AUTOMORPHISM INVARIANT MEASURES ON HOMOGENEOUS STRUCTURES

Thesis

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Chapter 1

Introduction

The main motivation of this work is as follows. Let $\mathcal{A}$ be an infinite structure with a finite relational language. We say that $\mathcal{A}$ has the *finite model property* iff each first order formula true in $\mathcal{A}$ is also true in a finite substructure of $\mathcal{A}$. One important example for such a structure is the Rado graph (countably infinite random graph). As it is well known, the finite model property of the Rado graph can be proved by a probabilistic argument (for further details we refer to the beginning of section 4.1 of the present work which is based on Lemma 7.4.6 of [2]). Further, because of its simplicity, it is natural to try to adapt the probabilistic argument to a more general setting and establish the finite model property for other structures, or other classes of structures - even for structures that don't have any direct link to probability theory. To carry out such a plan one has to handle the difficulties (A) and (B) below:

Difficulty (A): If we have a first order structure $\mathcal{A}$, then how could we find a well behaved probability measure on the underlying set A of $\mathcal{A}$?

For (A) we note that there is a great tradition of studying probability measures on first order structures. Such investigations can be traced back at least to the related work of Keisler (carried out in the 1960's). Studying probability measures on first order structures received reneved impetus in the last few years when in their celebrated paper [3] Hrusovski, Krupiński and Pillay constructed automorphism invariant measures on the automorphism group $Aut(\mathcal{A})$ of $\mathcal{A}$, on the Stone spaces of $\mathcal{A}$ and on the underlying set A of $\mathcal{A}$. Their intention was different from the finite model property and they used a rather abstract setting. Therefore, in the present work we will follow a different approach. If the automorphism group $Aut(\mathcal{A})$ of $\mathcal{A}$ is large, then (by the results of e.g. [5]) there exist automorphism invariant probability measures on (some dense subgroups of) $Aut(\mathcal{A})$, which can be transferred to the underlying set A of $\mathcal{A}$ by known methods. The idea behind the construction of such measures on $Aut(\mathcal{A})$ is somewhat similar to the construction of Haar measures. Firstly, there is a natural way to make $Aut(\mathcal{A})$ to a topological group. However, the topology thus obtained is not locally compact in general, so the technical details of constructing Haar measures and invariant measures on $Aut(\mathcal{A})$ are different.

The so obtained measures μ on A are finitely additive only. However, the original probabilistic argument establishing the finite model property of the Rado graph intensively uses the measure theoretic products (powers) of μ in order to measure certain subsets of direct powers of A. Further, products of finitely additive measures are not so well behaved: as they are described in [4], the usual product operation of finitely additive measures is not associative.

Difficulty (B): How to form product (power) measures of finitely additive measures for more than two factors?

This work is devoted to manage these difficulties. In Chapter 2 we recall the product construction of finitely additive measures from [4]. The main result of this section is an example exhibiting that the product operation proposed and studyed in [4] is not associative. Let $n \in \mathbb{N}$. In Section 2.2 we introduce the notion of "operation schemes" which can be used to describe all the ways a product measure can be formed by applying the non-associative product operation to obtain an n-factor product. In Chapter 3 we isolate a large enough family of subsets of ${}^nA$ whose measures are the same with respect to all product measures (regardless which operation scheme we used). The main results of Chapter 3 are Theorem 3.13 and Corollary 3.16. Finally, in Chapter 4 we apply the machinery developed in Chapter 3 for products of finitely additive probability measures, and study the finite model property in a fairly general setting. The main result of Section 4 is Corollary 4.13, which is the finitely additive analogue of Theorem 6.2 of [6] (which applies to countably additive measures).

The results obtained in this work are summarized in the manuscript [7] which we intend to publish in a research journal.

We close this Chapter by summing up our system of notation and presenting further preliminaries we need in later sections of this work.

1.1 Notation

Throughout this work $\mathbb{N}$ denotes the set of natural numbers and for every $n \in \mathbb{N}$ we have $n = \{0, 1, ..., n-1\}$. In addition, $\mathbb{R}$ and $\mathbb{R}^+$ denote the set of real numbers and the set of non-negative real numbers, respectively.

Let A and B be sets. Then ${}^AB$ denotes the set of functions whose domain is A and whose range is a subset of B. Moreover, for an ordinal α, ${}^{<\alpha}A$ is defined to be ${}^{<\alpha}A = \bigcup_{\beta<\alpha} {}^{\beta}A$. Similarly, for a cardinal α, $[A]^\alpha$ denotes the set of subsets of A which are of cardinality α. In addition, $|A|$ denotes the cardinality of A and $P(A)$ denotes the power set of A, that is, $P(A)$ consists of all subsets of A.

We use the delimeter $\langle\cdot\rangle$ for (finite or infinite) sequences. Occasionally, as an abuse of notation, by a finite tuple $\bar{x} \in X$ we mean a tuple that comprises of elements of X.

We use function composition in such a way that the rightmost factor acts first. That is, for functions f, g we define $f \circ g(x) = f(g(x))$. If $f : A \to B$ is a function and $X \subseteq {}^nA$ for some $n \in \mathbb{N}$, then we define

$$f[X] = \{\langle f(x_0), ..., f(x_{n-1})\rangle : \ \langle x_0, ..., x_{n-1}\rangle \in X\}.$$

Further, Id_A denotes the identity function on A and if $C \subseteq A$ then $f|_C$ denotes the restriction of f to C.

For any non-empty set A, $Sym(A)$ denotes the symmetric group of A (the group of all permutations of A). If $\mathcal{G}$ is a group or semigroup (with underlying set G) and $f_0, ..., f_{n-1} \in G$ then $\langle f_0, ..., f_{n-1} \rangle$ denotes the subgroup (or subsemigroup) of $\mathcal{G}$ generated by $\{f_0, ..., f_{n-1}\}$. We warn the reader that sometimes, as in the previous paragraph, $\langle f_0, ..., f_{n-1} \rangle$ simply denotes the sequence with terms $f_0, ..., f_{n-1}$. It will always be clear from the context if we mean the substructure generated by the f_i.

If $\mathcal{G}$ is a group acting on a set X and $a \in X$, then $O_{\mathcal{G}}(a)$ denotes the orbit of a with respect to the action of $\mathcal{G}$.

If $\mathcal{A}$ and $\mathcal{B}$ are structures, then $\mathcal{A} \leq \mathcal{B}$ denotes the fact that $\mathcal{A}$ is a substructure of $\mathcal{B}$. Structures will be denoted by calligraphic letters and their underlying sets will be denoted by the corresponding latin letter (in the case of groups usually we don't make such a strict distinction between the group itself and its underlying set and simply denote both by latin letters). In addition, $Aut(\mathcal{A})$ denotes the automorphism group of $\mathcal{A}$.

Crucially, throughout this work by "**measure**" we mean a finitely additive bounded measure unless otherwise specified. That is, a bounded function $\mu : A \to \mathbb{R}^+$ is a measure given that all the below stipulations are satisfied:

- A is a Boolean set algebra over a set X,
- $\mu(\emptyset) = 0$,
- μ is finitely additive, meaning that for all disjoint $E, F \in A$ we have $\mu(E \cup F) = \mu(E) + \mu(F)$.

Throughout this work, our notion of an integral is the same as in [4]. That is, the "usual" one, e.g. defined in Section III.2.2 of [1]. We will slightly elaborate on this in Chapter 2.

Chapter 2

The finitely additive product measure

In this chapter we will recall some notions from [4] which will play a central role in our investigations throughout this work. In the later parts of this chapter, we will confront with a technical difficulty which we will resolve in Section 3.2. More concretely, we will present the product measure of finitely additive measures studied in [4]. In Section 2.1 we will see that this construction does not yield an associative operation. In Corollary 2.16 this will be illustrated with a simple counterexample, which is the main result of this chapter. In Section 2.2 we will give a formal framework for describing the order of operations through which we can construct these product measures. This will be a useful notion throughout the rest of this work. After this, we will show that all of these product measures do behave nicely for at least the family of measurable rectangles.

Notation 2.1. Throughout this Chapter and Chapter 3, we will refer to sections of a function in the way below. Let $f : X \times Y \to Z$. Then, for fixed $x \in X$ we define the x-section f_x of f as usual:

$$f_x : Y \to Z;$$
$$f_x(y) = f(x, y).$$

Symmetrically, for fixed $y \in Y$ we have $f_y(x) = f(x, y)$.

Now we recall some definitions and results from [4].

Definition 2.2. Let X, Y be sets. A function $f : X \times Y \to \mathbb{R}$ is defined to be a **DLC function** if for any $\langle x_i \in X : i \in \mathbb{N}\rangle$ and $\langle y_j \in Y : j \in \mathbb{N}\rangle$ if both

$$\lim_{j\to\infty} \lim_{i\to\infty} f(x_i, y_j) \text{ and } \lim_{i\to\infty} \lim_{j\to\infty} f(x_i, y_j)$$

exist, then they are equal.

Definition 2.3. Let A be a Boolean set algebra over X. A is an **SP algebra** if it "separates points": for any $x \neq y \in X$ there are $E, F \in A$ so that $x \in E$, $y \in F$ and $E \cap F = \emptyset$.

Definition 2.4. Let A be a Boolean set algebra over X. A real-valued function f on X is A-**continuous** if for all $\varepsilon > 0$ there exists a finite partition $\langle E_i : i < N\rangle$ of X such that for all $i < N$ we have $E_i \in A$ and $|f(x) - f(y)| < \varepsilon$ holds for all $x, y \in E_i$.

Definition 2.5. Let A and B be SP algebras over X and Y. A bounded function $f : X \times Y \to \mathbb{R}$ is defined to be a Stone space function **(S-function)** with respect to A and B, if

1. f_y is A-continuous for all $y \in Y$;
2. f_x is B-continuous for all $x \in X$;
3. f is a DLC function.

We recall the notion of the integral with respect to the measure μ. Here, we give simplified reminders of the definitions of the notions needed.

Definition 2.6. Let A be an SP-algebra on a set X and let $\mu : A \to \mathbb{R}$ be a measure. A function $f : X \to \mathbb{R}$ is a simple A-measurable function, if it is of form

$$f(x) = \sum_{i \in n} a_i \cdot \chi_{E_i},$$

where for all $i \in n$ we have $a_i \in \mathbb{R}$ and $E_i \in A$ are pairwise disjoint sets. The μ-integral of a function of this form is as below:

$$\int\limits_X f \,\mathrm{d}\mu = \sum_{i \in n} a_i \cdot \mu(E_i).$$

A function f is μ-integrable provided there exists a sequence $\langle f_i : i \in \mathbb{N} \rangle$ of simple A-measurable functions which converges uniformly to f. Then the μ-integral of f is defined as

$$\int\limits_X f \,\mathrm{d}\mu = \int\limits_X \lim_{i \to \infty} f_i \,\mathrm{d}\mu = \lim_{i \to \infty} \int\limits_X f_i \,\mathrm{d}\mu.$$

We note that this result does not depend on the choice of the uniformly converging sequence.

Definition 2.7. Let A and B be SP algebras over X and Y. Let μ and ν be measures on A and B respectively.
The **limit product algebra** of A and B (denoted by $A * B$) is

$$A * B = \{E \subset X \times Y : \chi_E \text{ is an S-function with respect to A and B}\}.$$

The **finitely additive product measure** of μ and ν on $A * B$ (denoted by $\mu * \nu$) is defined as

$$\mu * \nu(E) = \int\limits_X \int\limits_Y \chi_E \,\mathrm{d}\nu \,\mathrm{d}\mu, \text{ for all } E \in A * B.$$

Theorem 2.8. *According to Theorem 5.4 of [4], $A * B$ is indeed a Boolean set algebra, and $\mu * \nu$ is indeed a measure on it.*

Remark 2.9. Going forward, in this work, unless otherwise specified, any algebra over a set X will be the full poweset algebra $P(X)$, and any measure not obtained from a product measure will be defined on all subsets of X.

2.1 The non-associativity of the construction

In this section we will exhibit through an example that the product measure construction presented in 2.7 is not associative, even in simple cases.

A ternary function $f : A \times B \times C \to \mathbb{R}$ can be considered as a binary function in multiple ways. For example, f can be regarded as a function over $(A \times B) \times C$ (in this case, its first variable ranges over $A \times B$). Similarly, f can be regarded as a function over $A \times (B \times C)$ (in this case, its second variable ranges over $B \times C$).

Next, we will define the characteristic function of a set X. When being considered as a binary function in one way, this function will be an S-function. When being considered as a binary function in another way, it will not be an S-function.

Proposition 2.10. *Based on the partition of its variables, the set X which has the function f below as its characteristic function, can be measurable or not measurable in the appropriate limit product algebras.*

$$f : \mathbb{N}^3 \to \{0, 1\},$$

$$f(x, y, z) = \begin{cases} 0, & \textit{if } x = y \textit{ and } z < x; \\ 1, & \textit{otherwise.} \end{cases}$$

Lemma 2.11. *Consider f as a function of form $f : \mathbb{N}^2 \times \mathbb{N} \to \{0, 1\}$. Then f is not a DLC function, therefore X is not measurable with respect to $P(\mathbb{N}^2) * P(\mathbb{N})$.*

Proof. Let $(x_i, y_i) = (i, i)$ for all $i \in \mathbb{N}$, and let $z_j = j$ for all $j \in \mathbb{N}$.

$$\text{Then } \lim_{j\to\infty} \lim_{i\to\infty} f(x_i, y_i, z_j) = \lim_{j\to\infty} 0 = 0 \neq 1 = \lim_{i\to\infty} 1 = \lim_{i\to\infty} \lim_{j\to\infty} f(x_i, y_i, z_j).$$

□

Consider f as a function of form $f : \mathbb{N} \times \mathbb{N}^2 \to \{0, 1\}$. In the next Lemmas we will show that this way f is an S-function.

Lemma 2.12. *For any $x_0 \in \mathbb{N}$, the section f_{x_0} is $P(\mathbb{N}^2)$-continuous.*

Proof. For any $\varepsilon > 0$, the sets

$$E_1 = \{(y, z) \in \mathbb{N}^2 : y = x_0,\ z < x_0\} \text{ and } E_2 = \mathbb{N}^2 \setminus E_1$$

form a good partition. To see this, observe that for all $(y, z) \in E_1$ we have $f(x_0, y, z) = 0$ and for all $(y, z) \in E_2$ we have $f(x_0, y, z) = 1$. □

Lemma 2.13. *For any $(y_0, z_0) \in \mathbb{N}^2$, the section $f_{(y_0,z_0)}$ is $P(\mathbb{N})$-continuous.*

Proof. Similarly, if $z_0 < y_0$, then

$$E_1 = \{x \in \mathbb{N} : x = z_0\} \text{ and } E_2 = \mathbb{N} \setminus E_1$$

form a good partition for any $\varepsilon > 0$. Otherwise $E_1 = \mathbb{N}$ is a good choice. □

Lemma 2.14. *$f : \mathbb{N} \times \mathbb{N}^2 \to \{0, 1\}$ is a DLC function.*

Proof. Assume, seeking a contradiction, that f is not a DLC function. Then there exist sequences

$$\langle x_i \in \mathbb{N} : i \in \mathbb{N} \rangle \quad \text{and} \quad \langle (y_j, z_j) \in \mathbb{N}^2 : j \in \mathbb{N} \rangle$$

such that both

$$\lim_{i\to\infty} \lim_{j\to\infty} f(x_i, y_j, z_j) \quad \text{and} \quad \lim_{j\to\infty} \lim_{i\to\infty} f(x_i, y_j, z_j)$$

converge, one of them to 1, the other to 0.

For any fixed x, $\lim_{j\to\infty} f(x, y_j, z_j) = \begin{cases} 0, & \text{if } \exists N \text{ such that } \forall j > N \ (y_j = x \text{ and } z_j < x), \\ 1, & \text{if } \exists N \text{ such that } \forall j > N \ (y_j \neq x \text{ or } z_j \geq x). \end{cases}$

$$\lim_{i\to\infty} \lim_{j\to\infty} f(x_i, y_j, z_j) = \begin{cases} 0, & \text{if } \exists N \text{ such that } \forall i > N \ (\lim_{j\to\infty} f(x_i, y_j, z_j) = 0), \\ 1, & \text{if } \exists N \text{ such that } \forall i > N \ (\lim_{j\to\infty} f(x_i, y_j, z_j) = 1). \end{cases}$$

Similarly, for fixed (y, z) we have

$$\lim_{i\to\infty} f(x_i, y, z) = \begin{cases} 0, & \text{if } \exists N \text{ such that } \forall i > N \ (x_i = y \text{ and } z < x_i), \\ 1, & \text{if } \exists N \text{ such that } \forall i > N \ (x_i \neq y \text{ or } z \geq x_i). \end{cases}$$

$$\lim_{j\to\infty} \lim_{i\to\infty} f(x_i, y_j, z_j) = \begin{cases} 0, & \text{if } \exists N \text{ such that } \forall j > N \ (\lim_{i\to\infty} f(x_i, y_j, z_j) = 0), \\ 1, & \text{if } \exists N \text{ such that } \forall j > N \ (\lim_{i\to\infty} f(x_i, y_j, z_j) = 1). \end{cases}$$

By symmetry , we may assume that

$$(1) \ \lim_{j\to\infty} \lim_{i\to\infty} f(x_i, y_j, z_j) = 0 \qquad \text{and} \qquad (2) \ \lim_{i\to\infty} \lim_{j\to\infty} f(x_i, y_j, z_j) = 1.$$

From (1) we obtain

$$\exists N_1 \ \forall j > N_1 \ \exists N_{1,j} \ \forall i > N_{1,j} \text{ we have } x_i = y_j \text{ and } z_j < x_i. \tag{3}$$

Similarly, from (2) we obtain

$$\exists N_2 \ \forall i > N_2 \ \exists N_{2,i} \ \forall j > N_{2,i} \text{ we have } x_i \neq y_j \text{ or } z_j \geq x_i. \tag{4}$$

For large enough i and j this is contradictory. In more detail, for a moment fix $j > N_1$ arbitrarily. Then, on one hand, by (3),

$$\text{for large enough } i, \text{ we have } x_i = y_j \text{ and } z_j < x_i. \tag{$\star$}$$

In particular, there exists x such that for all large enough i we have $x_i = x$ because of the following: choosing $j := N_1 + 1$ and $x := y_{N_1+1}$, from ($\star$) we obtain that for large enough i we have $x_i = y_{N_1+1} = x$. In addition, ($\star$) also implies that $z_j < x$ holds for $j > N_1$. Further, as ($\star$) holds for arbitrary $j > N_1$, we get that for large enough j we also have $y_j = x$.

On the other hand, for large enough $i > N_2$ we have $x_i = x$ and for large enough $j > N_{2,i}$ we have $z_j < x$ and $y_j = x$. Specifically, $x_i = x = y_j$ and $z_j < x = x_i$.

However, from (4) we get $x_i \neq y_j$ or $z_j \geq x_i$. This contradiction completes the proof. □

Corollary 2.15. *$f : \mathbb{N} \times \mathbb{N}^2 \to \{0, 1\}$ is an S-function, therefore X is measurable with respect to $P(\mathbb{N}) * P(\mathbb{N}^2)$.*

Corollary 2.16. *The $*$ operation of forming product measures is not associative.*

2.2 Rectangles

For our purposes, it is desirable to study only sets that are measurable irrespective to the order of operations through which we have constructed the underlying product algebra. That is, sets whose characteristic functions are S-functions regardless of which way we partition their variables. The next definition gives a precise notation for this order of operations.

Definition 2.17. We define **binary operation schemes** recursively.
For $i \in I$, i is a binary operation scheme and $\rho(i) = \{i\}$ is the set of indices it ranges over.
If $\tau_1,\ \tau_2$ are binary operation schemes and $\rho(\tau_1) \cap \rho(\tau_2) = \emptyset$, then $\langle \tau_1, \tau_2 \rangle$ is an operation scheme. Further, $\rho(\langle \tau_1, \tau_2 \rangle) = \rho(\tau_1) \cup \rho(\tau_2)$.
For an operation scheme τ and sets X_i for $i \in \rho(\tau)$ let

$$\prod_{\tau} X_i = \begin{cases} X_i, & \text{if } \tau = i; \\ \prod\limits_{\tau_1} X_i \times \prod\limits_{\tau_2} X_i, & \text{if } \tau = \langle \tau_1, \tau_2 \rangle. \end{cases}$$

For an operation scheme τ and SP algebras A_i let

$$\prod_{\tau} A_i = \begin{cases} A_i, & \text{if } \tau = i; \\ \prod\limits_{\tau_1} A_i * \prod\limits_{\tau_2} A_i, & \text{if } \tau = \langle \tau_1, \tau_2 \rangle. \end{cases}$$

For an operation scheme τ and SP algebras A_i, let μ_i be measures on them respectively. Then we define the measure $\mu^\tau : \prod\limits_{\tau} A \to \mathbb{R}$ as

$$\mu^\tau(x) = \begin{cases} \mu_i(x), & \text{if } \tau = i; \\ (\mu^{\tau_1} * \mu^{\tau_2})(x), & \text{if } \tau = \langle \tau_1, \tau_2 \rangle. \end{cases}$$

Example 2.18. For example, $(A_0 * A_1) * A_2$ and $A_0 * (A_1 * A_2)$ can be described by the schemes $\langle\langle 0, 1\rangle, 2\rangle$ and $\langle 0, \langle 1, 2\rangle\rangle$ respectively. For a more complex example, let $\tau = \langle\langle 1, \langle 2, 3\rangle\rangle, \langle 4, 5\rangle\rangle$. Then

$$\prod_{\tau} \mathbb{N} = (\mathbb{N} \times (\mathbb{N} \times \mathbb{N})) \times (\mathbb{N} \times \mathbb{N}).$$

Intuitively one can regard these operation schemes as "the skeletons" of strictly binary trees with finite depths. Borrowing from the language of computer science, they can be considered as fold operations, as they condense a list of objects into a single object by repeatedly using a binary operation. For example, the above defined τ could be represented as the tree of Figure 2.1.

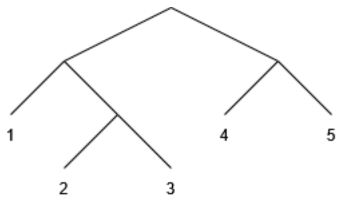

Figure 2.1: τ

Remark 2.19. The symbol $\prod\limits_{\tau}$ will be used both for the direct product of sets and for the limit product $(*)$ of algebras over the scheme τ. This should cause no confusion.

We will now show that the product measure defined in 2.7 behaves well when applied to the measurable rectangles.

Lemma 2.20. *Let $n \in \mathbb{N}$ and X_i be sets for all $i \in n$. Let A_i be SP algebras on X_i respectively. If for all $i \in n$ the sets $U_i \subset X_i$ are measurable (that is, $U_i \in A_i$), then $\prod\limits_{\tau} U_i$ is measurable ($\prod\limits_{\tau} U_i \in \prod\limits_{\tau} A_i$) for any operation scheme τ where $\rho(\tau) = n$.*
For any such τ, $\mu^{\tau}(\prod\limits_{\tau} U_i) = \prod\limits_{i \in n} \mu_i(U_i)$.

Proof. We apply induction on n.
First assume $n = 1$. Then U is measurable by assumption.
Turning to the inductive step, let $\tau = \langle \tau_1, \tau_2 \rangle$. Then $|\rho(\tau_i)| < n$. By induction $\prod\limits_{\tau_1} U_i$ and $\prod\limits_{\tau_2} U_i$ are measurable. As $\prod\limits_{\tau_1} U_i \times \prod\limits_{\tau_2} U_i$ is a rectangle, by 5.4 of [4] it is measurable.
Similarly,

$$(\mu_i^{\tau_1} * \mu_i^{\tau_2})(\prod_{\tau_1} U_i \times \prod_{\tau_2} U_i) \overset{5.4 \text{ of } [4]}{=} \mu_i^{\tau_1}(\prod_{\tau_1} U_i) \cdot \mu_i^{\tau_2}(\prod_{\tau_2} U_i) \overset{induction}{=} \prod_{i \in n} \mu_i(U_i).$$

□

Remark 2.21. The measurable rectangles of finite dimensions are measurable regardless of the order of operations with which they were constructed, that is, regardless of the operation scheme used to construct the product space. Their measures are the same in all of these spaces. In Section 3.2 we will exhibit a considerably larger family of sets with the above property.

Chapter 3

Solving the associativity problem

Let's assume we have a set X and a measure μ on it. Our aim in this chapter is to investigate the possibilities of deriving a measure μ^n on X^n. For any operation scheme τ for which $\rho(\tau) = n$, we have a construction for μ^τ. By only investigating sets that are measurable when looked at through the lense of all of these schemes, we get one step closer to this ideal measure μ^n. However, by assuring that a set is measurable "from any direction", we cannot simply assume that its measure is the same in any of these spaces. Our goal is to give sufficient conditions for some sets to have the same measure, regardless of which operation scheme was used in creating them. A family of sets that have this property is the set of measurable rectangles, as seen in Lemma 2.20.

In Section 3.1 we will introduce a family of functions that includes the characteristic functions of sets that are measurable with respect to every appropriate operation scheme. In Section 3.2 we will combine characteristics of these functions and a Fubini-style theorem from [4] to prove that these functions are well behaved with respect to integration (Theorem 3.13). Using this, we will define and prove the associativity of a (finite dimensional) product measure in Definition 3.17 and Corollary 3.16 respectively. In Chapter 3.3 we will introduce a way of transferring measures from groups to sets they act on, and vice versa. As an aside, we will show a different strategy for computing the measures of certain sets in Theorem 3.34.

3.1 Hereditary S-functions

In this section we will introduce a class of functions that will play a significant role. We start by recalling some results from [4] on which we will build.

Fact 3.1. *(Part of Lemma 4.2. of [4].) Let A be a Boolean set algebra on the set X. Let f be a real-valued function on X. Then the following are equivalent:*

- *f is A-continuous;*
- *if μ is a measure on A, then f is μ-integrable;*
- *f is a uniform limit of simple, A-measurable functions.*

Fact 3.2. *(Part of Theorem 4.4. of [4].) Let A and B be SP algebras on the sets X and Y respectively. Let μ and ν be measures on A and B respectively. Let f be an S-function on $X \times Y$. Then the function*

$$\Psi(y) = \int_X f_y \, \mathrm{d}\mu$$

is B-continuous. Similarly, the function $\Phi(x) = \int_Y f_x \, \mathrm{d}\nu$ is A-continuous.

Fact 3.3. *(Lemma 5.2.(C) of [4].) Let A and B be SP algebras on the sets X and Y respectively. If $\langle f_n : n \in \mathbb{N} \rangle$ is a sequence of S-functions on $X \times Y$ which converges uniformly to a function f, then f is an S-function.*

Definition 3.4. For all operation schemes τ with range $\rho(\tau)$ and sets X_i for $i \in \rho(\tau)$ we define the function $c_\tau : \prod_\tau X_i \to \prod_{\rho(\tau)} X_i$ by recursion, as follows. With the aid of these functions we can naturally convert a "bracketed" expression into its usual direct product version.

If $\tau = i$, then $c_\tau(x)(i) = x$.
If $\tau = \langle \tau_1, \tau_2 \rangle$, then $c_\tau = c_{\tau_1} \cup c_{\tau_2}$.

Remark 3.5. In order to make our text more reader-friendly, we will often use the following notation. If $\tau = \langle \tau_1, \tau_2 \rangle$, then

$$c_\tau(x_1, x_2) = (c_{\tau_1}(x_1), c_{\tau_2}(x_2)).$$

Definition 3.6. Let τ be an operation scheme with range $\rho(\tau)$. For all $i \in \rho(\tau)$ let X_i be a set endowed with the SP algebra A_i. We define a function $f : \prod_\tau X_i \to \mathbb{R}$ as a **hereditary S-function** (with respect to $\prod_\tau A_i$) recursively based on τ.
If $\tau = i$, f is a hereditary S-function if it is A_i-continuous.
If $\tau = \langle \tau_1, \tau_2 \rangle$, f is a hereditary S-function if the stipulations below are satisfied:

- f is $\prod_\tau A_i$-continuous;
- f is an S-function with respect to $\prod_{\tau_1} A_i$ and $\prod_{\tau_2} A_i$;
- for all $x \in \prod_{\tau_1} X_i$, the x-section f_x of f is a hereditary S-function (with respect to $\prod_{\tau_2} A_i$);
- for all $y \in \prod_{\tau_2} X_i$, the y-section f_y of f is a hereditary S-function (with respect to $\prod_{\tau_2} A_i$).

Remark 3.7. The condition of $\prod_{\tau} A_i$-continuity is to ensure integrability, in reference to Fact 3.1. Throughout the rest of this work, we will work with the full powerset as the underlying algebra $A = P(X)$. In this case, any bounded one variable function is a hereditary S-function. The following propositions are likely true for arbitrary SP algebras A, but for brevity's sake, we will only prove them with respect to $P(X)$, as that is sufficient for this work.

Proposition 3.8. *Let $n \geq 2$. For all $i \in n$, let X_i be a set endowed with the SP algebra $P(X_i)$. Let f be a real-valued function on $\prod_n X_i$ such that for all operation schemes τ with range $\rho(\tau) = n$ the function $f \circ c_\tau : \prod_\tau X_i \to \mathbb{R}$ is an S-function.*
Then for all $\tau = \langle \tau_1, \tau_2 \rangle$ with $\tau_2 = \langle \eta_1, \eta_2 \rangle$ and for all $x \in \prod_{\tau_1} X_i$, the x-section $g := (f \circ c_\tau)_x : \prod_{\tau_2} X_i \to \mathbb{R}$ is an S-function with respect to $\prod_{\eta_1} P(X_i)$ and $\prod_{\eta_2} P(X_i)$.
Symmetrically, the same is true for $g' := (f \circ c_\tau)_y : \prod_{\tau_1} X_i \to \mathbb{R}$.

Proof. We have to prove the following three assertions for g to be an S-function:

1. g_y is $\prod_{\eta_2} P(X_i)$-continuous for all $y \in \prod_{\eta_1} X_i$;
2. g_z is $\prod_{\eta_1} P(X_i)$-continuous for all $z \in \prod_{\eta_2} X_i$;
3. g is a DLC function.

1. Let $\tau' = \langle \langle \tau_1, \eta_1 \rangle, \eta_2 \rangle$. We claim that $g_y = (f \circ c_{\tau'})_{\langle x,y \rangle}$. To see this, let $z \in \prod_{\eta_2} X_i$. Then we have

$$g_y(z) = \left((f \circ c_\tau)_x\right)_y (z) = (f \circ c_\tau)_x(y, z) = f \circ c_\tau(x, \langle y, z \rangle) = f\left(c_{\tau_1}(x), c_{\tau_2}(y, z)\right) =$$
$$= f\left(c_{\tau_1}(x), c_{\eta_1}(y), c_{\eta_2}(z)\right) = f \circ c_{\tau'}(\langle x, y \rangle, z) = (f \circ c_{\tau'})_{\langle x,y \rangle}(z).$$

As $f \circ c_{\tau'}$ is an S-function by our assumption, for any $(x, y) \in \prod_{\langle \tau_1, \eta_1 \rangle} X_i$ the section $(f \circ c_{\tau'})_{\langle x,y \rangle}$ is $\prod_{\eta_2} P(X_i)$-continuous. Thus g_y is $\prod_{\eta_2} P(X_i)$-continuous as well.

2. Similarly to 1., this is easily verified with the choice of $\tau' = \langle \langle \tau_1, \eta_2 \rangle, \eta_1 \rangle$.

3. Let's assume that for the sequences $\langle a_k \in \prod_{\eta_1} X_i : k \in \mathbb{N} \rangle$ and $\langle b_j \in \prod_{\eta_2} X_i : j \in \mathbb{N} \rangle$ both of the double limits

$$\lim_{k\to\infty} \lim_{j\to\infty} g(a_k, b_j) \text{ and } \lim_{j\to\infty} \lim_{k\to\infty} g(a_k, b_j) \text{ exist.}$$

Then we need to prove their equality. As in 1., let $\tau' = \langle \langle \tau_1, \eta_1 \rangle, \eta_2 \rangle$. Then we have

$$\lim_{k\to\infty} \lim_{j\to\infty} g(a_k, b_j) = \lim_{k\to\infty} \lim_{j\to\infty} (f \circ c_\tau)_x(a_k, b_j) = \lim_{k\to\infty} \lim_{j\to\infty} f \circ c_\tau(x, \langle a_k, b_j \rangle) =$$
$$= \lim_{k\to\infty} \lim_{j\to\infty} f(c_{\tau_1}(x), c_{\eta_1}(a_k), c_{\eta_2}(b_j)) = \lim_{k\to\infty} \lim_{j\to\infty} f \circ c_{\tau'}(\langle x, a_k \rangle, b_j).$$

Similarly, we have

$$\lim_{j\to\infty}\lim_{k\to\infty} g(a_k, b_j) = \lim_{j\to\infty}\lim_{k\to\infty} f\circ c_{\tau'}(\langle x, a_k\rangle, b_j).$$

By our assumption $f\circ c_{\tau'}$ is an S-function, so it is a DLC function. Thus

$$\lim_{k\to\infty}\lim_{j\to\infty} f\circ c_{\tau'}(\langle x, a_k\rangle, b_j) = \lim_{j\to\infty}\lim_{k\to\infty} f\circ c_{\tau'}(\langle x, a_k\rangle, b_j).$$

From this it follows that

$$\lim_{k\to\infty}\lim_{j\to\infty} g(a_k, b_j) = \lim_{j\to\infty}\lim_{k\to\infty} g(a_k, b_j),$$

and thus g is a DLC function. □

Proposition 3.9. *Let $n \in \mathbb{N}$. For all $i \in n$ let X_i be a set endowed with the SP algebra $P(X_i)$. Let f be a real-valued bounded function on $\prod\limits_n X_i$ such that for all operation schemes τ with range $\rho(\tau) = n$ the function $f\circ c_\tau : \prod\limits_\tau X_i \to \mathbb{R}$ is a $\prod\limits_\tau P(X_i)$-continuous S-function. Then for all such τ, the function $f \circ c_\tau$ is a hereditary S-function.*

Proof. We apply induction on n. First assume n=1. By definition any bounded function is a hereditary S-function with respect to $P(X_i)$.
Turning to the inductive step, let $\tau = \langle\tau_1, \tau_2\rangle$. We have to prove the following assertions for $f\circ c_\tau$ to be a hereditary S-function:

1. $f\circ c_\tau$ is $\prod\limits_\tau P(X_i)$-continuous;
2. $f\circ c_\tau$ is an S-function;
3. $(f\circ c_\tau)_x$ is a hereditary S-function for all $x \in \prod\limits_{\tau_1} X_i$;
4. $(f\circ c_\tau)_y$ is a hereditary S-function for all $y \in \prod\limits_{\tau_2} X_i$.

1. and 2. are true by our assumptions. 3. and 4. are symmetrical, thus it is sufficient to prove 3.
If $|\rho(\tau_2)| = 1$, since f is bounded, so is $(f\circ c_\tau)_x$, thus it is a hereditary S-function with respect to $P(X_i)$.
If $|\rho(\tau_2)| \geq 2$, by induction it is sufficient to prove that $g := (f\circ c_\tau)_x \circ c_{\tau_2}^{-1} : \prod\limits_{\rho(\tau_2)} X_i \to \mathbb{R}$ is a function which meets the conditions of this theorem. More precisely, we want to prove that for all operation schemes η for which $\rho(\eta) = \rho(\tau_2)$, the function $g\circ c_\eta$ is $\prod\limits_\eta P(X_i)$-continuous and an S-function. Let $\tau' = \langle\tau_1, \eta\rangle$. We claim that $g\circ c_\eta = (f\circ c_{\tau'})_x$.
To see this, let $y \in \prod\limits_\eta X_i$. Then we have

$$(g\circ c_\eta)(y) = ((f\circ c_\tau)_x \circ c_{\tau_2}^{-1} \circ c_\eta)(y) = (f\circ c_\tau)(x, c_{\tau_2}^{-1}(c_\eta(y))) = f(c_{\tau_1}(x), c_{\tau_2}(c_{\tau_2}^{-1}(c_\eta(y)))) =$$
$$= f(c_{\tau_1}(x), c_\eta(y)) = (f\circ c_{\tau'})(x, y) = (f\circ c_{\tau'})_x(y).$$

Since f satisfies the conditions of Proposition 3.8, $(f\circ c_{\tau'})_x$ is an S-function, and thus so is $g\circ c_\eta$. Similarly, $(f\circ c_{\tau'})$ is an S-function by our assumption, therefore $(f\circ c_{\tau'})_x$ is $\prod\limits_\eta P(X_i)$-continuous. □

Proposition 3.10. *Let $n \in \mathbb{N}$. For all $i \in n$ let X_i be a set endowed with the SP algebra $P(X_i)$. Let μ_i be measures on $P(X_i)$ respectively. Let f be a real-valued bounded function on $\prod_n X_i$ such that for all operation schemes τ with range $\rho(\tau) = n$ the function $f \circ c_\tau : \prod_\tau X_i \to \mathbb{R}$ is a $\prod_\tau P(X_i)$-continuous S-function. Then for any such τ with $\tau = \langle \tau_1, \tau_2 \rangle$ the function $\Psi(y) = \int_{\prod_{\tau_1} X_i} (f \circ c_\tau)_y \, \mathrm{d}\mu^{\tau_1} : \prod_{\tau_2} X_i \to \mathbb{R}$ is a hereditary S-function.*

Symmetrically, the same is true for $\Psi'(x) = \int_{\prod_{\tau_2} X_i} (f \circ c_\tau)_x \, \mathrm{d}\mu^{\tau_2} : \prod_{\tau_1} X_i \to \mathbb{R}$.

Proof. The function f meets the conditions of Proposition 3.9, thus $f \circ c_\tau$ is a hereditary S-function. If $|\rho(\tau_2)| = 1$, Ψ is bounded and thus a hereditary S-function. If $|\rho(\tau_2)| \geq 2$, because of Proposition 3.9, it is sufficient to prove that for all operation schemes η for which $\rho(\eta) = \rho(\tau_2)$, the function $\Psi \circ c_{\tau_2}^{-1} \circ c_\eta$ is a $\prod_\eta P(X_i)$-continuous S-function. Let $\tau' = \langle \tau_1, \eta \rangle$ and let $y \in \prod_\eta X_i$. We have

$$\Psi \circ c_{\tau_2}^{-1} \circ c_\eta(y) = \Psi(c_{\tau_2}^{-1}(c_\eta(y))) = \int_{\prod_{\tau_1} X_i} (f \circ c_\tau)_{c_{\tau_2}^{-1}(c_\eta(y))} \, \mathrm{d}\mu^{\tau_1} =$$

$$\int_{\prod_{\tau_1} X_i} f_{c_{\tau_2}(c_{\tau_2}^{-1}(c_\eta(y)))} \circ c_{\tau_1} \, \mathrm{d}\mu^{\tau_1} = \int_{\prod_{\tau_1} X_i} f_{c_\eta(y)} \circ c_{\tau_1} \, \mathrm{d}\mu^{\tau_1} = \int_{\prod_{\tau_1} X_i} (f \circ c_{\tau'})_y \, \mathrm{d}\mu^{\tau_1}.$$

By our assumption, $f \circ c_{\tau'}$ is an S-function. Thus, by Fact 3.2, $\int_{\prod_{\tau_1} X_i} (f \circ c_{\tau'})_y \, \mathrm{d}\mu^{\tau_1}$ is $\prod_\eta P(X_i)$-continuous. Combining this with Fact 3.1, it follows that it is the uniform limit of simple $\prod_\eta P(X_i)$-measurable functions. As these simple functions are clearly S-functions as well, $\Psi \circ c_{\tau_2}^{-1} \circ c_\eta$ is the uniform limit of S-functions, and therefore by Fact 3.3 it is an S-function. □

3.2 Finite dimensional canonical products of finitely additive measures

In this section we will prove that under some conditions, some type of sets have the same measure irrespective of which operation scheme was used in creating the space. More precisely, we will show that functions that are hereditary S-functions with respect to every appropriate operation scheme are well behaved with respect to integration. Therefore sets that have such functions as characteristic functions have the same measures in all the appropriate limit product algebras. Using such sets we will present measures that can act as construction independent multidimensional products of finitely additive measures. We will call these *canonical product measures.*

We recall a Fubini-style theorem from [4] that will be crucial in our proof.

Fact 3.11. *(A "Fubini" Theorem, 6.1 of [4].) Let A and B be SP-algebras over the sets X and Y. Let μ and ν be measures on A and B respectively. Let f be a real-valued bounded function on $X \times Y$. If f is the uniform limit of simple $A * B$-measurable functions, then*

$$\int_X \int_Y f \, \mathrm{d}\mu \, \mathrm{d}\nu = \int_Y \int_X f \, \mathrm{d}\nu \, \mathrm{d}\mu = \int_{X \times Y} f \, \mathrm{d}(\mu * \nu).$$

Remark 3.12. If f is a hereditary S-function, it is by definition $A * B$-continuous. Then, by 3.1 it is the uniform limit of simple measurable functions, and thus 3.11 applies to it.

Theorem 3.13. *Let $n \in \mathbb{N}$ and for all $i \in n$ let X_i be sets. Let $\mu_i : P(X_i) \to \mathbb{R}$ be (bounded, finitely additive) measures. Let f be a real-valued bounded function on $\prod_n X_i$ such that for all operation schemes τ with range $\rho(\tau) = n$ the function $f \circ c_\tau : \prod_\tau X_i \to \mathbb{R}$ is a hereditary S-function. Then for all such operation schemes τ we have*

$$\int_{\prod_\tau X_i} f \circ c_\tau \, \mathrm{d}\mu^\tau = \int_{X_0} \dots \int_{X_{n-1}} f \, \mathrm{d}\mu_{n-1} \dots \mathrm{d}\mu_0.$$

We note that the order of these integrals can be changed without impacting this value.

Proof. We apply induction on n. For $n = 1$ the hypotesis clearly holds, as by definition $\prod_\tau X_i = X_0$ and $\mu^\tau = \mu_0$.
Turning to the inductive step, let $\tau = \langle \tau_1, \tau_2 \rangle$. Starting on the left side of the equation, we have

$$\int_{\prod_\tau X_i} f \circ c_\tau \, \mathrm{d}\mu^\tau = \int_{\prod_{\tau_1} X_i \times \prod_{\tau_2} X_i} f \circ c_\tau \, \mathrm{d}(\mu^{\tau_1} * \mu^{\tau_2}).$$

As $f \circ c_\tau$ is a hereditary S-function, Fact 3.11 aplies, thus we have

$$\int_{\prod_{\tau_1} X_i \times \prod_{\tau_2} X_i} f \circ c_\tau \, \mathrm{d}(\mu^{\tau_1} * \mu^{\tau_2}) = \int_{\prod_{\tau_1} X_i} \int_{\prod_{\tau_2} X_i} f \circ c_\tau \, \mathrm{d}\mu^{\tau_2} \, \mathrm{d}\mu^{\tau_1}.$$

Let $g : \prod_{\tau_1} X_i \to \mathbb{R}$ be the function for which $g(x) := \int\limits_{\prod_{\tau_2} X_i} (f \circ c_\tau)_x \, \mathrm{d}\mu^{\tau_2}$. By Proposition 3.10, g is a hereditary S-function. Similarly, for any operation scheme η with range $\rho(\eta) = \rho(\tau_1)$, the function $g \circ c_{\tau_1}^{-1} \circ c_\eta : \prod_\eta X_i \to \mathbb{R}$ is a hereditary S-function. Combined with the fact that $\rho(\tau_1) \subsetneq \rho(\tau)$, it is clear that we can apply the inductive hypothesis for $g \circ c_{\tau_1}^{-1}$. Thus we have

$$\int\limits_{\prod_{\tau_1} X_i} \int\limits_{\prod_{\tau_2} X_i} f \circ c_\tau \, \mathrm{d}\mu^{\tau_2} \, \mathrm{d}\mu^{\tau_1} = \int\limits_{\prod_{\tau_1} X_i} g \circ c_{\tau_1}^{-1} \circ c_{\tau_1} \, \mathrm{d}\mu^{\tau_1} \underset{induction}{=} \underbrace{\int\limits_{X_i} \dots \int\limits_{X_i}}_{i \in \rho(\tau_1)} g \circ c_{\tau_1}^{-1} \underbrace{\mathrm{d}\mu_i \dots \mathrm{d}\mu_i}_{i \in \rho(\tau_1)} . \quad (\star)$$

Observe that by induction the order of these repeated integrals can be changed.

By definition, all sections of a hereditary S-function are hereditary S-functions as well. Thus for all $x \in \prod_{\tau_1} X_i$, the functions $h_x = (f \circ c_\tau)_x : \prod_{\tau_2} X_i \to \mathbb{R}$ are hereditary S-functions. Similarly, for any operation scheme ξ with range $\rho(\xi) = \rho(\tau_2)$ we have $h_x \circ c_{\tau_2}^{-1} \circ c_\xi = (f \circ c_{\tau'})_x$ where $\tau' = \langle \tau_1, \xi \rangle$. Since $f \circ c_{\tau'}$ is a hereditary S-function by our assumption, so is $(f \circ c_{\tau'})_x$ and thus $h_x \circ c_{\tau_2}^{-1} \circ c_\xi$ is one as well. Combined with the fact that $\rho(\tau_2) \subsetneq \rho(\tau)$, this means that we can apply the inductive hypothesis for $h_x \circ c_{\tau_2}^{-1}$. Thus we have

$$\int\limits_{\prod_{\tau_2} X_i} (f \circ c_\tau)_x \, \mathrm{d}\mu^{\tau_2} = \int\limits_{\prod_{\tau_2} X_i} h_x \circ c_{\tau_2}^{-1} \circ c_{\tau_2} \, \mathrm{d}\mu^{\tau_2} \underset{induction}{=} \underbrace{\int\limits_{X_i} \dots \int\limits_{X_i}}_{i \in \rho(\tau_2)} h_x \circ c_{\tau_2}^{-1} \underbrace{\mathrm{d}\mu_i \dots \mathrm{d}\mu_i}_{i \in \rho(\tau_2)} =$$

$$= \underbrace{\int\limits_{X_i} \dots \int\limits_{X_i}}_{i \in \rho(\tau_2)} (f \circ c_\tau)_x \circ c_{\tau_2}^{-1} \underbrace{\mathrm{d}\mu_i \dots \mathrm{d}\mu_i}_{i \in \rho(\tau_2)} .$$

We may notice that this means that

$$g \circ c_{\tau_1}^{-1}(x) = \underbrace{\int\limits_{X_i} \dots \int\limits_{X_i}}_{i \in \rho(\tau_2)} (f \circ c_\tau)_{c_{\tau_1}^{-1}(x)} \circ c_{\tau_2}^{-1} \underbrace{\mathrm{d}\mu_i \dots \mathrm{d}\mu_i}_{i \in \rho(\tau_2)} = \underbrace{\int\limits_{X_i} \dots \int\limits_{X_i}}_{i \in \rho(\tau_2)} f_x \underbrace{\mathrm{d}\mu_i \dots \mathrm{d}\mu_i}_{i \in \rho(\tau_2)} .$$

Combining this with $(\star)$, we get

$$\underbrace{\int\limits_{X_i} \dots \int\limits_{X_i}}_{i \in \rho(\tau_1)} g \circ c_{\tau_1}^{-1} \underbrace{\mathrm{d}\mu_i \dots \mathrm{d}\mu_i}_{i \in \rho(\tau_1)} = \underbrace{\int\limits_{X_i} \dots \int\limits_{X_i}}_{i \in \rho(\tau_1)} \underbrace{\int\limits_{X_i} \dots \int\limits_{X_i}}_{i \in \rho(\tau_2)} f \underbrace{\mathrm{d}\mu_i \dots \mathrm{d}\mu_i}_{i \in \rho(\tau_2)} \underbrace{\mathrm{d}\mu_i \dots \mathrm{d}\mu_i}_{i \in \rho(\tau_1)} = \int\limits_{X_0} \dots \int\limits_{X_{n-1}} f \, \mathrm{d}\mu_{n-1} \dots \mathrm{d}\mu_0 .$$

Without specifics we note that the order of these integrals can be changed. The idea behind this is twofold. First, by the above induction, for any two operation schemes that "have their indices in the same order" these integrals are the same (e.g. for $\langle 0, \langle 1, 2\rangle\rangle$ and $\langle\langle 0, 1\rangle, 2\rangle$). Second, by Fact 3.11, switching the order of two operation schemes in the recursive definition of a scheme τ does not change these integrals (e.g. in the case of $\langle 0, \langle 1, 2\rangle\rangle$ and $\langle 0, \langle 2, 1\rangle\rangle$). Combining these repeatedly produces any permutation of these integrals. □

Definition 3.14. Let $n \in \mathbb{N}$. For all $i \in n$ let X_i be sets endowed with the SP-algebras $P(X_i)$. The **canonical product algebra** of X_i is defined to be

$$C_n(X_i) = \{Y \in P(\prod_n X_i) : \chi_Y \circ c_\tau \text{ is an S-function for all operation schemes } \tau \text{ with range } \rho(\tau) = n\}.$$

Similarly, if for all $i \in n$ the sets $X_i = X$, then we call $C_n(X)$ the **canonical power algebra** of X.

Remark 3.15. It is easy to see that

$$C_n(X_i) = \bigcap_{\tau : \rho(\tau) = n} \{c_\tau(Y) : Y \in \prod_\tau P(X_i)\}.$$

Corollary 3.16. *Let $n \in \mathbb{N}$. For all $i \in n$ let X_i be a set and let μ_i be a measure on $P(X_i)$. Let $Y \in C_n(X_i)$ be arbitrary. Then, for each operation scheme τ with range $\rho(\tau) = n$ we have*

$$\mu^\tau(c_\tau^{-1}(Y)) = \int_{X_0} \dots \int_{X_{n-1}} \chi_Y \,\mathrm{d}\mu_{n-1} \dots \mathrm{d}\mu_0.$$

In particular, $\mu^\tau(c_\tau^{-1}(Y))$ does not depend on τ.

Proof. Clearly $\chi_Y \circ c_\tau$ is the characteristic function of $c_\tau^{-1}(Y)$. If the function $\chi_Y \circ c_\tau$ is an S-function, then by Definition 2.7, $c_\tau^{-1}(Y) \in \prod_\tau P(X_i)$. Thus $\chi_Y \circ c_\tau$ is $\prod_\tau P(X_i)$-measurable. Since for all appropriate operation schemes τ the function $\chi_Y \circ c_\tau$ is a $\prod_\tau P(X_i)$-measurable S-function, therefore by Proposition 3.9 it is a hereditary S-function as well. Thus we have

$$\mu^\tau(c_\tau^{-1}(Y)) \overset{2.7}{=} \int_{\prod_{\tau_1} X_i} \int_{\prod_{\tau_2} X_i} \chi_Y \circ c_\tau \,\mathrm{d}\mu^{\tau_2} \,\mathrm{d}\mu^{\tau_1} \overset{3.11}{=} \int_{\prod_\tau X_i} \chi_Y \circ c_\tau \,\mathrm{d}\mu^\tau \overset{3.13}{=} \int_{X_0} \dots \int_{X_{n-1}} \chi_Y \,\mathrm{d}\mu_{n-1} \dots \mathrm{d}\mu_0.$$

□

Definition 3.17. Let $n \in \mathbb{N}$ and for all $i \in n$ let X_i be sets endowed with the measures $\mu_i : P(X_i) \to \mathbb{R}$. The **canonical product measure** $\prod_{i \in n} \mu_i$ is defined as the set function $\prod_{i \in n} \mu_i : C_n(X_i) \to \mathbb{R}$ for which

$$\prod_{i \in n} \mu_i(E) = \mu^\tau(c_\tau^{-1}(E)).$$

Specifically, if for all $i \in n$ $\mu_i = \mu$ over a set X, we call μ^n the n-dimensional **canonical power measure** of μ. It follows from Corollary 3.16 that this is well-defined.

Remark 3.18. $C_n(X_i)$ is indeed (the underlying set of) a Boolean set algebra and μ^n is indeed a finitely additive measure.

Proof. $E \in C_n(X_i)$ if and only if $c_\tau^{-1}(E) \in P(X_i)^\tau$ for all operation schemes τ. As $P(X_i)^\tau$ is a Boolean set algebra for every τ, so is $C_n(X_i)$. More precisely:

- $\prod_{i\in n} X_i \in C_n(X)$ as $c_\tau^{-1}(\prod_{i\in n} X_i) \in P(X_i)^\tau$;
- if $E \in C_n(X_i)$, then $\prod_{i\in n} X_i - E \in C_n(X_i)$ as $c_\tau^{-1}(E) \in P(X_i)^\tau$ and $P(X_i)^\tau$ is closed under complementation;
- if $E_1, E_2 \in C_n(X_i)$, then $E_1 \cup E_2 \in C_n(X_i)$ as $c_\tau^{-1}(E_1), c_\tau^{-1}(E_2) \in P(X_i)^\tau$ and $P(X_i)^\tau$ is closed under binary unions.

Similarly, as μ^τ is finitely additive for each τ, so is μ^n. □

3.3 Computing the canonical measure in a special case

In this section we will provide an interesting alternative way of computing the canonical power measure of certain sets. First we will recall a construction due to Tarski in Section 3.3.1, which can be used in transferring a measure from a group to a set upon which the group acts. Similarly, we will recall a reversal of this construction, discussed in Section 5 of [6] and Remark 3.14 of [5]. The idea behind these constructions is fixing a point of the underlying set, and examining which group elements transform it into which set elements. We will discuss how a similar construction can be used in transferring other functions from a set to a group and vice versa. These constructions will be useful in Section 4.1 as well. In Section 3.3.2, supposing an additional property of our starting invariant measure, we will prove the equality of a multidimensional integral and an integral over the set of cosets of a certain stabilizer subgroup. The contents of Section 3.3.2 are not in the main direction of our investigation, however, we are providing them as an interesting aside. In some instances, the result of Theorem 3.34 could be used to easily calculate canonical measures.

3.3.1 Measures and group actions

Let G be a group acting transitively on a set X. Let $x_0 \in X$, let G_{x_0} be the stabilizer subgroup of G with respect to x_0, that is,

$$G_{x_0} = \{g \in G : g(x_0) = x_0\}.$$

The set of cosets of G_{x_0} will be denoted by G/G_{x_0}.
Let $x \in X$. Then $\{g \in G : g(x_0) = x\}$ is a coset of G_{x_0}, thus an element of G/G_{x_0}. Similarly, if $Y \subset X$, then $\{g \in G : g(x_0) \in Y\}$ is the union of some cosets of G_{x_0}, as there exists $H \subset G/G_{x_0}$ so that $\{g \in G : g(x_0) \in Y\} = \cup H$.

Notation 3.19. Let $A_{x_0} = \{\cup H : H \subset G/G_{x_0}\}$.

It is easy to see that this is (the underlying set of) a Boolean set algebra.

Definition 3.20. Let G be a group acting on a set X and let A be a Boolean set algebra on X. We say that a measure $\mu : A \to \mathbb{R}$ is G-invariant, if for all $g \in G$ and $Y \in A$ we have $g(Y) \in A$ and $\mu(Y) = \mu(g(Y))$.

Now we recall Tarski's construction which can be used to transfer an invariant measure from a group G to a set G is acting upon.

Definition 3.21. Let G be a group acting on a set X and let $x_0 \in X$. Let $A \subset P(G)$ be a Boolean set algebra over X such that $A_{x_0} \subset A$. Let $\mu : A \to \mathbb{R}$ be a G-invariant measure. Then $\mu_{x_0} : P(X) \to \mathbb{R}$ is defined as the function for which

$$\mu_{x_0}(Y) = \mu(\{g \in G : g(x_0) \in Y\}).$$

Remark 3.22. μ_{x_0} is a G-invariant measure on the algebra $P(X)$. Further information can be found in Fact 5.1 of [6].

From [6] we recall the converse of this construction.

Definition 3.23. Let $\mu : P(X) \to \mathbb{R}$ be a G-invariant measure. Then keeping the prior notation of A_{x_0}, let

$$\mu^{x_0} : A_{x_0} \to \mathbb{R},$$

$$\mu^{x_0}(Y) = \mu(\{g(x_0) : g \in Y\}).$$

Remark 3.24. According to Lemma 5.4. of [6], μ^{x_0} is a measure on the algebra A_{x_0}.

Theorem 3.25. *(Theorem 5.5 of [6].) Suppose μ is a G-invariant measure defined on $P(X)$ and let $x_0 \in X$. Then for any $Y \subset X$ we have*

$$(\mu^{x_0})_{x_0}(Y) = \mu(Y).$$

Theorem 3.26. *Similarly, let $x_0 \in X$ and let G be a group acting on X. Let $A \subset P(G)$ be a set algebra on G for which $A_{x_0} \subset A$. Suppose $\mu : A \to \mathbb{R}$ is a G-invariant measure of G. Then for any $H \in A_{x_0}$*

$$(\mu_{x_0})^{x_o}(H) = \mu(H).$$

Proof. First, observe that if H is the union of some cosets of G_{x_0}, then

$$\{g \in G : g(x_0) \in \{h(x_0) : h \in H\}\} = H.$$

Using this observation in the last step, we obtain

$$(\mu_{x_0})^{x_o}(H) = \mu_{x_0}(\{h(x_0) : h \in H\}) = \mu(\{g \in G : g(x_0) \in \{h(x_0) : h \in H\}\}) = \mu(H).$$

□

Assuming that G acts transitively on X, we can define a bijection between X and G/G_{x_0}, and thus between functions of type $X \to Y$ and functions of type $G/G_{x_0} \to Y$, where Y is an arbitrary set.

Definition 3.27. Let X be a set with a group G acting on it. Let $x_0 \in X$ and Y be a set. Let $f : G/G_{x_0} \to Y$ and $h : X \to Y$. Then

$$f_{x_0} : X \to Y,$$
$$f_{x_0}(x) = f(gG_{x_0}), \text{ where } g(x_0) = x.$$

Similarly,

$$h^{x_0} : G/G_{x_0} \to Y,$$
$$h^{x_0}(gG_{x_0}) = h(g(x_0)).$$

Clearly, these functions are well-defined.

Remark 3.28. Indeed, with these definitions, the following are true:

$(f_{x_0})^{x_0} = f,$
$(g^{x_0})_{x_0} = g,$
$(Id_X)^{x_0} : G/G_{x_0} \to X$ is a bijection between G/G_{x_0} and X, where Id_X is the identity function of X.

Definition 3.29. We will use a partial version of this construction as well. Let X and Y be sets with the group G acting on them both. Let $x_0 \in X$, $y_0 \in Y$ and Z be a set. For a function $f : X \times Y \to Z$ we define

$$f^{x_0} : G/G_{x_0} \times Y \to Z,$$

$f^{x_0}(gG_{x_0}, y) = f(g(x_0), y)$ and similarly,

$$f^{y_0} : X \times G/G_{y_0} \to Z,$$

$$f^{y_0}(x, gG_{y_0}) = f(x, g(x_0)).$$

Notation 3.30. Repeating this construction in both variables yields the same result regardless of which order we apply it in, hence we use the notation

$$f^{x_0,y_0} = (f^{x_0})^{y_0} = (f^{y_0})^{x_0}.$$

Remark 3.31. We occasionally use these notions with a finite tuple $\bar{x} \in X$ instead of a single element x_0. In these instances we mean the pointwise (with respect to $\bar{x}$) application of these constructions.

3.3.2 Computing the canonical measure

Definition 3.32. Let G be a group acting on a set X. Let $\mu : P(G) \to [0, 1]$ be a G-invariant probability measure (that is, $\mu(G) = 1$). We call μ a **stabilizer independent** measure if for all tuples $\bar{x}, \bar{y} \in X$ we have

$$\mu(G_{\bar{x},\bar{y}}) = \mu(G_{\bar{x}} \cap G_{\bar{y}}) = \mu(G_{\bar{x}}) \cdot \mu(G_{\bar{y}}).$$

The following proposition will be useful in a later proof, as well as help us get acquinted with this notion.

Proposition 3.33. *Let X be a set with a group G acting on it such that for all finite tuples $\bar{x} \in X$, the set of cosets $G/G_{\bar{x}}$ is finite. Let $\alpha = \langle \beta, \gamma \rangle$ be an operation scheme and $\bar{x} = \langle x_1, x_2 \rangle \in \prod_{\alpha} X$. Assume further that for all $g_1 G_{x_1} \in G/G_{x_1}$ and $g_2 G_{x_2} \in G/G_{x_2}$ there exists*

$$g \in G \text{ such that } gG_{\bar{x}} = g_1 G_{x_1} \cap g_2 G_{x_2}. \qquad (**)$$

Let $\mu : P(G) \to \mathbb{R}$ be a G-invariant stabilizer independent probability measure. Let $f : \prod_{\alpha} X \to \mathbb{R}$ be a hereditary S-function. Then

$$\int_{G/G_{x_1} \times G/G_{x_2}} f^{x_1,x_2} \,\mathrm{d}(\mu * \mu) = \int_{G/G_{\bar{x}}} f^{\bar{x}} \,\mathrm{d}\mu.$$

For completeness we note that $(**)$ holds if G is the automorphism group of certain Fraïssé limits or of certain stable structures. We do not go into details, as these issues are not in the main direction of this work.

Proof. Since G/G_{x_1} and G/G_{x_2} are both finite, we can rewrite the first integral as

$$\int_{G/G_{x_1} \times G/G_{x_2}} f^{x_1,x_2} \,\mathrm{d}(\mu * \mu) =$$

$$= \sum_{\substack{g_1 G_{x_1} \in G/G_{x_1} \\ g_2 G_{x_2} \in G/G_{x_2}}} (\mu * \mu)(g_1 G_{x_1} \times g_2 G_{x_2}) \cdot f^{x_1,x_2}(g_1 G_{x_1}, g_2 G_{x_2}).$$

As $g_1 G_{x_1} \times g_2 G_{x_2}$ are all measurable rectangles, 2.20 applies. Thus we obtain

$$\sum_{\substack{g_1 G_{x_1} \in G/G_{x_1} \\ g_2 G_{x_2} \in G/G_{x_2}}} (\mu * \mu)(g_1 G_{x_1} \times g_2 G_{x_2}) \cdot f^{x_1,x_2}(g_1 G_{x_1}, g_2 G_{x_2}) =$$

$$= \sum_{\substack{g_1 G_{x_1} \in G/G_{x_1} \\ g_2 G_{x_2} \in G/G_{x_2}}} \mu(g_1 G_{x_1}) \cdot \mu(g_2 G_{x_2}) \cdot f^{x_1,x_2}(g_1 G_{x_1}, g_2 G_{x_2}).$$

As μ is stabilizer independent, we can further transform this, obtaining

$$\sum_{\substack{g_1 G_{x_1} \in G/G_{x_1} \\ g_2 G_{x_2} \in G/G_{x_2}}} \mu(g_1 G_{x_1}) \cdot \mu(g_2 G_{x_2}) \cdot f^{x_1,x_2}(g_1 G_{x_1}, g_2 G_{x_2}) =$$

$$= \sum_{\substack{g_1 G_{x_1} \in G/G_{x_1} \\ g_2 G_{x_2} \in G/G_{x_2}}} \mu(g_1 G_{x_1} \cap g_2 G_{x_2}) \cdot f^{x_1,x_2}(g_1 G_{x_1}, g_2 G_{x_2}) = \ (\star).$$

By $(**)$, for all $g_1G_{x_1} \in G/G_{x_1}$ and $g_2G_{x_2} \in G/G_{x_2}$ we have $gG_{\bar{x}}$ such that

$$gG_{\bar{x}} = g_1G_{x_1} \cap g_2G_{x_2}.$$

Similarly, for these we have

$$f^{x_1,x_2}(g_1G_{x_1}, g_2G_{x_1}) = f(g_1(x_1), g_2(x_2)) = f(g(\bar{x})) = f^{\bar{x}}(gG_{\bar{x}}).$$

Using these facts, we obtain

$$(\star) \;=\; \sum_{gG_{\bar{x}} \in G/G_{\bar{x}}} \mu(gG_{\bar{x}}) \cdot f^{\bar{x}}(gG_{\bar{x}}) = \int\limits_{G/G_{\bar{x}}} f^{\bar{x}} \,\mathrm{d}\mu.$$

□

Theorem 3.34. *Let X be a set with a group G acting on it transitively such that for all finite tuples $\bar{x} \in X$, the set of cosets $G/G_{\bar{x}}$ is finite. Let $\mu : P(G) \to [0,1]$ be a G-invariant stabilizer independent probability measure. Let α be an arbitrary operation scheme with $\rho(\alpha) = n$ and for all $i \in n$ let $x_i \in X$ and $\mu_i = \mu_{x_i}$. Let $\bar{x} \in \prod_\alpha X$ so that $c_\alpha(\bar{x}) = (x_0, ..., x_{n-1})$. If a function $f : \prod_\alpha X \to \mathbb{R}$ is a hereditary S-function, then*

$$\int\limits_{\prod_\alpha X} f \,\mathrm{d}\mu^\alpha = \int\limits_{G/G_{\bar{x}}} f^{\bar{x}} \,\mathrm{d}\mu.$$

Proof. First, by Fact 3.1, these integrals exist. We will prove their equality by induction on n. First assume n = 1, thus $\alpha = 0$. Then we have $f : X \to \mathbb{R}$ and we need to show

$$(*) \int\limits_X f \,\mathrm{d}\mu_{x_0} = \int\limits_{G/G_{x_0}} f^{x_0} \,\mathrm{d}\mu.$$

Since G acts transitively on X, for all $x \in X$ there exists a $g_x \in G$ for which $g_x(x_0) = x$ holds. Define the function $\Phi : X \to G/G_{x_0}$ so that $\Phi(x) = g_xG_{x_0}$ holds for all x. (Similarly to Remark 3.28, $\Phi = (Id_{G/G_{x_0}})_{x_0}$, where $Id_{G/G_{x_0}}$ is the identity function of G/G_{x_0}.) Clearly Φ is a bijection between X and G/G_{x_0}. Moreover, Φ preserves measures in the following sense: for any $Y \subset G/G_{x_0}$

$$\mu(\cup Y) = \mu_{x_0}(\{\Phi^{-1}(y) : y \in Y\}).$$

Thus Φ is an isomorphism between the measure spaces $(X, P(X), \mu_{x_0})$ and $(G/G_{x_0}, P(G/G_{x_0}), \mu)$. Further, $f = f^{x_0} \circ \Phi$. From this we can see that $(*)$ holds.

Turning to the inductive step, let $\alpha = \langle \beta, \gamma \rangle$. For $\bar{x} \in \prod_\alpha X$ we use the notation $\bar{x} = \langle x', x'' \rangle$. In this case, we can regard f as a binary function over $\prod_\beta X \times \prod_\gamma X$. For clarity's sake, keep in mind that β ranges over the first variable and γ ranges over the second variable. Thus the variable we are integrating with respect to is always clear. Starting with the left side of the equation, we have

$$\int\limits_{\prod\limits_{\alpha} X} f(x,y)\,\mathrm{d}\mu^{\alpha} = \int\limits_{\prod\limits_{\langle\beta,\gamma\rangle} X} f(x,y)\,\mathrm{d}(\mu^{\beta} * \mu^{\gamma}) \overset{3.11}{=} \int\limits_{\prod\limits_{\beta} X}\int\limits_{\prod\limits_{\gamma} X} f(x,y)\,\mathrm{d}\mu^{\gamma}\,\mathrm{d}\mu^{\beta} = (I_1).$$

As $|\rho(\gamma)| < n$ and for any fixed $x \in \prod\limits_{\beta} X$ we can use the inductive hypothesis on $\int\limits_{\prod\limits_{\gamma} X} f(x,y)\,\mathrm{d}\mu^{\gamma}$, we get

$$(I_1) = \int\limits_{\prod\limits_{\beta} X} \left(\int\limits_{G/G_{x''}} f^{x''}(x,h)\,\mathrm{d}\mu \right) \mathrm{d}\mu^{\beta}_{x_i} = (I_2).$$

Similarly, as $F = \int\limits_{G/G_{x''}} f^{x''}(x,h)\,\mathrm{d}\mu$ is a function of form $F : \prod\limits_{\beta} X \to \mathbb{R}$ with $|\rho(\beta)| < n$ and by Proposition 3.10 it is a hereditary S-function, we can use the inductive hypothesis again. Then we obtain

$$(I2) = \int\limits_{G/G_{x'}} \left(\int\limits_{G/G_{x''}} f^{x''}(g,h)\,\mathrm{d}\mu \right)^{x'} \mathrm{d}\mu = \int\limits_{G/G_{x'}}\int\limits_{G/G_{x''}} f^{x',x''}(g,h)\,\mathrm{d}\mu\,\mathrm{d}\mu = (I_3).$$

As f is a hereditary S-function, so is $f^{x',x''}$. Thus, we can use Fact 3.11 again, and we get

$$(I_3) = \int\limits_{G/G_{x'} \times G/G_{x''}} f^{x',x''}(g,h)\,\mathrm{d}(\mu * \mu).$$

As μ is stabilizer independent, by Proposition 3.33 this is equal to

$$\int\limits_{G/G_{\bar{x}}} f^{\bar{x}}(g)\,\mathrm{d}\mu.$$

□

Chapter 4

The finite model property

Recall that an infinite structure $\mathcal{A}$ has the **finite model property** iff each first order formula true in $\mathcal{A}$ is also true in a finite substructure of $\mathcal{A}$. More formally, for any first order formula φ, if $\mathcal{A} \models \varphi$, then there exists a finite substructure $\mathcal{A}_0$ of $\mathcal{A}$ for which $\mathcal{A}_0 \models \varphi$ still holds. An important and nontrivial example for a structure having the finite model property is the Rado graph $\mathcal{G}_R = \langle V, E \rangle$ (that is, the random graph in the Erdős-Rényi sense with countably infinitely many vertices).

Our goal in this chapter will be twofold. First we will present some information regarding the natural occurence of automorphism invariant (finitely additive) measures in Section 4.1. Then, in Section 4.2 we will present a sketch of a proof for the finite model property of the Rado graph. Using this proof as a motivational example, we will examine a generalization of the ideas presented. We will show such a generalization (Theorem 6.2 of [5]) that uses countably additive measures as a tool for proving the finite model property of some structures. Finally, in Corollary 4.13 we will show an analogue of that proof to establish the finite model property of certain structures that have an automorphism invariant (finitely additive) measure.

4.1 On the existence of automorphism invariant measures

Studying the existence of (not necessarily automorphism invariant) finitely additive probability measures defined on the underlying sets of certain first order structures has a great tradition. Such investigations go back at least to the related work of H. J. Keisler (carried out in the 1960's). In the last few years these investigations received renewed impetus.

More concretely, there are two sources of finding automorphism invariant measures on the underlying sets: Stone spaces and the automorphism group of a given first order structure. One such source is the celebrated paper [3], where in Sections 2, 3, and 4 therein, the authors show the existence of automorphism invariant measures in a rather general setting. However, their intention and the methods they apply are quite different from our present investigations. Hence we follow the approach of [5] which is based on more elementary investigations (which are essentially group theoretical, combined with countable combinatorics).

Recall that an elementary mapping is one which preserves the truth value of each (first order) formula. By a finite elementary mapping on a structure $\mathcal{A}$ we mean an elementary mapping which has finite substructures of $\mathcal{A}$ both as its domain and its codomain.

Definition 4.1. A structure $\mathcal{A}$ is strongly $\aleph_0$-homogeneous if each finite elementary mapping of $\mathcal{A}$ extends to an automorphism of $\mathcal{A}$.

Let $\mathcal{A}$ be a countable strongly homogeneous structure. Endow A with the discrete topology and ${}^A A$ with the product topology. As it is well known, this topology on ${}^A A$ is the same as the pointwise convergence topology induced by the discrete topology on A. Endow $Aut(\mathcal{A})$ with the subspace topology inherited from the above topology on ${}^A A$. This way $Aut(\mathcal{A})$ becomes a topological group, its operations will be continuous. In fact, with the above topology $Aut(\mathcal{A})$ is a Polish space (that is, separable and metrizable by a complete metric), which is a particularly well behaved and well understood class of topological spaces. From now all topological notions of $Aut(\mathcal{A})$ should be understood with respect to the topology described above.

Definition 4.2. (Definition 2.2 of [5].) A topological group G is defined to be σ^*-compact if there exists an increasing sequence $\langle G_n : n \in \mathbb{N}\rangle$ of subgroups of G such that G_n is compact for all $n \in \mathbb{N}$ and $G = \bigcup_{n\in\mathbb{N}} G_n$.

Recall that if G is a group of permutations of a set A (that is, $G \leq Sym(A)$), then the G-orbit of an element $a \in A$ is denoted by

$$O_G(a) = \{g(a) : g \in G\}.$$

Similarly, for any finite tuple $\bar{g} \in G$ and element $a \in A$, the orbit $O_{\bar{g}}(a)$ is the orbit of a with regards to the subgroup of G generated by $\bar{g}$.

Definition 4.3. (Definition 2.3 of [5].) Let $\mathcal{A}$ be a first order structure. For any $n \in \mathbb{N}$ we define

$$Aut_n^{Fin}(\mathcal{A}) := \{\bar{g} \in {}^n Aut(\mathcal{A}) : \forall a \in A \text{ we have } |O_{\bar{g}}(a)| < \aleph_0\}.$$

Fact 4.4. *(Lemma 2.6.1 of [5].) Let $\mathcal{A}$ be a countable strongly $\aleph_0$-homogeneous structure. Suppose $Aut_n^{Fin}(\mathcal{A})$ is dense in ${}^n Aut(A)$ for all $n \in \mathbb{N}$. Then there exists a dense σ^*-compact subgroup of $Aut(A)$. For Fraïssé limits in particular, the converse of this statement holds as well. For further details see Remark 4.5 of [5].*

We note that the density of $Aut_n^{Fin}(\mathcal{A})$ in ${}^n Aut(\mathcal{A})$ is known to be equivalent with interesting and thoroughly studied combinatorial properties of finite substructures of $\mathcal{A}$. In particular, Hrushovski's extension property can be characterized in terms of density conditions on $Aut_n^{Fin}(\mathcal{A})$. We don't recall the particulars here, as these details are not in the main direction of the topic of this work. For further information, see for example Lemma 4.3 of [5].

Fact 4.5. *(Lemma 3.10 of [5].) Suppose A is a countable set and $G \leq Sym(A)$. If G is σ^*-compact then there exists a G-invariant finitely additive probability measure μ on G such that $dom(\mu) = Borel(G)$.*

Corollary 4.6. *Let $\mathcal{A}$ be a countable strongly $\aleph_0$-homogeneous structure and suppose $Aut_n^{Fin}(\mathcal{A})$ is dense in ${}^nAut(\mathcal{A})$ for all $n \in \mathbb{N}$. Then there exists a dense σ^*-compact subgroup G of $Aut(\mathcal{A})$ and a G-invariant finitely additive probability measure μ on G such that $dom(\mu) = Borel(G)$.*

Proof. By Fact 4.4, there exists a a dense σ^*-compact subgroup G of $Aut(\mathcal{A})$ and by Fact 4.5 there exists a G-invariant finitely additive probability measure μ on G such that $dom(\mu) = Borel(G)$. □

In Proposition 4.12, the measure used is not on the automorphism group, but on the (underlying set of the) structure itself. This is not a problem, as a measure on a group can be easily turned into a measure on the structure and vice versa, as we have shown in Section 3.3.1. With the aid of Remark 3.22 we can construct automorphism invariant probability measures on the underlying set of certain strongly $\aleph_0$-homogeneous structures as follows.

Theorem 4.7. *Let $\mathcal{A}$ be a countable strongly $\aleph_0$-homogeneous structure and suppose $Aut_n^{Fin}(\mathcal{A})$ is dense in ${}^nAut(\mathcal{A})$ for all $n \in \mathbb{N}$. Then there exists a finitely additive probability measure ν on A which is invariant under a dense subgroup of $Aut(\mathcal{A})$ such that $dom(\nu) = P(A)$.*

Proof. By Corollary 4.6 there exist a dense σ^*-compact subgroup G of $Aut(\mathcal{A})$ and a G-invariant finitely additive probability measure μ on G such that $dom(\mu) = Borel(G)$. Let $x_0 \in A$ be arbitrary and define the set function ν to be $\nu = \mu_{x_0}$. Observe that G_{x_0} is an open subset of G. It follows that $A_{x_0} \subseteq Borel(G)$. Therefore Remark 3.22 applies and we obtain that $\nu = \mu_{x_0}$ is a G-invariant finitely additive probability measure with $dom(\nu) = P(A)$. □

4.2 Finitely additive measures and the finite model property

The well known proof that the Rado graph has the finite model property can be summarized as follows. Enumerate the set of vertices of $\mathcal{G}_\mathcal{R}$ as $V = \{a_n : n \in \omega\}$ and denote by $\mathcal{G}_n$ the subgraph of $\mathcal{G}_R$ spanned by $\{a_k : k < n\}$. It is well known that the first order theory of $\mathcal{G}_R$ (more generally, the first order theory of any Fraïssé limit in a finite relational language) can be axiomatized by $\forall\exists$-formulas. Suppose φ is a $\forall\exists$-formula true in $\mathcal{G}_R$. Then a short and relatively straightforward calculation shows that the probability $P(\mathcal{G}_n \models \varphi)$ converges to 1 as n tends to infinity. In particular, the probability $P(\mathcal{G}_n \models \varphi)$ is strictly positive for large enough (finite) n; therefore, for some (in fact, all large enough) finite n there exists an n-element substructure $\mathcal{A}_n$ of $\mathcal{G}_R$ such that $\mathcal{A}_n \models \varphi$. Versions of this proof can be found in several sources. For further details we refer to Lemma 7.4.6 of [2] where the above idea is presented in a somewhat more general setting.

Let $\mathcal{A}$ be any countable first order structure. One strategy for proving the finite model property of $\mathcal{A}$ could be adapting the probabilistic argument from the Rado graph. If there exists an automorphism invariant (countably or finitely additive) measure on the underlying set A of $\mathcal{A}$, the adaptation is straightforward but not completely obvious. In the case of the Rado graph the invariant measure we use is defined on the set of all edges of the graph, however it is more natural to find automorphism invariant measures on the automorphism group $Aut(\mathcal{A})$ or the underlying set A of $\mathcal{A}$. We presented some situations in which these exist in Section 4.1. The approach based on this adaptation has been carried out in [6] using countably additive measures. We will present the authors' findings shortly, but first we will examine the concept of *fat* formulas.

Roughly, a formula $\varphi(\overline{v}, w)$ is defined to be *fat* if the μ-measures of the slices $\{||\varphi(\overline{v}, w)||_{\overline{a}} : \overline{a} \in A\}$ are bounded from below with a positive number ε. To provide an intuitive explanation, let ε be any positive real number and introduce the quantifiers $\exists_\varepsilon w$ with intended meaning "there exist at least μ-measure ε many w such that...". We define this by stipulating

Definition 4.8. $\mathcal{A} \models \exists_\varepsilon w\varphi(\overline{a}, w)$ iff $\mu(\{b \in A : \mathcal{A} \models \varphi(\overline{a}, b)\}) \geq \varepsilon$.

Thus, φ is a fat formula iff there exists $\varepsilon > 0$ such that $\mathcal{A} \models \forall\overline{v}\exists_\varepsilon w\varphi(\overline{v}, w)$. Precisely:

Definition 4.9. Let $\mathcal{A}$ be a structure endowed with a measure $\mu : P(A) \to \mathbb{R}$. Let $\varphi(x_0, ..., x_{r-1}, y)$ be a formula with free variables $x_0, ..., x_{r-1}, y$. We say that φ is fat, if there exists $\varepsilon > 0$ such that for all $a_0, ..., a_{r-1} \in A$ we have

$$\mu(\{b \in A : \mathcal{A} \models \varphi(a_0, ..., a_{r-1}, b)\}) > \varepsilon.$$

Assume $\mathcal{A}$ is a Fraïssé limit (of its finite substructures). Theorem 6.2 of [6] can be rephrased as follows. Suppose μ is an automorphism invariant countably additive probability measure with $dom(\mu) = P(A)$. For an arbitrary quantifier free formula φ if there exists $\varepsilon > 0$ such that

$$\mathcal{A} \models \forall\overline{v}\exists_\varepsilon w\varphi(\overline{v}, w),$$

then there exists a finite substructure $\mathcal{A}_0$ of $\mathcal{A}$ for which

$$\mathcal{A}_0 \models \forall\overline{v}\exists w\varphi(\overline{v}, w) \text{ still holds.}$$

In other words, if a $\forall\exists$-formula $\forall\overline{v}\exists w\varphi(\overline{v}, w)$ is true in $\mathcal{A}$ in the strong sense that for all $\overline{a}$, the μ-measure of the set $\{b \in A : \mathcal{A} \models \varphi(\overline{a}, b)\}$ is larger than a positive constant ε (which does not depent on $\overline{a}$), then $\forall\overline{v}\exists w\varphi(\overline{v}, w)$ is also true in a finite substructure of $\mathcal{A}$. Consequently, if the theory of $\mathcal{A}$ can be axiomatized by $\forall\exists$-formulas such that the quantifier free parts of these formulas are fat, then $\mathcal{A}$ has the finite model property.

In [6] the above result (Theorem 6.2 of [6]) was examined for countably additive probability measures. Here, in Proposition 4.12 and Corollary 4.13 below we prove an analogous result for finitely additive measures. On one hand, this is a considerable achievement, because weakening the condition of "being countably additive" to "being finitely additive" makes it easier to find invariant measures. However, the adaptation of the probabilistic argument presented above requires dealing with large (measure theoretic) powers of the measure given on the underlying set of the structure we are examining. As we have seen in earlier sections, constructing well behaved powers of finitely additive probability measures requires further work. This was done in Sections 2. and 3. with the finite model property and with the approach for proving it as described above as the motivation.

Definition 4.10. Let $\mathcal{A}$ be a structure and let $\mu : P(A) \to \mathbb{R}$ be an automorphism invariant (finitely additive) measure. We call $\mathcal{A}$ canonically measurable if each of its definable subsets belongs to the appropriate dimensional canonical power algebra of A.

As an example, we note that by a yet unpublished result of Sági if $\mathcal{A}$ is stable, then it is canonically measurable. Here we do not recall the definition of stability, as it would require much technical preparation not in scope of the topic of this work. For further information, see Section 6.7 of [2].

Notation 4.11. Suppose $\mathcal{A}$ is a relational structure and $\bar{s} \in A$ is a finite tuple. By $\mathcal{A}|_{\bar{s}}$ we mean the substructure of $\mathcal{A}$ which comprises of the elements of $\bar{s}$.

Proposition 4.12. *Let $\mathcal{A}$ be a countable canonically measurable relational structure endowed with a (finitely additive) probability measure $\mu : P(\mathcal{A}) \to [0,1]$. Let $\varphi(x_0, ..., x_{r-1}, y)$ be a fat formula. Then*

$$\lim_{n\to\infty} \mu^n(\{\bar{s} \in {}^nA : \mathcal{A}|_{\bar{s}} \nvDash (\forall x_0 ... \forall x_{r-1})(\exists y)\varphi\}) = 0.$$

We note that if in addition μ is $Aut(\mathcal{A})$-invariant (or at least G-invariant for some dense subgroup G of $Aut(\mathcal{A})$), then in many cases one can show that certain formulas are fat. We postpone presenting related investigations for further papers.

Proof. For each $\bar{a} \in {}^rA$ let

$$Z_{\bar{a}} := \{b \in A : \mathcal{A} \vDash \neg\varphi(\bar{a}, b)\} = A - \{b \in A : \mathcal{A} \vDash \varphi(\bar{a}, b)\}.$$

Since φ is a fat formula, there exists $\varepsilon > 0$ such that $\mu(Z_{\bar{a}}) \leq 1 - \varepsilon$ for all $\bar{a}$.
Let $n \in \mathbb{N}$ so that $r < n$. For any $i \in [n]^r$ let

$$X_i := \{\bar{s} \in {}^nA : \mathcal{A} \vDash \neg(\exists j \in n)\varphi(\bar{s}|_i, s_j)\}.$$

For each $\bar{a} \in {}^rA$, we can look at the $\bar{a}$-section of X_i. Let's use the notation

$$(X_i)_{\bar{a}} := \{\bar{s} \in {}^{n-i}A : \mathcal{A} \vDash \neg(\exists j \in (n-i))\varphi(\bar{a}, s_j)\}.$$

Clearly, $(X_i)_{\bar{a}} \subset {}^{n-i}Z_{\bar{a}}$. We remark that ${}^{n-i}Z_{\bar{a}}$ is a rectangle, thus it is μ^{n-r}-measurable even without the condition of $\mathcal{A}$ being canonically measurable. Using the product measure μ^{n-r} on $(X_i)_{\bar{a}}$ we get

$$\mu^{n-r}((X_i)_{\bar{a}}) \leq \mu^{n-r}({}^{n-i}Z_{\bar{a}}) \overset{2.20}{=} (\mu(Z_{\bar{a}}))^{n-r} \leq (1-\varepsilon)^{n-r}.$$

Thus we have

$$\mu^n(X_i) = (\mu^r * \mu^{n-r})(X_i) = \int_{A^r}\int_{A^{n-r}} \chi_{X_i}\, \mathrm{d}\mu^{n-r}\, \mathrm{d}\mu^r \leq \int_{A^r} (1-\varepsilon)^{n-r}\, \mathrm{d}\mu^r \leq (1-\varepsilon)^{n-r}.$$

Let $Y_n := \{\bar{s} \in {}^nA : \mathcal{A}|_{\bar{s}} \not\models \forall x_0, ..\forall x_{r-1} \exists y \varphi(x_0, ..., x_{r-1}, y)\}$. Then we have

$$P_n := \mu^n(Y_n) = \mu^n(\bigcup_{i \in [n]^r} X_i) \leq \sum_{i \in [n]^r} \mu^n(X_i) \leq \binom{n}{r}(1-\varepsilon)^{n-r}.$$

Thus

$$0 \leq \lim_{n\to\infty} P_n \leq \lim_{n\to\infty} \binom{n}{r}(1-\varepsilon)^{n-r} \leq \lim_{n\to\infty} n^r(1-\varepsilon)^n = 0.$$

□

Corollary 4.13. *Let $\mathcal{A}$ be a countable canonically measurable relational structure endowed with a (finitely additive) probability measure $\mu : P(\mathcal{A}) \to [0,1]$. Suppose $\mathcal{A}$ can be axiomatized by a set Σ of $\forall\exists$ formulas such that their quantifier free parts are fat. Then $\mathcal{A}$ has the finite model property.*

Proof. Let φ be a formula such that $\mathcal{A} \models \varphi$. Then by the compactness theorem we have a finite $\Sigma_\varphi \subset \Sigma$ such that $\Sigma_\varphi \models \varphi$. For each $\psi \in \Sigma_\varphi$ let its quantifier free part be ψ'. The conditions of Proposition 4.12 are satisfied by ψ', so it is true that

$$\lim_{n\to\infty} \mu^n(\{\bar{s} \in {}^nA : \mathcal{A}|_{\bar{s}} \not\models \psi\}) = 0.$$

Thus we have

$$\lim_{n\to\infty} \mu^n(\{\bar{s} \in {}^nA : \mathcal{A}|_{\bar{s}} \not\models \Sigma_\varphi\}) \leq \sum_{\psi \in \Sigma_\varphi} \lim_{n\to\infty} \mu^n(\{\bar{s} \in {}^nA : \mathcal{A}|_{\bar{s}} \not\models \psi\}) = \sum_{\psi \in \Sigma_\varphi} 0 = 0.$$

Essentially, this means that for large enough n most substructures $\mathcal{A}_0$ of $\mathcal{A}$ for which $|A_0| = n$, we have $\mathcal{A} \models \Sigma_\varphi$. Thus φ remains true in some finite $\mathcal{A}_0$, therefore $\mathcal{A}$ has the finite model property. □

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I, Édua Boróka Kun, hereby declare that I did not use any AI tools in the making of this work.