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On Two Kakeya-type Problems

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Contents

1 Acknowledgement

First of all, I would like to express my gratitude to my supervisor, Tamás Keleti, whose contribution to my studies goes far beyond the scope of this thesis. His seminar sparked my interest in the topic of geometric measure theory.

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Last but not least, I would like to thank my teachers for sharing their deep knowledge and inspiring style throughout my BSc years.

2 Introduction

In 1917, Soichi Kakeya posed the following question now known as the Kakeya needle problem. What is the minimal area required to rotate a unit line segment continuously by 180 degrees within the plane? This question was answered by Besicovitch in his paper [1], where he showed that no such minimum exists—in other words, for any given positive number, there exists a rotation of the segment that sweeps an area no greater than that number.

A central concept in Besicovitch's proof was the notion of a Besicovitch set.

Definition 2.1. A *Besicovitch set* in $\mathbb{R}^n$ is a set that contains a unit line segment in every direction.

Besicovitch demonstrated that there exist Besicovitch sets of Lebesgue measure zero in two dimensions. It follows readily from this that Besicovitch sets of measure zero also exist in all dimensions $n \geq 2$. Nevertheless, the celebrated Kakeya conjecture asserts that, with respect to Hausdorff dimension, such sets cannot be small: they must have full dimension in $\mathbb{R}^n$.

Conjecture 2.2. *Besicovitch sets in $\mathbb{R}^n$ have Hausdorff dimension n.*

The conjecture is trivially true for $n = 1$, was proven for $n = 2$ by Davies [5] in 1971, and, in 2025, Hong Wang and Joshua Zahl [10] announced a proof for the case $n = 3$.

The topic gained broader attention in the 1970s when Fefferman [7] used similar ideas to disprove a major conjecture in Fourier analysis. He showed that so-called *ball multiplier operators* are unbounded on $\mathbb{R}^n$ for $n \geq 2$. A closely related conjecture, the *Bochner–Riesz conjecture*, remains open and is also intimately connected to the theory of Kakeya sets.

Since the original formulation of the problem, considerable attention has been given to questions concerning how objects can be moved in such a way that they sweep only a small area during the motion. Many open problems remain in this area to this day. The results presented in this thesis were motivated by a result of Cunningham.

Cunningham refers to a certain type of line segment as a *bird*: the segment is divided into three parts, with the two outer parts of length w referred to as the *wings*, and the short middle part of length b referred to as the *body*. Cunningham proved the following theorem:

Theorem 2.3 (Cunningham [3]). *Given a bird as described above and any bounded set S, and $\varepsilon > 0$, there exists a continuous motion of the bird such that its body passes over every point of S while both its wings stay in a set K of area less than ε.*

The main goal of this thesis is to prove two theorems similar in nature to the one above. The similarity lies in the fact that in both cases, we construct a motion in which one object is constrained to sweep a small area, while another object sweeps a large area.

To this end, we first prove a lemma and a theorem in Sections 3 and 4, respectively. Then, in Sections 5 and 6, we present two applications of the theorem from Section 4.

In Section 5, we describe a motion of a square and study the area swept by the initially vertical sections of the square. We show that during a certain motion, the areas swept by these segments can differ greatly from one another.

In Section 6, we again describe the motion of a square; however, in this case, we consider the area swept by certain individual points of the square. We show that the fact that during a motion a point sweeps a set of positive measure does not imply that the set swept by the surrounding points also has positive measure.

The results presented in this thesis are original, unless stated otherwise. In particular, the main theorems in Sections 5 and 6 are new. The result in Section 6 was developed in collaboration with Márk Kökényesi, whose contribution is gratefully acknowledged.

3 Generalization of a construction of Talagrand

3.1 Notations

In this section, we consider C^2 curves of the form $t \mapsto (g(t), t)$ such that g is Lipschitz. We denote the family of these curves by Δ. When we say that a curve $\gamma \in \Delta$ is d-Lipschitz, we mean that the function g in its representation $t \mapsto (g(t), t)$ is d-Lipschitz. For every $\gamma \in \Delta$, γ_s denotes $\gamma + (s, 0)$. For every $p \in \mathbb{R}^2$ there exists a unique $s \in \mathbb{R}$ such that $p \in \gamma_s$. We denote by $\alpha_\gamma(p)$ the angle between the derivative of γ_s at the point p and the x-axis. For any $K \subseteq \mathbb{R}^2$ we define $h_\alpha(K)$ as the Lebesgue measure of the orthogonal projection of the set K in direction α. With a slight abuse of notation we also define

$$h_\gamma(K) = \lambda_1\left(s : \gamma_s \cap K \neq \emptyset\right).$$

This can be intuitively interpreted as the projection of K onto the x-axis along curves parallel to γ. We say that a set is *elementary* if it is the finite union of convex polygons.

3.2 Lemma

In [9] Talagrand proved the following theorem:

Theorem 3.1 (Talagrand, 1980)**.** *For every upper semi-continuous, π-periodic function $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ there exists a compact set $K \subseteq \mathbb{R}^2$ such that $h_\alpha(K) = f(\alpha)$ for every $\alpha \in \mathbb{R}$.*

While the classical theorem concerns linear projections, this section aims to prove a lemma involving projections along curves.

Lemma 3.2. *Let K be an elementary compact set, $F \subseteq (0, \pi)$ a closed set and $U \subseteq (0, \pi)$ an open set containing F. Let $\Gamma \subset \Delta$ be a set of curves with uniformly bounded curvature and a universal Lipschitz constant d. Let $\varepsilon > 0$. Then there exists an elementary compact $L \subseteq K$, such that for every $\gamma \in \Gamma$*

(i) $\alpha_\gamma(K) \subseteq F \Rightarrow h_\gamma(L) \leq \varepsilon$,

(ii) $\alpha_\gamma(K) \cap U = \emptyset \Rightarrow h_\gamma(L) \geq h_\gamma(K) - \varepsilon$.

We use the following lemma, whose proof follows from elementary geometry and is therefore omitted.

Lemma 3.3. *Let K be an elementary compact set, $\alpha \in [0, \pi]$ and $\varepsilon > 0$. There exists a finite union of rectangles $L \subseteq K$ such that*

(i) the longer sides of the rectangles form an angle α with the x-axis;

(ii) the total length of the shorter sides of the rectangles is less than ε;

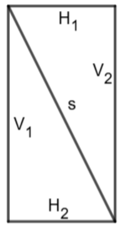

Figure 1: Upper bound of $h_\gamma(S)$

(iii) for every $\gamma \in \Delta$*, if* $|\alpha_\gamma(p) - \alpha| > \varepsilon$ *for every* $p \in K$*, then* $h_\gamma(L) > h_\gamma(K) - \varepsilon$*.*

The following lemma provides a construction that plays a crucial role in the proof of Lemma 3.2.

Lemma 3.4. *Let* R *be a closed rectangle with sides of length* l *and* L *such that* $L > l$ *and let* ξ*,* ξ' *be real numbers such that* $0 < \xi < \xi' < \pi/2$*. Let* $\alpha \in [-\pi/2, \pi/2)$ *denote the angle between the sides of length* L *and the* x*-axis. Let* $\Gamma \subset \Delta$ *denote a set of curves with uniformly bounded curvature and a universal Lipschitz constant* d*. There exists an elementary compact set* $K \subseteq R$ *such that for every* $\gamma \in \Gamma$*:*

(i) $\alpha_\gamma(R) \subseteq [\alpha, \alpha + \xi] \Rightarrow h_\gamma(K) \leq 18l(2 + 2d)$*,*

(ii) $\alpha_\gamma(R) \cap [\alpha, \alpha + \xi'] = \emptyset \Rightarrow h_\gamma(K) = h_\gamma(R)$*.*

We also need the following result.

Claim 3.5. *Let* γ *be a* C^1 *curve of the form* $y \mapsto (g(y), y)$ *such that* g *is* $d-$*Lipschitz. Let* S *be a line segment of length* l*. Then*

$$h_\gamma(S) \leq l(2 + 2d).$$

Proof. Let us take an axis-aligned rectangle with two of its vertices being the endpoints of the segment S (see Figure 1). If a curve intersects S it intersects a side of this rectangle. We denote the vertical sides by V_1, V_2 and the horizontal ones by H_1, H_2. By elementary calculations:

$$h_\gamma(V_1) = h_\gamma(V_2) \leq dl.$$

Therefore

$$h_\gamma(S) \leq h_\gamma(H_1) + h_\gamma(H_2) + h_\gamma(V_1) + h_\gamma(V_2) \leq 2l + 2dl = l(2 + 2d).$$

□

Proof of Lemma 3.4. First we show that if $\tan \xi' \leq \frac{l}{L}$, then $K = R$ is an appropriate set. In this case if a curve intersects the rectangle, then it also intersects a segment obtained by extending the shorter sides of the rectangle by a length l in both direction. Hence, by Claim 3.5 we get that $h_\gamma(R) \leq 6l(2 + 2d)$

Therefore we can assume that there is a constant $0 < \theta < 1/2$ such that

$$\tan\xi < \frac{l}{2\theta L} < \tan\xi'.$$

We present an algorithm to generate a sequence of parallelograms from any P parallelogram. For $P = ABCD$ we define a parallelogram $f_1(P) = A_1B_1C_1D_1$, where

$$A_1 = A, \quad B_1 = A + \left(\frac{1}{2} + \theta\right)\overrightarrow{AB} + \frac{\overrightarrow{AD}}{2},$$
$$C_1 = B_1 + \frac{\overrightarrow{AD}}{2}, \quad D_1 = A_1 + \frac{\overrightarrow{AD}}{2}.$$

We define $f_2(P)$ as a translation of $f_1(P)$ by $(1/2-\theta)\overrightarrow{AB}$ and we denote its vertices by $A_2B_2C_2D_2$, correspondingly.

Let us define by induction a family $\mathcal{L}_n$ of finite unions of parallelograms as follows. We set:

$$\mathcal{L}_0(P) = \{P\}, \quad \text{and} \quad \mathcal{L}_{n+1}(P) = \{f_i(Q),\, Q \in \mathcal{L}_n(P),\, i = 1, 2\}.$$

We observe that all elements of $\mathcal{L}_n$ are translated copies of each other. Let t be a line perpendicular to AD. We denote by τ_n the slope (with respect to t) of the sides of the parallelograms in $\mathcal{L}_n$ that are not perpendicular to t. (That is, τ_n denotes the slope of segment when t is considered as the horizontal reference direction.) For the rest of the proof, when we say slope of a line, we mean the slope compared to t. Let a_n denote the length of the side of a parallelogram in $\mathcal{L}_n$ that is perpendicular to t and let b_n denote the length of the orthogonal projection of an other side to t. By elementary calculations:

(a) The slope of the sides of $f_1(P)$ and $f_2(P)$ that are not perpendicular to t is

$$\tau_{n+1} = \tau_n + \left(\frac{a_n}{(1+2\theta)b_n}\right);$$

(b) Let $I = A_2D_2 \cup B_1D_1 \cup A_{22}D_{22} \cup B_{21}D_{21} \cup A_{12}D_{12} \cup B_{11}D_{11}$ (see Figure 2). Observe that every curve γ whose slope lies in $[\tau_0, \tau_2]$ and which intersects P, also intersects I. Hence

$$\begin{aligned} h_\gamma\left(\bigcup \mathcal{L}_2(P)\right) &= h_\gamma(I) \\ &\le 2|BC|(2+2d), \text{ where} \end{aligned}$$

the inequality follows from Claim 3.5, as the total length of the segments in I is $2|BC|$.

(c) The slope of A_2C_1 is equal to $\tau + \left(\frac{a_0}{2\theta b_0}\right)$.

From the last condition it follows that if the slope of a curve is greater than $\tau + \left(\frac{a_0}{2\theta b_0}\right)$ or negative and the curve intersects AB and CD, then it intersects either A_1B_1 and C_1D_1, or it intersects A_2B_2 and C_2D_2.

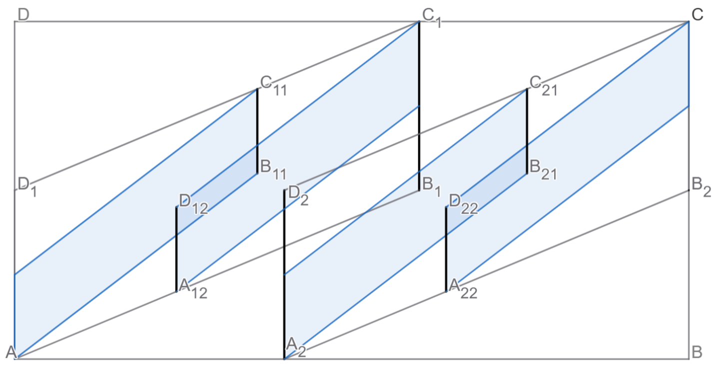

Figure 2: $\mathcal{L}_2(R)$ and the set I.

Now we consider the case with R. We consider $\mathcal{L}_n(R)$ for an arbitrary n, whose exact value will be determined later. Let t_0 be a line parallel to the side of R of length L. Based on the properties described previously, the length of their side that is perpendicular to t_0 is $a_n = \frac{l}{2^n}$, and the length of their orthogonal projection to t_0 is $b_n = L\left(\frac{1}{2}+\theta\right)^n$. The slope of their sides that are not perpendicular to t_0:

$$\tau_n = \frac{l}{2\theta L}\left(1-\left(\frac{1}{(1+2\theta)}\right)^n\right).$$

Since $\lim_{n\to\infty}\tau_n = \frac{l}{2\theta L} > \tan\xi$, we can set n such that $\tau_{n-2} \geq \tan\xi$. Let $L(R)$ be the union of the elements of $\mathcal{L}_n(R)$, and define $K(R)$ as the union of $L(R)$ and the two squares $S_1(R), S_2(R) \subseteq R$ located at the ends of R (each of the shorter sides of R is a side of $S_1(R)$ or $S_2(R)$). Observe that if a curve intersects $S_1(R)$ it intersects one of its side, hence by Claim 3.5 we get that $h_\gamma(S_1(R)) = h_\gamma(S_2(R)) \leq 4l(2+2d)$.

Observe that a curve γ with $\alpha_\gamma(R) \leq \alpha$ or $\alpha_\gamma(R) > \xi'$ which intersects R without intersecting $S_1(R)$ or $S_2(R)$, must intersect both sides of R that are parallel to t_0. Given that

$$\tau_i + \frac{a_i}{2\theta b_i} = \frac{l}{2\theta L} \leq \tan\xi'$$

for every $i \leq n$, we conclude by induction that it intersects a parallelogram in $\mathcal{L}_n(R)$ on both of its sides with slope τ_n. Hence, point (ii) is established.

We define

$$c := \min_{0\leq k\leq n-1}(|\tau_{k+1}-\tau_k|).$$

Since the curvature is equally bounded in Γ, there is an r, such that if $||x-y|| < r$ than $|\alpha_\gamma(x)-\alpha_\gamma(y)| < c$.

Now we show that $K(R) \subseteq R$ has property (i) for every R with diameter less than c.

Take a $\gamma \in \Gamma$ such that $0 \leq \alpha_\gamma(R) \leq \xi$ and $m < n-1$, so that $\tau_m \leq \alpha_\gamma(R) \leq \tau_{m+2}$. Then we have:

$$h_\gamma(K(R)) \leq h_\gamma(S_1(R)) + h_\gamma(S_2(R)) + h_\gamma(L(R)) \leq 8l(2+2d) + h_\gamma(L(R)).$$

There are 2^m parallelograms in $\mathcal{L}_m(R)$, and $\mathcal{L}_{m+2}(R)$ is the set of $\mathcal{L}_2(Q)$ for $Q \in \mathcal{L}_m(R)$. $h_\gamma(\mathcal{L}_2(Q)) \leq 2a_m(2+2d) = 2^{-m+1}l(2+2d)$ as we have seen in (b). Since L is included in the union of $\mathcal{L}_{m+2}(R)$, we have

$$h_\gamma(L(R)) \leq 2^{-m+1}l(2+2d)2^m = 2l(2+2d).$$

Hence

$$h_\gamma(K(R)) \leq 8l(2+2d) + 2l(2+2d).$$

Now we extend the construction to an arbitrary rectangle R. Choose a positive integer N such that the diameter of each rectangle in the $N \times N$ subdivision of R is less than c. We partition R into $N \times N$ congruent rectangles, ensuring that each has diameter less than c. From this grid, we select the rectangles along the main diagonal, denoted by $R_1, R_2, \ldots, R_N$. Define

$$K'(R) = \bigcup_{i=1}^{N} K(R_i) \cup (S_1(R) \cup S_2(R)).$$

We show that $K'(R)$ is the desired set. First we take a γ such that $0 \leq \alpha_\gamma(R) \leq \xi$.

$$\begin{aligned} h_\gamma(K'(R)) &\leq \sum_{i=1}^{N} h_\gamma(K(R_i)) + h_\gamma(S_1(R)) + h_\gamma(S_1(R)) \\ &\leq N \cdot 10\frac{1}{N}l(2+2d) + 8l(2+2d) \\ &= 18l(2+2d). \end{aligned}$$

On the other hand, if $\xi' \leq \alpha_\gamma(R) < \pi$ and γ_s intersect R for some s, than either it intersects $S_1(R) \cup S_2(R)$ or there is an index i such that it also intersects R_i, thus it also intersects $K(R_i)$. In either case, it intersects $K'(R)$. $\square$

Proof of Lemma 3.2. Since there exists a finite union of intervals of the type $[\alpha, \alpha+\xi]$, with $\xi < \pi/2$, containing F and contained in U, we reduce Lemma 3.2 to the case where $F = F_1 \cup \ldots F_n$ such that for every $i \leq n$ F_i is a closed interval. We denote $F_i = [\alpha_i, \alpha_i + \xi_i]$ and for every i we choose ξ_i' such that $[\alpha, \alpha_i + \xi_i'] \cap U = \emptyset$.

We apply Lemmas 4 and 5 iteratively. We define a sequence of elementary sets $K = L_0 \supseteq L_1 \supseteq \cdots \supseteq L_n = L$ as follows.

For every $i \leq n$ by Lemma 3.3 there is an L_i' set which is a union of N rectangles and

(i) the longer sides of the rectangles form an angle α_i with the x-axis;

(ii) the total length of the shorter sides of the rectangles is less than $\frac{\varepsilon}{18(2+2d)}$;

(iii) for every $\gamma \in \Delta$, if $|\alpha_\gamma(p) - \alpha| > \varepsilon$ for every $p \in K$, then $h_\gamma(L) > h_\gamma(K) - \varepsilon/N$.

Let $L_i' = R_1^i \cup \cdots \cup R_N^i$ and we denote the length of the shorter side of R_k^i by l_k^i. By Lemma 3.4 for every k there exists an elementary compact set $M_k^i \subseteq R_k^i$ such that

(i) $\alpha_\gamma(R_k^i) \subseteq [\alpha_i, \alpha_i + \xi_i] \Rightarrow h_\gamma(M_k^i) \leq 18 l_k^i (2 + 2d)$,

(ii) $\alpha_\gamma(R_k^i) \cap [\alpha, \alpha + \xi'] = \emptyset \Rightarrow h_\gamma(M_k^i) = h_\gamma(R_k^i)$.

We define

$$L_{i+1} = \bigcup M_k^i.$$

Observe that for every $i \leq n$ and for every $\gamma \in \Gamma$

(i) $\alpha_\gamma(L_i) \subseteq \bigcup_{r=1}^{i} F_i \Rightarrow h_\gamma(L_i) \leq \varepsilon$,

(ii) $\alpha_\gamma(K) \cap U = \emptyset \Rightarrow h_\gamma(L) \geq h_\gamma(K) - (i/N)\varepsilon$.

For $i = n$ we get the statement of Lemma 3.2. $\square$

4 Duality based constructions

4.1 Notations

Let V denote the set of planes in $\mathbb{R}^3$ that forms $45°$ angle with the plane $\{x = 0\}$. For a set $H \subset \mathbb{R}^2$ and $p \in \mathbb{R}$, let $\operatorname{Sec}^p(H)$ denote the 1-dimensional Lebesgue measure of the section of H consisting of points whose first coordinate is p; that is,

$$\operatorname{Sec}^p(H) := \lambda\left(\{y \in \mathbb{R} : (p, y) \in H\}\right).$$

For a vector $v \in \mathbb{R}^2$ let $p_v(S)$ denote the orthogonal projection of K onto the line parallel to v. That is,

$$p_v(S) = \frac{v \cdot S}{||v||},$$

where $\cdot$ denotes the scalar product.

4.2 Construction of a Set with Localized Large Sections via Duality

The main objective of this section is to prove Theorem 4.1. Lemma 4.2 serves as a preliminary result for Lemma 4.4, which in turn is a key step toward establishing Theorem 4.1. We will begin by stating the theorem and the lemmas, and provide their proofs afterward.

Theorem 4.1. *For every $\varepsilon > 0$ real numbers $0 < a \leq b < 1$, and angle interval $I \subset [0, \pi)$, there exists a closed set $A \subset \mathbb{R}^3$, which is the union of planes from V, all lying at a bounded distance from the origin, such that:*

(i) for every $h \in [a, b]$, the Lebesgue measure of $A \cap (\{h\} \times [0, 2] \times [0, 2])$ is greater than $4 - \varepsilon$;

(ii) for every $h \in [0, 1] \setminus [a - \varepsilon, b + \varepsilon]$, the Lebesgue measure of $A \cap (\{h\} \times [-2, 4] \times [-2, 4])$ is less than ε;

(iii) all planes from V that form the set A intersect the plane $\{x = 0\}$ in such a way that the angle their intersection makes with the y-axis lies in the interval I.

Lemma 4.2. *Let $R = ABCD$ be a rectangle such that the side AB is parallel to the x-axis, let $x_0 < x_1$ be given real numbers and $\varepsilon > 0$. Suppose that $\tau_1, \tau_2 \in \mathbb{R}$ are given with $\tau_1 < \tau_2$. Then there exists a closed set $K \subseteq \mathbb{R}^2$, which is the union of non-horizontal lines whose slopes lie in the interval $[\tau_1, \tau_2]$, such that:*

(i) for every $t \in [x_0, x_1]$, we have $\operatorname{Sec}^t(K \cap R) > \lambda(AD) - \varepsilon$;

(ii) for every $t \notin [x_0, x_1]$, we have $\operatorname{Sec}^t(K \cap R) = 0$.

There are several results similar to the lemma above. In [4], Davies proved that every measurable set $A \subseteq \mathbb{R}^2$ can be covered with a Borel set of lines, such that the Lebesgue measure of the union of the lines is equal to $\lambda(A)$. He also proved that the statement holds if the slopes of the lines are required to be in a given interval. Later, in [2], Csörnyei showed that for every open set $A \subset \mathbb{R}^2$ and point $x \in \mathbb{R}^2$ not belonging to A, there exists a Borel set of lines $\mathcal{L}$ such that $\mathcal{L}$ contains residually many lines through each point of A, and $\mathcal{L} \cap A$ intersects each line through x in a set of (Lebesgue) measure zero. If we choose x to be an ideal point we get a similar result as Lemma 4.2. The difference between these results and our lemma is that our lemma requires the line set to be a closed set.

For a set $S \subset \mathbb{R}^2$ we denote $S + (\{0\} \times (-t, t))$ by S_t and for a set $H \subset \mathbb{R}$ we define $H_t = H + (-t, t)$. Let K be a closed set as in Lemma 4.2 for given $\tau_1 < \tau_2 \in \mathbb{R}$, and let $c = \max(|\tau_1|, |\tau_2|)$. For every $t, d > 0$ and $p, q \in \mathbb{R}$, such that $|p - q| < d$, we have

$$\mathrm{Sec}^p(K_{t+cd} \cap R) \geq \mathrm{Sec}^q(K_t \cap R),$$

since the set corresponding to the left-hand side contains the set corresponding to the right-hand side. Also, for a given p, the function $\mathrm{Sec}^p(K_{t+cd} \cap R)$ is continuous in d. Hence with $d \to 0$ we get that for every $t \geq 0$, the function $\mathrm{Sec}^p(K_t \cap R)$ is upper semi-continuous in p.

Furthermore, for every $p \notin [x_0, x_1]$, it is clear that $\mathrm{Sec}^p(K_t \cap R) \xrightarrow{t \to 0} 0$. It is not difficult to see that if a sequence of upper semi-continuous functions converges pointwise from above to a continuous function on a compact set, then the convergence is uniform. Thus we get the following claim.

Claim 4.3. *Let K be a closed set given by Lemma 4.2. Let $\varepsilon > 0$. There is a δ, such that for every $p \notin [x_0, x_1]_\varepsilon$ and $d \leq \delta$, we have*

$$\mathrm{Sec}^p(K_d \cap R) < \varepsilon.$$

In the following, we introduce some notation and make some observations that will be useful in the next lemma.

For given $x, y, s \in \mathbb{R}$ define the curve $C_{x,y,s}(u) := (s - yu - x\sqrt{1+u^2},\ u)$. Note that varying the parameter s translates the curves along the x-axis. If $p = C_{x,y,s}(t)$ is a given point on the curve, then we denote by $\tau_{x,y}(p)$ the slope of the tangent of $C_{x,y,s}$ at the point p. Observe that for $p \in \mathbb{R}^2$ the value of $\tau_{x,y}(p)$ is well defined, since for every $x, y \in \mathbb{R}$ there is a unique $s \in \mathbb{R}$, such that $p \in C_{x,y,s}$.

By elementary calculations, one obtains:

$$\tau_{x,y}(a, b) = \frac{1}{-y - x\frac{b}{\sqrt{1+b^2}}} = \frac{1}{-y - xg(b)}, \tag{4.1}$$

where $g(b) = \frac{b}{\sqrt{1+b^2}}$. If $xg(b) + y = 0$ we say that $\tau_{x,y}(a, b) = \infty$.

It is easy to see that the function g is increasing, bijective, and bounded, with $|g(b)| < 1$ for all $b \in \mathbb{R}$. Note that g^{-1} exists on the interval $(-1, 1)$. It is clear that if $y + xg(b) = y' + x'g(b)$ for some $x, x', y, y', b \in \mathbb{R}$ then for every $a \in \mathbb{R}$ we have $\tau_{x,y}(a,b) = \tau_{x',y'}(a,b)$.

We also define for $x, y \in \mathbb{R}$ and $F \subseteq \mathbb{R}$ the function

$$h_{x,y,F}(K) = \lambda(\{s \in F \colon C_{x,y,s} \cap K \neq \emptyset\}).$$

Intuitively $h_{x,y,F}(K)$ means the Lebesgue measure of the projection of K to $F \times \{0\}$ along curves of the form $C_{x,y,-}$. Notice that for every set $K \subseteq \mathbb{R}^2$ and $x, y \in \mathbb{R}$ we have $h_{x,y,\mathbb{R}}(K) = h_{x,y}(K)$.

Lemma 4.4. *For every $0 < a \leq b < 1$, every $\varepsilon > 0$, and every pair of real numbers $y_0 < y_1$, there exists a compact set $K \subset \mathbb{R}^2$ such that the following hold:*

(i) For every $x \in [a,b]$ we have

$$\lambda(\{y \in [0,2] : h_{x,y,[0,2]}(K) \geq 2 - \varepsilon\}) \geq 2 - \varepsilon.$$

(ii) For every $x \in [0, a-\varepsilon] \cup [b+\varepsilon, 1]$ we have

$$\lambda\left(\{y \in [-2,4] : h_{x,y,[-2,4]}(K) \geq \varepsilon\}\right) \leq \varepsilon.$$

(iii) The second coordinate of every point in K lies between y_0 and y_1.

We now proceed to prove the statements above.

Proof of Theorem 4.1 using Lemma 4.4. The proof follows a similar strategy to that in [8], with the key difference being that Lemma 4.4 is stronger than the corresponding result used by Kökényesi. The argument relies on a duality method, not in the plane but in $\mathbb{R}^3$. For further examples illustrating this technique, see [6]. This method is commonly used in geometric measure theory to construct sets with given properties. Rather than directly constructing such a set, one defines a suitable dual for each set and then attempts to construct the dual instead. We now demonstrate how this approach applies in our context.

To every point $(a,b,c) \in \mathbb{R}^3$ we assign the plane $\{a + bx + cy = z\}$ and we denote it by $Pl(a,b,c)$. Notice that Pl is a bijective map between the points of $\mathbb{R}^3$ and the non-vertical planes in $\mathbb{R}^3$. By elementary calculations, we find that a plane forms an angle of 45° with the plane $\{x = 0\}$ if and only if for the corresponding point (a,b,c), we have

$$b^2 - c^2 = 1.$$

We define the *plane set* of $K \subseteq R^3$ as

$$Pl(K) = \bigcup_{p \in K} Pl(p).$$

It is not hard to see that if a set K is closed, then its plane set is also closed.

Notice that $(x,y,z) \in Pl(K) \Leftrightarrow z \in (K \cdot (1,x,y))$, therefore for every set $F \subseteq \mathbb{R}$ and $x, y \in \mathbb{R}$, we have

$$\lambda(\{z \in F\colon (x,y,z) \in Pl(K)\}) = \lambda(F \cap (K \cdot (1,x,y))).$$

From Fubini's theorem we get that for every measurable sets $F_1, F_2 \subseteq \mathbb{R}$ and $t \in \mathbb{R}$, we have

$$\lambda_2((\{t\} \times F_1 \times F_2) \cap Pl(K)) = \int_{F_1} \lambda_1(F_2 \cap K \cdot (1,t,y))dy.$$

Our goal is to construct a closed set $K \subseteq \mathbb{R}^3$ (in coordinates denoted by a, b, c) such that $K \subseteq \{b^2 - c^2 = 1\}$ and K has the following three properties:

1. For every $t \in [a,b]$, we have
$$\int_{[0,2]} \lambda([0,2] \cap (K \cdot (1,t,y)))dy > 4 - \varepsilon.$$

2. For every $t \in [0,1] \setminus [a,b]_\varepsilon$, we have
$$\int_{[-2,4]} \lambda([-2,4] \cap (K \cdot (1,x,t)))dx < \varepsilon.$$

3. Every point in K has a positive second coordinate.

The plane set of such a K would satisfy properties (i) and (ii) stated in the lemma. As for property (iii), we will not express it explicitly here; however the reader may verify its validity at the appropriate point in the proof applying part (iii) of Lemma 4.4.

Suppose that we have such a set K. By elementary calculations for any $s, x, y \in R$, using that $K \subseteq \{b^2 - c^2 = 1\}$, we have that

$$s \in K \cdot (1,x,y) \Leftrightarrow \{(s - yu \pm x\sqrt{u^2+1}, \mp\sqrt{u^2+1}, u) \in \mathbb{R}^3\colon u \in \mathbb{R}\} \cap K \neq \emptyset.$$

Define

$$\begin{aligned} P : \mathbb{R}^3 &\to \mathbb{R}^2 \\ (a,b,c) &\mapsto (a,c). \end{aligned}$$

Let $K' = P(K)$. We get that

$$s \in K \cdot (1,x,y) \Leftrightarrow \{(s - yu - x\sqrt{u^2+1}, u) \in \mathbb{R}^2\colon u \in \mathbb{R}\} \cap K' \neq \emptyset.$$

Therefore, for every set $F \subset \mathbb{R}$

$$\lambda(K \cdot (1,x,y) \cap F) = \lambda(\{s \in F : C_{x,y,s} \cap K' \neq \emptyset\}) = h_{x,y,F}(K'). \tag{4.2}$$

By Lemma 4.4 there exists a closed set $K' \subseteq \mathbb{R}^2$ such that the following holds:

(i) For every $x \in [a, b]$, we have

$$\lambda(\{y \in [0,2] : h_{x,y,[0,2]}(K') \geq 2 - \varepsilon/4\}) \geq 2 - \varepsilon/4,$$

(ii) For every $x \in [0, a-\varepsilon] \cup [b+\varepsilon, 1]$, we have

$$\lambda\left(\{y \in [-2,4] : h_{x,y,[-2,4]}(K') \geq \varepsilon/12\}\right) \leq \varepsilon/12.$$

Let $K = P^{-1}(K') \cap \{b \geq 0\}$, which guarantees the third required property of K. Since P is a continuous function and the set K' is closed, K is also closed. We show that K has the desired properties.

First, if $x \in [a, b]$ then, using (4.2) and property (i) from the definition of K', we get that

$$\begin{aligned}
\int_{[0,2]} \lambda_1([0,2] \cap K \cdot (1,x,y))dy &= \int_{[0,2]} h_{x,y,[0,2]}(K')dy \\
&\geq (2-\varepsilon/4)\lambda(\{y \in [0,2] \colon h_{x,y,[0,2]}(K') \geq 2-\varepsilon/4\}) \\
&\geq (2-\varepsilon/4)(2-\varepsilon/4) \\
&\geq 4-\varepsilon.
\end{aligned}$$

Thus we obtained the first required property of K.

If $x \notin [a-\varepsilon, b+\varepsilon]$ then, using (4.2), property (ii) from the definition of K', and the fact that $h_{x,y,[-2,4]}(K') \leq 6$, we get

$$\begin{aligned}
\int_{[-2,4]} \lambda_1([-2,4] \cap K \cdot (1,x,y))dy &= \int_{[-2,4]} h_{x,y,[-2,4]}(K')dy \\
&\leq 6\lambda(\{y \in [-2,4] \colon h_{x,y,[-2,4]}(K') \geq \varepsilon/12\})+ \\
&\qquad + \frac{\varepsilon}{12}\lambda(\{y \in [-2,4] \colon h_{x,y,[-2,4]}(K') \leq \varepsilon/12\}) \\
&\leq \frac{6\varepsilon}{12} + \frac{\varepsilon 6}{12} \\
&\leq \varepsilon.
\end{aligned}$$

Thus we obtained the second required property of K. □

Proof of Lemma 4.4 using Lemma 4.2. We first provide an informal overview to build intuition, and then proceed to the formal construction.

We build our set as a compact subset of a rectangle $R = [-N, N] \times [y_0, y_1]$, where N is great enough such that for each $x \in [0,1],\ y \in [0,2]$ and $s \in [0,2]$ the curve $C_{x,y,s}$ meets both horizontal sides of R. We consider very thin horizontal stripes $R_1, R_2, \ldots, R_M$ of the initial rectangle and we manage them independently. We define a function F that determines a subset $F(R_i)$ of each stripe R_i, such that the following holds for some sets $S_i \subseteq T_i \subseteq [0,1] \times [-2,4]$:

$$(x,y) \in S_i \Rightarrow h_{x,y,[0,2]}(F(R_i)) \geq 2 - \varepsilon, \tag{4.3}$$

$$(x,y) \notin T_i \Rightarrow h_{x,y,[-2,4]}(F(R_i)) \leq \varepsilon/M. \tag{4.4}$$

We define $K = \bigcup_{i=1}^{M} F(R_i)$. It is sufficient to produce $(S_i)_{i=1}^{M}$ and $(T_i)_{i=1}^{M}$ such that

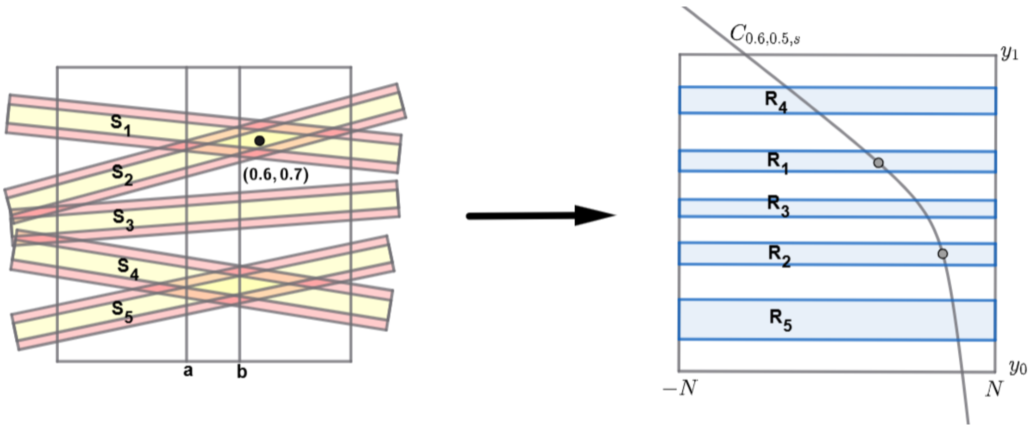

Figure 3: Rectangles S_i and the corresponding rectangles R_i.

(a) $x_0 \in [a,b] \Rightarrow \mathrm{Sec}^{x_0}(\bigcup S_i \cap ([0,1] \times [0,2])) > 2 - \varepsilon$,

(b) $x_0 \notin [a,b]_\varepsilon \Rightarrow \mathrm{Sec}^{x_0}(\bigcup T_i \cap ([0,1] \times [-2,4])) \leq \varepsilon$.

We choose the sets S_i to be rectangles and T_i to be exactly twice as wide rectangles. Furthermore, each rectangle S_i is centered within the corresponding T_i (see Figure 3).

During the proof, first we choose the sets S_i and T_i as above. The slope of center line of S_i determines the second coordinate of the center of the stripe R_i. The height of R_i is responsible for property (4.4), so it depends only on ε, M and on the second coordinate of the center line of R_i.

Choosing sets S_i and T_i. By Lemma 4.2 we define a closed set $L \subseteq \mathbb{R}^2$ such that L is the union of lines with slope between $-g(y_1)$ and $-g(y_0)$ and

- for every $t \in [a,b]$, we have
$$\mathrm{Sec}^t(L \cap ([0,1] \times [0,2])) > 2 - \varepsilon; \tag{4.5}$$
- for every $t \notin [a,b]$, we have
$$\mathrm{Sec}^t(L \cap ([0,1] \times [-2,4])) = 0.$$

By Claim 4.3 there is a δ such that for every $r < \delta$ and $p \notin [a,b]_\varepsilon$, we get
$$\mathrm{Sec}^p(L_r \cap ([0,1] \times [-2,4])) < \varepsilon. \tag{4.6}$$

Fix $d < \delta/2$. Observe that L_d is a union of open stripes (since L is a union of lines and we thicken each line to a stripe). The union of these stripes covers the compact set $L \cap ([0,1] \times [-2,4])$, thus there is a finite subcover. We choose a finite subcover and cut off the end of these stripes (we only consider finite parts of the stripes that are inside $[-2,4] \times [-2,4]$), thus we obtain rectangles. These are the rectangles S_i and we define rectangles T_i to be twice as wide rectangles as S_i with the same center line (see the red parts is Figure 3). Using (4.8) and the fact that $L \subseteq \bigcup S_i$, we establish property (a). Observe that $\bigcup T_i \subseteq L_{2d}$ thus, applying (4.6), for every $p \notin [a,b]_\varepsilon$, we get

$$\mathrm{Sec}^p(([0,1]\times[-2,4]) \cap \bigcup T_i) < \varepsilon.$$

Therefore, we established property (b).

We denote the slope of the center line of the rectangle T_i by τ^i. Since $-g(y_1) < \tau^i < -g(y_0)$ we can define $c_i = g^{-1}(-\tau^i)$. Observe that $y_0 < c_i < y_1$. The equation of the center line of the rectangle T_i is $(-\tau_i)x + y = m_i$ for some $m_i \in \mathbb{R}$. The set T_i is a bounded subset of $\{(x,y) \in \mathbb{R}^2 : \; |(-\tau_i)x + y - m_i| \le 2d\}$. Furthermore,

$$T_i \supseteq \{(x,y) \in [0,1]\times\mathbb{R} : \; g(c_i)x + y \in [m_i - 2d, m_i + 2d]\}, \tag{4.7}$$

while the rectangle S_i is a bounded subset of

$$\{(x,y) \in \mathbb{R}^2 : \; g(c_i)x + y \in [m_i - d, m_i + d]\}. \tag{4.8}$$

Setting the second coordinates and the heights of the rectangles R_i. We set the second coordinate of the center line of the rectangle R_i as c_i. Now we determine the height of R_i and we denote it by h_i. We would like to choose h_i such that for every $-h_i < t < h_i$:

$$|g(c_i + t) - g(c_i)| \le d/3. \tag{4.9}$$

Since g is continuous such an h_i exists.

Determining $F(R_i)$. We define

$$U_i = \left\{ \frac{1}{-q} : \; q \notin \left[m_i - \frac{4d}{3}, m_i + \frac{4d}{3} \right] \right\} \quad F_i = \left\{ \frac{1}{-q} : \; q \notin \left(m_i - \frac{5d}{3}, m_i + \frac{5d}{3} \right) \right\}.$$

Observe that F_i is a closed set and U_i is an open set containing F_i. From elementary calculations we get that for $(x,y) \in [0,1]\times[-2,4]$ the curvature of $C_{x,y,-}$ is uniformly bounded. Using this fact we can apply Lemma 3.2 for the closed set $\tan^{-1}(F_i)$, open set $\tan^{-1}(U_i)$, and rectangle R_i, we obtain a compact subset $F(R_i)$ of each R_i such that:

(i) $\tau_{x,y}(R_i) \subseteq F_i \Rightarrow h_{x,y}(F(R_i)) \le \varepsilon/M$,

(ii) $\tau_{x,y}(R_i) \cap U_i = \emptyset \Rightarrow h_{x,y}(F(R_i)) \ge h_{x,y}(R_i) - \varepsilon$.

From property (ii), it easily follows that $h_{x,y,[0,2]}(F(R_i)) \ge 2 - \varepsilon$.

We have to check (4.3) and (4.4).

Fix $(x,y) \in S_i$. From (4.8), (4.9) and $|x| \le 1$ we obtain that for every $t \in [-h_i, h_i]$, we have

$$(g(c_i + t)x + y) \in \left[m_i - \frac{4d}{3}, m_i + \frac{4d}{3} \right].$$

For every $p \in R_i$, the second coordinate of p is between $c_i - h_i$ and $c_i + h_i$. That means that for every $p \in R_i$, from (4.1) we have

$$\tau_{x,y}(p) \in \left\{ \frac{1}{-(g(c)x+y)} : c \in [c_i - h_i, c_i + h_i] \right\}.$$

Hence $\tau_{x,y}(p) \notin U_i$ for every $p \in R_i$. Therefore, from the definition of $F(R_i)$ property (4.3) is established.

Fix $(x,y) \notin T_i$. Then from (4.7), (4.9) and $|x| \leq 1$ we obtain that for every $t \in [-h_i, h_i]$, we have

$$(g(c_i + t)x + y) \notin \left[m_i - \frac{5d}{3}, m_i + \frac{5d}{3} \right].$$

Hence $\tau_{x,y}(p) = \frac{1}{-(g(c)x+y)} \in F$ (where c is the second coordinate of p) for every $p \in R_i$. Therefore, from the definition of $F(R_i)$, property 4.4 is established.

□

Proof of Lemma 4.2. The proof is based on the method of duality.

For every point $(a,b) \in \mathbb{R}^2$ we define $L(a,b) \subset \mathbb{R}^2$ as the set of points on the line $\{ax + b = y\}$. For any set $S \subset \mathbb{R}^2$ we define the *line set* $L(S)$ as $\bigcup_{p \in S} L(p)$. We show a set $K \subseteq \mathbb{R}^2$ such that $L(K) \subseteq \mathbb{R}^2$ has the desired properties.

Observe that $(t,y) \in L(K) \Leftrightarrow y \in (K \cdot (t,1))$. Without loss of generality, we can assume that $R = [0,1]^2$. Then for $t \in [0,1]$

$$\mathrm{Sec}^t(L(K) \cap R) = \lambda_1(L(K) \cap (\{t\} \times [0,1])) = \lambda_1((K \cdot (t,1)) \cap [0,1]).$$

The set $(K \cdot (t,1))$ can be considered as the orthogonal projection of K to the line $\{x/t = y\}$ magnified by $\sqrt{1+t^2}$, that is $K \cdot (t,1) = |(t,1)| \cdot p_{(t,1)}(K)$. Since for $t \in [0,1]$ we have $1 \leq \sqrt{1+t^2} < 2$, it is enough to show a set K such that for every $t \in [x_0, x_1]$, we have

$$\lambda(p_{(t,1)}(K) \cap [0,2]) > 2 - \varepsilon$$

and for every $t \notin [x_0, x_1]$, we have

$$p_{(t,1)}(K) \leq \varepsilon.$$

Furthermore, $K \subset [\tau_1, \tau_2] \times \mathbb{R}$ so the line set of K contains only lines with slope between τ_1 and τ_2.

In [9] Talagrand proved the following lemma.

Lemma 4.5. *For every $\varepsilon > 0$, closed interval $[a,b] \subset \mathbb{R}$ and closed rectangle R, there is an elementary closed set $L \subseteq R$ such that*

(i) $t \in [x_0, x_1] \Rightarrow \lambda(p_{(t,1)}(R)) - \lambda(p_{(t,1)}(L)) \leq \varepsilon$

(ii) $t \notin [x_0, x_1]_\varepsilon \Rightarrow \lambda(p_{(t,1)}(L)) \leq \varepsilon.$

Using the Lemma above and Lemma 3.3 we construct a proper set K. Let $[\tau_1, \tau_2] \times [0, N] = K_0$, where N is large enough that for every $t \in [0,1]$, we have $[0,2] \subseteq p_{(t,1)}(K_0)$. We define K_0' as a subset of K_0 as in Lemma 3.3 for $\varepsilon/2$ (in this case, we consider lines in Lemma 3.3). We define K_1 as the union of a subset of each rectangle in K_0 as in Lemma 4.5 for $\varepsilon/(4N_0)$, where N_0 is the number of rectangles in K_0. We define K_1' as a subset of K_1 as in Lemma 3.3 for $\varepsilon/8$.

We iterate this algorithm. In step i we define K_i as the union of a subset of each rectangle in K_{i-1}' as in Lemma 3.3 for $2^{-2i}\varepsilon/(N_i)$, where N_i denotes the number of rectangles in set K_{i-1}. We also define K_i' as a subset of K_i as in Lemma 4.5 for $2^{-2i-1}\varepsilon$.

Observe that

$$K_0 \supseteq K_1 \supseteq K_1' \supseteq K_2 \supseteq \ldots .$$

For every $i \in \mathbb{N}$, the set K_i is compact; therefore

$$L = \bigcap_{i \in \mathbb{N}} K_i$$

is compact. By induction, we have the following.

(i) $t \in [x_0, x_1] \Rightarrow \lambda(p_{(t,1)}(K_0)) - \lambda(p_{(t,1)}(K_i)) \leq \varepsilon_0 - \varepsilon_0/2^{-2i}$

(ii) $t \notin [x_0, x_1]_{\varepsilon_0/2^{-2i}} \Rightarrow \lambda(p_{(t,1)}(K_i)) \leq \varepsilon_0/2^{-2i}$

Since for every $t \in [x_0, x_1]$ the set $p_{(t,1)}(K_0)$ contains $[0,2]$, we have

(i) $t \in [x_0, x_1] \Rightarrow \lambda([0, \sqrt{t^2+1}] \setminus p_{(t,1)}(L)) \leq \varepsilon$

(ii) $t \notin [x_0, x_1] \Rightarrow \lambda(p_{(t,1)}(L)) \leq \varepsilon$.

Therefore, we proved Lemma 4.2.

$\square$

5 Motion of the square $[0,1]^2$

Definition 5.1 (Motion of the plane)**.** A function $M\colon [0,1] \to \mathrm{Isom}^+(\mathbb{R}^2)$ is called a *motion* if it is continuous, where $\mathrm{Isom}^+(\mathbb{R}^2)$ denotes the set of orientation preserving isometries of $\mathbb{R}^2$. For simplicity, we denote $\mapsto M(t)(p)$ by $M_t(p)$.

In [1], Besicovitch proved that any unit line segment can be fully rotated in an arbitrarily small area, thereby answering the Kakeya needle problem. Later, in [5], Davies strengthened this result by showing that finitely many parallel segments can be rotated simultaneously in a small area. Márk Kökényesi further extended this result: in [8], he proved that a unit square can be fully rotated such that every initially vertical line segment sweeps an area less than ε.

In this section, we prove the following theorem. Its main significance lies in the fact that it demonstrates that the results of Davies and Kökényesi do not follow directly from Besicovitch's result.

Theorem 5.2. *For every $0 < a < b < 1$ and $\varepsilon > 0$, there exists a continuous motion of the square $[0,1]^2$ such that every initially vertical line segment with $x-$coordinate in $[0, a-\varepsilon] \cup [b+\varepsilon, 1]$ sweeps an area less than ε and every initially vertical line segment with $x-$coordinate in $[a, b]$ sweeps an area greater than 1.*

The next statement follows quickly from the theorem.

Corollary 5.3. *For every $0 < a < b < 1$ and $\varepsilon > 0$, the unit square can be fully rotated such that every initially vertical line segment with $x-$coordinate in $[0, a-\varepsilon] \cup [b+\varepsilon, 1]$ sweeps an area less than ε and every initially vertical line segment with $x-$coordinate in $[a, b]$ sweeps an area greater than 1.*

Proof. By the Theorem 5.2, the unit square can be moved in such a way that every initially vertical segment whose first coordinate lies in $[a, b]$ sweeps an area of at least 1, while every initially vertical segment whose first coordinate lies in $[0,1] \setminus [a,b]_\varepsilon$ sweeps an area of at most $\varepsilon/2$.

Kökényesi [8] showed that for every $\varepsilon > 0$, the unit square can be fully rotated such that every initially vertical segment sweeps an area of at most ε.

Therefore, we can realize the motion described in the statement as follows. First, we move the square according to the construction given in the first paragraph. Then, we reverse this motion, returning the square to its initial position. Finally, we apply Kökényesi's result to perform a full rotation of the square, ensuring that every segment sweeps an area of Lebesgue measure at most $\varepsilon/2$. $\square$

Notation 5.4. For a set $S \subseteq \mathbb{R}^3$ we denote $S + (\{0\} \times B_r(0))$ by S_r, where $B_r(0)$ denotes the open disk of radius r centered at the origin.

We need the following claim.

Claim 5.5. *Let $A \subseteq \mathbb{R}^3$ be a closed set that is the union of planes from V, defined at the beginning of Section 4, and $F \subseteq \mathbb{R}$ be a compact set and let $N, \varepsilon > 0 \in \mathbb{R}$ be arbitrary real numbers. Furthermore, let us assume that for every $x_0 \in F$ the set $A \cap \{x = x_0\}$ has 2-dimensional Lebesgue measure less than $\varepsilon/2$. Then, there exists a $\delta > 0$ such that for every $0 < r < \delta$ and $x_0 \in F$, we have*

$$\lambda_2\left(A_r \cap [-N, N]^3 \cap \{x = x_0\}\right) \leq \varepsilon.$$

Proof. The proof of this claim is similar to that of Claim 4.3. We define the function

$$f_r(x) = \lambda_2\left(A_r \cap [-N, N]^3 \cap \{x = x_0\}\right),$$

which is upper semi-continuous in x for every r. Moreover, we have $f_r(x) \to f_0(x) \leq \varepsilon/2$ as $r \to 0$, monotonically from above. By compactness of F it is not hard to see that there exists $\delta > 0$ such that for all $r < \delta$ and all $x \in F$, we have $f_r(x) \leq \varepsilon$. $\square$

In the following, we apply Theorem 4.1 and Claim 5.5 to prove Theorem 5.2.

Proof of Theorem 5.2. We construct the desired motion using two different types of moves. The first is called a *filling* motion and the second a *setting* motion.

First we construct finitely many distinct filling motions $M^1, M^2, \ldots, M^n$ of the square such that:

(i) if $x_0 \in [a, b]$, then the union of the regions swept by the initially vertical segment at x_0 during the motions $M^1, M^2, \ldots, M^n$ has area at least 1;

(ii) if $x_0 \notin [a, b]_\varepsilon$, then the union of the regions swept by the initially vertical segment at x_0 during the motions $M^1, M^2, \ldots, M^n$ has area at most $\varepsilon/2$.

Then we link these motions with setting motions - during which every initially vertical segment sweeps an area with Lebesgue measure at most $\varepsilon/(2n)$. Therefore, the whole motion looks as follows. The square moves according to M^1, then applying a setting motion, the square moves to position M_0^2, then it does the M^2 motion and so on. The square alternates between filling motions and setting motions. (Since motions are defined as functions from $[0, 1]$ to $\mathrm{Isom}^+(\mathbb{R}^2)$, every motion in question has to be reparameterized such that applying them after each other we get a motion.)

In the following, we construct the setting motions and the filling motions. The case of the setting motion is simpler since they are independent of each other.

Setting motion. The next theorem proves the existence of the setting motion.

Theorem 5.6. *For every $\varepsilon > 0$ and two congruent copies S_1, S_2 of the square $[0, 1]^2$, there exists a motion M, such that $M_0([0, 1]^2) = S_1$, $M_1([0, 1]^2) = S_2$ and every initially vertical segment sweeps an area less than ε.*

In [8], Kökényesi essentially proved this theorem, however, he did not state it in this sense.

Proof of Theorem 5.6. As Kökényesi showed, a square can be rotated by an arbitrary angle during which segments sweep a small area. Using Pál joins, the square can also be translated in that manner, in the following way

Originally, Pál joins make it possible to move any segment to a translated copy, such that during the motion the segment sweeps a small area. First we translate the segment far away in direction parallel with itself, next we tilt by a small angle, so when we translate it back, it exactly arrives at the end of its copy. Finally we tilt it back to its original position (see Figure 4).

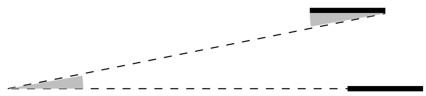

Figure 4: Illustration of the Pál join construction

This movement can be done not with a segment, but with a whole square.

Therefore, to get a motion as in Theorem 5.6 we rotate S_1 to a parallel position as S_2, then with Pál joins we move it to S_2. $\square$

We say that a setting motion has parameter ε, if every initially vertical segment sweeps an area less than ε.

Filling motion. Let A be a closed set as in Theorem 4.1. By Claim 5.5, there exists δ, such that for every $r < \delta$ and $x_0 \notin [0,1] \setminus [a,b]_\varepsilon$, we have

$$\lambda_2(A_r \cap [-2,4]^3 \cap \{x = x_0\}) \leq \varepsilon/2.$$

Fix such an $r < \delta$. The set A_r is the union of thickened planes. Since $A \cap [-2,4]^3$ is compact, we can choose finitely many planes (contained in A), whose union we denote by B, such that

$$(A \cap [-2,4]^3) \subseteq B_r.$$

For every $x_0 \in [0,1]$ we denote the set $B_r \cap \{x = x_0\}$ by B^{x_0}. We denote the number of planes in B with n.

Now we give n filling motions $M^1, M^2, \dots M^n$. Fix $x_0 \in [0,1]$. We give the filling motions not as motions of the square, but as motions of the segment $s = \{x_0\} \times [0,1]$. Such a motion naturally expands to a motion of the square.

Let P be an orthogonal projection onto the plane $\{x = 0\}$, that is

$$\begin{aligned} P\colon \mathbb{R}^3 &\to \mathbb{R}^2 \\ (x,y,z) &\mapsto (y,z). \end{aligned}$$

We denote $P(B^{x_0}) \cap [-2,4]^2$ by B'. The set B' is the union of n trapezoids, $T_1, T_2 \dots, T_n$.

We define the motion M^i as a motion during which the segment s sweeps precisely those parts of the trapezoid T^i that it can reach while remaining parallel to the bases of T^i, and such that every point of the unit square remains within the square $[-2,4]^2$ throughout the motion.

Observe that during the motion M^i, the segment s sweeps over the set $B' \cap [0,2]^2$, regardless of the choice x_0.

Notice that an initially vertical segment with $x-$coordinate $x' \in [0,1]$, sweeps an area that is a subset of $P(B^{x'})$. Moreover, it sweeps completely the set $P(B^{x'}) \cap [0,2]^2$. Therefore, after the motions $M^1, M^2, \ldots, M^n$ for every $x' \in [0,1]$, the initially vertical segment with $x-$coordinate x' sweeps over $P(B \cap (\{x'\} \times [0,2] \times [0,2]))$ but remains inside $P(B_r \cap (\{x'\} \times [-2,4] \times [-2,4]))$.

From the definition of set B and B_r we get that for every $x' \in [a,b]$, the corresponding segment sweeps at least an area $4 - \varepsilon > 1$ and for every $x' \notin [a,b]_\varepsilon$, the segment sweeps over an area less than $\varepsilon/2$.

Finally, we link the filling motions with setting motions with parameter $\varepsilon/(2n)$. Therefore, segments in $[a,b]$ sweep an area with Lebesgue measure at least 1, and segments in $[0,1] \setminus [a,b]_\varepsilon$ sweep an area with Lebesgue measure at most ε. $\square$

6 The strong Peano curve

Definition 6.1 (Peano curve)**.** The *Peano curve* is a continuous and surjective map from $[0, 1]$ to $[0, 1]^2$.

The interesting property of this function, which might be counter-intuitive at first, is that while its domain is 1-dimensional, its image has a non-zero 2-dimensional Lebesgue measure. The main objective of this section is to prove the following theorem.

Theorem 6.2. *There exists a motion M of the plane such that*

(i) $\lambda(M[(0.5, 0.5)]) \geq 1/2$,

(ii) $p \in [0, 1]^2 \setminus \{(0.5, 0.5)\} \Rightarrow \lambda(M[p]) = 0$,

where $M[p]$ denotes the set swept by the point p during the motion M.

In some sense, this theorem gives an example for a so-called strong Peano curve. While the middle of the square is Peano-like (however it does not necessarily fill the square $[0, 1]^2$), the image of any other point in the square $[0, 1]^2$ has Lebesgue measure zero.

First, we give a construction of the Peano curve, to illustrate the method which is used for the construction of the strong Peano curve.

Theorem 6.3. *There exists a Peano curve.*

Proof. We construct a closed subset $F \subseteq [0, 1]$ and a continuous surjective map $m\colon F \to [0, 1]^2$. Then, by Tietze's Extension Theorem, m extends to a continuous map $\tilde{m}\colon [0, 1] \to [0, 1]^2$ such that $\tilde{m}|_F = m$.

First, we define by induction a sequence $\mathcal{S}_i$ of collections of subsets of $[0, 1]^2$. We set $\mathcal{S}_1 = \{[0, 1]^2\}$. To construct $\mathcal{S}_{i+1}$ from $\mathcal{S}_i$, divide each set $S \in \mathcal{S}_i$ into n closed (not necessarily disjoint) subsets of diameter at most $\frac{\mathrm{diam}(S)}{2}$, for some integer $n \geq 1$. The resulting sets form $\mathcal{S}_{i+1}$, see Figure 5.

Each set in $\mathcal{S}_i$ is indexed by an integer vector $\mathbf{v}$, and for any $S_{\mathbf{v}} \in \mathcal{S}_i$, we label its children in $\mathcal{S}_{i+1}$ as $S_{(\mathbf{v}|1)}, \ldots, S_{(\mathbf{v}|n)}$, where $(\mathbf{v}|j)$ denotes the concatenation of $\mathbf{v}$ with j. Let I_i denote the set of all such index vectors of length i. For each $i \in \mathbb{N}$, we have

$$\bigcup_{S \in \mathcal{S}_i} S = [0, 1]^2.$$

Similarly, we define a sequence $\mathcal{F}_i$ of collections of disjoint closed intervals in $[0, 1]$, starting with $\mathcal{F}_1 = \{[0, 1]\}$. Each interval in $\mathcal{F}_i$ is indexed by an element of I_i, in such

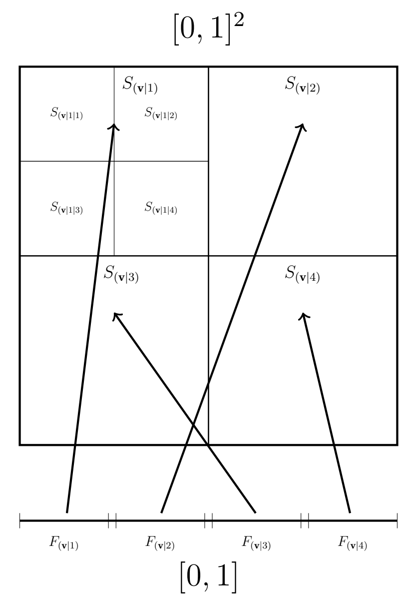

Figure 5: $\mathcal{F}_i, \mathcal{S}_i$ and a part of $\mathcal{S}_{i+1}$.

a way that for any $(\mathbf{v}|j) \in I_i$, the corresponding interval $F_{(\mathbf{v}|j)} \in \mathcal{F}_i$ is a subinterval of $F_{\mathbf{v}} \in \mathcal{F}_{i-1}$. Define

$$F^i := \bigcup_{F \in \mathcal{F}_i} F, \quad \text{and} \quad F := \bigcap_{i \in \mathbb{N}} F^i.$$

Then F is a closed subset of $[0,1]$, and for every $x \in F$, there exists a unique nested sequence of intervals $F_{\mathbf{v}^i} \in \mathcal{F}_i$ such that $x \in F_{\mathbf{v}^i}$ for all i.

We now define $m\colon F \to [0,1]^2$ by

$$m(x) := \bigcap_{i \in \mathbb{N}} S_{\mathbf{v}^i}.$$

Since the diameters of the sets $S_{\mathbf{v}^i}$ tend to zero and they are nested, the intersection contains exactly one point, thus $m(x)$ is well-defined.

It is straightforward to check that m is continuous and surjective. Therefore, the extended map $\tilde{m}\colon [0,1] \to [0,1]^2$ is a continuous surjection—this is the desired Peano curve. $\square$

Before proving Theorem 6.2, we introduce some notation and prove an auxiliary lemma that will be used throughout the argument.

For any vector $\mathbf{v} \in \mathbb{R}^2$ and angle $\alpha \in [-\pi, \pi]$ let $\mathrm{rot}_\alpha(\mathbf{v})$ denote the vector we obtain by rotating $\mathbf{v}$ by angle α (counterclockwise, around the origin). For any set $H \subseteq \mathbb{R}^2$ define

$$H(\alpha, \mathbf{v})\colon = H + \mathrm{rot}_\alpha(\mathbf{v}),$$

where $+$ denotes the translation of the set by the given vector.

Lemma 6.4. *Let $T \subseteq [0,1]^2$ be a rectangle, and let $\varepsilon > 0$ be a real number. Then there exist finitely many rectangles $T_1, T_2, \ldots, T_n \subset T$, real numbers $r_1, r_2, \ldots, r_n \in (-\varepsilon/2, \varepsilon/2)$, and* *a real number* $0 < \varepsilon' < \varepsilon/2$*, such that the following holds for any vector* $\mathbf{w} = (w_1, w_2) \in [-1/2, 1/2]^2$ *with* $|w_1| > \varepsilon$*:*

(i)

$$\lambda\left(\bigcup_{k=1}^{n} \bigcup_{\alpha\in[-\varepsilon',\varepsilon']} T_k(r_k + \alpha, \mathbf{w})\right) < 2\varepsilon,$$

(ii)

$$\lambda\left(\bigcup_{1\le k\le n} T_k\right) > \lambda(T) - \varepsilon.$$

Proof. By Theorem 4.1 there is a closed set A in $\mathbb{R}^3$ that is the union of planes from V, such that the following hold:

(i) The Lebesgue measure of $A \cap (\{1/2\} \times [0, 2] \times [0, 2])$ is greater than $4 - \varepsilon/2$;

(ii) for every $h \in [0, 1] \setminus (1/2 - \varepsilon/2, 1/2 + \varepsilon/2)$, the Lebesgue measure of $A \cap (\{h\} \times [-2, 4] \times [-2, 4])$ is less than ε;

(iii) all planes from V that form the set A intersect the plane $\{x = 0\}$ in such a way that the angle their intersections make with the z-axis lie in the interval $(-\varepsilon/2, \varepsilon/2)$.

Let $\delta > 0$ be a real number as in Claim 5.5, for $F = [0, 1] \setminus (1/2 - \varepsilon/2, 1/2 + \varepsilon/2)$ and $N = 2$. Let $d \in \mathbb{R}$ be such that $d < \delta/2$. The set A_d is the union of thickened planes that cover the compact set $A \cap [-2, 2]^3$, where A_d is the set derived from A as described in Notation 5.4. Since $A \cap [-2, 2]^3$ is compact and the covering is open, there exists a finite subcover. We denote the closure of these finitely many thickened planes by $Q_d^1, Q_d^2, \dots, Q_d^m$, their union by B, and the original planes by $Q^1, Q^2, \dots, Q^n$. We define ε' to be less than $\min(d, \varepsilon/2)$. From the definition of d and ε', for every $t \in F$ we have that

$$\lambda_2\Big((B \cap \{x = t\}) + (\{0\} \times B_{\varepsilon'}(0))\Big) \le \varepsilon. \tag{6.1}$$

Define P as the orthogonal projection in the direction x that is

$$P(x, y, z) = (y, z).$$

Observe that $P(B \cap \{x = 1/2\}) = P(B^{1/2})$ is the union of finitely many stripes. We now label these stripes as $R_1, R_2, \dots, R_m$, such that $R_i = P((Q_d^i) \cap \{x = 1/2\})$. Let $T_1, T_2, \dots, T_n \subseteq T$ be rectangles, such that they have the following properties:

(i) Each rectangle T_i is a subset of one of the stripes R_j.

(ii) For every $1 \le i \le n$, we have $\operatorname{diam}(T_i) \le \operatorname{diam}(T)/2$.

(iii) $\lambda\left(\bigcup_{k=1}^{n} T_k\right) > \lambda(P(B^{1/2}) \cap T) - \varepsilon/2$.

These will serve as the rectangles appearing in the statement of the lemma. It remains to determine the real numbers r_i, and to verify that the rectangles satisfy the required properties.

Let r_i be defined as follows. There exists an index j such that the stripe R_j contains T_i. Then r_i is the angle between the z-axis and the line $Q^j \cap \{x = 0\}$. From the third property in the definition of set A, it is clear that r_i lies in the interval $(-\varepsilon/2, \varepsilon/2)$.

First, we check the second property in Lemma 6.4. The union of the rectangles in $P(B^{1/2})$ covers the square $[0, 2]^2$ with an error of at most $\varepsilon/2$, thus, using property (iii) in the definition of T_i rectangles, $\bigcup T_i$ covers $T \subset [0, 1]^2$ with an error of at most ε.

Finally, we show the first property. By the definition of T_i, there exists R_j, such that $T_i \subseteq R_j$. Therefore, we have

$$T_i + \mathrm{rot}_{r_i}(\mathbf{w}) \subseteq (R_j + \mathrm{rot}_{r_i}(\mathbf{w})) \cap [-2, 2]^2. \tag{6.2}$$

Since the line $Q^j \cap \{x = 0\}$ forms an angle r_i with the $z-$axis, we have that $\mathrm{rot}_{r_i}((0, 1))$ is parallel with $P(Q^j \cap \{x = 1/2\})$. Therefore, $R_j + \mathrm{rot}_{r_i}((0, 1)) = R_j$. Furthermore,

$$R_j + \mathrm{rot}_{r_i}(\mathbf{w}) = R_j + \mathrm{rot}_{r_i}((w_1, 0)). \tag{6.3}$$

The following statement comes from the fact that the translation of R_j by a vector $\mathbf{v}$ perpendicular to its center line coincides either with $P(B^{1/2+||\mathbf{v}||})$ or with $P(B^{1/2-||\mathbf{v}||})$:

$$R_j + \mathrm{rot}_{r_i}((w_1, 0)) \subseteq P(B^{1/2+w_1}) \cup P(B^{1/2-w_1}). \tag{6.4}$$

Using the definition of $T_i(r_i, \mathbf{w})$, (6.2), (6.3), and (6.4), we get

$$T_i(r_i, \mathbf{w}) \subseteq [-2, 2] \cap \left(P(B^{1/2+w_1}) \cup P(B^{1/2-w_1})\right). \tag{6.5}$$

Observe that for any vector $\mathbf{w} \in [-1/2, 1/2]^2$ and angles α, β we have

$$|| \mathrm{rot}_\alpha(\mathbf{w}) - \mathrm{rot}_\beta(\mathbf{w})|| < |\alpha - \beta| \cdot ||\mathbf{w}||. \tag{6.6}$$

Let $q_1, q_2, \ldots, q_n \in (-\varepsilon/2, \varepsilon/2)$ be arbitrary real numbers. From (6.6) and the fact that $||w|| \leq 1$, we have

$$\bigcup_{i=1}^{n} \bigcup_{\alpha \in [-\varepsilon', \varepsilon']} T_i(q_i + \alpha, \mathbf{w}) \subseteq \left(\bigcup_{1 \leq i \leq n} T_i(q_i, \mathbf{w}) + B_{\varepsilon'}(0) \right) \subseteq \left(\bigcup_{1 \leq i \leq n} T_i(q_i, \mathbf{w}) \right) + B_{\varepsilon'}(0). \tag{6.7}$$

Therefore, using (6.7) and (6.5), we have

$$\begin{aligned} \bigcup_{k=1}^{n} \bigcup_{\alpha \in [-\varepsilon', \varepsilon']} T_k(r_k + \alpha, \mathbf{w}) &\subseteq \left(\bigcup_{1 \leq k \leq n} T_k(r_k, \mathbf{w}) \right) + B_{\varepsilon'}(0) \\ &\subseteq [-2, 2]^2 \cap \left(P(B^{1/2+w_1}) \cup P(B^{1/2-w_1})\right) + B_{\varepsilon'}(0). \end{aligned} \tag{6.8}$$

From (6.1) and the fact that $\varepsilon/2 < |w_1| < 1/2$, we have

$$\lambda\left(\left(P(B^{1/2\pm w_1}) + B_{\varepsilon'}(0)\right) \cap [-2,2]^2\right) \le \varepsilon$$

Therefore, using (6.8),

$$\lambda\left(\bigcup_{k=1}^{n} \bigcup_{\alpha\in[-\varepsilon',\varepsilon']} T_k(r_k+\alpha, \mathbf{w})\right) \le 2\varepsilon,$$

thus we obtained the first property in Lemma 6.4. □

Remark 6.5. By a change of coordinates argument, Lemma 6.4 also holds if in the statement of the lemma we assume that the first coordinate of $\mathbf{w}$ is at least ε.

Now, we are ready to prove Theorem 6.2.

Proof of Theorem 6.2. The method of the proof will resemble that of Theorem 6.3.

We will define the motion by specifying, at each moment in time, the position of the center of the unit square and its angle of rotation. These two parameters will first be given continuously on a closed set $F \subset [0,1]$, whose complement is the union of countably many disjoint open intervals. On each of these intervals, we will then extend the motion linearly, both in terms of the position of the center and the angle of rotation. As a result, during the extension, each point of the square will only sweep sets of Lebesgue measure zero.

We define by induction a sequence $\mathcal{S}_i$ of collections of sub-rectangles of $[0,1]^2$ and simultaneously a sequence $\mathcal{R}_i$ of set of real numbers from $[-\pi,\pi]$. We also define a sequence of real numbers ε_i, starting with $\varepsilon_1 = 1/4$. We set $\mathcal{S}_1 = \{[0,1]^2\}$ and $\mathcal{R}_1 = \{0\}$, and index the elements of $\mathcal{S}_i$ and $\mathcal{R}_i$ with i dimensional integer vectors (just as in the proof of Theorem 6.3). The index set of $\mathcal{S}_i$ coincides with that of $\mathcal{R}_i$ and we denote it by $\mathcal{I}_i$. We construct $\mathcal{S}_{i+1}$ from $\mathcal{S}_i$ using Lemma 6.4 if i is odd and Remark 6.5 if i is even.

Similarly, we define a sequence $\mathcal{F}_i$ of collections of disjoint closed intervals in $[0,1]$, starting with $\mathcal{F}_1 = \{[0,1]\}$. Each interval in $\mathcal{F}_i$ is indexed by an element of $\mathcal{I}_i$, in such a way that for any $(\mathbf{v}|j) \in \mathcal{I}_i$, the corresponding interval $F_{(\mathbf{v}|j)} \in \mathcal{F}_i$ is a subinterval of $F_{\mathbf{v}} \in \mathcal{F}_{i-1}$. Define

$$F^i := \bigcup_{F\in\mathcal{F}_i} F, \quad \text{and} \quad F := \bigcap_{i\in\mathbb{N}} F^i.$$

We present the construction only in the case when i is odd; the case when i is even is handled in a very similar way.

We consider each set $S_{\mathbf{v}} \in \mathcal{S}_i$ in a coordinate system rotated by an angle $r_{\mathbf{v}} \in \mathcal{R}_i$. In this coordinate system, we apply Lemma 6.4 with $T = S_{\mathbf{v}} \subset [0,1]^2$ and $\varepsilon = \varepsilon_i/|\mathcal{I}_i|$. This way we obtain rectangles $T_1, T_2, \ldots, T_n$, real numbers $r_1, r_2, \ldots, r_n$ and $\varepsilon_{\mathbf{v}}$, such that

(i)

$$\lambda\left(\bigcup_{k=1}^{n}\bigcup_{\alpha\in[-\varepsilon_{\mathbf{v}},\varepsilon_{\mathbf{v}}]} T_k(r_k+\alpha,\mathbf{w})\right) < 2\varepsilon_i/|\mathcal{I}_i|,$$

(ii)

$$\lambda\left(\bigcup_{1\le k\le n} T_k\right) > \lambda(S_{\mathbf{v}})-\varepsilon_i.$$

Let $\varepsilon_{i+1}=min\{\varepsilon_{\mathbf{v}}\colon \mathbf{v}\in\mathcal{I}_i\}$ and let $S_{(\mathbf{v}|i)}=T_i$ and $r_{(\mathbf{v}|i)}=r_{\mathbf{v}}+r_i$ for every $1\le i\le n$. We denote the number of rectangles by $n(\mathbf{v})=n$. It is not hard to see that for every $i\in\mathbb{N}$ and $\mathbf{v}\in\mathcal{I}_i$ we have $r_{\mathbf{v}}\in(-\varepsilon_1,\varepsilon_1)$, that is

$$r_{\mathbf{v}}\in(-1/4,1/4). \tag{6.9}$$

Observe that if i is odd, then

$$\lambda\left(\bigcup_{S\in\mathcal{S}_{i+1}} S\right) > \lambda\left(\bigcup_{S\in\mathcal{S}_i} S\right)-\varepsilon_i. \tag{6.10}$$

As a consequence of (6.9), for every $\mathbf{v}\in\mathcal{I}_i$ we have $\cos(r_{\mathbf{v}})>1/2$. Therefore, for every $\mathbf{w}\in[0,1]^2$ with $|w_1|>2\varepsilon_i$, the absolute value of the first coordinate of $\mathrm{rot}_{r_{\mathbf{v}}}(\mathbf{w})$ is greater than ε_i, thus, from the construction of $\mathcal{S}_{i+1}$, we obtain

$$\lambda\left(\bigcup_{n=1}^{n(\mathbf{v})}\bigcup_{\alpha\in[-\varepsilon_{i+1},\varepsilon_{i+1}]} S_{(\mathbf{v}|n)}(r_{\mathbf{v}}+\alpha,\mathbf{w})\right)\le 2\varepsilon_i/|I_i|$$

($r_{\mathbf{v}}$ appears because previously we looked at $S_{\mathbf{v}}$ in a tilted coordinate system). Therefore, from the definition of $\mathcal{S}_{i+1}$ we have

$$\lambda\left(\bigcup_{S_{\mathbf{v}}\in\mathcal{S}_{i+1}}\bigcup_{\alpha\in[-\varepsilon_{i+1},\varepsilon_{i+1}]} S_{\mathbf{v}}(r_{\mathbf{v}}+\alpha,\mathbf{w})\right)\le 2\varepsilon_i. \tag{6.11}$$

Simultaneously, we chose n closed, disjoint sub-intervals of $F_{\mathbf{v}}\in\mathcal{F}_i$ and denote them by $F_{(\mathbf{v}|1)},F_{(\mathbf{v}|2)},\dots,F_{(\mathbf{v}|n)}$. On the limit set $F=\bigcap_i F^i$, each point $t\in F$ lies in a unique nested sequence of intervals

$$t\in F_{\mathbf{v}^{(1)}}\supset F_{\mathbf{v}^{(2)}}\supset\cdots,$$

with corresponding rectangles

$$S_{\mathbf{v}^{(1)}}\supset S_{\mathbf{v}^{(2)}}\supset\cdots,$$

and angles

$$r_{\mathbf{v}^{(1)}},\ r_{\mathbf{v}^{(2)}},\ \dots.$$

Since the diameters of the rectangles $S_{\mathbf{v}^{(i)}}$ tend to zero and the differences $|r_{\mathbf{v}^{(i+1)}} - r_{\mathbf{v}^{(i)}}| \leq \varepsilon_i \leq 2^{-i}$, both the centers and the rotation angles converge. We define the position of the center of the square at time $t \in F$ as the limit of the centers of $S_{\mathbf{v}^{(i)}}$, and the angle as the limit of the $r_{\mathbf{v}^{(i)}}$. We extend this motion to the interval $[0,1]$ as explained at the beginning of the proof.

We need to verify that every point $p \in [0,1]^2$, with $p \neq (1/2,1/2)$, sweeps a set of Lebesgue measure zero and $(1/2,1/2)$ sweeps a set with Lebesgue measure at least $1/2$ during the motion. As explained above, it is sufficient to show that the images of p and $(1/2,1/2)$ under the motion restricted to F have the desired measure.

The image of $(1/2,1/2)$ during the motion restricted to F is the set

$$\bigcap_{i=1}^{n} \bigcup_{S \in \mathcal{S}_i} S.$$

Using (6.10) and the fact that sets $\bigcup_{S \in \mathcal{S}_i} S$ are nested, we have

$$\lambda\left(\bigcap_{i=1}^{n} \bigcup_{S \in \mathcal{S}_i} S\right) > \lambda(\bigcup \mathcal{S}_1) - \sum_{i=1}^{n} \varepsilon_i \geq 1 - 1/2,$$

since $\mathcal{S}_1 = \{[0,1]^2\}$ and $\varepsilon_{i+1} < \varepsilon_i/2$ for every $i \in \mathbb{N}$, with $\varepsilon_1 = 1/4$.

Finally, we show that every point $p \in [0,1]^2$ with $p \neq (1/2,1/2)$ sweeps an area of Lebesgue measure 0. For that, fix an arbitrary $\varepsilon > 0$. We will construct a set of Lebesgue measure at most ε such that the image of p under the restricted motion lies entirely within this set. Let $p = (1/2,1/2) + \mathbf{w}$. By symmetry, we can assume that the first coordinate of $\mathbf{w}$ is non-zero.

Let i be an even number such that $\varepsilon_{i-1} < \min(\varepsilon/2, |w_1/2|)$. During the time $F_{\mathbf{v}}$ with $\mathbf{v} \in \mathcal{I}_i$, the image of $(1/2,1/2)$ is completely in $S_{\mathbf{v}}$, and the square is rotated with an angle that lies in $(r_{\mathbf{v}} - \varepsilon_i, r_{\mathbf{v}} + \varepsilon_i)$. Therefore, the image of $p = (1/2,1/2) + \mathbf{w}$ is in

$$\bigcup_{\alpha \in [-\varepsilon_i, \varepsilon_i]} S_{\mathbf{v}}(r_{\mathbf{v}} + \alpha, \mathbf{w}). \tag{6.12}$$

We defined $\mathcal{S}_i$ using Lemma 6.4 (or Remark 6.5). Using (6.12) and (6.11) we get that

$$\lambda\left(M[p]|_F\right) \leq \lambda\left(\bigcup_{S_{\mathbf{v}} \in \mathcal{S}_i} \bigcup_{\alpha \in [-\varepsilon_i, \varepsilon_i]} S_{\mathbf{v}}(r_{\mathbf{v}} + \alpha, \mathbf{w})\right) \leq 2\varepsilon_{i-1} < \varepsilon,$$

□

which completes the proof of Theorem 6.2.

References

[1] Besicovitch, A. S. On Kakeya's problem and a similar one. *Mathematische Zeitschrift 27* (1928), 312–320.

[2] Csörnyei, M. How to make Davies' result visible. *Bulletin of the London Mathematical Society 33* (2001), 59–66.

[3] Cunningham, F. Three Kakeya problems. *The American Mathematical Monthly 6*, 81 (1971), 582–592.

[4] Davies, R. O. On accessibility of plane sets and differentiation of functions of two real variables. *Mathematical Proceedings of the Cambridge Philosophical Society 48* (1952), 215–232.

[5] Davies, R. O. Some remarks on the Kakeyya problem. *Mathematical Proceedings of the Cambridge Philosophical Society 69* (1971), 417–421.

[6] Falconer, K. *Fractal Geometry: Mathematical Foundations and Applications*. John Wiley & Sons, 2013.

[7] Fefferman, C. The multiplier problem for the ball. *Annals of mathematics 94*, 2 (1971), 330–336.

[8] Kökényesi, M. An improvement on a result of Davies. *arXiv preprint arXiv:2411.11083* (2024).

[9] Talagrand, M. Sur la mesure de la projection d'un compact et certaines familles de cercles. *Bulletin des Sciences Mathématiques 104*, 2 (1980), 225–231.

[10] Wang, H., and Zahl, J. Volume estimates for unions of convex sets, and the Kakeya conjecture in three dimensions. *arXiv preprint arXiv:2502.17655* (2025).

MI nyilatkozat

Alulírott Fleiner Zsigmond nyilatkozom, hogy szakdolgozatom elkészítése során az alább felsorolt feladatok elvégzésére a megadott MI alapú eszközöket alkalmaztam:

Feladat	**Felhasznált eszköz**	**Felhasználás helye**	**Megjegyzés**
Nyelvhelyesség ellenőrzése	GPT-4o	Teljes dolgozat	
Ábra készítése	GPT-4o	4–5. ábra	
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