About intersections of convex bodies

THESIS

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1. Introduction

In this thesis, we consider two loosely related topics. First, we study the volume of intersection of convex bodies. The motivation came from studying quantitative Helly theorem by Bárány, Katchalski, Pach, [BKP82, BKP84].

In [Sch70], Schneider generalized the difference body in the following way.

$$D^n(K) = \left\{(x_1, x_2, \dots, x_{n-1}) \mid (C_1 + x_1) \cap (C_2 + x_2) \cap \dots \cap (C_{n-1} + x_{n-1}) \cap C_n \neq \emptyset\right\},$$

which, for $n = 2$ is clearly the set $C_2 - C_1$.

Here, we introduce a quantitative variant, where instead of requiring the intersection to be non-empty, we require it to have a certain volume.

Definition 1.0.0.1. *($\mathcal{K}$ set) Let $C_1, C_2, \dots, C_n \in \mathbb{R}^d$ be convex bodies, $x_1, x_2, \dots, x_n \in \mathbb{R}^d$ we define the $\mathcal{K}$-set by,*

$$\mathcal{K}(C_1, C_2, \dots, C_n) = \left\{(x_1, x_2, \dots, x_{n-1}) \mid \lambda((C_1+x_1)\cap(C_2+x_2)\cap\dots\cap(C_{n-1}+x_{n-1})\cap C_n) \geq 1\right\},$$

where λ denotes the Lebesgue measure.

The first section is about this set and the properties of the set. The main result is that this set is convex. This result is not from the literature.

It is a natural isoperimetric type problem to find the maximum of $\lambda(\mathcal{K}(C_1, C_2))$ for all pairs of convex bodies $C_1, C_2 \subset \mathbb{R}^2$ where $\lambda(C_1) = \lambda(C_2) = a \in \mathbb{R}^+$. We show that surprisingly, this maximum is not the disk for $a > 5.83$. The question for values of a close to 1 remains open. We use analytic tools to solve these questions.

Second, we discuss a classical open question, the inscribed square problem.

Let C be a Jordan curve. A polygon P is inscribed in C, if all vertices of P belong to C.

Problem 1.0.1. *Is there an inscribed square in every Jordan curve in* $\mathbb{R}^2$*?*

The problem was proposed by Otto Toeplitz in 1911. The question for an arbitrary curve is still open. Nevertheless, it is solved for many curves with special properties, for example piecewise analytic curves, locally monotone curves, and Lipschitz graphs. The problem with the general answer is that the sequence of squares can converge to a one-point quadrilateral. The special properties are needed to avoid this scenario. Here we will present the solution of the problem for locally monotone curves by Stromquist. [Str89]

The structure of the thesis is the following. In Chapter 2 we discuss the properties of the $\mathcal{K}$ set and show some examples and remarks. In Chapter 3, we talk about homology based on László Fehér's lecture notes [Feh24] to be able to present the proof of the inscribed square problem for locally monotone curves. Chapter 4 contains this proof for the inscribed square problem.

2. The $\mathcal{K}$ set

In this section, we study the $\mathcal{K}$ set.

Definition 2.0.0.1. *(Convex set) Let C be a set in $\mathbb{R}^n$, C is convex, if for every $x, y \in C$ and $\lambda \in [0,1]$, we have that $\lambda x + (1-\lambda)y \in C$*

Definition 2.0.0.2 ($\mathcal{K}$ function)**.** *Let $C_1, C_2, \ldots, C_n \subset \mathbb{R}^d$ be convex bodies, $f_1 = \chi_{C_1}$, $f_2 = \chi_{C_2}, \ldots f_n = \chi_{C_n}$. We define the function $\mathcal{K} : \mathbb{R}^{d\times(n-1)} \to \mathbb{R}$. as*

$$\mathcal{K}(x_1, x_2, \ldots, x_{n-1}) = \int_{\mathbb{R}^d} f_1(x_1 - y)f_2(x_2 - y)\ldots f_n(y)dy,$$

where χ denotes the characteristic function of a set. We note that

$$\mathcal{K}(x_1, x_2, \ldots, x_{n-1}) = \lambda(C_1 + x_1 \cap C_2 + x_2 \cap ... \cap C_{n-1} + x_{n-1} \cap C_n).$$

The $\mathcal{K}$-set of $C_1, \ldots, C_n$ is

$$\mathcal{K}(C_1, C_2, \ldots, C_n) = \{(x_1, x_2, \ldots, x_{n-1}) \in (\mathbb{R}^d)^{n-1} : \mathcal{K}(x_1, x_2, \ldots, x_{n-1}) \ \geq 1\} \subset (\mathbb{R}^d)^{n-1},$$

which is a superlevel of the $\mathcal{K}$ function.

Theorem 2.0.1. *(The $\mathcal{K}$ set is convex.)*

For any $C_1, C_2, ..., C_n \subset \mathbb{R}^n$ convex bodies $\mathcal{K}(C_1, C_2, ..., C_n)$ is convex.

Proof. First, we introduce the following notation.
For $x = (x_1, x_2, \ldots, x_n) \in (\mathbb{R}^d)^n, y \in \mathbb{R}^n, F(x,y) = f_1(x_1 - y)f_2(x_2 - y)\ldots f_n(y)$. Observe that

$$\mathcal{K}(x_1, x_2, \ldots, x_n) = \int_{\mathbb{R}^d} F(x,y)dy$$

It is sufficient to show that, $\mathcal{K}(x_1, x_2, \ldots, x_n)$ is logarithmically concave. That is because $\mathcal{K}$ is the superlevel of the function. Suppose that $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^{d\times(n-1)} \times \mathbb{R}^d$

$x_1 = (x_{1,1}, x_{1,2}, \ldots, x_{1,n})$, $x_2 = (x_{2,1}, x_{2,2}, \ldots, x_{2,n})$
We need to prove that $F(x, y)$ is logarithmically concave. The value of the function is either 0 or 1. We need to show that: For any $\lambda \in [0, 1]$

$$F\left((1-\lambda)(x_1, y_1) + \lambda(x_2, y_2)\right) \geq F(x_1, y_1)^{1-\lambda} F(x_2, y_2)^{\lambda}$$

When the right side is equal to 0, the inequality is trivial so, we need to check the case, when the right side is 1. We have to prove that the left side will always be 1, if the right is 1. Assume that the 0 vector is in every C_i. Then $x_1, y_1 \in C_i + x_{1_i}$ and $x_2, y_2 \in C_i + x_{2_i}$ In that case for every λ, $(1-\lambda)y_1 + \lambda y_2$ is in the intersection of the convex bodies. We know that:

$$y_1 \in C_i + x_{2,i},$$

and

$$y_1 \in C_i + x_{1,i}.$$

After multiplying the first formula with $(1-\lambda)$ and the second formula with λ and adding them together we get

$$(1-\lambda)y_1 + \lambda y_2 \in (1-\lambda)C_i + \lambda C_i + (1-\lambda)x_{1,i} + \lambda x_{2,i}$$

which is exactly what we wanted, because for every convex set C we know that

$$(1-\lambda)C_i + \lambda C_i \;=\; C_i$$

So the final form is

$$(1-\lambda)y_1 + \lambda y_2 \in C_i + (1-\lambda)x_{1,i} + \lambda x_{2,i}.$$

So we proved that $F(x, y)$ is logarithmically concave. By the Prékopa-Leindler inequality ([AAGM15] Chapter 1, Theorem 1.4.1), $\int_{\mathbb{R}^d} F(x, y)$ is also logarithmically concave. Using the property of log-concave functions, the $\mathcal{K}$ set is convex, completing the proof of Theorem 2.0.1. □

Lemma 2.0.2. *(Every convex set can be obtained as a $\mathcal{K}$ set)*

Let A be an arbitrary convex set in $\mathbb{R}^d$, then there exist two convex sets $B, C \subset (\mathbb{R}^n)$ such that $\mathcal{K}(B, C) = A$.

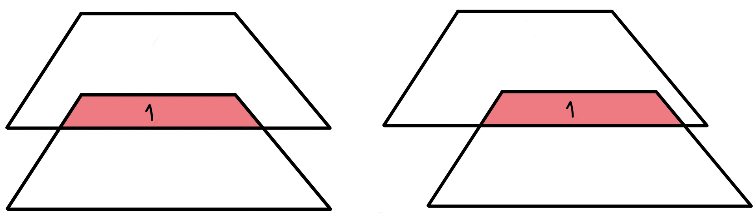
Proof. First take an arbitrary convex set B of measure 1 . We claim that, $\mathcal{K}(B, A+B) = A$. With the choice of B, we have that $A \cap (B + A)$ must equal to B, so B rolls freely in $A + B$ and thus, $\mathcal{K}(B, A + B) = A$.

□

Remark 2.0.2.1. *Let A, B, C, D be sets in $\mathbb{R}^d$, where $\{A, B\} \neq \{C, D\}$. We show that $\mathcal{K}(A, B)$ may be equal to $\mathcal{K}(C, D)$.*

We will show continuum many pairs of sets with the same $\mathcal{K}$. Consider a rectangle in $\mathbb{R}^2$ of measure 1. Let A be the convex hull of that rectangle and its translation by the standard basis vector e_1. Let B be the same rectangle, just with a translation by e_2 . It is easy to see that $\mathcal{K}(A, B)$ is a unit square, and choosing different rectangles will not affect the set. There are continuum different rectangles of measure 1, so we showed that many pair of convex sets with the same $\mathcal{K}$ set.

Remark 2.0.2.2. *If C is a convex body in $\mathbb{R}^2$, then $\mathcal{K}(C, C)$ is not necessarily strictly convex.*



Indeed, let C be a trapezoid. As one can see in the figure, when we shift the first trapezoid vertically as much as we can the intersection will be another trapezoid of measure 1. As we shift the trapezoid horizontally, the intersection remains the same, so the boundary of $\mathcal{K}(C, C)$ is a line segment, so the set is not strictly convex.

This example can be extended to any dimensions.

Indeed, take the same trapezoid in $\mathbb{R}^n$, which is in the span of e_1 and e_2, let C be the Minkowski sum of the trapezoid and $l_i = \{p = \alpha e_i \ , 0 \leq \alpha \leq 1\}$ for $i = \{3, 4, .., n\}$, then $\lambda(C)$ is 1, and $\mathcal{K}(C, C)$ has that same line segment in its boundary as the 2 dimensional trapezoids.

Definition 2.0.2.1. *(Inscribed square)*

A square is inscribed in a curve $\omega \subset \mathbb{R}^2$, if all of its vertices lie on ω.

Lemma 2.0.3. *For every bounded convex set $C \in \mathbb{R}^2$,if there is an inscribed square in it ,then there exists a similarity transformation, S such that $dim(\mathcal{K}(S(C), Q)) \leq 1$, where Q denotes the unit square.*

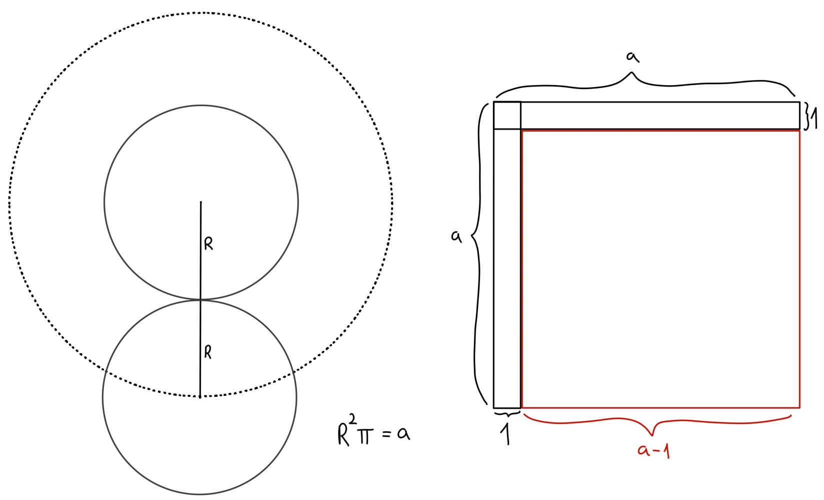
Note that the converse does not hold. Indeed, consider the convex hull of the unit square and a point outside of the square that is, C= $conv([-\frac{1}{2}, \frac{1}{2}]^2 \cup (1,1))$. The unit square just fits in but it cannot move. So, the $\mathcal{K}$ set will be a singleton, but it is not an inscribed square.

The following isoperimetric question is natural. For which C_1, C_2 convex sets on the plane is the area of $\mathcal{K}(C_1, C_2)$ maximal? One natural candidate is if $C_1 = C_2 = D$, where D is the unit disk. However, this is not true as shown by the following example.

Example 2.0.1. *Fix $0 < a \in \mathbb{R}$, let C_1, C_2 be convex sets of measure a and D_a be the disk of measure a in $\mathbb{R}^2$. We discuss the possible values of $\lambda(\mathcal{K}(C_1, C_2))$.*

When $a \leq 1$, then trivially $\lambda(\mathcal{K}(C_1, C_2)) = 0$ for every C_1, C_2, because the set cannot be other than the empty set or just one point when $C_1 = C_2$ and $a = 1$.

When $a > 1$, then the minimal measure of $\mathcal{K}(C_1, C_2)$ is always 0, because if we take a thin rectangle of measure a and the other set is the same rectangle rotated with 90 degrees, then the intersection cannot be measure 1.



Thus, the interesting question is the maximum of $\mathcal{K}(C_1, C_2)$ over all pairs of planar convex bodies of area a. We show that for a>5.83, this maximum is larger than $\mathcal{K}(D_a, D_a)$, where D_a denotes the disk of measure a. First we consider two disks of measure a and a $1 \times a$ rectangle and its rotated copy as the figure shows. The measure of the $\mathcal{K}$ set of the rectangles is $(a-1)^2$. As for the disks, we approximate the $\mathcal{K}$ set with the disk with two times bigger radius. This is bigger than the real $\mathcal{K}$ of the disk, because when we shift the disk in any direction by the diameter,the intersection is just a point. The measure of this approximation is $4a$. so the question is:

$$4a \ < \ (a-1)^2$$

From this we get:

$$a \geq 5.83$$

Note that by computing the area of $\mathcal{K}(D_a, D_a)$ more precisely, we would obtain a better bound for a.

Question 2.0.1. *Let $\epsilon > 0$, is there C_1, C_2 convex sets of measure a for every ϵ, such that $\lambda(\mathcal{K}(D_a, D_a)) \leq \lambda(\mathcal{K}(C_1, C_2))$?*

3. Homology

In this section we discuss the basics of homology. The main part is to understand the definition of homology groups and some of its properties.

3.1 Simplicial homology

Definition 3.1.0.1. *(Simplex)*

We call the convex hull of n+1 points in $\mathbb{R}^n$ an n-dimensional simplex. Let $t_0, ..., t_n \in \mathbb{R}^n$ be vectors.

$$\Delta^n = \left\{ \sum_{i=0}^{n} \alpha_i t_i \ , 0 \leq \alpha_i \leq 1, \sum_{i=0}^{n} \alpha_i = 1 \right\}$$

When $t_i = e_i$ then the simplex is called standard n-simplex

Definition 3.1.0.2. *(Abstract simplicial complex)*

Let Λ be a non-empty finite set. Then a family Λ of subsets of V is an abstract simplicial complex *if for every $A \in \Lambda$ and $B \subset A$ we have $B \in \Lambda$.*

Definition 3.1.0.3. *(Simplicial complex) Let X be a topological space. Then a simplicial complex on X is the following. We take an abstract simplicial complex Λ and for every $A \in \Lambda$, we fix an injective, continuous map $\eta_A : \Delta^n \to X$, where $|n| = |A| - 1$ such that if $B \subset A$ then $\eta_B = \eta_A|_B$, where $\eta_A|_B$ denotes the restriction of the map η_A to B.*

We can think of simplicial complexes as just simplexes glued together. We define Δ_n, the n faces of the simplicial complex Λ

$$\Delta_n := \{A \in \Lambda : |A| = n + 1\}$$

.

Definition 3.1.0.4. *(Realization of a simplicial complex)*

Let Λ be an abstract simplicial complex, then the realization of Λ, $R(\Lambda)$ is $\bigcup\limits_{A\in\Lambda} Im(\eta_A)$.

Exercise 3.1.1. *Give a homeomorphism between $R(\Lambda)$ and S^n, where $R(\Lambda)$ is the triangulation of S^n, which is basically the faces of the n+1 simplex.*

Proof. Place $R(\Lambda)$ inside S^n and take a point inside the convex hull of $R(\Lambda)$. From that point project $R(\Lambda)$ to S^n. The projection is continuous and is a bijection because $R(\Lambda)$ in this exercise is convex. □

Definition 3.1.0.5. *(n-chaines)*

An n-chain is a free abelian group generated from the n-faces of a simplicial complex Λ:

$$C_n := \left\{ \sum_{f\in\Lambda_n} n_f f y : n_n \in \mathbb{Z} \right\}$$

Definition 3.1.0.6. *(Boundary homomorphism)*

$$\partial_n : C_n(\Lambda) \to C_{n-1}(\Lambda)$$

$$\partial_n[v_0, ..., v_n] := \sum_{i=0}^{n} (-1)^i [v_0, ..., \hat{v}_i, ..., v_n],$$

where $\hat{v}_i$ means that v_i is left out of the vertices.

Lemma 3.1.1. $\partial_n \partial_{n-1} = 0$.

Proof. We expand $\partial_n\partial_{n-1}$

$$\partial_n\partial_{n-1}[v_0, ..., v_n] = \sum_{i<j} (-1)^i (-1)^j [v_0, ..., \hat{v}_i, ..., \hat{v}_j, ..., v_n] +$$

$$\sum_{i>j} (-1)^i (-1)^{j-1} [v_0, ..., \hat{v}_j, ..., \hat{v}_i, ..., v_n]$$

If we switch i and j, it is easy to see that the result is the negative of the first, so they cancel out each other. □

Definition 3.1.1.1. *(Homology group) Let $C_1, \ldots, C_n$ be Abelian groups and $\partial_n : C_n \to C_{n-1}$ homeomorphism for every $n \in \mathbb{Z}$. We say that $C = (C_n, \partial_n)$ is a chain-complex, if $\partial_n \partial_{n-1} = 0$ for every $n \in \mathbb{Z}$*

$$... \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow ... \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

We define $H_n(C) = Ker(\partial_n)/Im(\partial_{n+1})$ as the nth homology group of C.

Definition 3.1.1.2. *(Betti number)*

The rank of the $H_n(C(X))$ group is called the nth Betti number of X.

3.2 Singular homology

Definition 3.2.0.1. *(Singular n-chains)*

Let X be a topological space, $\sigma : \Delta_n \to X$, a continuous function is an n-simplex. We define $C_n(X)$ as a free Abelian group generated by the n-simplexes. An element of $C_n(X)$ is an n-chain.

The boundary homomorphism is the following function

$$\partial_n(\sigma) := \sum_{i=0}^{n} (-1)^i \sigma|[v_0, ..., \hat{v_i}, ..., v_n].$$

$C(X) = (C_n(X), \partial_n)$ is a chain-complex and $H_n(x) = H_n(C(X))$ is the nth singular homology group of X.

In the definition below, if we change $\mathbb{Z}$ to an arbitrary Abelian group G, $H_n(X; G)$ is called the homology with coefficient G. We will use this later, when $G = \mathbb{Z}_2$ in the proof of the inscribed square problem for locally monotone curves.

Exercise 3.2.1. *Prove that $H_0(X) \cong \mathbb{Z}^k$, where k is the number of path connected components of X.*

Proof. Recall that $H_0(X) = \frac{Ker\partial_0}{Im\partial_1}$. It is obvious that $Ker(\partial_0) = C_0$ because ∂_0 goes from C_1 to 0 and it is a homomorphism. From the definition $Im\partial_1$ is $v_i - v_j$. After we have quotient out every vertex in the simplicial complex, two vertices is equivalent if there is a path from one to the other. This is exactly the definition of a path connected component. □

3.3 Properties of homology

Definition 3.3.0.1. *(Chain map)*

Let $f : X \to Y$ be a continuous map between the two topological space X and Y. f will induce the homomorphism $f_\sharp : C_n(X) \to C_n(Y)$.

$$f_\sharp(\sum_i n_i\sigma_i) = \sum_i n_i f_\sharp(\sigma_i) = \sum_i n_i f\sigma_i.$$

Observe that,

$$f_\sharp\partial(\sigma) = f_\sharp(\sum_i (-1)^i \sigma|[v_0, ..., \hat{v}_i, ..., v_n]) = \sum_i (-1)^i f\sigma|[v_0, ..., \hat{v}_i, ..., v_n]) = \partial f_\sharp(\sigma).$$

This shows us that the diagram below is commutative, because every composition of maps from a point in the diagram to another is equal.

$$\begin{array}{ccccccccc}
... & \longrightarrow & C_{n+1}(X) & \xrightarrow{\partial_{n+1}} & C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \longrightarrow & ... \\
 & & \downarrow f_\sharp & & \downarrow f_\sharp & & \downarrow f_\sharp & & \\
... & \longrightarrow & C_{n+1}(Y) & \xrightarrow{\partial_{n+1}} & C_n(Y) & \xrightarrow{\partial_n} & C_{n-1}(Y) & \longrightarrow & ...
\end{array}$$

Lemma 3.3.1. *Let X, Y, Z be topological spaces and $f : X \to Y$, $g : Y \to Z$ be continuous maps. A basic property of induced homomorphism is: $(fg)_* = f_*g_*$ $X \xrightarrow{f} Y \xrightarrow{g} Z$ the claim comes from the associativity of the compositions. $\Delta_n \xrightarrow{\sigma} X \xrightarrow{f} Y \xrightarrow{g} Z$*

Theorem 3.3.2. *If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphism $f_* = g_* : H_n(X) \to H_n(Y)$*

The proof is written in Alan Hatcher's book ([Hat02], Chapter 2, Theorem 2.10).

4. Inscribed square problem for locally monotone curves

Problem 4.0.1. *Let C be a Jordan curve. A polygon P is inscribed in C, if all vertices of P belong to C. Is there an inscribed square for every Jordan curve in $\mathbb{R}^2$.*

This question is still open; we do not know the answer for every Jordan curve. There are results for specific curves, for example: Piecewise analytic curves, Locally monotone curves, Lipschitz graphs. Here, I will represent the inscribed square problem for locally monotone curves the proof was made by Walter Stromquist in 1989 .

4.1 Inscribed rhombuses

Definition 4.1.0.1. *(simple closed curve)*

A simple closed curve is a continuous function $\omega : \mathbb{R} \to \mathbb{R}^n$ which satisfies $\omega(x) = \omega(y)$ if, and only if $x - y$ is an integer. Here we will talk about curves with domain $[0, 1]$, and we call it closed if $\omega(0) = \omega(1)$.

Definition 4.1.0.2. *(Inscribed quadrilateral)*

A quadrilateral is inscribed in ω if all of its vertices lie on ω. In $\mathbb{R}^2$ quadrilaterals can be inscribed despite having points outside the curve.

Definition 4.1.0.3. *We want to talk about quadrilaterals so let Q denote the set of quadrilaterals in ω.*

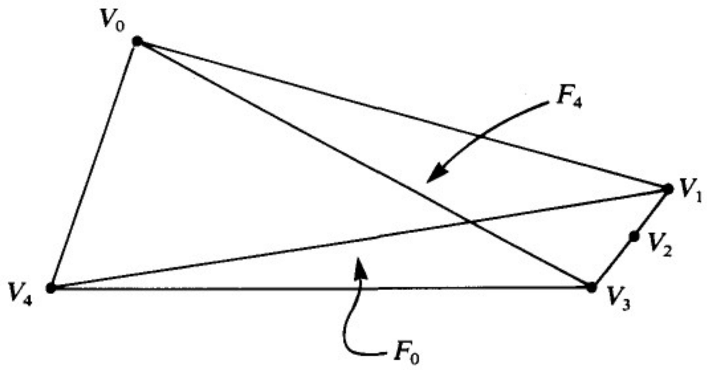
$$Q = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | 0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 1\}$$

Q is a four simplex with vertices

$$v_0 = (1,1,1,1), v_1 = (0,1,1,1)$$

$$v_2 = (0,0,1,1), v_3 = (0,0,0,1), v_4 = (0,0,0,0)$$

and faces $F_0, ..., F_4$ where every F_i is opposite v_i.



Every point $x \in Q$ represents a quadrilateral with vertices $(\omega(x_1), \omega(x_2), \omega(x_3), \omega(x_4))$. This is not a one-to-one representation, because multiple point in Q can represent the same quadrilateral. For example every point in F_1 has it's pair in F_4.

$$(0, \omega(x), \omega(y), \omega(z)) \sim (\omega(x), \omega(y), \omega(z), 1)$$

They are the same, all they differ is the numbering of the vertices. Some of them may be just one point or have sides with zero length. Note that all of v_i is the same one point quadrilateral. Now to make sure that we get the wanted quadrilateral we need to define a few more tools.

Definition 4.1.0.4. *For each* $i = (1, 2, 3, 4)$ *define:*

$$s_i(x) = \|\omega(x_{i+1}) - \omega(x_i)\|$$

(When talking about $\omega(x_{i+1})$ *when* $i = 4$ *then it we mean* $\omega(x_1)$*). This is the i-th side if the quadrilateral. This is a continuous function on* Q*. For each* i *let* Q_i *be:*

$$Q_i = \{x \in int(Q) \mid s_i(x) = max\ s_j(x),\ j = 1, 2, 3, 4\}$$

Where $int(Q)$ *is the interior of* Q*.*

Now Q_i is the set of quadrilaterals whose i-th side is the longest. A quadrilateral in ∂Q is in Q_i if it is the limit of quadrilaterals in $int(Q)$ with the same property. By this each Q_i is closed and $\bigcup Q_i = Q$. We defined it this way to avoid one point quadrilaterals being in every Q_i

Lemma 4.1.1. *If* ω *is a smooth curve, then each one-point quadrilateral is contained in only one set* Q_i*. In particular,* $v_i \in Q_i$ *for* $i = 1, 2, 3$ *and* $x \in Q_4$ *for each* x *on the edge connecting* v_0 *and* v_4

There will be a stronger statement, that will prove this special case as well. There, smoothness will not be neccessary.

For every Q_i , $i = 1, 2, 3$ $v_i \notin Q_i$ and $F_i \cap Q_i = \emptyset$ and for Q_4 it includes the segment from v_0 to v_4 and $Q_4 \cap (F_4 \cap F_0) = \emptyset$, $R = \bigcap_i Q_i$, every point $x \in \mathbb{R}$ is a rhombus, an inscribed quadrilateral which sides are equal and nonzero.

$$d_{13}(x) = \|\omega(x_3) - \omega(x_1)\|$$

and

$$d_{24}(x) = \|\omega(x_4) - \omega(x_2)\|$$

A rhombus is called a square-like quadrilateral, if it satisfies the following property:

$$d_{13}(x) = d_{24}(x)$$

So it has sides with equal length and diagonals with equal length. A rhombus is called *thin*, if $d_{13} \geq d_{24}$ for its diagonals and is called fat, if $d_{13} \leq d_{24}$. We will denote the set of thin and fat rhombuses by R_{THIN} and R_{FAT}. From the definition we get that $R = R_{THIN} \bigcup R_{FAT}$. Another smaller statement is that if $x \in F_0$ is a fat rhombus then $h(x)$ is a thin rhombus and vice versa. In the next section we want to understand the rhombuses on F_0 and F_4 deeper and take a look at the degrees of the set of thin and fat rhombuses.

4.2 Degree of a set

This part of the proof we will define the degree of a set. We will use homology groups over $\mathbb{Z}_2$.This will denote the "parity" of the set. First of all we have to define a helpful function on a simplex. Let A be an n-simplex with v_i vertices, F_i faces. Closed subsets A_i $i = 0, ..., n$ a cover of A where for every i $v_i \in A_i$ and$F_i \cap A_i = \emptyset$. Let A_i be a cover and $K \subset \bigcap A_i$ which is both open and closed. There exist such sets because $K = \emptyset, K = \bigcap A_i$ are both open and closed in $\bigcap A_i$ Now we can define the function mentioned above, it is a reversing map for the cover A_i:

$$f : (A \setminus \bigcap A_i) \longrightarrow \partial A$$

$$f(x) = \sum_i \frac{d(x, A_i)}{\sum_j d(x, A_j)} v_i$$

where $d(x, A_i)$ is the distance from x to A_i. This is a convex combination of the vertices and one vertex will always have weight one because A_i is a cover so every x is in at least one A_i. From this fact we know the image will be on ∂A cause it is the convex combination of n vertices. We had to exclude $\bigcap A_i$ from the domain because there $\sum_j d(x, A_j) = 0$. Here every reversing map is equivalent up to homotopy. Every $x \in A$ is in at least in one A_i and that x point goes to the intersection of F_i for the same i's and we have to give a homotopy on the ∂A. Now let $L = \bigcap A_i \setminus K$. L is obviously both open and closed in $\bigcap A_i$. Because K has the same properties, so L and K are both just the union of a few connected component.

Take a triangulation of $\bigcap A_i$. We can refine the triangulation so that there exists a union of simplexes that covers K and none of them touches L. This covers boundary separates K from L. This boundary represents a homology class $\gamma \in H(A \setminus \bigcap A_i)$. We will call it γ_k. Also the induced map:

$$f_* : H_n(A \setminus \bigcap A_i) \to H_n(\partial A)$$

Now everything is given to define the degree of A_i around K. The degree of A_i around K is the image of $f(\gamma_k)$ in $H_n(\partial A)$ and f is any reversing map for A_i. It does not matter which reversing map we choose because all of them are equivalent for homotopy. We want the homology group to be $\mathbb{Z}_2$ so every homology group will be over it. It's value is not dependent on the choices we made before. From now on the cover A_i is fixed so it will be denoted as deg$K$. Moreover it can be defined on any B simplex with an $f : (A \setminus \bigcap A_i) \longrightarrow \partial B$ which sends every A_i to different faces of B. Then the degree can be defined the same way $f_*(\gamma_k)$ in $H_n(\partial B)$, because there is an isomorphism g from B to A. where gf is a reversing map and $(gf)_*(\gamma_k) = f_*(\gamma_k)$

Lemma 4.2.1. *$deg\emptyset = 0$*

Proof. When K is the emptyset then the cover can be anything that does not touch $\bigcap A_i$ and it is equivalent to the trivial class. □

Lemma 4.2.2. *If $K = K_1 \cup K_2$ then $degK = degK_1 + degK_2$*

Lemma 4.2.3. *$deg\bigcap A_i = 1$*

Proof. Let S^{n-1} be the n-1 dimensional sphere and g a continuous map without fixed points, from the sphere to itself. This will be homotopic to the antipodal map and also a homeomorphism. From this $g_* : H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$ is the identity. In this case ∂A is homeomorphic to S^{n-1} . f restricted to ∂A is a function from ∂A to itself without fixed points, so it induces the identity map. But ∂A is surrounds $\bigcap A_i$, so $f_*(\gamma_{\bigcap A_i}) = 1$. □

Theorem 4.2.4. *If ω is a smooth curve, then ω has an inscribed quadrilateral with equal sides and equal diagonals in it.*

Proof. Firstly we suppose that there is no intersection of R_{THIN} and R_{FAT} which means that R can be written as the disjoint union of the two. The face F_0 is a simplex, it has a cover $F_0 \cap Q_i$. It is a cover because Q_i was a cover of Q. The rhombuses on F_0 can be written as:

$$F_0 \cap R = (F_0 \cap R_{THIN}) \sqcup (F_0 \cap R_{FAT})$$

From this and lemma 3 and 4 we get:

$$1 = deg(F_0 \cap R) = deg(F_0 \cap R_{THIN}) + deg(F_0 \cap R_{FAT})$$

because here $\cap A_i$ is R.

$$deg(F_0 \cap R_{THIN}) \neq deg(F_0 \cap R_{FAT})$$

After this we will show that both side is equal to $deg(F_4 \cap R_{THIN})$. The map h from F_0 to F_4 is a homeomorphism and also an isomorphism between the covers $\{F_0 \cap Q_i\}$ and $\{F_4 \cap Q_i\}$. When we switch the coordinates then the length of the sides do not change, just the numbering shifts. When a quadrilaterals X first side was the longest then $h(X)$'s fourth side will be the longest. Therefore

$$deg_{\{F_0 \cap Q_i\}}(F_0 \cap R_{FAT}) = deg_{\{F_4 \cap Q_i\}}(h(F_0 \cap R_{FAT})) = deg_{\{F_0 \cap Q_i\}}(F_4 \cap R_{THIN})$$

Now the other side. Let $f : (Q - R) \to \partial F$:

$$f(x) = \sum_{i=1}^{4} \frac{d(x, Q_i)}{\sum_{j=1}^{4} d(x, Q_j)} v_i$$

Where $d(x, Q_i)$ is the distance from x to Q_i. f restricted to F_0 is a reversing map for the cover $F_0 \cap Q_i$ and is a reversing map when restricted to F_4. We assumed that R_{THIN} and R_{FAT} is disjoint so we can triangulate Q finely enough to be able to take a Δ 4-chain of simplices that separate R_{THIN} from R_{FAT}. $\partial\Delta$ is a 3-chain in $(Q - R)$. From this take the simplices which are not is F_0 or F_4. This is a "tube" from F_0 to F_4. $\square$

The smoothness was required in lemma 1, but we can give a weaker statement that is sufficient for this lemma. This will be called "Condition A".

Definition 4.2.4.1. *A curve ω satisfies Condition A if each point $\omega(y)$ of the curve had a neighbourhood $U(y)$ in $\mathbb{R}^n$ such that no two chords in $U(y)$ are perpendicular.*

This definition has a few equivalent form. For example geometrically it means that every point in the curve has a neighbourhood that every chord differ less than 90 degrees. Also it can be said precisely formulated: ω satisfies Condition A, if every $y \in \omega, \exists \mu$ such that $\forall x_1, x_2, x_3, x_4, \in (y - \mu, y + \mu)$:

$$(\omega(x_2)) - \omega(x_1)) \cdot (\omega(x_4)\omega(x_3)) > 0$$

Because of the periodically of ω , μ can be chosen independently of y.

Lemma 4.2.5. *If ω satisfies Condition A, then each one-point quadrilateral in Q is contained in exactly one set Q_i. $v_i \in Q_i$ for $i = 1, 2, 3$ and $y \in Q_4$ for every y on the edge connecting v_0 and v_4.*

Proof. Let us take a one point quadrilateral y from the edge connecting v_0 and v_4 and check that it is in exactly one Q_i. The goal is to show that every point in it has a neighbourhood such that all x in that neighbourhood and also in $int(Q)$ the fourth side of x is the unique longest side. Now consider an element from the neighbourhood of y and in $int(Q)$. Let z_1, z_2, z_3, z_4 represent the vectors of the sides of the quadrilateral x. $z_i = \omega(x_{i+1}) - \omega(x_i)$. We have to show that z_4 is the longest side. For example show that $z_4 > z_2$. We know the following:

$$z_4 = z_1 + z_2 + z_3$$

Take the dot product with z_2.

$$z_4 \cdot z_2 = z_1 \cdot z_2 + z_2 \cdot z_2 + z_3 \cdot z_2$$

But from this z_4 would have a bigger component in the direction of z_2 than z_2 itself, this means that $z_4 > z_2$. One can do the same for each z_i. This means that each x sufficiently near to y is in Q_4, so y also is in Q_4. □

4.3 Inscribed quadrilaterals

Theorem 4.3.1. *If ω satisfies Condition A, then ω admits a an inscribed quadrilateral with equal sides and equal diagonals.*

In this section we will talk about curves in $\mathbb{R}^2$. There we will define a weaker smoothness statement and prove that these kinds of curves admit an inscribed quadrilateral.

Smooth curves, polygons, convex curves, etc. satisfy this condition.

Definition 4.3.1.1. *(Segment of a curve) A segment of a curve is the image of an interval in the parameter space. To be precise, we identify the two endpoints of the full interval in the parameter space, so basically the parameter space is S^1. The length of this segment is $(b-a)$ measured in the parameter space. It is monotone in direction u if the function $f = \omega(x) \cdot u$ where $x \in (a, b)$ is an increasing function.*

Definition 4.3.1.2. *(Locally monotone curve) The curve ω is locally monotone if, for every $y \in \mathbb{R}$ there is an interval $(y-\mu, y+\mu)$ and a direction $u(y)$, such that the segment $(y-\mu, y+\mu)$ is monotone in the direction $u(y)$.*

When μ is satisfies this property then the periodicity allows us to choose μ as a constant. So every segment with maximum length 2μ is monotone in some direction. This is called locally monotone with constant μ. An intuition is looking at this definition geometrically. This means that we take a point from the curve y and a direction $n(y)$, the curve is locally monotone, if for every y there is a neighbourhood and a direction such that there is no chord perpendicular with $n(y)$.

Theorem 4.3.2. *If μ is locally monotone curve in $\mathbb{R}^2$, then ω admints an inscribed square.*

Proof. Assume that ω is a locally monotone with constant μ. We will approximate ω with smooth curves ω_ϵ which contain an inscribed square, what we know from the first theorem. This way we will make a subsequence of inscribed square which will converge to an inscribed square in ω. The main part is to show that the limit square is not a one point quadrilateral. Firstly we will show that ω_ϵ is locally monotone with constant at least $\frac{1}{2}\mu$, that will establish a lower bound for the size of square in ω_ϵ.

Let $\|x\|$ define the minimum length of an interval (in the parameter space) which covers all four vertices $\omega(x_1), \omega(x_2), \omega(x_3), \omega(x_4)$. This can be written as the minimum of : $(x_4 - x_1), ((1 + x_3) - x_4), ((1 + x_2) - x_3), ((1 + x_1) - x_2)$. Meg lehetne magyarazni!!!!

Let $\epsilon > 0$, and $\delta > 0$ such that $|x - y| < \delta$ implies that $\|\omega(x) - \omega(y)\| < \epsilon$ In any case choose $\delta < \frac{1}{2}\mu$.

Define $\omega_\epsilon : \mathbb{R} \to \mathbb{R}^2$ by:

$$\omega_\epsilon(x) = \frac{1}{\delta} \int_{t=0}^{\delta} \omega(x+t)dt$$

By choosing ω_ϵ this way we got $\|\omega_\epsilon(x) - \omega(x)\| < \epsilon$ for all x.

$$\omega'_\epsilon = \frac{1}{\delta}(\omega(x+\delta) - \omega(x)$$

We apply the first theorem to find the inscribed square in the S_ϵ in ω_ϵ.

Now we have to show that ω_ϵ is locally monotone with constant $\frac{1}{2}\mu$. We can show it by writing down ω_ϵ using ω. So take $y \in \mathbb{R}$ and $u(y)$ is the direction in which ω is locally monotone with constant μ. Let x_1, x_2 be two points in $(y - \frac{1}{2}\mu, y + \frac{1}{2}\mu$ where $x_1 < x_2$.

$$(\omega_\epsilon(x_2) - \omega_\epsilon(x_1)) \cdot u(y) = \frac{1}{\delta} \int_{t=0}^{\delta} (\omega(x_2+t) - \omega(x_1+t)) > 0$$

The monotonicity provides that the integrand is strictly positive. So the chords cannot be perpendicular to $u(y)$ which is exactly what we needed. So the inscribed square cannot have sides less than μ because then it's vertices would be in an interval with length μ and there ω is monotone. From this we get that the series of squares have to converge to another square which is not a one point square.

$\square$

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I, Sándor Fazekas, declare that I have not used any AI-based tools in the preparation of my thesis.