unique point:

$$y_{j0} = a_1 - a_{j1}, \quad y_{j1} = a_0 - a_{j0},$$

so the corresponding point in $\mathbb{CP}^1$ is $(a_1 - a_{j1}, a_0 - a_{j0}) = f_j(a_0, a_1, 1) = f_j(X_i)$, as desired. □

Corollary 3.2.17. $B \setminus \bigcup_{i=1}^t E_i = B \cap (U \times \mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1) = G$.

Definition 3.2.13. (Blow-up of a projective plane curve). Let $X_1, \ldots, X_t \in \mathbb{CP}^2$ be distinct points, and let $\pi : B \to \mathbb{CP}^2$ be the blow-up of $\mathbb{CP}^2$ at these points. Let $P \subset \mathbb{CP}^2$ be a projective plane curve. Then we define the blow-up of the curve P at $X_1, \ldots, X_t$ as

$$\widetilde{P} := \pi^{-1}(P \cap U) = \pi^{-1}(P \setminus \{X_1, \ldots, X_t\}),$$

where $U = \mathbb{CP}^2 \setminus \{X_1, \ldots, X_t\}$.

Without proof, we state an important proposition.

Proposition 3.2.18. [Rüd15] If $X_1, \ldots, X_t$ are the only singular points of P, and they are all ordinary, then the blow-up curve $\widetilde{P}$ is nonsingular, and $\widetilde{P}$ and P are birational.

Hence, we have the following vital corollary:

Corollary 3.2.19. Let P be any projective plane curve. Then P is birational to a nonsingular projective curve.

We remark that this can be generalized to any field of characteristic 0:

Theorem 3.2.20. (Hironaka's resolution theorem). [Hir64] Every singular algebraic variety over a field of characteristic zero admits a resolution of singularities by a sequence of blow-ups of smooth subvarieties of codimension ≥ 2.

Remark 3.2.21. In positive characteristic, the resolution of singularities remains an open problem in dimensions greater than two. It is, however, known to hold for algebraic curves, and has been partially established for surfaces.

4 Puiseux's expansion

The goal of this section is to prove that any algebraic plane curve can be locally parametrized near a locally irreducible singular point by formal power series in fractional powers of x, called Puiseux series. The method goes back to Newton, who showed that one can consider y as an implicit function of x, and then solve $f(x, y) = 0$ by expanding y as a series in powers of $x^{1/n}$.

In certain cases, when $\frac{\partial f}{\partial y}(0, 0) \neq 0$, the classical implicit function theorem (Theorem 1.8) applies and yields a convergent power series solution $y = \varphi(x)$. In other cases, we want a solution using Puiseux series.

Throughout this section, we will work with formal power series, so let

$$f = \sum_{(\mu,\nu)\in\mathbb{N}^2} a_{\mu\nu} x^{\mu} y^{\nu} \in \mathbb{C}[[x, y]] \qquad (\spadesuit)$$

be a given power series satisfying $f(0, 0) = 0$. After a suitable change of coordinates, we may assume that the equation is general in y of order $k > 1$.

4.1 Puiseux's formal theorem

In this subsection, we will mostly rely on the notations, definitions and propositions of subsection 1.10. We will work with formal power series defined as in Equation $\spadesuit$.

Definition 4.1.1. (Puiseux series). Let n be a positive integer. A Puiseux series in x is a formal power series of the form

$$\varphi(x) = \sum_{r\geq r_0} a_r x^{r/n}, \quad \text{with } a_r \in \mathbb{C},\ r_0 \in \mathbb{N},$$

that is, a power series in fractional powers of x. The set of all such series for a fixed n is denoted

$$\mathbb{C}[[x^{1/n}]] := \left\{ \sum_{r\geq r_0} a_r x^{r/n} \right\}.$$

Definition 4.1.2. (Ring of Puiseux series). The union over all n is called the ring of Puiseux series:

$$\mathbb{C}\{\{x\}\} := \bigcup_{n=1}^{\infty} \mathbb{C}[[x^{1/n}]].$$

Remark 4.1.1. [Fis01] $\mathbb{C}\{\{x\}\}$ is an integral domain

Definition 4.1.3. (Order). For $\varphi(x) = \sum_{r\geq r_0} a_r x^{r/n}$, with $a_r \in \mathbb{C}$, $r_0 \in \mathbb{Z}$ we define its order by the rational number $\operatorname{ord}\varphi := \min\{m/n \mid a_m \neq 0\} \geq 0$.

Without proof we list a few important results connected to $\mathbb{C}\{\{x\}\}$.

Proposition 4.1.2. [Fis01] Every normalised polynomial polynomial in $\mathbb{C}\langle x\rangle[y]$ splits into linear factors in the ring $\mathbb{C}\{\{x\}\}$.

Proposition 4.1.3. [Fis01] The quotient field of $\mathbb{C}\{\{x\}\}$ is algebraically closed.

Proposition 4.1.4. [Fis01] For a Weierstrass polynomial $f \in \mathbb{C}\langle x\rangle[y]$ of degree d, there is a factorisation $(y-\varphi_1)\dots(y-\varphi_d)$ of f such that $\varphi_i \in \mathbb{C}\{\{x\}\}$ has positive order. If f is irreducible, then all φ_i have the same order.

Definition 4.1.4. (Carrier). The carrier of f (defined as in Equation ♠) is the set of exponent pairs corresponding to non-zero coefficients:

$$\operatorname{carr}(f) := \{(\mu, \nu) \in \mathbb{N}^2 \mid a_{\mu\nu} \neq 0\}.$$

Definition 4.1.5. (Quasihomogeneous polynomial). Let $f(x, y) \in \mathbb{C}[x, y]$ and let $(p, q) \in \mathbb{N}^2_{>0}$. The polynomial f is called quasihomogeneous of type (p, q) and weighted degree d if all monomials $x^a y^b$ appearing in f satisfy

$$pa + qb = d.$$

Proposition 4.1.5. Let $f(x, y)$ be a quasihomogeneous polynomial of type (p, q) and weighted degree d. Then:

(a) $f(t^q x, t^p y) = t^d f(x, y)$.

(b) The set of exponent pairs (a, b) with $pa + qb = d$ lies on a straight line of slope $-\frac{p}{q}$.

Proof. For part a) simply substitute:

$$f(t^p x, t^q y) = \sum c_{a,b}(t^p x)^a (t^q y)^b = \sum c_{a,b} t^{pa+qb} x^a y^b.$$

Since each $pa + qb = d$, this becomes:

$$\sum c_{a,b} t^d x^a y^b = t^d \sum c_{a,b} x^a y^b = t^d f(x, y),$$

which proves the first part.

For part b), the equation $pa + qb = d$ is a linear equation in the integer variables a and b. The set of exponent pairs (a, b) that satisfy this equation lie on the line

$$b = -\frac{p}{q} a + \frac{d}{q},$$

which has slope $-\frac{p}{q}$. Therefore, the set of exponent pairs corresponding to monomials in f lie on a straight line of slope $-\frac{p}{q}$, as claimed. □

Proposition 4.1.6. Let $f \in \mathbb{C}[x, y]$ be a quasihomogeneous polynomial with weights p and q, and suppose that f is general in y of order $k \geq 1$. Then there exists at least one $\lambda \in \mathbb{C}$ such that

$$f(t^q, \lambda t^p) = 0.$$

Moreover, if the carrier of f contains at least two points, then λ can be chosen such that $\lambda \neq 0$.

Proof. [Fis01] We begin by introducing the substitution $x = t^q, \quad y = \lambda t^p$, so that computations are understood formally with fractional exponents. Substituting into f, we get

$$f(x, y) = f(t^q, \lambda t^p) = \sum_{p\mu+q\nu=kp} a_{\mu\nu}(t^q)^\mu (\lambda t^p)^\nu = \sum_{p\mu+q\nu=kp} a_{\mu\nu} \lambda^\nu t^{q\mu+p\nu}.$$

Since all terms satisfy $p\mu + q\nu = kp$, the exponent of t in each term is kp, so we can factor:

$$f(t^q, \lambda t^p) = t^{kp} \sum_{p\mu+q\nu=kp} a_{\mu\nu} \lambda^\nu =: t^{kp} g(\lambda),$$

where $g(\lambda) \in \mathbb{C}[\lambda]$ is a polynomial of degree $k \geq 1$, so it has at least one complex root $\lambda \in \mathbb{C}$. Moreover, if the carrier of f contains a monomial $x^\mu y^\nu$ with $\nu < k$, then $g(\lambda)$ has a non-zero root. □

Example 4.1.1. Consider the curve $x^2 - y^3$ (see Figure 2), which is general in order 3. Its equation is clearly a quasihomogeneous polynomial with weights $q = 3$ and $p = 2$. Now, we have

$$x = t^q = t^3, \quad y = \lambda t^p = \lambda t^2,$$

and obtain

$$f(t^3, \lambda t^2) = t^6(1 - \lambda^3).$$

Hence, $f(t^3, \lambda t^2) = 0$ if and only if $\lambda^3 = 1$. Therefore, the admissible values of λ are the three cube roots of unity:

$$\lambda \in \left\{1, e^{2\pi i/3}, e^{4\pi i/3}\right\}.$$

Proposition 4.1.7. Let $f = \sum_{\mu,\nu} a_{\mu\nu} x^\mu y^\nu \in \mathbb{C}[[x, y]]$ be general in y of order $k \geq 1$, and assume that $y^k \nmid f$. Then there exist relatively prime positive integers $p, q \in \mathbb{N}$ with $q \neq 0$, such that:

a) for all $(\mu, \nu) \in \operatorname{carr}(f)$, we have $q\mu + p\nu \geq pk$;

b) there exists at least one $(\mu, \nu) \in \operatorname{carr}(f)$ such that $\mu \geq 1$, $\nu < k$, and $q\mu + p\nu = pk$.

Proof. (Based on [Fis01]) Let us visualise the carrier of f, which is general in y, and fix a coordinate system with μ-axis horizontal and ν-axis vertical. Since $y^k \nmid f$, there exists at least one point $(\mu, \nu) \in \operatorname{carr}(f)$ with $\nu < k$. Fix such a point with minimal slope through $(0, k)$ to some point of the carrier. That is, consider the set of lines through $(0, k)$ that intersect $\operatorname{carr}(f)$, and choose the one that hits a point (μ, ν) with the smallest positive slope. Since $\nu < k$ and $\mu > 0$, this slope is well-defined and rational. Now write the line as $q\mu + p\nu = pk$ with relatively prime $p, q \in \mathbb{N}$, and observe that all other points of the carrier lie on or above this line. This gives us the desired inequality for all points in the carrier, and equality for at least one such point with $\mu \geq 1$ and $\nu < k$. □

Definition 4.1.6. (Quasihomogeneous initial polynomial). Continuing from the proposition above, we define the quasihomogeneous initial polynomial of f as

$$\tilde{f} := \sum_{q\mu + p\nu = pk} a_{\mu\nu} x^\mu y^\nu \in \mathbb{C}[x, y].$$

This polynomial consists of all terms of f lying on the line of slope $-p/q$ through the point $(0, k)$, and it is itself general in y of order k.

Remark 4.1.8. we obtain the decomposition

$$f = \tilde{f} + h, \qquad \text{where} \quad h = \sum_{q\mu + p\nu \geq pk+1} a_{\mu\nu} x^\mu y^\nu \in \mathbb{C}[x, y].$$

The series h contains the higher-order terms with respect to the weighted order determined by the line $q\mu + p\nu = pk$.

Now we define an iteration process which will help us to construct a solution for the Puiseux problem.

Definition 4.1.7. (Iteration step). Let $f = \tilde{f} + h \in \mathbb{C}[[x, y]]$ be general in y of order $k \geq 1$, and assume that $y^k \nmid f$. Let $p, q \in \mathbb{N}$ be relatively prime integers as in Proposition 4.1.7, and suppose that $\lambda \in \mathbb{C}$ satisfies

$$\tilde{f}(x, \lambda x^{p/q}) = 0.$$

We then set $x = x_1^q, \quad y = \lambda x_1^p + x_1^p y_1$, and substitute into f. This gives:

$$f(x, y) = f(x_1^q, \lambda x_1^p + x_1^p y_1) = x_1^{pk} f_1(x_1, y_1),$$

where $f_1 \in \mathbb{C}[x_1, y_1]$ is a new series. In particular,

$$f_1(0, y_1) = g(\lambda + y_1),$$

where $g(\lambda) = \sum a_{\mu\nu}\lambda^\nu$ as in the previous section.

Proposition 4.1.9. Let $f_1 \in \mathbb{C}[x_1, y_1]$ be obtained from f via the above substitution. Then:

a) f_1 is general in y_1 of some order k_1, with $1 \leq k_1 \leq k$;

b) If $k_1 = k$, then $q = 1$.

Proof. [Fis01] For part **a)** let us define $\gamma(y_1) := g(\lambda + y_1) \in \mathbb{C}[y_1]$, where $\lambda \in \mathbb{C}^*$ satisfies $g(\lambda) = 0$. Then $f_1(0, y_1) = \gamma(y_1)$, and the order of f_1 in y_1 is $k_1 := \operatorname{ord}_{y_1}(\gamma)$, which satisfies $1 \leq k_1 \leq \deg g = k$, since λ is a root of g.

For **b)**, if $k_1 = k$, then $\gamma(y_1) = c(\lambda + y_1)^k$ for some $c \in \mathbb{C}^*$, so

$$g(\lambda) = \sum_{q\mu + p\nu = pk} a_{\mu\nu}\lambda^\nu = a_{0k}\lambda^k + a_{\mu,k-1}\lambda^{k-1} + \cdots .$$

Now, using the binomial expansion, the coefficient of y_1^{k-1} in $\gamma(y_1)$ is $-ck\lambda$, and hence for this to be non-zero in the original f, we must have $a_{\mu,k-1} \neq 0$ for some $\mu > 0$ with

$$q\mu + p(k-1) = pk \quad \Rightarrow \quad q\mu = p.$$

Since $\mu \in \mathbb{N}$, this implies $q \mid p$, and as $\gcd(p, q) = 1$, it follows that $q = 1$. $\square$

Theorem 4.1.10. (Formal Puiseux parametrization). There exists a natural number n and a formal power series

$$\varphi(x) = \sum_{r \geq r_0} a_r x^{r/n} \in \mathbb{C}[x^{1/n}]$$

such that

$$f(x, \varphi(x)) = 0 \quad \text{in } \mathbb{C}[x^{1/n}].$$

Proof. [Fis01] We define the iteration recursively. Start with

$$f_0 = f, \quad x_0 = x, \quad y_0 = y, \quad k_0 = k.$$

At the i-th step, if $y_i^{k_i} \mid f_i$, then $y_i = 0$ is a solution of $f_i(x_i, y_i) = 0$, and we stop. Otherwise, by the previous results, we can write

$$f_i = \tilde{f}_i + h_i$$

with $\tilde{f}_i$ quasihomogeneous and general in y_i of order k_i, and the equation of its carrier is $q_i\mu + p_i\nu = k_i p_i$ for relatively prime $p_i, q_i \in \mathbb{N}$. Then there exists $\lambda_i \in \mathbb{C}^*$ such that

$$\tilde{f}_i(x_i, \lambda_i x_i^{p_i/q_i}) = 0.$$

We define the next variables by:

$$x_{i+1} = x_i^{1/q_i}, \qquad y_i = \lambda_i x_i^{p_i/q_i} + x_i^{p_i/q_i} y_{i+1} = x_{i+1}^{p_i}(\lambda_i + y_{i+1}).$$

Substituting into f_i, we get:

$$f_i(x_i, y_i) = x_{i+1}^{k_i p_i} f_{i+1}(x_{i+1}, y_{i+1}),$$

where f_{i+1} is again general in y_{i+1} of order $k_{i+1} \leq k_i$ by Proposition 4.1.9.

By iterating this process, we obtain sequences $\{p_i\}, \{q_i\}, \{k_i\} \subset \mathbb{N}$ with $k_0 \geq k_1 \geq k_2 \geq \cdots \geq 1$, and by Proposition 4.1.9 we have $q_j = 1$ for all $j \geq N$ for some finite $N \in \mathbb{N}$. Now set $x = t^n, \quad x_i = t^{n_i}$ for $0 \leq i \leq N$, where $n_i = p_0 q_1 \cdots q_i$. Then all fractional exponents can be avoided, and the iteration yields:

$$y = \sum_{i=0}^{\infty} \lambda_i t^{m_i}, \qquad m_0 = p_0 q_1 \cdots q_N, \quad m_{i+1} = m_i + p_{i+1} \prod_{j=i+2}^{N} q_j.$$

Since $p_i \geq 1$, the exponents $m_i \to \infty$, and thus the series

$$\varphi(t) = \sum_{i=0}^{\infty} \lambda_i t^{m_i}$$

is a well-defined formal Puiseux series in $\mathbb{C}\{\{t\}\}$.

Finally, substituting back, we get:

$$f(x, \varphi(x)) = f(t^n, \varphi(t)) = x_0^r f_{N+1}(x_{N+1}, y_{N+1}) = 0,$$

since f_{N+1} is divisible by $y_{N+1}^{k_{N+1}}$ and $y_{N+1} = 0$. Therefore, $f(x, \varphi(x)) = 0$. $\square$

Example 4.1.2. [Fis01]Let

$$f(x, y) = y^4 - 2y^2 x - 4y^2 x^2 - 3y^2 x^3 + x^2 + 4x^3 + 7x^4 + 6x^5 + 2x^6.$$

We apply the Puiseux iteration procedure. The leading quasihomogeneous part of f is

$$\widetilde{f_0} = y^4 - 2xy^2 + x^2,$$

which is quasihomogeneous of weighted degree $k_0 = 4$ with weights 2 and 1. Solving $\widetilde{f_0}(x, \lambda x^{1/2}) = 0$ gives $\lambda_0 = -1$. We make the substitution:

$$x = x_1^2, \quad y = x_1(\eta_1 + \lambda_0) = x_1(\eta_1 - 1),$$

and define the transformed function $f_1(x_1, \eta_1) := f(x_1^2, x_1(\eta_1 - 1))$. A direct computation gives:

$$\begin{aligned} f_1(x_1, \eta_1) = \eta_1^4 x_1^4 - 4\eta_1^3 x_1^4 + 4\eta_1^2 x_1^4 - 4x_1^2 \eta_1^2 + 8x_1^2 \eta_1 - 3x_1^4 \eta_1^2 \\ + 6x_1^4 \eta_1 + 4x_1^4 + 6x_1^6 + 2x_1^8. \end{aligned}$$

The new leading part is

$$\widetilde{f_1} = 4\eta_1^2 x_1^2 + 8x_1^2 \eta_1 + 4x_1^4,$$

which again has a solution $\lambda_1 = -1$ for $\widetilde{f_1}(x_1, \lambda_1) = 0$. We now substitute:

$$x_1 = x_2, \quad \eta_1 = \eta_2 + \lambda_1 = \eta_2 - 1,$$

and obtain the parametrization:

$$x = x_2^2, \quad y = x_2(-1 + (\eta_2 - 1)) = x_2(-1 - \eta_2).$$

Setting $\eta_2 = -x_2$ gives a full parametrization:

$$x = t^2, \quad y = -t - t^3.$$

This means the Puiseux expansion terminates after the second iteration:

$$f(t^2, -t - t^3) = 0.$$

Without proof, we state an important fact:

Theorem 4.1.11. (Convergence of Puiseux expansions). [Fis01] In Theorem 4.1.10 if f is convergent, then so is φ.

Definition 4.1.8. (Branch at a singular point). Let $F \subset \mathbb{C}^2$ be a plane algebraic curve defined by a reduced polynomial $f(x,y) \in \mathbb{C}[x,y]$, and let $S \in F$ be a singular point. A branch of F at S is an equivalence class of irreducible analytic components of the germ of F at S.

Equivalently, each branch corresponds to a distinct Puiseux expansion $y = \varphi(x)$ centred at S (after a suitable change of coordinates so that $S = (0,0)$ and $f(0,0) = 0$), where $\varphi(x) \in \mathbb{C}[[x^{1/n}]]$ is a convergent Puiseux series satisfying $f(x, \varphi(x)) = 0$.

We state a few interesting properties of branches related to what we have seen so far:

Remark 4.1.12. The number of branches at S is the number of distinct points in the inverse image $\pi^{-1}(S)$ under the normalisation map $\pi : \widetilde{C} \to C$ defined as in subsection 3.1.

Remark 4.1.13. Blowing up the plane at a singular point S separates the branches of a curve F passing through S with different tangent lines. That is, the blow-up curve of F intersects the exceptional divisor in as many distinct points as there are different tangent lines at S.

Remark 4.1.14. [Kir92] The branches at the origin are described by the Puiseux expansions

$$y = g_j\left(e^{2\pi i s/m_j} x^{1/m_j}\right)$$

for $1 \leq j \leq l$ and $1 \leq s \leq m_j$, where $g_j(t) \in \mathbb{C}\{t\}$ are holomorphic functions satisfying $g_j(0) = 0$. Each essentially different function g_j corresponds to a distinct branch.

Example 4.1.3. Consider the alpha curve (Figure 5) defined by $y^2 = x^2(x+1)$. To find the branches of the curve at the origin, we solve for y using Puiseux expansions:

$$y = \pm x\sqrt{1+x}.$$

Now expand the square root into a convergent power series:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots \quad \Rightarrow \quad y = \pm x\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots\right).$$

Thus, the curve admits two distinct Puiseux expansions at the origin:

$$y = x + \frac{1}{2}x^2 - \frac{1}{8}x^3 + \cdots, \quad y = -x - \frac{1}{2}x^2 + \frac{1}{8}x^3 - \cdots.$$

These are not conjugate under root-of-unity transformations, and they define two distinct branches of the curve at the origin.

4.2 Newton polygon

In this subsection, we will mostly rely on the notations, definitions and propositions of subsection 1.10.

Definition 4.2.1. (Newton polygon). The Newton polygon of f is the convex hull of the union of the first quadrant rays translated to start at the points in $\mathrm{carr}(f)$. That is, define

$$\Delta(f) := \mathrm{Conv}\left(\bigcup_{(\mu,\nu)\in \mathrm{carr}(f)} \{(\mu+u, \nu+v) \mid u, v \in \mathbb{R}_{\geq 0}\} \right),$$

and the Newton polygon is the lower boundary of $\Delta(f)$, consisting of finitely many line segments (and possibly half-lines).

Example 4.2.1. [Kir92] Consider $f(x,y) = y^4 - 2x^5y^2 - 4x^8y + x^{10} - x^{11}$. In this case, the Newton polygon consists of a single line segment (not including the axes).

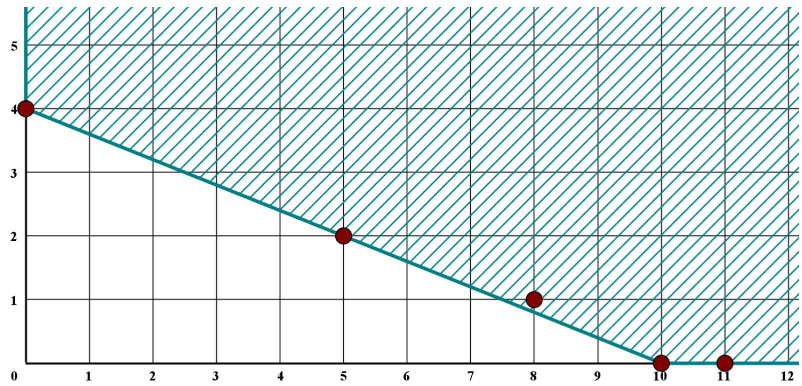

Figure 15
Newton polygon of $y^4 - 2x^5y^2 - 4x^8y + x^{10} - x^{11}$ illustrated with [Int19].

In the previous subsection we have seen how to construct the Puiseux expansion. Now, we shall see another method involving the Newton polygon.

Definition 4.2.2. (Newton polygon–based Puiseux iteration). Let $f(x,y) \in \mathbb{C}[[x,y]]$ be a convergent power series with $f(0,0) = 0$.

1) Construct the Newton polygon of f.

2) Choose the lowest edge (initial segment), that is select the edge of the Newton polygon with the smallest slope. Let p/q be its slope (with p, q being coprime positive integers).

3) Solve the initial polynomial $f_0(t)$: Write the initial form (or truncated polynomial) corresponding to this edge:

$$\widetilde{f}_0(x,y) = \sum_{(a,b) \text{ on edge}} c_{ab} x^a y^b.$$

Substitute $x = x$, $y = \lambda x^{p/q}$ into $\widetilde{f}_0(x,y)$, and solve for λ:

$$\widetilde{f}_0(x, \lambda x^{p/q}) = x^w f_0(\lambda),$$

where $f_0(\lambda)$ is a polynomial in λ. Choose any root λ_0 of f_0.

4) Define new variables:

$$x = x_1^q, \quad y = x_1^p(\lambda_0 + \eta_1).$$

Substitute into f:

$$f(x_1^q, x_1^p(\lambda_0 + \eta_1)) =: f_1(x_1, \eta_1),$$

and factor out the maximal power of x_1:

$$f_1(x_1, \eta_1) = x_1^{w_1} \cdot \bar{f}_1(x_1, \eta_1).$$

5) Repeat the process for f_1: Construct the Newton polygon of $\bar{f}_1$, choose the steepest edge, compute its slope p_1/q_1, solve the corresponding initial polynomial for λ_1, and define

$$\eta_1 = \lambda_1 + \eta_2, \quad x_1 = x_2^{q_1}.$$

Then proceed recursively.

We obtain a sequence of positive rationals $\mu_0 = \frac{p_0}{q_0}$, $\mu_1 = \frac{p_1}{q_1}$, $\mu_2 = \frac{p_2}{q_2}, \ldots$ and complex numbers $t_0, t_1, t_2, \ldots$ and successive approximate solutions $(x, y) = (x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots$ to the equation $p(x, y) = 0$ related by

$$x = x_1^{1/q_0}, \quad x_1 = x_2^{1/q_1}, \quad x_2 = x_3^{1/q_2}, \quad \ldots$$

and

$$y = x^{\mu_0}(t_0 + y_1), \quad y_1 = x_1^{\mu_1}(t_1 + y_2), \quad y_2 = x_2^{\mu_2}(t_2 + y_3), \quad \ldots$$

We wish to show that the Puiseux expansion

$$\begin{aligned} y &= t_0 x^{\mu_0} + t_1 x_1^{\mu_1} x^{\mu_0} + t_2 x_2^{\mu_2} x_1^{\mu_1} x^{\mu_0} + \cdots \\ &= t_0 x^{\mu_0} + t_1 x^{\mu_0 + \mu_1/q_0} + t_2 x^{\mu_0 + \mu_1/q_0 + \mu_2/(q_0 q_1)} + \cdots \end{aligned}$$

is a genuine solution near the origin.

Proposition 4.2.1. Let $f(x, y) \in \mathbb{C}[[x, y]]$ and let $\bar{f}_1(x_1, \eta_1)$ be the polynomial obtained in step 4) of the Newton polygon–based Puiseux iteration defined above. Denote by

$$\beta_0 := \min\{b \mid \exists a \text{ with } c_{ab} \neq 0 \text{ and } a + \mu b = \nu_0\},$$

the smallest exponent of y occurring in the initial form, and let

$$\beta_1 := \min\{\beta \mid \exists \alpha \text{ with } \bar{f}_1(x_1, \eta_1) \text{ contains } \eta_1^\beta \text{ with non-zero coefficient}\}.$$

Then either $\beta_1 < \beta_0$, or the Newton–Puiseux process terminates trivially with $q = 1$.

Proof. [Kir92] Recall that in step 4), we write $f(x_1^q, x_1^p(\lambda_0 + \eta_1)) = x_1^{\nu_0} \cdot \bar{f}_1(x_1, \eta_1)$, and the expansion of $\bar{f}_1$ in η_1 has constant term zero (otherwise $f(0, 0) \neq 0$). Now suppose that $\beta_1 = \beta_0$. Then

$$\bar{f}_1(x_1, \eta_1) = c_{\alpha, \beta_0} x_1^\alpha \eta_1^{\beta_0} + \cdots,$$

and we must have

$$\alpha + \mu(\beta_0 - 1) = \nu_0 = \mu\beta_0,$$

which implies

$$\mu = \alpha.$$

Since $\mu = p/q$, it follows that $\mu \in \mathbb{Z}$, so $q = 1$.

Thus, unless $q = 1$, the exponent β_1 must be strictly smaller than β_0, ensuring that each iteration introduces a new, smaller exponent and the expansion progresses. □

Corollary 4.2.2. The Puiseux expansion defined as in Definition 4.2.2 can be expressed as a formal power series in $x^{1/n}$ for some $n \in \mathbb{N}$.

Proof. We have $q_i = 1$ unless $\beta_{i-1} > \beta_i$, where $\beta_0 \geq \beta_1 \geq \beta_2 \geq \cdots$ is a decreasing sequence of positive integers. So there are at most finitely many i with $q_i > 1$. If n denotes the product of the q_i values, then the Puiseux expansion may be expressed as a formal power series

$$y = \sum_{r \geq 1} a_r x^{r/n}$$

in $x^{1/n}$. □

Proposition 4.2.3. Let $f(x, y) \in \mathbb{C}[[x, y]]$ be a convergent power series with $f(0, 0) = 0$. Then the Puiseux expansion defined as in Definition 4.2.2 is a power series in $x^{1/n}$ which converges for x sufficiently close to 0, and satisfies

$$f\left(x, \sum_{r \geq r_0} a_r x^{r/n}\right) = 0.$$

Proof. [Kir92] From Corollary 3.1.4 it follows there are holomorphic functions $g_1, \ldots, g_l$ defined near 0 and positive integers $m_1, \ldots, m_l$ such that $m_1 + \cdots + m_l = d$ and a non-zero constant K such that

$$f(x, y) = K \prod_{1 \leq j \leq l} \prod_{1 \leq s \leq m_j} \left(y - g_j(e^{2\pi i s/m_j} x^{1/m_j})\right)$$

for all y and all sufficiently small x. We can expand each $g_j(t)$ as a convergent power series

$$g_j(t) = \sum_{r \geq 0} a_r^{(j)} t^r$$

near 0. Let N be the least common multiple of $m_1, \ldots, m_l$ and n. Then the series

$$g_j(e^{2\pi i s/m_j} x^{1/m_j}) = \sum_{r \geq r_0} a_r^{(j)} e^{2\pi i r s/m_j} x^{r/m_j}$$

and the Puiseux expansion $\sum_{r \geq r_0} a_r x^{r/n}$ can all be regarded as elements of the ring $\mathbb{C}\{x^{1/N}\}$ of formal power series in $x^{1/N}$. If $Q(y)$ is a polynomial in y with coefficients in $\mathbb{C}\{x^{1/N}\}$ which satisfies $Q(c) = 0$ for some $c \in \mathbb{C}\{x^{1/N}\}$ and can be expressed in the form $Q(y) = K(y - c_1) \cdots (y - c_d)$ for some $K \in \mathbb{C} \backslash \{0\}$ and $c_1, \ldots, c_d \in \mathbb{C}\{x^{1/N}\}$, then $c = c_j$ for some $j \in \{1, \ldots, d\}$. Therefore it suffices to show that, as a formal power series in $x^{1/N}$, the Puiseux expansion satisfies

$$f\left(x, \sum_{r \geq r_0} a_r x^{r/n}\right) = 0.$$

For then the Puiseux expansion must coincide with one of the series

$$\sum_{r \geq r_0} a_r^{(j)} e^{2\pi i r s/m_j} x^{r/m},$$

and hence must converge for sufficiently small x.

The construction of the Puiseux expansion (Definition 4.2.2) shows that the exponent of the smallest power of $x^{1/N}$ occurring in the polynomial

$$f\left(x, \sum_{r=r_0}^{M} a_r x^{r/n}\right)$$

is at least $p_0\beta_0 + p_1\beta_1 + \cdots + p_M\beta_M$ (as seen in Definition 4.2.2, Proposition 4.2.1 and Corollary 4.2.2), which tends to infinity as $M \to \infty$, since each p_j and β_j is a positive integer. This tells us that every coefficient in the formal power series

$$f\left(x, \sum_{r\geq r_0} a_r x^{r/n}\right)$$

is zero, i.e. the whole expression is zero in $\mathbb{C}\{x^{1/N}\}$, as required. □

Example 4.2.2. [Kir92] Let's continue with Example 4.2.1. We take $\mu_0 = \frac{5}{2}$ and $\nu_0 = 10$. Then the initial polynomial is

$$f_0(t) = \sum_{\alpha+\mu_0\beta=\nu_0} c_{\alpha\beta} t^\beta = 1 - 2t^2 + t^4 = (1-t^2)^2,$$

which has roots $t_0 = \pm 1$. Thus our first approximate solutions are $y = \pm x^{5/2}$. We now substitute $x = x_1^2$ and $y = x_1^5(\pm 1 + y_1)$ into $f(x, y)$, and factor out the term x_1^{20}, to get:

$$\begin{aligned} f_1(x_1, y_1) &= (\pm 1 + y_1)^4 - 2(\pm 1 + y_1)^2 - 4x_1(\pm 1 + y_1) + 1 - x_1^2 \\ &= y_1^4 \pm 4y_1^3 + 4y_1^2 - 4x_1y_1 - x_1^2 \pm 4x_1. \end{aligned}$$

Again, the Newton polygon of f_1 is a single line segment. We take $\mu_1 = \frac{1}{2}$ and $\nu_1 = 1$, to get our next approximate solution: $y_1 = t_1 x_1^{1/2}$, where t_1 is a root of $4t^2 \pm 4 = 0$, i.e. $t_1 = \pm 1$ when $t_0 = 1$, and $t_1 = \pm i$ when $t_0 = -1$. In each case it is easy to check that the approximate solution $y_1 = t_1 x_1^{1/2}$ is in fact a genuine solution to the equation $f_1(x_1, y_1) = 0$. Thus the procedure stops at this point and the Puiseux expansions

$$y = x^{5/2} \pm x^{11/4} \qquad \text{and} \qquad y = -x^{5/2} \pm i x^{11/4}$$

are solutions to the equation $f(x, y) = 0$.

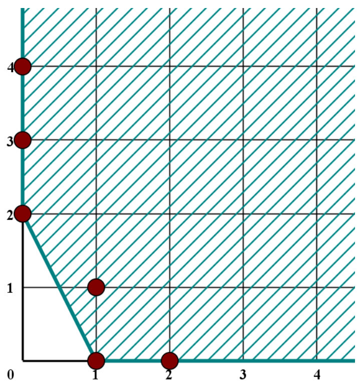

Figure 16
Newton polygon of $y^4 \pm 4y^3 + 4y^2 - 4xy - x^2 \mp 4x$ illustrated with [Int19].

5 Noether's formula

In this section, we introduce and prove the general genus formula for algebraic curves. Along the way, we will also explore important invariants of singularities as well.

5.1 Ramifications

In this subsection, we will mostly rely on the notations, definitions and propositions of subsection 1.3 and subsection 2.1.

Given a curve $P \in \mathbb{CP}^2$ defined by the homogeneous polynomial $p(x, y, z)$ of degree $d > 1$. By Proposition 1.1.7, we may assume that $[0 : 1 : 0] \notin P$.

Definition 5.1.1. (The φ map). Define $\varphi : P \to \mathbb{CP}^1$ by $[x : y : z] \mapsto [x : z]$.

Definition 5.1.2. (Ramification index). The ramification index at a point $[a : b : c] \in P$ is the order of the zero of the polynomial $p(a, y, c)$ in y at $y = b$. It is denoted by $\nu[a : b : c]$. A point is called a ramification point if $\nu[a : b : c] > 1$.

Remark 5.1.1. $\nu[a : b : c] > 0$ if and only if $[a : b : c] \in P$.

Proposition 5.1.2. For any $[a : c] \in \mathbb{CP}^1$, $\varphi^{-1}([a : c])$ contains exactly

$$d - \sum_{Q \in \varphi^{-1}([a:c])} (\nu(Q) - 1)$$

points.

Proof. [Kir92] Note that a point of P lies in $\phi^{-1}([a : c])$ if and only if it is of the form $[a : b : c]$ where $b \in P$ satisfies $p(a, b, c) = 0$. We may assume that $p(0, 1, 0) = 1$ since $p(0, 1, 0) \neq 0$. Then $p(a, y, c)$ is a monic polynomial of degree d in y, so

$$p(a, y, c) = \prod_{1 \leq i \leq k} (y - \omega_i)^{m_i},$$

where $\omega_1, \dots, \omega_k$ are distinct complex numbers and $m_1, \dots, m_k$ are positive integers satisfying

$$m_1 + \cdots + m_k = d.$$

Thus, $\phi^{-1}([a, c]) = \{[a : b\omega_i : c] : 1 \leq i \leq k\}$, and the ramification index at $[a : \omega_i : c]$ is

$$\nu[a : \omega_i : c] = m_i.$$

Combining these, we obtain the statement. □

Definition 5.1.3. (Branch locus and branched cover). Let $\mathcal{R}$ denote the set of ramification points. The image $\varphi(\mathcal{R})$ is called the branch locus, and $\varphi : P \to \mathbb{CP}^1$ is called a branched cover of $\mathbb{CP}^1$.

Proposition 5.1.3. Suppose that P is nonsingular. Then there are at most $d(d-1)$ ramification points.

Proof. [Kir92] Since P is nonsingular, it is irreducible, since if $p = qr$ then by Proposition 2.1.1 there exists a point $[a : b : c]$ for which $q(a, b, c) = r(a, b, c) = 0$ and thus $[a : b : c]$ is a singular point. By assumption $[0 : 1 : 0] \notin P$, so the coefficient $p(0, 1, 0)$ of y^d in $p(x, y, z)$ is non-zero. Thus the homogeneous polynomial

$$\frac{\partial p}{\partial y}(x, y, z)$$

is not identically zero and has degree $d-1$, so it cannot be divisible by $p(x,y,z)$. Hence the projective curve Q of degree $d-1$ defined by this polynomial has no component in common with P. Thus the statement follows from Proposition 2.1.2, because the set $\mathcal{R}$ of ramification points of P is the intersection of P and Q.

□

Proposition 5.1.4. Suppose that P is nonsingular. If $\nu[a:b:c] \leq 2$ for all $[a:b:c] \in P$, then P has exactly $d(d-1)$ ramification points.

Proof. [Kir92] By Corollary 2.1.8, it suffices to show that if $[a:b:c]$ lies in $P \cap Q$, then $[a:b:c]$ is a nonsingular point of Q, and the tangent lines to P and Q at $[a:b:c]$ are distinct. Indirectly suppose that $[a:b:c]$ satisfies

$$p(a,b,c) = 0 = p_y(a,b,c),$$

because it lies in P and Q, and the vector

$$(p_{xy}(a,b,c), p_{yy}(a,b,c), p_{zy}(a,b,c))$$

is either zero or a scalar multiple of the vector

$$(p_x(a,b,c), p_y(a,b,c), p_z(a,b,c)).$$

This implies that

$$p(a,b,c) = 0 = p_y(a,b,c) = p_{yy}(a,b,c),$$

that is

$$\nu[a:b:c] > 2.$$

Contradiction. □

Proposition 5.1.5. Suppose that P is nonsingular.

a) If $d \geq 2$, then P has at most $3d(d-2)$ points of inflection.

b) If $d \geq 3$, then P has at least one point of inflection.

Proof. [Kir92] Since H_p is homogeneous of degree $3(d-2)$, and provided that it is not constant (when $d > 2$ this means not identically zero), it defines a projective curve in $\mathbb{CP}^2$. We know by Proposition 2.1.3 that a nonsingular curve is irreducible, so if p and H_p would have a nonconstant common factor, then p would divide H_p, so every point of P would be a point of inflection. However by Proposition 1.3.4, this is a contradiction. Thus p and H_p have no nonconstant common factor for $d > 1$. The statement now follows from Proposition 2.1.1 and Proposition 2.1.2. □

5.2 The genus formula

In this subsection, we will mostly rely on the notations, definitions and propositions of subsection 1.6 and subsection 1.11.

Lemma 5.2.1. Let $q(z,w)$ be a polynomial with complex coefficients in z and w such that for any fixed $z \in \mathbb{C}$ the polynomial $q(z,w)$ in w is monic of degree n. Define $\phi : V(q) \to \mathbb{C}$ by

$$\phi(z,w) = z.$$

Then any $z_0 \in V(q)$ has an open neighbourhood U in $V(q)$ such that each connected component of $\phi^{-1}(U)$ contains at most one point of $\phi^{-1}(\{z_0\})$.

Proof. [Kir92] If $\phi^{-1}(\{z_0\}) = \{(z_0, w_1), \dots, (z_0, w_k)\}$ then

$$q(z_0, w) = \prod_{1 \le i \le k} (w - w_i)^{m_i}$$

where $m_1, \dots, m_k$ are positive integers such that $m_1 + \cdots + m_k = n$. Choose $\varepsilon > 0$ such that $|w_i - w_j| > 2\varepsilon$ if $i \neq j$. Then by Corollary 1.8.3 there is some $\delta > 0$ such that if $|z - z_0| < \delta$ the polynomial $A(z, w)$ in w has at least m_i roots in the disc

$$D_i = \{w \in \mathbb{C} : |w - w_i| < \varepsilon\}$$

when $1 \le i \le k$. Since the discs D_i are disjoint and the sum of the m_i is n, this means that if $|z - z_0| < \delta$ then all the roots of $q(z, w)$ lie in $D_1 \cup \cdots \cup D_k$, and hence

$$\phi^{-1}(\{z \in \mathbb{C} : |z - z_0| < \delta\}) \subset C \times (D_1 \cup \cdots \cup D_k).$$

Therefore every connected component of

$$\phi^{-1}(\{z \in \mathbb{C} : |z - z_0| < \delta\})$$

is a subset of $V(q) \times D_i$ for some $1 \le i \le k$, and hence contains at most one point of $\phi^{-1}(\{z_0\})$. □

Proposition 5.2.2. Given any triangulation (V, E, F) of $\mathbb{CP}^1$ such that the branch locus $\psi(R)$ of ψ is contained in the set of vertices V.

a) there is a triangulation $(\widetilde{V}, \widetilde{E}, \widetilde{F})$ of $\widetilde{P}$ such that

$$\widetilde{V} = \psi^{-1}(V), \quad \#\widetilde{E} = d\,\#E, \quad \#\widetilde{F} = d\,\#F,$$

where d is the degree of ψ;

b)

$$\#\widetilde{V} = d\#V - \sum_{R \in \pi(\mathcal{R})} (\nu_\phi(R) - 1) + \sum_{S \in \mathrm{Sing}(P)} (\#\pi^{-1}\{S\} - 1).$$

Proof. [Kir92] We will use Theorem 3.1.8, that is the map $\pi : \widetilde{P} \to P$ is continuous and surjective, $\mathcal{R} \supset \pi^{-1}(\mathrm{Sing}(P))$ and $\pi : \widetilde{P} - \pi^{-1}(\mathrm{Sing}(P)) \to P - \mathrm{Sing}(P)$ is a homeomorphism. For a) We must show that $\widetilde{V}, \widetilde{E}, \widetilde{F}$ satisfy the conditions of the definition of a triangulation (Definition 1.11.1), and that the formulas for $\#\widetilde{V}, \#\widetilde{E}$ and $\#\widetilde{F}$ are correct. For this, let $\psi = \varphi \circ \pi : \widetilde{P} \to \mathbb{CP}^1$ and let (V, E, F) be a triangulation of $\mathbb{CP}^1$ such that the branch locus $\psi(\mathcal{R})$ is contained in the set of vertices V. We will now use Corollary 1.11.9. Also, by the previous lemma, Proposition 1.11.8 can be applied to the map φ. Thus, if $f \in F$ and $q \in \widetilde{P}$ and $\psi(q) = f(t)$ for some $t \in \Delta$ not equal to any of the vertices $(0,0), (1,0), (0,1)$, then there is a unique continuous map $\widetilde{f} : \Delta \to \widetilde{P}$ such that $\psi \circ \widetilde{f} = f$ and $\widetilde{f}(t) = q$. By Proposition 5.1.2, $\psi^{-1}(f(t))$ consists of exactly d points of $\widetilde{P}$ (because $f(t)$ does not belong to the branch locus $\psi(\mathcal{R})$), so we can deduce that there are exactly d continuous maps $\widetilde{f} : \Delta \to \widetilde{P}$ such that $\psi \circ \widetilde{f} = f$. This means that $\#\widetilde{F} = d\#F$.

We can also deduce that

$$\widetilde{P} - \psi^{-1}(V) = \{\varphi^{-1}(t) : f \in F, t \in \Delta, t \notin \{(0,0), (1,0), (0,1)\}\}$$

which can be written as

$$\{\widetilde{f}(t) : \widetilde{f} \in \widetilde{F}, t \in \Delta, t \notin \{(0,0), (1,0), (0,1)\}\}.$$

In particular, let

$$G = \bigcup_{\widetilde{f} \in \widetilde{F}} \widetilde{f}(\Delta).$$

Then G contains $\widetilde{P} - \psi^{-1}(V)$ and is therefore dense in $\widetilde{P}$ because $\psi^{-1}(V)$ is finite by Proposition 5.1.2. Since $\widetilde{P}$ is compact, G is compact, hence closed in $\widetilde{P}$, so $G = \widetilde{P}$. This implies that

$$\psi^{-1}(V) = \{\widetilde{f}(t) : \widetilde{f} \in \widetilde{F}, t \in \{(0,0),(1,0),(0,1)\}\}.$$

If $\widetilde{e} \in \widetilde{E}$ then $\psi \circ \widetilde{e} \in E$, so either $\psi \circ \widetilde{e} = e$ or $\psi \circ \widetilde{e} = \sigma_i \circ e$ for some $i \in \{1,2,3\}$ where $\sigma_1, \sigma_2, \sigma_3$ are as in the definition of the triangulation (iii)). Thus, if $\widetilde{e} \in \widetilde{E}$ then either $\widetilde{e}$ or $\sigma_i \circ \widetilde{e}$ lies over $e \in E$. Let

$$\widetilde{e}(t) \in \{\widetilde{f}(0) : \widetilde{f} \in \widetilde{F}, \psi(\widetilde{f}(0)) = e(0) \in V\}.$$

This tells us that

$$\psi^{-1}(V) = \{\widetilde{e}(0) : \widetilde{e} \in \widetilde{E}\} \cup \{\widetilde{e}(1) : \widetilde{e} \in \widetilde{E}\}.$$

That is, condition i) of Definition 1.11.1 is satisfied. It follows from Proposition 1.11.8 that if $e \in E$ and $q \in \widetilde{P}$ and $\psi(q) = e(t)$ for some $t \in (0,1)$, then there is a unique continuous map $\widetilde{e} : [0,1] \to \widetilde{P}$ such that $\psi \circ \widetilde{e} = e$ and $\widetilde{e}(t) = q$. Moreover, by Proposition 1.11.7, the restriction of $\widetilde{e}$ to $(0,1)$ is a homeomorphism onto its image in $\widetilde{P}$. So condition ii) of the definition follows using the uniqueness of $\widetilde{e}$.

Therefore, if

$$\Gamma = \bigcup_{e \in E} e([0,1]) \cup V \cup \{e(t) : e \in E, t \in (0,1)\}$$

then

$$\psi^{-1}(\Gamma) = \psi^{-1}(V) \cup \{\widetilde{e}(t) : \widetilde{e} \in \widetilde{E}, t \in (0,1)\} = \widetilde{\Gamma}$$

where

$$\widetilde{\Gamma} = \bigcup_{\widetilde{e} \in \widetilde{E}} \widetilde{e}([0,1]).$$

Furthermore, by Proposition 5.1.2 yet again, if $t \in (0,1)$ and $e \in E$, then $\psi^{-1}(e(t))$ consists of exactly d points of $\widetilde{P}$ (because $e(t)$ does not belong to $\psi(R)$), so there are exactly d continuous maps $\widetilde{e} : [0,1] \to \widetilde{P}$ such that $\psi \circ \widetilde{e} = e$. Thus, $\#\widetilde{E} = d\#E$. Also, by Proposition 1.11.7, if $\widetilde{f} \in \widetilde{F}$ then the restriction of $\widetilde{f}$ to Δ° is a homeomorphism onto its image, which is a connected component of

$$\psi^{-1}(f(\Delta^\circ)) = \varphi^{-1}(f(\Delta^\circ)) \subset P^\circ,$$

and since $f(\Delta^\circ)$ is a connected component of $\mathbb{CP}^1 - \psi(\mathcal{R})$, it follows that $\widetilde{f}(\Delta^\circ)$ is a connected component of

$$\widetilde{P} - \psi^{-1}(R) = \widetilde{P} - \pi^{-1}(\mathrm{Sing}(P)).$$

This shows that the first half of condition iii) is satisfied; we have already noted that the second half is true. Conditions iv) and v) follow easily from what we have already done. Thus it remains to show that

$$\#\widetilde{V} = d\#V - \sum_{R \in \mathcal{R}} (\nu(R) - 1).$$

Luckily, this follows immediately from Proposition 5.1.2 since V contains $\psi(\mathcal{R})$.

Now let us show part b). By Proposition 5.1.2, the inverse image under $\varphi : P \to \mathbb{CP}^1$ of any $Q \in \mathbb{CP}^1$ contains exactly

$$d - \sum_{Q' \in \varphi^{-1}\{Q\}} (\nu(Q') - 1)$$

points. Moreover $\nu(Q') = 1$ if $Q' \notin \pi(\mathcal{R})$ and $\varphi^{-1}(V) \supset \pi(\mathcal{R})$. Thus,

$$\#\varphi^{-1}(V) = d\#V - \sum_{R\in\pi(\mathcal{R})} (\nu(R) - 1).$$

Since

$$\pi : \widetilde{P} - \pi^{-1}(\mathrm{Sing}(P)) \to P - \mathrm{Sing}(P)$$

is a bijection and $\varphi^{-1}(V)$ contains $\mathrm{Sing}(P)$ it follows that

$$\#\psi^{-1}(V) = \#\pi^{-1}\varphi^{-1}(V) = d\#V - \sum_{R\in\pi(\mathcal{R})} (\nu(R) - 1) + \sum_{S\in\mathrm{Sing}(P)} (\#\pi^{-1}\{S\} - 1)$$

as required. □

We shall continue to use Theorem 3.1.8 as before.

Proposition 5.2.3. Suppose that coordinates are chosen on $\mathbb{CP}^2$ so that $[0:1:0]$ does not lie on P or on the tangent line to P at any of (the finitely many) points of $P - \mathrm{Sing}(P)$ which are inflection points on P. Then if $R \in \pi(\mathcal{R})$ and $R \notin \mathrm{Sing}(C)$ we have

$$\nu(R) = 2 \qquad \text{and} \qquad I\left[p, \frac{\partial p}{\partial y}\right](R) = 1.$$

Proof. This immediately follows from Proposition 1.6.2's part f) and Proposition 5.1.4. □

Corollary 5.2.4. [Kir92] Suppose that coordinates are chosen on $\mathbb{CP}^2$ so that $[0:1:0]$ does not lie on P or on the tangent line to P at any of (the finitely many) points of $P - \mathrm{Sing}(P)$ which are inflection points on P. Then the Euler number $\chi(\widetilde{P})$ of $\widetilde{P}$ is given by

$$\chi(\widetilde{P}) = d(3-d) + \sum_{S\in\mathrm{Sing}(P)} \left(I\left[p, \frac{\partial p}{\partial y}\right](S) - \nu(S) + \#\pi^{-1}\{S\}\right).$$

Proof. By definition

$$\chi(\widetilde{C}) = \#\widetilde{V} - \#\widetilde{E} + \#\widetilde{F}$$

where $(\widetilde{V}, \widetilde{E}, \widetilde{F})$ is any triangulation of $\widetilde{C}$. Therefore by proposition Proposition 5.2.2,

$$\chi(\widetilde{C}) = d(\#V - \#E + \#F) - \sum_{R\in\pi(\mathcal{R})} (\nu(R) - 1) + \sum_{S\in\mathrm{Sing}(P)} (\#\pi^{-1}\{S\} - 1)$$

where (V, E, F) is a triangulation of $\mathbb{CP}^1$. Since $\chi(\mathbb{CP}^1) = 2$ (by Proposition 1.11.5), we have $\#V - \#E + \#F = 2$. Then by the previous proposition

$$\sum_{R\in\pi(\mathcal{R})-\mathrm{Sing}(C)} (\nu(R) - 1) = \sum_{R\in\pi(\mathcal{R})-\mathrm{Sing}(C)} I\left[p, \frac{\partial p}{\partial y}\right](R).$$

Since $\pi(\mathcal{R})$ is the intersection of the curve P in $\mathbb{CP}^2$ defined by $p(x,y,z)$ and the curve in $\mathbb{CP}^2$ defined by $\frac{\partial p}{\partial y}(x,y,z)$ and $\mathrm{Sing}(P) \subset \pi(R)$, it follows from Bézout's theorem (Theorem 2.1.7) that

$$\sum_{R\in\pi(\mathcal{R})-\mathrm{Sing}(P)} I[p, \frac{\partial p}{\partial y}](R) = d(d-1) - \sum_{S\in\mathrm{Sing}(P)} I[p, \frac{\partial p}{\partial y}](S).$$

Combining these equalities gives the required formula for $\chi(\widetilde{P})$. □

Definition 5.2.1. ($\delta(S)$). Define the delta number at S by

$$\delta(S) := \frac{1}{2}(I[p, \frac{\partial p}{\partial y}](S) - \nu(S) + \#\pi^{-1}\{S\}).$$

Remark 5.2.5. [BK86] It can be shown that $\delta(S)$ is always a positive integer and is invariant of coordinate change.

Definition 5.2.2. (Milnor number). The Milnor number of P at S is defined as

$$\mu(S) := I\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right](S),$$

Remark 5.2.6. [BK86] The Milnor number is an analytic invariant of the singularity.

The two invariants are strongly related:

Theorem 5.2.7. (Milnor-Jung formula). [Mil68] $2\delta(S) = \mu(S) + \#\pi^{-1}\{S\} - 1$.

Now let's turn back to the main topic:

Definition 5.2.3. (Genus of singular curve). The genus of an irreducible singular projective curve is the genus of its resolution or normalisation (which is a smooth curve).

Theorem 5.2.8. (Noether's formula). The genus g of an irreducible projective curve P of degree d in $\mathbb{CP}^2$ is

$$\frac{1}{2}(d-1)(d-2) - \sum_{S \in \mathrm{Sing}(P)} \delta(S).$$

Proof. Simply use Corollary 5.2.4 and substitute $\chi(\widetilde{P}) = 2 - 2g$. □

Corollary 5.2.9. The Euler number of a nonsingular curve of degree d in $\mathbb{CP}^2$ is $d(3-d)$, while its genus is $\frac{1}{2}(d-1)(d-2)$.

Remark 5.2.10. This formula may seem familiar. In fact, if each singularity is an ordinary multiple point, Noether's formula becomes the pseudo genus defined as in Definition 3.2.9.

Example 5.2.1. Consider the alpha curve (Figure 5) defined by $x^3 + x^2 - y^2$. From Example 4.1.3, we conclude that it has a node singularity at the origin, so from Noether's formula its genus is $\frac{2 \cdot 1}{2} - 1 = 0$. In fact, it looks like a punctured torus, but why? Consider the smooth nonsingular curve $x^3 - x - y^2$ (Figure 3) . Its genus is 1 by Corollary 5.2.9, so it is a torus. Note that this curve has equation $y^2 - (x+1)x(x-1)$. Now, consider the curve $y^2 - (x+1)x(x-\epsilon)$ and see what happens when ϵ approaches 0. Clearly, the curve approaches the alpha curve. However, in the mean time, the meridian of the torus originally containing $[0:0:1]$ and $[1:0:1]$ (the origin and $(1,0)$ in the affine real plane shown in Figure 3) gets smaller and smaller, and finally shrinks to a point: the singular origin. Thus, a meridian of the torus collapsed to a point, which is topologically a sphere with two horns touching. The normalisation separates the two branches at this point, which gives us a sphere (S^2). This is another way of seeing that the genus is, in fact, 0.

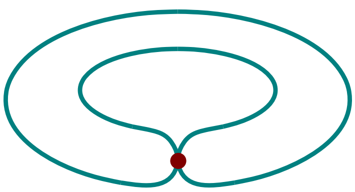

Figure 17
Sketch of the alpha curve.

Theorem 5.2.11. (Clebsch's formula). Suppose that the curve $P \in \mathbb{CP}^2$ of degree d has only nodes and cusps as singularities. Then the genus of P is

$$\frac{1}{2}(d-1)(d-2) - r - s,$$

where r is the number of node points and s is the number of cusps.

Proof sketch. [BK86] We must show that the delta invariants are 1 for both nodes and cusps. For $x^2 - y^2$, we have that $I = 2$, $\nu = 2$ and $\#\pi^{-1} = 2$, so $\delta = \frac{1}{2}(2-2+2) = 1$. For $x^3 - y^2$, we have that $I = 3$, $\nu = 2$, $\#\pi^{-1} = 1$, so $\delta = \frac{1}{2}(3-2+1) = 1$. $\square$

Proposition 5.2.12. For any non-negative integer g there exists an algebraic curve in $\mathbb{CP}^2$ with genus g.

Proof. (Based on [BK86]) Consider the plane projective curve defined by the equation

$$p_* = y^2 - x^{2g+2} - 1$$

which has homogenisation

$$p = y^2 z^{2g} - x^{2g+2} - z^{2g+2}.$$

As $\frac{\partial p}{\partial x} = x^{2g+1}$, $\frac{\partial p}{\partial y} = 2yz^{2g}$, $\frac{\partial p}{\partial z} = 2g \cdot y^2 z^{2g-1} - (2g+2)z^{2g+1}$, it has one singular point, namely $S = [0 : 1 : 0]$. We will prove that it has genus g. Substitute $y = 1$:

$$z^{2g} - x^{2g+2} - z^{2g+2} = z^{2g}(1 - z^2) - x^{2g+2}.$$

We can remove the factor $1 - z^2$ as it clearly does not affect the local singularity type. So we have

$$z^{2g} - x^{2g+2}.$$

Let us use now Theorem 5.2.8. From Definition 5.2.1, we need to calculate $I[p, \frac{\partial p}{\partial x}](S)$, $\nu(S)$ and $\#\pi^{-1}\{S\}$ (note that in the original definition $\frac{\partial p}{\partial y}$ appears instead of $\frac{\partial p}{\partial x}$, but now we have to project on one of the other coordinates, and we have chosen x for now). The easiest one is last one, the number of branches. Note that $z^{2g} - x^{2g+2} = (z^g - x^{g+1})(z^g + x^{g+1})$, there are two distinct branches intersecting at the origin. Also, we have

$$I[z^{2g} - x^{2g+2}, (2g+1)x^{2g+1}](S) = I[z^{2g} - x^{2g+2}, x^{2g+1}](S) = (2g+1) \cdot 2g = 4g^2 + 2g.$$

As we have chosen the x coordinate for the projection, $\nu(S) = 2g + 2$. Therefore,

$$\delta(S) := \frac{1}{2}(I[p, \frac{\partial p}{\partial x}](S) - \nu(S) + \#\pi^{-1}\{S\}) = \frac{1}{2}\left(4g^2 + 2g - 2g - 2 + 2\right) = 2g^2.$$

Hence, the genus is

$$\frac{1}{2}(d-1)(d-2) - \sum_{S \in \mathrm{Sing}(P)} \delta(S) = \frac{1}{2}(2g+1) \cdot 2g - 2g^2 = g.$$

$\square$

Remark 5.2.13. We also could have used Theorem 5.2.7, as the Milnor number can be easily calculated to be $\mu(S) = (2g+1)(2g-1) = 4g^2 - 1$, implying

$$\delta(S) = \frac{1}{2}\left(\mu(S) + \#\pi^{-1}\{S\}) - 1\right) = \frac{1}{2}\left(4g^2 - 1 + 2 - 1\right) = 2g^2.$$

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Nyilatkozat

a Mesterséges Intelligencia használatáról

Alulírott *Bán-Szabó Áron* nyilatkozom, hogy szakdolgozatom elkészítése során az alább felsorolt feladatok elvégzésére a megadott MI alapú eszközöket alkalmaztam:

Feladat	**Felhasznált eszköz**	**Felhasználás helye**	**Megjegyzés**
Nyelvtani ellenőrzés	ChatGPT-4o	Teljes dolgozat	-
Megfogalmazási javaslatok	ChatGPT-4o	Köszönetnyilvánítás, absztrakt, 2-5. fejezetek	Egyes bevezető és átvezető szövegrészek stílusának csiszolása
Források gyűjtése	ChatGPT-4o	Irodalomjegyzék	A 3., 4. és 5. fejezetek témáihoz kapcsolódó szakirodalom keresése, *BibTeX*-hivatkozások összeállítása
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A felsoroltakon túl más MI alapú eszközt nem használtam.

Aláírás

Budapest, 2025-06-02