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Arrangements of pseudocircles

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Contents

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1 Introduction

Arrangements of curves have long played a central role in geometry. In particular, the study of *pseudocircles* — simple closed curves in the plane or on the sphere that pairwise intersect exactly twice — offers a powerful generalization of both arrangements of straight lines and arrangements of proper circles. This topic was first studied by Branko Grünbaum in the 1970s [13], who introduced arrangements of pseudocircles and stated many open problems, that have been researched ever since. These problems include the combinatorial structure of such arrangements, such as the maximum and minimum number of k-cells, in particular, triangles and digons (which are 2-cells).

After looking at the key definitions in Section 2; we will take a look at Grünbaum's digon conjecture in Section 3, which states that there are at most $2n-2$ digons in pairwise intersecting arrangements of pseudocircles, which was fully settled last year in [2]. Snoeyink and Hershberger adapted sweeping for arrangements of pseudolines and arrangements of pseudocircles, which we look at in Section 4, to show that any digon-free, intersecting arrangement of n pseudocircles must have at least $4n/3$ triangular cells [19]; which is at the beginning of Section 5, which looks at lower and upper bounds regarding the number of triangles in mostly pairwise intersecting arrangements. Another recent result is that Felsner and Scheucher constructed families with only $\lceil 4n/3 \rceil$ triangles, thereby disproving Grünbaum's conjectured $2n-4$ lower bound [12]. Another topic of recent research is the topic of circularizability [11], which we take a brief look at in Section 6. We conclude the thesis with a list of open problems in Section 7.

2 Key definitions and basic properties

We present the definition of an arrangement of pseudocircles and after that discuss some of its basic properties.

Definitions

A *family of pseudocircles* is a collection of simple closed curves $\mathcal{A} = \{C_1, C_2, \ldots, C_n\}$ such that for every pair C_i, C_j $(i \neq j)$, the intersection $C_i \cap C_j$ is empty or consists of exactly one touching point or exactly two crossing points where the curves pass through each other transversally. We call the simple closed curves C_i *pseudocircles*. Two pseudocircles are said to *touch* if they meet at a single point without crossing and they are said to *cross* if they intersect at two points.

We call the cell decomposition of the plane induced by a family of pseudocircles an *arrangement of pseudocircles*. A *k-cell* is a cell whose boundary contains k crossing and touching points and we denote the number of k-cells in the arrangement by $p_k(\mathcal{A})$ or p_k if the arrangement is clear from the context. A 2-cell is called a *digon* and it is a *lens*, if it is on the inside of both pseudocircles incident to it, otherwise it is called a *lune*. A 3-cell is called a *triangle*, a 4-cell is a *quadrangle* and a 5-cell is a *pentagon* (following Grünbaum's naming convention [13]).

The *intersection graph* of a pseudocircle family is the graph where each vertex corresponds to a pseudocircle, and an edge connects two vertices if and only if the corresponding curves intersect. An arrangement is *connected* if its intersection graph is connected. It is called *(pairwise) intersecting* if every pair of pseudocircles intersects.

An arrangement is *simple* if no three pseudocircles intersect at a common point. This is often a natural assumption, or in other cases it can be achieved by perturbing some of the pseudocircles around the points where at least three meet.

An arrangement is *cylindrical* if there exist two cells of the arrangement that are separated by every pseudocircle. We can construct a cylindrical arrangement from a projective arrangement of pseudolines, but we will need to define arrangements of pseudolines for that.

A *(projective) family of pseudolines* is a collection of closed curves in the projective plane, each pair crossing exactly once and we call the induced cell decomposition of the plane a *(projective) arrangement of pseudolines*. An euclidean family is a restriction of a projective family with the usual $\mathbb{R}^2 \subset \mathbb{RP}^2$ embedding, such that each pair of curves still cross in $\mathbb{R}^2$.

Construction 2.1 (Arrangement of pseudocircles from pseudolines [11])**.** *Let $\mathcal{A}$ be a projective arrangement of pseudolines. Draw $\mathcal{A}$ in a disk such that every pseudoline of $\mathcal{A}$ connects two opposite points of the boundary of the disk. Project the disk to the northern hemisphere of a sphere S^2, so that the boundary of the disk becomes the equator of S^2. Then, using a projection through the center of S^2, copy the arrangement to the southern hemisphere. This construction connects the two images of every pseudoline to a pseudocircle and each pair of pseudocircles cross exactly twice, once on the northern and once on the southern hemispher or twice on the equator.*

The family of these pseudocircles is intersecting and has twice the number of vertices, edges and faces of every type as $\mathcal{A}$. The arrangements obtained by this construction are cylindrical, the two images of any face from the pseudoline arrangement is a suitable pair.

In fact, these intersecting arrangements have the stronger property: if two pseudocircles don't cross on a third pseudocircle, then the third pseudocircle separates the two crossing points of the first two. This property was used by Felsner and Scheucher [11] to define *arrangements of great-pseudocircles* as it generalizes arrangements of great-circles on the sphere.

An arrangement of pseudocircles is *circularizable* if there is an isomorphic arrangement of circles. Here, isomorphic means that there is a bijection between the two arrangements which maps all k-cells to k-cells and incidencies remain the same.

Basic properties and observations

Any pairwise intersecting arrangement of n pseudocircles has $\binom{n}{2}$ intersecting pairs, and thus at most $2\binom{n}{2}$ crossing points in total. In a pairwise intersecting simple arrangement, every pair intersects in exactly two crossings and no three curves intersect at the same point. Each crossing point belongs to exactly two pseudocircles and contributes to the boundary of four cells. Euler's formula can be applied to the cell decomposition of the plane induced by a pseudocircle arrangement to relate the number of cells, edges, and vertices.

Proposition 2.2. *The number of cells in a pairwise intersecting simple arrangement of $n \geq 1$ pseudocircles is $n(n-1)+2$.*

Proof. The number of vertices is $v = 2\binom{n}{2} = n(n-1)$ because every pair of pseudocircles has two crossing points. The number of edges is $e = 2n(n-1)$, because every psudocircle is cut into $2(n-1)$ parts by its $2(n-1)$ crossing points. Thus, the total number of cells is

$$c = e - v + 2 = 2n(n-1) - n(n-1) + 2 = n(n-1) + 2.$$

Another way to prove this proposition is by considering how the number of regions change as we add the pseudocircles one by one. There are 2 cells after adding the first pseudocircle. The kth pseudocircle is cut into $2(k-1)$ pieces when added and every one of its pieces cuts an existing cell into two. Therefore the number of cells after every pseudocircle has been added is

$$c = 2 + \sum_{k=2}^{n} 2(k-1) = 2 + n(n-1).$$

□

3 Grünbaum’s digon conjecture

The problem of determining the maximum number of digons in terms of the total number of curves n was posed by Grünbaum in 1972 and remained open for over fifty years.

Conjecture 3.1 (Grünbaum, 1972 [13])**.** *In any simple arrangement of $n > 2$ pairwise intersecting pseudocircles, the number of digons satisfies*

$$p_2 \leq 2n - 2.$$

Grünbaum provided examples for arrangements of $n \geq 4$ pseudocircles with exactly $2n-2$ digons, which can be seen in Figure1 [13, Figure 3.28]. An isomorphic arrangement can be achieved by true circles, as well. It is important to mention that the usual digons can be made touchings (and vice versa) by contracting the digon into one point (or in the opposite case expanding it). The digons that cannot be made touchings are the infinite cells and a digon formed by one pseudocircle with two touching points, they cannot happen with $n \geq 4$ in this chapter because the arrangements are intersecting.

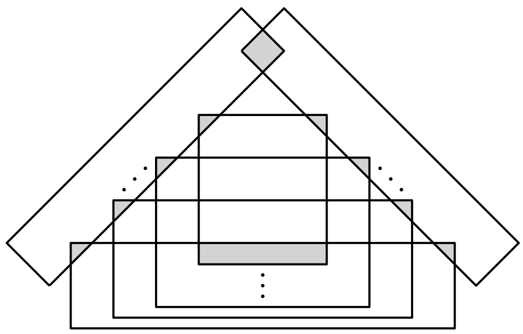


Figure 1: An arrangement of $n \geq 4$ pairwise intersecting pseudocircles with exactly $2n - 2$ digons. Digons are highlighted gray.

Trivially, the lower bound on the number of digons in a simple, intersecting arrangement of $n \geq 3$ circles is 0. For example, a simple arrangement of great-circles on a sphere cannot contain a digon, and using a stereographic projection, it will be a simple, intersecting arrangement of circles on the plane without digons.

For arrangements that are not necessarily intersecting, the current best bounds on the maximum number of digons are $\Omega(n^{4/3})$ from a construction of Erdős of n lines and n points with this amount of point-line incidencies, replacing points with tiny circles and lines with really large circles (see [16]); and $O(n^{3/2} \log n)$ from Marcus and Tardos [15]. Counting the number of touching points for unit circles is equivalent to the famous Erdős unit distance problem, because in this case the problem becomes placing as many as possible centers of unit circles with distance 2. The current best results for that are $\Omega(n^{1+c/\log\log n})$ [7] and $O(n^{4/3})$ [20].

Over time, Grünbaum's conjecture was confirmed in progressively more general settings. In 2004 Agarwal et. al. [3] showed that for *cylindrical* (i.e. there exist two cells of the arrangement that are separated by every pseudocircle) arrangements, the conjecture is true. An *intersecting arrangement of pseudoparabolas* is a collection of infinite x-monotone curves, called *pseudoparabolas*, where each pair of them either have a single touching or intersect in exactly two points where they cross. These arrangements of pseudoparabolas have a bijection with cylindrical, intersecting arrangements of pseudocircles and they can be drawn on the surface of a cylinder. The two separated cells correspond to the top and bottom cells. Agarwal et. al. [3, Theorem 2.4] showed that in intersecting arrangements of pseudoparabolas, there are at most $2n-4$ touchings and in an intersecting cylindrical arrangement of pseudocircles, the number of touchings is at most $2n - 3$.

Using this above result and dividing any intersecting arrangement into constantly many cylindrical arrangements, they prove that the number of digons in intersecting arrangements is in $O(n)$. The division is made by the existence of a stabbing number k, which means that for any intersecting arrangement in the plane, there is a set of k points, with at least one inside any pseudocircle [Corollary 2.8] and this gives a decomposition into k cylindrical arrangements.

Felsner, Roch, and Scheucher [10] proved that Grünbaum's conjecture is true if there are three pseudocircles, each pair of which intersects in a digon. Their proof is based on the local study of the arrangement near these three pseudocircles, they use some ideas from Agarwal et. al. [3] and in the end they create a planar, bipartite graph, where they prove planarity using the strong Hanani-Tutte theorem. Independent edges are edges with four different endpoints in total.

Theorem 3.2 (Strong Hanani–Tutte [18][Section 3)**.** *A graph is planar if and only if it can be drawn such that every pair of independent edges cross an even number of times.*

Grünbaum's digon conjecture has been studied for proper circles, Alon et. al. [4] in 2001, proved that there are at most $20n - 2$ digons, specifically at most $18n$ lenses and at most $2n - 2$ lunes in arrangements of proper circles. Pinchasi [17] in 2002, proved that there are at most $n + 3$ digons, specifically at most n lenses and at most 3 lunes in pairwise intersecting arrangements of unit circles. In 2024, Ackerman et. al. proved Grünbaum's bound for geometric circles:

Theorem 3.3 (Ackerman et. al., 2024 [1])**.** *Every non-trivial simple arrangement of n pairwise intersecting circles has at most $2n - 2$ digons.*

The same authors proved the general conjecture in their next paper, 52 years after Grünbaum stated it:

Theorem 3.4 (Ackerman et. al., 2024 [2])**.** *Every simple arrangement of $n > 2$ pairwise intersecting pseudocircles has at most $2n - 2$ digons.*

Their proof is using ideas from their previous paper [1] and ideas from Agarwal et. al. [3]. The main idea is to draw the bipartite double cover of the arrangement's digon graph in the plane, using strong Hanani-Tutte. The digon graph is the graph, where every vertice corresponds to a circle and edges connect two vertices if and only if their respective circles form a digon, the bipartite double cover is doubling the vertice set, such that each vertex x splits into x_{in} and x_{out} and every xy edge becomes $x_{in}y_{out}$ and $x_{out}y_{in}$. A planar bipartite graph with $2n$ vertices has at most $4n - 4$ edges. Drawing the previously described graph in the plane and using that they doubled each edge from the digon graph, it corresponds to an upper bound of $2n - 2$ of the number of digons in the arrangement.

4 Sweeping

Sweeping a collection of points with a straight line, not parallel to any of the lines determined by the points, encounters the points one by one. This method and its generalizations have proven to be useful for examining arrangements of lines or curves. It is also a really important tool in geometric algorithms; for many problems, the most effective algorithms use some type of sweeping. Sweeping was introduced to pseudolines and pseudocircles by Snoeyink and Hershberger in their 1991 article [19].

Definition 4.1. *A bi-infinite curve in $\mathbb{R}^2$ is a restriction of a simple closed curve in $\mathbb{RP}^2$ with the usual $\mathbb{R}^2 \subset \mathbb{RP}^2$ embedding. The curve is only bi-infinite if it had at least one point in $\mathbb{RP}^2 \setminus \mathbb{R}^2$ (otherwise it's a closed curve).*

Definition 4.2 (Sweep of a Curve Arrangement)**.** *Let $\mathcal{A}$ be a finite set of simple, bi-infinite or closed curves in the plane (or on the sphere), each pair intersecting in at most k points which we will call the k-intersection property. Fix one curve $c \in \mathcal{A}$, it partitions the surface into two regions (inside/outside or above/below). A sweep of the portion of $\mathcal{A}$ lying above c is a family $\{c_t\}_{t\geq 0}, \quad c_0 = c$ such that:*

- *Each c_t is simple, they are all closed or all bi-infinite.*
- *If $t' > t$, then $c_{t'}$ lies entirely in the above–region of c_t.*
- *Every point in the original above–region of c lies on exactly one c_t (except one point if the swept component is bounded).*
- *For each t, $\mathcal{A} \cup \{c_t\}$ still satisfies the exact k–intersection property.*

We say that we can sweep $\mathcal{A}$ above c if such a family exists; similarly one can sweep below, inside and outside c. If both directions succeed, we say $\mathcal{A}$ can be swept starting from c.

We are especially interested in the k-intersection property for $k = 2$, because families of pseudocircles are the families of closed curves with the 2-intersection property. Pseudolines are said to have the exact 1-intersection property, which means that each pair of curves must cross exactly once. Similarly, intersecting arrangements of pseudocircles have the exact 2-intersection property. The main result of Snoeyink and Hershberger is the following theorem.

Theorem 4.3 (Sweeping Theorem [19, Theorem 3.1])**.** *Let $\mathscr{A}$ be a finite set of simple, bi-infinite and closed curves in the plane satisfying the exact k–intersection property. Then $\mathscr{A}$ can be swept from any curve $c \in \mathscr{A}$ if $k = 1$ or $k = 2$. For $k \geq 3$ there exist arrangements that are unsweepable.*

In any period excluding a finite number of moments, the combinatorial structure of the arrangement remains the same, that is the order of the intersection points with the other curves reamins the same.

These finite number of moments can be reduced to the following operations:

- *Passing a triangle*, meaning that a triangle incident to the sweeping curve becomes a point, where three curve cross and then reappears on the other side of the curve. In Figure 2, u, v, w become the same point where the red, blue and black curves cross and then v, w reappears on the sweeping line, but in the opposite order than before.

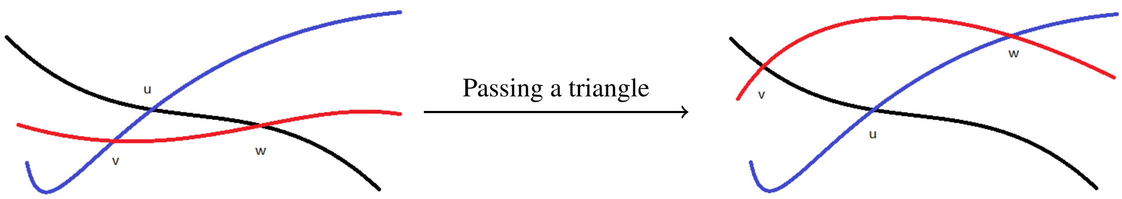


Figure 2: The red sweeping curve makes progress by passing a triangle.

- *Passing a hump*, meaning that the sweeping curve forms a digon with another curve and after this operation, they no longer intersect (there is one moment when they touch each other), see Figure 2.

- *Taking a loop*, meaning the opposite of passing a hump, the sweeping curve doesn't intersect another curve before this move and they form a digon after completing the move.

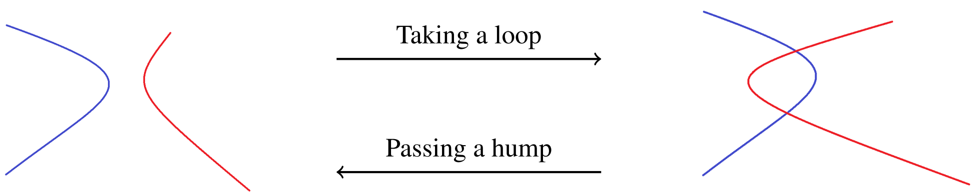


Figure 3: The red sweeping curve makes progress by taking a loop or passing a hump.

The authors' proof of this theorem is indirect, they examine a counterexample where the sweep cannot make progress with a minimum number of curves. They prove that one of the operations is possible or a counterexample with a smaller number of curves exists.

Sweeping has proven especially useful for proving bounds on the number of triangles and digons. For instance, we will use sweeping to prove lower bounds on triangles for intersecting pseudocircle arrangements and for intersecting pseudocircle arrangements without digons.

Sweeping can be used to prove an extension theorem for pseudocircle arrangements. This technique generalizes Levi's extension lemma for pseudolines to pseudocircles. Snoeyink and Hershberger proved this theorem in the same paper, where they introduced sweeping.

Lemma 4.4 (Levi's extension lemma, 1926 [14]). *Given a pseudoline arrangement and two points not on the same line, we can add a pseudoline through the two points to the arrangement.*

Levi proved this lemma for pseudoline arrangements, which are the families of curves with the exact 1-intersection theory, Snoeyink and Hershberger proved the same for families of curves with the 1-intersection or 2-intersection property.

Theorem 4.5 (Extension Theorem [19, Theorem 5.1]). *Let $\mathscr{A}$ be a finite set of simple curves in the plane with the k–intersection property, and let $P = \{p_1, \ldots, p_{k+1}\}$ be $k + 1$ points not all on a single curve of $\mathscr{A}$. If $k = 1$ or $k = 2$, then there exists a new curve c passing through every point of P so that $\mathscr{A} \cup \{c\}$ still satisfies the k–intersection property. For $k \geq 3$, there are counterexamples to the theorem.*

5 Triangle bounds in arrangements

A central question raised by Grünbaum was the number of k-cells in intersecting arrangements of pseudocircles, we have already seen his digon conjecture in Section 3, now we turn to bounds on the number of triangles.

Lower bounds

We will prove lower bounds for intersecting arrangements of circles, but first let us look at why we need the pseudocircles to be (pairwise) intersecting. For instance, we can take two families of disjoint pseudocircles, we will call them red and blue, without touching points. This means that the intersection graph is even and the cells in the arrangement will always have an even number of sides, because the sides need to alternate between red and blue arcs, see Figure 4. This means that in these types of arrangements there are no triangles

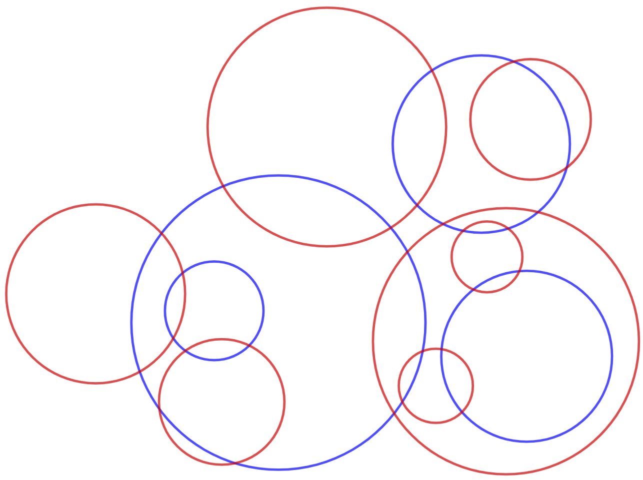


Figure 4: A connected arrangement without triangles.

First, we prove an important local proposition for the number of digons and triangles incident to a given pseudocircle with the use of sweeping. We will use this result to prove global lower bounds for the number of triangles.

Proposition 5.1 (Snoeyink, Hershberger [19])**.** *In an intersecting arrangement of pseudocircles, every pseudocircle is incident to two cells that are digons or triangles on either side.*

Proof. We will first prove this proposition for the inside of a given pseudocircle. We can map the given pseudocircle and its inside to a circle C and its inside D with a self-homeomorphism of the plane using the Jordan-Schönflies theorem [6].

Then we forget the part of the arrangement outside the circle C and extend the curves to infinity (from the crossing points on the circle) without crossing. Now we have an arrangement of bi-infinite and closed curves with the 2-intersection property and we can sweep inside c according to Theorem 4.3. Our original pseudocircle arrangement was intersecting, therefore there is no curve that does not intersect c and the sweep cannot proceed by taking a loop, because it would violate the 2-intersection property. The sweep can only proceed by passing a triangle or a hump, so there is a triangle or digon on the inside of c.

We can cut c inside that digon or triangle and extend both ends to infinity without crossing. Now we have an arrangement of bi-infinite curves, so we can still sweep the originally inside part of c. The sweep still cannot proceed by taking a hump, which means there is another digon or triangle on the original inside of c and therefore on the inside of our original given pseudocircle.

Using an inversion with respect to a circle whose center is in the inside of our given pseudocircle and the center is also not on any pseudocircles, we can swap the inside and outside part of a pseudocircle, so the above proof applies to the outside of any pseudocircle, too.

□

Corollary 5.2. *In an intersecting arrangement of n pseudocircles*

$$2p_2 + 3p_3 \geq 4n.$$

Proof. This follows from Proposition 5.1, there are at least 4 cells incident to every pseudocircle that are digons or triangles and with this method of counting, we count each digon twice and each triangle three times, we count them once for each side. □

The lower bound for triangles in digon-free arrangements follows directly

Theorem 5.3 (Lower Bound (digon-free) Snoeyink, Hershberger [19])**.** *If an arrangement of pseudocircles is intersecting and digon-free, then*

$$p_3 \geq \frac{4n}{3}.$$

Grünbaum conjectured that digon-free arrangements of n intersecting pseudocircles have at least $2n-4$ triangles. This is not true, as Felsner and Scheucher first constructed a family with $\lceil \frac{16n}{11} \rceil$ triangles in [12]. Then, together with Roch, they proved that $\lceil \frac{4n}{3} \rceil$ is the correct lower bound.

Theorem 5.4 (Felsner, Roch, Scheucher [12])**.** *For every $n \geq 6$, there exists a simple, intersecting, digon-free arrangement of n pseudocircles $\mathscr{A}_n$ with*

$$p_3(\mathscr{A}_n) = \lceil \tfrac{4n}{3} \rceil.$$

Their proof is recursive, starting with an arrangement of 3 pseudocircles forming 8 triangles and after that, in each step replacing one pseudocircle with a bundle of 4 pseudocircles. The first step is exceptional, there are 6 pseudocircles forming 8 triangles and no digons, but after this step, each following step replaces one pseudocircle with a bundle of four, generating four more triangles. This provides an example for $n = 3k$ with $4k = \frac{4}{3}n$ triangles. For $n = 3k+1, 3k+2$ they create a suitable example from the example for $n = 3k$.

All of the counterexamples for Grünbaum's triangle conjecture provided in [10, 12] are non-circularizable, therefore, Grünbaum's conjecture was restated for euclidean circles.

Conjecture 5.5 (Weak Grünbaum triangle conjecture)**.** *Every simple digon-free arrangement of n pairwise intersecting circles has at least $2n-4$ triangles.*

According to Levi [14], there are at least n triangles in projective arrangements, this means that in arrangements of great-pseudocircles (Construction 2.1) there are at least $2n$ triangles. Felsner and Kriegel [9] proved that there are at least $n-2$ triangles in euclidean arrangements of pseudolines. Using this result, if we can add a pseudocircle to an intersecting arrangement, such that it separates the two intersection points of any pair of pseudocircles, then there are at least $2n-4$ triangles. This follows from the simple fact that such a pseudocircle would cut the arrangement into two euclidean arrangements of pseudolines.

Next, we allow digons and prove a general lower bound on triangles.

Lemma 5.6 ([12]). *In a simple, intersecting arrangement of n $\geq$ 3 pseudocircles, every digon incident to a given pseudocircle lie on its same side.*

Proof. Suppose indirectly that the pseudocircle C forms a digon on its inside with the pseudocircle D and on its outside with E. If $D = E$, there is a pseudocircle, which is not C or $D = E$ (because $n \geq 3$), and that pseudocircle has to intersect D, which would ruin one of the digons formed by C and D, they cannot intersect on C, because the arrangement is simple.

If $D \neq E$, they have to intersect each other, because the arrangement is intersecting, but they cannot cross in the inside of C because that would ruin the digon formed by C and D, they cannot cross on the outside, because that would ruin the digon formed by C and E and they cannot cross on C, beacuse the arrangement is simple. □

Theorem 5.7 (Felsner, Scheucher [12]). *For any simple, intersecting arrangement of n pseudocircles (digons permitted)*

$$p_3 \geq \frac{2n}{3}$$

Proof. There cannot be digons on both sides of a pseudocircle according to our lemma, so on one of its sides, every pseudocircle has to have at least two triangles, because of Proposition 5.1. Counting the triangles on every circle, we get at least $2n$, but each triangle is counted three times, which means that there are at least $\frac{2n}{3}$ triangles in the arrangements. □

Small arrangments can be enumerated up to $n = 7$ using computers. Based on this, the correct lower bound is $p_3 \geq n-1$ for $3 \leq n \leq 7$, which motivates the following conjecture.

Conjecture 5.8 (Felsner, Scheucher [12]). *Every simple arrangement of n pairwise intersecting circles has at least n − 1 triangles.*

Last in this section, we prove the proposition that by adding pseudocircles to an arrangement, $p_3 + p_2$ never decreases ($c = 1$ case of the following proposition). Because of Theorem 3.4 $p_2 \leq 2n - 2$ for intersecting arrangements, this bounds the number of triangles in an intersecting arrangement $\mathscr{A}$ using the number of triangles in any of its subarrangements $\mathscr{A}'$ by $p_3(\mathscr{A}') \leq p_3(\mathscr{A}) + 2n - 2$.

Proposition 5.9 ([12]). *For every subarrangement $\mathscr{A}'$ of an arrangement of pseudocircles $\mathscr{A}$ and any constant $1 \leq c \leq 2$,*

$$p_3(\mathscr{A}') + cp_2(\mathscr{A}') \leq p_3(\mathscr{A}) + cp_2(\mathscr{A}).$$

Proof. We will prove this proposition by removing any pseudocircle C, proving the proposition for that subarrangement $\mathscr{A}'$ and we get the proposition for smaller subarrangements by iterating this argument.

The subarrangement $\mathscr{A}'$ partitions the pseudocircle C into a finite number of arcs. We add them back one by one and check that for each step $cp_2 + p_3$ never decreases. If the added arc is outside every digon and triangle, $cp_2 + p_3$ cannot decrease.

If an arc is added inside a triangle, not through any of its vertices, it creates a triangle and a quadrangle, or a digon and a pentagon. If added through one of its vertices, it creates two triangles, or a digon and a quadrangle. When added through two virtices, the arc creates a digon and a triangle.

If an arc is added inside a digon not through any vertices, it creates two triangles or a digon and a quadrangle. When added through one of the vertices, it creates a digon and a triangle, and when added through two vertices, it creates two digons.

The two types operations where one of p_2 and p_3 decreases are when an arc is added to a triangle and it creates a digon and a pentagon or quadrangle, and when the arc creates two triangles when added inside a digon. We need $c \geq 1$, because of the first type of operation for $p_3 + cp_2$ not to decrease, and we need $c \leq 2$ because of the second type of operation.

□

Upper bounds

As we have already seen in Proposition 2.2, the total number of cells in a simple, intersecting arrangement of n pseudocircles is $2 + n(n-1)$, thus the following theorem means that at most $\frac{2}{3} + O(\frac{1}{n})$ of all cells are triangles.

Theorem 5.10 (Maximum number of triangles, Felsner, Scheucher [12]). *In any simple, pairwise intersecting arrangement of n pseudocircles, the total number of triangles satisfies*

$$p_3 \le \frac{4}{3}\binom{n}{2} + O(n).$$

Using the recently proved Theorem 3.4, which states that in simple, intersecting arrangements there are at most $2n - 2$ digons, we can prove a stronger upper bound.

Theorem 5.11. *In any simple, pairwise intersecting arrangement $\mathscr{A}$ of $n \ge 4$ pseudocircles, the total number of triangles satisfies*

$$p_3(\mathscr{A}) \le \frac{4}{3}\binom{n}{2} + \frac{2}{3}(n-1).$$

The following proof is from Felsner and Scheucher, except the very end, where we determine the $O(n)$ term. They used that the number of digons is in $O(n)$, which was the best available result at the time (Agarwal et. al. [3]). Figures 6 and 7 are copied from their paper [12].

Proof. We view $\mathscr{A}$ as a 4-regular plane graph, the arcs incident to digons become parallel lines, this also means that every vertex is incident to 4 cells. The vertex set X is the set of crossing points and edges are between consecutive crossing points on a pseudocircle, the number of vertices is $|X| = 2\binom{n}{2}$.

We begin the proof by showing that no vertex is incident to four triangles. Assume that a crossing of C_1 and C_2 is incident to four triangles. There is a pseudocircle C_3 which bounds all four triangles and intersects only C_1 and C_2 (see Figure 5). That is impossible, because the arrangement is intersecting and $n \ge 4$, so no vertex is incident to four triangles.

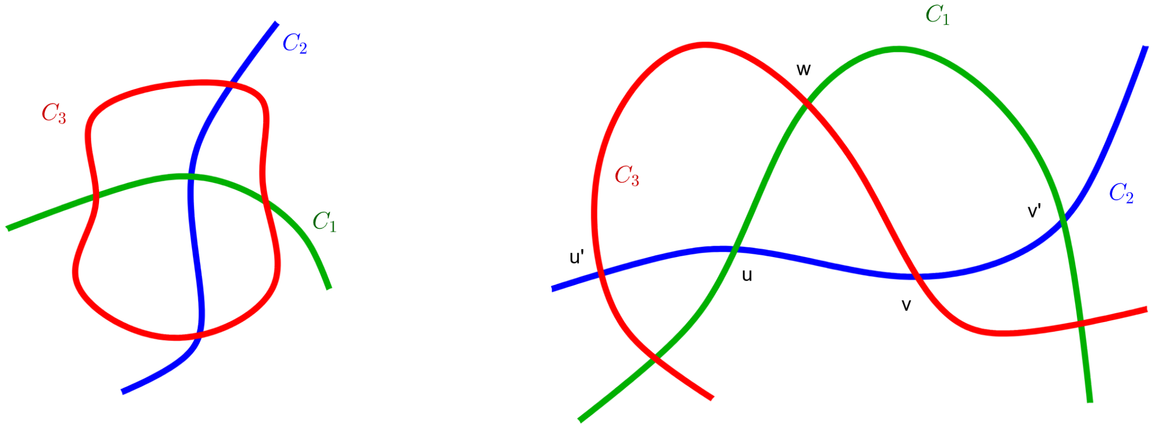


Figure 5: Illustrations for the first two steps of the proof.

Let $X' \subset X$ be the set of vertices incident to three triangles. Our goal will be to perturb a small number of pseudolines such that X' becomes empty. After that, each vertex will be incident to at most two triangles and each triangle is incident to three vertices, so the number of triangles at that point will be at most $\frac{2|X|}{3} = \frac{4}{3}\binom{n}{2}$. To empty X', first we need to show some of its properties.

Next, we will prove that two adjacent crossings u, v in X' share two triangles. u and v are both incident to four cells, three of which are triangles. It means that at least one triangle is incident to both of them, suppose for contradiction that the other is not (see Figure 5). uvw is the triangle incident to both of them. u is incident to three triangles, therefore uw bounds another triangle, let that be uwu'. Similarly vw has to bound another triangle, let that be vwv'. Both uu' and vv' has to bound another triangle, because u, v are incident to three triangles, but that means pseudocircles C_1, C_3 cross three times. That is impossible, therefore two adjacent crossings in X' share two triangles.

Let u, v, w be three distinct crossings in X'. If u is adjacent to both v and w, then v is adjacent to w. It's because u is incident to exactly three triangles and both segments uv and uw are incident to two triangles, therefore there is a uvw triangle. This means that X' is a disjoint union of complete graphs. There cannot be complete graphs with more than three points, because K_5 is not planar and the existence of a K_4-component would contradict 4-regularity of the original graph. Figure 6 shows the possible configurations of points in X'.

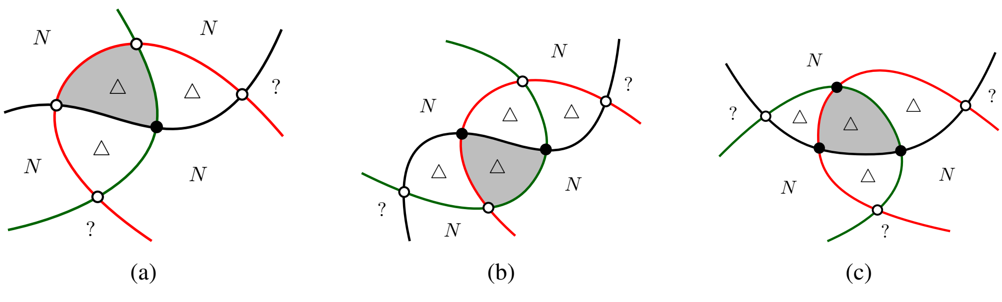


Figure 6: An illustration. In this figure △ marks a triangle, "N" marks a k-cell with $k \geq 4$ ("neither a triangle, nor a digon"), "?" marks an arbitrary cell. Crossings with 3 incident triangles are shown as black vertices (these are the crossings in X').

We note that every vertex in X can be at most once in these types of configurations. The (black) vertices in X' are obviously in exactly one of these configurations, because the configurations are created around them. The other (white) vertices in the configurations are crossings of pseudolines, which meet once more in these configurations. So in order to appear twice in these configurations, a pair of pseudocircles would need to appear in two configurations, one of the configurations appearing on one pair of arcs between their crossing points and the other configuration appearing on their other pair of arcs. In each configuration, there is only one crossing point on each arc of a pseudocircle between the two crossing points of a pair of pseudocircles. This means that both pseudocircles appearing twice in these configurations have only four crossing points, which means they intersect only two other pseudocircles. This is impossible beacuse $n \geq 4$ and the arrangement is intersecting.

For each configuration, the 'passing a triangle' operation (intruduced in Section 4) performed by one of the lines empties X' and creates two or three digons and never removes digons, see Figure 7. We perform these operations on every one of these configuratons and bound the original number of triangles.

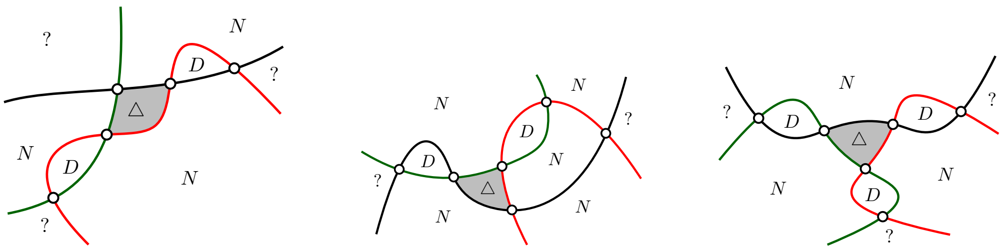


Figure 7: The configurations in (a), (b), and (c) are obtained by passing the gray triangle in the configuration from Figure 6(a), 6(b), and 6(c), respectively. The digons created by the flip are marked "D".

After performing these operations, every vertice is incident to at most two triangles and each triangle is incident to three vertices, so the number of triangles will be at most $\frac{2|X|}{3} = \frac{4}{3}\binom{n}{2}$ and we can improve this bound by $\frac{1}{3}$ for every vertex incident to only one triangle and by $\frac{2}{3}$ for every vertex incident to no triangles. Next, we count how many triangles are removed for each created digon, but we also correct for the number of vertices created by these operations which are incident to less than two triangles. It's important here for the correction, that each vertex could only be included in one configuration.

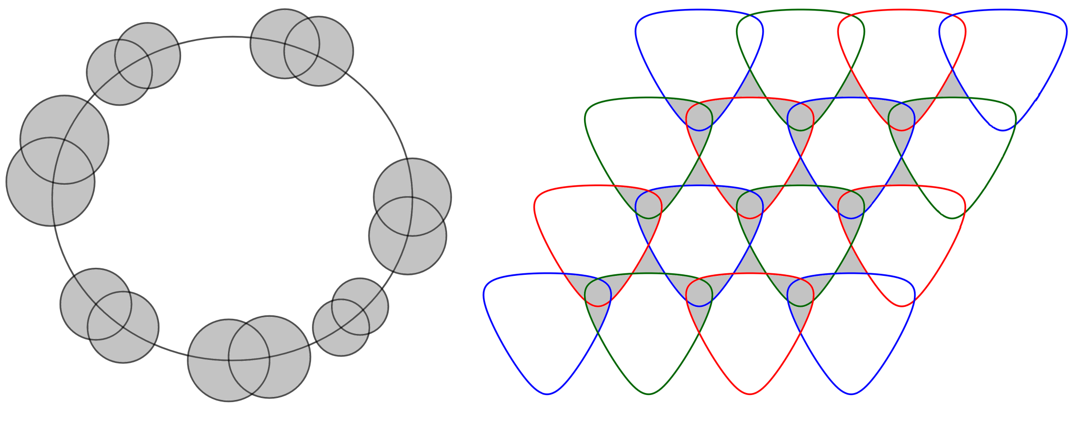
In an (a) type operation, we create 2 digons, lose at most 2 triangles (the created ? cell could be a triangle) and create 4 vertices which are incident to at most one triangle (because ? cells can be triangles). This is a loss of at most $2 - 4 \cdot \frac{1}{3} = \frac{2}{3}$ triangles compared to the upper bound with empty X' with the creation of 2 digons.

In a (b) type operation, we create 2 digons, lose 3 triangles and create 5 vertices incident to at most one triangle and another one incident to no triangles. This is a corrected loss of at $3 - 5 \cdot \frac{1}{3} - \frac{2}{3} = \frac{2}{3}$ triangles with the creation of 2 digons.

In a (c) type operation, we create 3 digons, lose 3 triangles and create 6 vertices incident to at most one triangle for a corrected loss of $3 - 6 \cdot \frac{1}{3} = 1$ triangle with the creation of 3 digons.

In each case, there is a corrected loss of $\frac{1}{3}$ triangles per every created digon. We can create at most $2n-2$ digons because of Theorem 3.4 for the total corrected loss of at most $\frac{1}{3}(2n-2) = \frac{2}{3}(n-1)$ triangles. Thus $p_3(\mathcal{A}) \leq \frac{4}{3}\binom{n}{2} + \frac{2}{3}(n-1)$, which completes the proof. □

As the total number of cells in a simple, intersecting arrangement of pseudocircles is $2 + n(n-1)$ according to Proposition 2.2, the previous theorem can be rewritten as the triangle-to-cell ratio is at most $\frac{2}{3} + O(\frac{1}{n})$ (we can calculate the $O(\frac{1}{n})$ term, which is $\frac{2(n-3)}{3(n^2-n+2)}$). The arguments can't be applied generally as we have an example for $n = 2k + 1$ ($k \geq 2$ and natural), where every cell except two are triangles, see Figure 8(a). This gives us a family of arrangements with triangle-to-cell ratio converging to 1. However, in simple arrangements where every pseudocircle intersects at least three other pseudocircles, we can prove a similar bound of $\frac{5}{6} + O(\frac{1}{n})$ and provide a family where the triangle to cell ratio converges to $\frac{5}{6}$, see Figure 8(b).



(a) (b)

Figure 8: Every triangular face is colored with grey

Theorem 5.12 (Felsner, Scheucher [12])**.** *For a simple arrangement of pseudocircles where every pseudocircle intersects at least three other pseudocircles, the triangle-to-cell ratio is at most* $\frac{5}{6} + O(\frac{1}{n})$

Proof. Everything, except the bound of the number of digons, still applies in this setting from the previous proof of Theorem 5. We used the intersecting property where we proved that no vertex is incident to four triangles and where we proved that every vertex is only in at most one configuration from Figure 6, but each time, the property that every pseudocircle intersects at least three others is enough for us.

We will use that every vertex in X' has at least two neighbors in $X \setminus X'$ as can be seen in Figure 6. We will also use that every vertex in $X \setminus X'$ has at most two neighbors from X', which can be seen in the same figure, applying that every vertex only occurs at most once in these types of configurations. We can discharge $\frac{1}{2}$ from every vertex in X' to two of its neighbors in $X \setminus X'$, getting at most 1 at each vertex in $X \setminus X'$. Thus $|X| \leq |X \setminus X'|$. By counting the number of triangles incident to all the points, we get

$$p_3 \leq \frac{3|X| + 2|X \setminus X'|}{3} \leq \frac{5|X|}{6}.$$

The number of edges in this 4-regular graph is $\frac{4|X|}{2} = 2|X|$, thus from Euler's formula, the number of cells is $c = e - v + 2 = 2|X| - |X| + 2 = |X| + 2$. The triangle to cell ratio is at most

$$\frac{5|X|}{6(|X| + 2)} = \frac{5}{6} + O(\frac{1}{|X|}).$$

This concludes the proof as $|X| \geq 3n$, because there are at least 6 vertices on each pseudocircle and each crossing gets counted twice. □

The maximum number of triangles in pseudoline and line arrangements is well-studied in the euclidean and projective plane, and based on those constructions, we can give lower bounds for the maximum number of triangles in the pseudocircle case.

First, we will use the well-known model of great-circles on a sphere for projective arrangements of lines. In that model, antipodal (opposite) pairs of points on the pseudocircle are the points of the projective plane. Thus, the arrangement of great-circles on the sphere corresponding to a projective arrangement of lines has twice as much vertices, edges and faces of any type as the projective arrangement.

Proposition 5.13 (Felsner, Scheucher [12])**.** *There are simple, intersecting arrangements of n* circles *with $p_3 = \frac{4}{3}\binom{n}{2}$ triangles for infinitely many $n \equiv 0 \mod 6$ and for infinitely many $n \equiv 4 \mod 6$.*

Proof. Blanc [5] (see also [8]) proved that there are projective arrangements of lines with

$\frac{2}{3}\binom{n}{2}$ triangles for infinitely many $n \equiv 0 \mod 6$ and for infinitely many $n \equiv 4 \mod 6$. Hence, their great-circle representations have $\frac{4}{3}\binom{n}{2}$ triangles and they are intersecting because every pair of projective lines had an intersection. Using a stereographic projection, we get an intersecting arrangement of circles in the plane with the same number of triangles. □

Blanc also gives projective arrangements of lines for infinitely many $n \equiv k \mod 6$ for $k = 1, 3, 5$ with $\frac{2}{3}\binom{n}{2} - O(n)$ triangles and projective arrangements of pseudolines for $n \equiv 2 \mod 6$ with $\frac{2}{3}\binom{n}{2} - O(n)$ triangles, which correspond to $\frac{4}{3}\binom{n}{2} - O(n)$ triangles in the corresponding arrangement of circles or in the corresponding arrangement of great-pseudocircles (see Construction 2.1).

Computational results show that for $n = 3$ the maximum number of triangles is 8, but $\lfloor\frac{4}{3}\binom{n}{2}\rfloor$ is only 4. For $n = 5, 6$ the maximum number of triangles is exactly $\lfloor\frac{4}{3}\binom{n}{2}\rfloor$ and for $n = 7, 8$, it is $\lfloor\frac{4}{3}\binom{n}{2}\rfloor + 1$, for $n = 9, 10$ we only know $p_3 \geq \lfloor\frac{4}{3}\binom{n}{2}\rfloor$ ([12]). It would be interesting to know if there are constructions with $p_3 = \frac{4}{3}\binom{n}{2}$ for all n and whether $p_3 \leq \frac{4}{3}\binom{n}{2} + O(1)$ is true, or even better, the precise maximum for large n.

6 Circularizability results and computational aspects

Classification results for small arrangements

Every arrangement of at most 4 pseudocircles is circularizable. Among all connected arrangements of five pseudocircles, exactly four are non-circularizable. Each of them fails to satisfy certain incidence theorems proven by Felsner and Scheucher. Among the 2131 digon-free, pairwise intersecting arrangements of six pseudocircles, exactly three are non-circularizable, these results and many more regarding circularizability can be found in [11].

Incidence theorems

In their study of circularizability of pseudocircle arrangements, Felsner and Scheucher proved incidence theorems similar to Miquel's theorem to show non-circularizability of certain arrangements [11].

Theorem 6.1 (Miquel's Theorem)**.** *Let C_1, C_2, C_3, C_4 four circles, with $C_1 \cap C_2 = \{a, x\}$; $C_2 \cap C_3 = \{b, y\}$; $C_3 \cap C_4 = \{c, z\}$ and $C_4 \cap C_1 = \{d, v\}$. If a, b, c, d lie on a circle, then x, y, z, v also lie on a circle.*

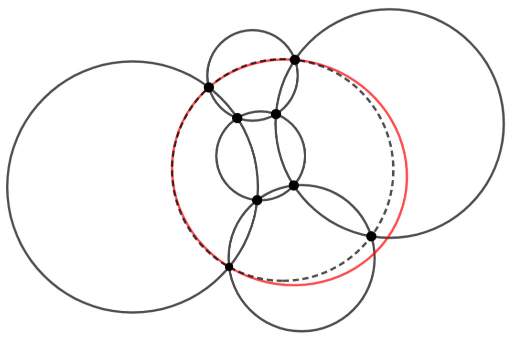


Figure 9: A non-circularizable arrangement, according to Miquel's Theorem

Miquel's theorem can be used to provide a non-circularizable arrangement. In Figure 9 the five black circles and the red curve is a non-circularizable arrangement, because if there was a circularizable drawing, Miquel's theorem would mean that the red curve needs to go through the eighth crossing of the four outer black circles, like the circle with the dashed line, but that would not be isomorphic to the initial arrangement.

This is really similar to the example of non-stretchable pseudolines Levi gave already in 1926 [14], which cannot be stretched due to Pappos's Theorem. Stretchability is the corresponding property of arrangements of pseudolines, it means that there is an isomorphic arrangement of lines. Figure 10 shows Levi's example, according to Pappos's Theorem, in an arrangement of lines, the red curve must go through the ninth crossing of the eight black lines, like tha dashed line, but it does not, so this is a non-stretchable arrangement of pseudolines (if the red curve intersects the top and bottom lines once, which is easily achievable outside the boundaries of the figure).

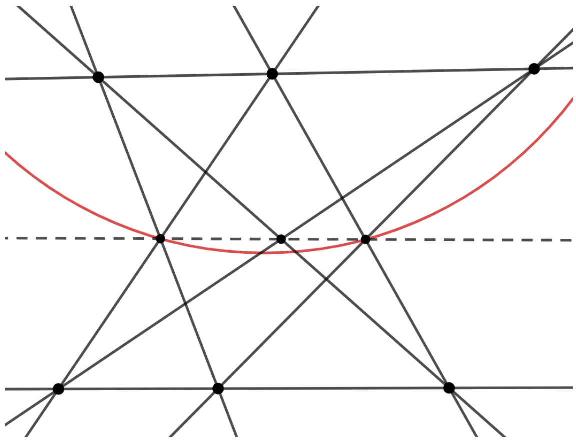


Figure 10: A non-stretchable arrangement, according to Pappos's Theorem

Software and computational aspects

Felsner and Scheucher wrote a recursive computer program to generate all arrangements of pseudocircles up to a given size, currently they have run it for up to six connected curves and seven intersecting curves. It starts with an arrangement of one pseudocircle and adds pseudocircles one by one in all possible ways. The data was saved by using the dual graph of an arrangement, which means original faces are edges and edges are pairs of faces which are incident in the original arrangement [11]. Data for all small arrangements are available on Felsner and Scheucher's pseudocircles webpage: `https://www3.math.tu-berlin.de/diskremath/pseudocircles/`

The use of computers have been helpful to check or formulate conjectures. For example the computer can find counterexamples to Grünbaum's triangle conjecture, see Theorem 5.4 and it helped formulate Conjecture 5.8.

However, we cannot expect many more results by using computers for enumeration. There are $2^{\Theta(n^2)}$ arrangements of pseudocircles and $2^{\Theta(n^2)}$ arrangements of circles [11]. This means that enumerating all arrangements of pseudocircles for larger n is impossible given our current technology.

7 Conclusion and open problems

In this thesis, we reviewed the most important techniques and results regarding pseudocircles, focusing mainly on results regarding triangle and digon bounds in intersecting arrangements of pseudocircles. We conclude the thesis by providing a list of open problems, most of which have already been mentioned.

1. **Problems concerning digons:** What is the maximum number of digons in non-intersecting arrangements of pseudocircles, circles or unit circles? See Section 3.
2. **What is the stabbing number for intersecting arrangements?** See [3, Corollary 2.8].
3. **Weak Grünbaum triangle conjecture:** Conjecture 5.5: Every simple digon-free arrangement of n pairwise intersecting circles has at least $2n-4$ triangles.
4. **Conjecture 5.8:** Every simple arrangement of n pairwise intersecting circles has at least $n-1$ triangles.
5. **What is the upper bound of triangles in intersecting arrangements?** See Theorem 5 and Proposition 5.13. Is it $\frac{4}{3}\binom{n}{2} + O(n)$?
6. **Can we give a condition of circularizability?**

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Use of Artificial Intelligence Statement

I, Máté Jánosik declare that during the preparation of my thesis, I used the AI-based tools listed below for the specified tasks:

Task	**Tool(s) Used**	**Location in Thesis (Chapter, Page)**
Writing this statement	ChatGPT (GPT-4)	here
Outline writing	ChatGPT (GPT-4)	Some section and subsection titles
Translating my words/sentences from Hungarian	DeepL	Entire thesis
Grammar and Language Check	Writefull (as the language check included on overleaf)	Entire thesis

Beyond the tools listed above, I did not use any other AI-based tools.

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