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Fractal Zeta Functions for p-adic Affine Varieties.

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Acknowledgements

I have reached the final steps of my getting a Bachelor's degree in Mathematics. But this isn't the end, this is merely a milestone I pass on this journey before the destination I do not yet know. I'd like to take the opportunity to reflect on these last few years, which flew by quickly, and to express my gratitude for those who helped me along this path, particularly those who contributed to the creation of this wonderous document.

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Contents

Chapter 1

Introduction

Lapidus and Van Frankenhuijsen [12] presented a theory of fractal strings set in $\mathbb{R}$ which we will partially discuss in Chapter 2. The concepts introduced by them most naturally generalize to metric spaces that are topological groups as well. According to a theorem by Birkhoff and Kakutani, a topological group G is metrizable if, and only if, G is Hausdorff, and the identity element has a countable neighbourhood basis. Moreover, if G is metrizable, G gives a compatible metric d, which is left-invariant: $d(xy, xz) = d(y, z)$. A Polish group is a topological group where G as a topological space is Polish[1]. It is known that every separable metrizable topological group can be embedded densely to a Polish group. [9]

In this paper we will focus on the special case when the group in question is $(\mathbb{Z}_p, +)$. If we equip $\mathbb{Z}$ with the metric induced by $|.|_p$ (Equation 3.1), then the Polish group in which it can be densely embedded is $\mathbb{Z}_p$. Since this is not merely a group but a ring, it comes with a large family of spectral strings that arise from affine varieties.

Note that this family only exists over $\mathbb{Z}_p$. Over $\mathbb{R}$ these sets do not give rise to fractal strings, except in dimension one, when they are trivial[2].

In the present paper, we generalize the one-dimensional real fractal zeta function introduced in [12] to the p-adic setting $(\zeta_{V(I)}(s))$. First we will concentrate on affine varieties for which we establish a number of interesting properties. In particular we show

1. the rationality of $\zeta_{V(I)}(s)$ (Theorem 3.3.1)

[1] A topological space is called Polish if it is separable and completely metrizable.

[2] Or we can say they do not exist even there, since there are only finitely many intervals that have to be left out.

2. that the abscissa of convergence of $\zeta_{V(I)}(s)$ coincides with the Minkowski dimension generalized from the Minkowski dimension of fractal strings (Theorem 4.2.6)

3. a closed formula of $\zeta_{V(I)}(s)$ as a p-adic integral (Theorem 5.2.1).

We then show how these concepts generalize to the p-adic projective space.

A quick outline of this note is as follows. In Chapter 2 we review the theory of fractal strings on the line and introduce the fractal zeta function. Then we illustrate the correlation between a fractal string's dimension and its zeta function's abscissa of convergence. After that in Chapter 3 we extend the definitions presented previously to p-adic affine varieties such as fractal strings, and fractal zeta functions. We then prove the rationality of said functions for the non-singular case. In Chapter 4 we compare different definitions of dimensions on p-adic affine varieties, both algebraic and analytic. We then go on by discussing the connections between $\zeta_{V(I)}(s)$ and Igusa's local zeta function, which enables us to give a closed formula for the p-adic fractal zeta function. We end with transfering our results of affine fractal strings to projective varieties.

Chapter 2

The Geometric Zeta Function of a Fractal String

The following definitions and statements, with their proofs, can be found in Lapidus and Van Frankenhuijsen [12].

2.1 The zeta function of a fractal string

A *fractal string* is a bounded open set (Ω) of the real line, and as such, it consists of countably many disjoint open intervals. Let us denote the complement of Ω with V_Ω. A fractal string may be represented by the lengths of these intervals which form a sequence: $\mathcal{L} = l_1, l_2, l_3, \ldots$. Without loss of generality we may assume that $l_1 \geq l_2 \geq \cdots > 0$. It is also known that the sum $\sum_{j=1}^{\infty} l_j$ is finite and equal to the Lebesgue measure of Ω.
The generalized Dirichlet-series

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s$$

is the geometric zeta function of the fractal string $\mathcal{L}$. Since $\zeta_{\mathcal{L}}(s)$ converges in 1, it gives a holomorphic function for $\operatorname{Re} s > 1$[1].

The study of this function is motivated by the question the Polish-American mathematician, Mark Kac asked: "Can one hear the shape of a drum?", i.e. can one recover (up to isometry) a domain from its spectrum of the Dirichlet Laplacian over Ω:

[1]Recall that for a positive real number a, and $s \in \mathbb{C}$, $a^s = e^{s \log a}$.

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

This question lead to numerous studies dealing with the subject and although it has been proved that there are non-isometric domains of $\mathbb{R}^n$ ($n \geq 4$) that are isospectral, one can recover much of its geometric properties by determening its spectrum. We may find the spectrum by finding the fractal string's spectral zeta function.

Every eigenvalue λ of the Dirichlet Laplaplacian gives a frequency $f = \frac{\sqrt{\lambda}}{\pi}$ of the fractal string. And since the frequency of an interval of length l yields the frequencies $l^{-1}, 2l^{-1}, 3l^{-1}, \ldots$, the frequencies of $\mathcal{L}$ are $k \cdot l_j^{-1}, k \in \mathbb{Z}^+$.

Definition 2.1.1. The spectral zeta function of $\mathcal{L} = \{l_j\}_{j=1}^{\infty}$ is

$$\zeta_\nu(s) = \sum_{k,j=1}^{\infty} (k \cdot l_j^{-1})^{-s}$$

Since

$$\zeta_\nu(s) = \sum_{k,j=1}^{\infty} (k \cdot l_j^{-1})^{-s} = \sum_{k,j=1}^{\infty} k^{-s} \cdot l_j^s = \sum_{k=1}^{\infty} k^{-s} \cdot \sum_{j=1}^{\infty} l_j^s = \zeta(s) \cdot \zeta_{\mathcal{L}}(s)$$

(where $\zeta(s)$ is the Riemann zeta function), we may obtain the spectral zeta function by finding the fractal zeta function of the fractal string, rendering it an interesting topic of research.

2.2 Properties of a fractal string's zeta function

The Minkowski dimension of the fractal string can be defined as the following:

$$D_{\mathcal{L}} = \inf\{\alpha \geq 0 : \mathrm{vol}(\varepsilon) = O(\varepsilon^{1-\alpha}) \text{ as } \varepsilon \to 0^+\}$$

(which is the inner Minkowski dimension of $\partial\Omega$), where $\mathrm{vol}(\varepsilon)$ is the volume of the outer ε neighbourhood of V_Ω:

$$\mathrm{vol}(\varepsilon) = \mathrm{vol}_1\{x \in \Omega : d(x, \partial\Omega) < \varepsilon\}$$

The Minkowski dimension of the boundry of Ω gives important information about the eigenvalues of the above mentioned Dirichlet Laplacian. The question may arise, why consider the Minkowski dimension instead of the more commonly known Hausdorff dimenion. Although the two dimensions often agree (like in the case of similitudes) the Hausdorff dimension takes into account the geometric representations of sets. The Minkowski dimension observes the inner tubes of the boundry and is invariant to the order and position of intervals, enabling us to calculate it without the knowledge of the fractal string's representation.

Definition 2.2.1. The abscissa of convergence for the fractal zeta function $\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s$ is the following:

$$\sigma_{\mathcal{L}} = \inf\{\alpha \in \mathbb{R} : \sum_{j=1}^{\infty} l_j^s \text{ converges if } \operatorname{Re} s > \alpha\}$$

This is well defined since $\sigma_{\mathcal{L}}$ exists[2] and $\forall z, s \in \mathbb{C}\ |z^s| = |z|^{\operatorname{Re} s}$, so if for $\alpha \in \mathbb{R}\ \sum_{j=1}^{\infty} l_j^\alpha$ converges, then based on the Weierstrass M-test and the fact that $\lim_{j\to\infty} l_j = 0$ the series $\sum_{j=1}^{\infty} l_j^s$ has to converge for any $s \in \mathbb{C}, \operatorname{Re} s > \alpha$. [15]

An interesting fact is that the Minkowski dimension coincides with the abscissa of convergence of a fractal string's zeta function:

Theorem 2.2.2. *[12]*

$$D_{\mathcal{L}} = \sigma_{\mathcal{L}}$$

We will illustrate this theorem through an example:

Example 2.2.3. The Cantor string is the complement of the ternary Cantor set in [0,1] (thus $l_1 = \frac{1}{3}, l_2 = l_3 = \frac{1}{9}; l_4 = l_5 = l_6 = l_7 = \frac{1}{27}, \ldots$).

$$V_C(\varepsilon) = \sum_{j:l_j<2\varepsilon} l_j + \sum_{j:l_j\geq 2\varepsilon} 2\varepsilon = 2\varepsilon \cdot (2^n - 1) + \sum_{k=n}^{\infty} 2^k 3^{-k-1} = 2\varepsilon \cdot 2^n + \left(\frac{2}{3}\right)^n - 2\varepsilon$$

if $0 < \varepsilon \leq \frac{1}{2}$ and $3^{-n} \geq 2\varepsilon > 3^{-n-1}$, because the discs covering the endpoints of the interval overlap if $l_j < 2\varepsilon$. To find the Minkowski dimension we may assume that $2\varepsilon = 3^{-n}$, because any other ε is between two of this form and thus through the squeeze theorem we get the same result:

$$V_C(\varepsilon) = 2\varepsilon \cdot 2^n + \left(\frac{2}{3}\right)^n - 2\varepsilon = 3 \cdot \left(\frac{2}{3}\right)^n - \frac{2}{3^n} = \frac{2}{3^n}(3 \cdot 2^{n-1} - 1)$$

[2] Since $\lim_{j\to\infty} l_j = 0$, for any $\alpha \in \mathbb{R}_{\leq 0}$ there exists j_α, that if $j > j_\alpha$, then $l_j^\alpha \geq 1$, which means that $\sum_{j=1}^{\infty} l_j^\alpha = \infty$, thus $\sigma_{\mathcal{L}} \geq 0$.

So to get the Minkowski dimension we take the $\log_\varepsilon$ of the tube of the fractal:

$$\frac{\log 2 - n\log 3 + \log(3\cdot 2^{n-1}-1)}{-n\log 3} \sim \frac{n\log 2 - (n-1)\log 3}{-n\log 3} \sim 1 - \frac{\log 2}{\log 3}$$

which makes $D_C = \log_3 2$.

The zeta function of the Cantor string is:

$$\zeta_C(s) = \sum_{k=1}^{\infty} 2^{k-1}3^{-ks} = \frac{1}{2}\sum_{k=1}^{\infty} 2^k 3^{-ks} < \infty \iff |\frac{2}{3^s}| < 1 \iff \frac{2}{3^{\operatorname{Re} s}} \iff \operatorname{Re} s > \log_3 2$$

so the abscissa of convergence is also $\log_3 2$.

In general, the geometric zeta function may not have an analytic continuation to all of $\mathbb{C}$, but we can introduce a screen S of the fractal string:

$$S : S(t) + it \quad (t \in \mathbb{R})$$

the contour of the fractal, where $S(t)$ is a continous function $S : \mathbb{R} \to [-\infty, D_{\mathcal{L}}]$. Then we say that

$$W = \{s \in \mathbb{C} | \operatorname{Re} s \geq S(\mathrm{Im} s)\}$$

is a window of the fractal string and we assume that the fractal zeta function has a meromorphic extension to a neighbourhood of W.

The set of poles of $\zeta_{\mathcal{L}}$ in the window ($\mathcal{D}_{\mathcal{L}} \subset W$) is called the visible complex dimensions of $\mathcal{L}$, or complex dimensions if $W = \mathbb{C}$ (so, if $\zeta_{\mathcal{L}}$ has a meromorphic extension to $\mathbb{C}$). Through Theorem 2.2.2 we know that for any s that has $\operatorname{Re} s > D_{\mathcal{L}}$ the fractal zeta function converges, $\zeta_{\mathcal{L}}$ is holomorphic on the $\operatorname{Re} s > D_{\mathcal{L}}$ half plane, thus

$$\mathcal{D}_{\mathcal{L}} \subset \{s \in \mathbb{C} | \operatorname{Re} s \leq D_{\mathcal{L}}\}.$$

Example 2.2.4. Let us take a look at the Cantor string and choose $W = \mathbb{C}$:

$$\zeta_C(s) = \sum_{k=1}^{\infty} 2^{k-1}3^{-ks} = \frac{3^{-s}}{1 - 2\cdot 3^{-s}}$$

thus the poles of the fractal zeta function are at $1 - 2\cdot 3^{-s} = 0$. Therefore, the complex dimensions of the Cantor string are:

$$\mathcal{D}_{\mathcal{L}} = \{D_{\mathcal{L}} + in\frac{2\pi}{\log 3} | n \in \mathbb{Z}\}$$

since $\frac{2\pi}{\log 3}$ is the period of oscillation. [12]

Chapter 3

p-adic Fractal Strings Associated to Affine Varieties

3.1 Background

Let us introduce a new absolute value on the integers and then extend it to the rational numbers[1] [11]. Let's fix a prime p and define for $a \in \mathbb{Z}$

$$|a|_p = p^{-k} \iff (p^k|a) \wedge (p^{k+1} \not| a) \tag{3.1}$$

Since for any $a, b \in \mathbb{Z} : |ab|_p = |a|_p|b|_p$:

$$\left|\frac{a}{b}\right|_p = \frac{|a|_p}{|b|_p}$$

is well defined on $\mathbb{Q}$. The completion of this metric space gives us the p-adic numbers: $\mathbb{Q}_p$ [10]. We use $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ as the notation for the unit ball in $\mathbb{Q}_p$. It is also the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$.

It will be convenient to use another realization of $\mathbb{Z}_p$. $\mathbb{Z}_p$ is isomorphic to the set that can be defined as $\{\sum_{k=0}^{\infty} a_k p^k : a_k \in \{0, 1, \ldots, p-1\} \forall k\}$ (formal sums) or the set of infinite sequences $x_1, x_2, \ldots$ where $\forall k \in \mathbb{N}\ x_{k+1} \equiv x_k \mod p^k$. For any $x, y \in \mathbb{Z}_p$:

$$|x - y|_p \leq p^{-k} \iff p^k|(x-y) \iff x \equiv y \mod p^k$$

So $\{\sum_{k=0}^{\infty} a_k p^k : a_k \in \{0, 1, \ldots, p-1\} \forall k, a_0 = b_0, \ldots, a_n = b_n\}$ is the representation of a ball in $\mathbb{Z}_p$ with p^{-n} radius.

[1] A theorem of Ostrowski shows that the possible absolute values on $\mathbb{Q}$ are equvivalent to either the usual $|.|$ or one of the $|.|_p$-s introduced here. [11]

Proposition 3.1.1. *$\mathbb{Z}_p$ is homeomorphic to the Cantor set.*

Proof. It follows from Brouwer's theorem [2] that every non-empty, perfect, compact, totally disconnected metric space is homeomorphic to the Cantor set, thus it is sufficient to prove that $\mathbb{Z}_p$ has these properties:

1. *perfect*: If $x \in \mathbb{Z}_p$ and $|x|_p = p^{-k}, k \in \mathbb{N}$, then $B_{p^{-k}}(x) \subset \mathbb{Z}_p$.

2. *compact*: It is enough to show that $\mathbb{Z}_p$ is sequential compact.
   Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{Z}_p$ be a sequence. Define $a_0 \in \{0, \dots, p-1\}$ such that $x_n \equiv a_0 \mod p$ for infinitely many x_n-s, and let x_{n_0} be one of them. Recursively define $a_k \in \{0, \dots, p-1\}$ such that $x_n \equiv \sum_{i=0}^{k} a_i p^i \mod p^{k+1}$ for infinitely many x_n-s, and let x_{n_k} be one of them.
   Then $\lim_{k\to\infty} x_{n_k} = x = \sum_{i=0}^{\infty} a_i p^i$, since $|x - x_{n_k}|_p \leq p^{-(k+1)}$.

3. *totally disconnected*: It is enough to see that $\forall x \in \mathbb{Z}_p$ has a neighbourhood base consisting of clopen subsets. Since $x \mapsto x + y$ is a homeomorphism in $\mathbb{Z}_p$ for any $y \in \mathbb{Z}_p$, we may assume that $x = 0$. $\{B_{p^{-k}}(0) | \forall k \in \mathbb{N}\}$ is a neighbourhood base of 0 with clopen subsets.

□

By Haar's theorem [5] on every locally compact topological group there exists a left-translation-invariant measure called left Haar measure. On a commutative topological group, this is right-translation-invariant as well and thus is simply called Haar measure. We define the Haar measure of $\mathbb{Z}_p$ to be 1: $\lambda(\mathbb{Z}_p) = 1$.

To determine the Haar measure on $\mathbb{Z}_p$, note that for any $k \in \mathbb{N}$ $\mathbb{Z}_p$ is the disjoint union of p^k balls of radius p^{-k} all of whom are translates of each other. Moreover any ball of radius p^{-k} is one of these balls and hence,

$$\lambda(B_{p^{-k}}(x)) = p^{-k}$$

for any x. Finally any ball agrees with a ball of radius p^{-k} for some k: if $p^{-(k+1)} < r \leq p^{-k}$ than $B_r(x) = B_{p^{-k}}(x)$ and so, the Haar measure of a ball with radius r has to be p^{-k} for the unique k such that $p^{-(k+1)} < r \leq p^{-k}$.

We make the product space $\mathbb{Z}_p^n$ a metric space induced by the maximum norm:

$$|(x_1, \dots, x_n)^T|_p = \max\{|x_1|_p, \dots, |x_n|_p\}$$

This also has a Haar measure and applying the same as before, the normalized Haar measure in $\mathbb{Z}_p^n$ is

$$\forall x \in \mathbb{Z}_p^n, p^{-(k+1)} < r \leq p^{-k} : \lambda(B_r(x)) = p^{-nk}.$$

3.2 Introducing the p-adic fractal zeta function

Definition 3.2.1. A p-adic fractal string (V) is the union of countably many disjoint open balls.

In every bounded metric space, we can call a bounded open set a fractal string if it is the disjoint union of countably many open balls. Because of the structure of $\mathbb{Z}_p$, all bounded open sets are fractal strings.

In the definition of such a decomposition of a p-adic fractal string, the radii of the balls are not unique, hence some care is needed when defining the fractal zeta function. Open balls in $\mathbb{Z}_p$ have neither a well defined center nor a well defined radius. It can happen that $B_r(x) = B_s(y)$ without $x = y$ or $r = s$. However this can only happen if $p^{-(k+1)} < r, s \leq p^{-k}$ and $|x - y|_p < p^{-k}$ [4]. The fact that any open ball of radius r is the union of p balls of radii r/p further complicates this issue. Thus a p-adic fractal string is not uniquely defined, we can give the same set as the disjoint union of countably many open balls in numerous ways.

Definition 3.2.2. Given an open set U of $\mathbb{Z}_p$ we will say that a ball $B_r(x) \subset U$ is maximal (with respect to U) if for any balls $B_s(y) \subset U$ if $B_r(x) \subset B_s(y)$ then $B_r(x) = B_s(y)$. Given a maximal ball in U $B_r(x)$ we define its radius to be p^{-k} with $p^{-(k+1)} < r \leq p^{-k}$.

We will require that in the collection that gives our fractal string no finite unions give another ball of $\mathbb{Z}_p$, thus, each ball present in the representation of the fractal string as a union of countably many disjoint open balls is maximal with respect to the fractal string.

To introduce the p-adic zeta function of varieties we will first prove that the complement of affine varieties are fractal strings by Definition 3.2.1, the union of countably many disjoint maximal balls.

Definition 3.2.3. Let $f_1, \ldots, f_k$ be a polynomials of n-variables with integer coefficients and I the ideal generated by them. Then

$$V(I) = \{x \in \mathbb{Z}_p^n : f(x) = 0, \forall f \in I\}$$

is the variety of $f_1, \ldots, f_k$.

Let us introduce the following notation: if $x = \sum_{k=0}^{\infty} a_k p^k$, then $[x]_m = \sum_{k=0}^{m-1} a_k p^k$. Let $N(p^k)$ denote the number of solutions $\mod p^k$ and set $N(p^0) = 1$.

Proposition 3.2.4. *$\mathbb{Z}_p^n \backslash V(I)$ is a fractal string and for each $k \in \mathbb{Z}^+$ there are $p^n N(p^{k-1}) - N(p^k)$ maximal balls of radius p^{-k} in the complement of $V(I)$.*

Lemma 3.2.5. *Let $f \in \mathbb{Z}_p[x_1, ..., x_n]$ and $x \in \mathbb{Z}_p^n$. Then $f(x) = 0$ if and only if $f([x]_k) \equiv 0 \mod p^k$ for all $k \in \mathbb{Z}^+$.*

Proof. The map $x \mapsto x \mod p^k$ is a ring homomorphism and $x \equiv [x]_k \mod p^k$. Thus if $\forall k f([x]_k) \equiv 0 \mod p^k$, then $\forall k f(x) \equiv 0 \mod p^k$, which means that for any positive integer k $|f(x)|_p < p^{-k}$, thus $f(x) = 0$.

If $f(x) = 0$, then $f(x) \equiv 0 \mod p^k$ for any $k \in \mathbb{Z}^+$ and since $x \equiv [x]_k \mod p^k$ we get that $\forall k f([x]_k) \equiv 0 \mod p^k$. $\square$

Using Lemma 3.2.5 we can now prove Proposition 3.2.4.

Proof. If $[x]_k$ is no longer a solution for $f([x]_k) = 0$ (where as $[x]_{k-1}$ is) then the p^{-k} ball around $[x]_k$ is in the complement of $V(f)$, because any point of $\mathbb{Z}_p^n$ can only differ from $[x]_k$ till the kth digit in their formal series representation. The complement is the disjoint union of such balls.

If $I = (f_1, \ldots, f_m)$ then $V(I) = \bigcap_{i=1}^m V(f_i)$ and in $\mathbb{Z}_p^n$ the finite intersection of the union of countably many disjoint balls is countably many disjoint balls.

All maximal balls in the complement of $V(I)$ of radius p^{-k} come from solutions $\mod p^{-(k-1)}$, thus there are $p^n N(p^{k-1}) - N(p^k)$ of them. $\square$

We generalize the definition presented in Chapter 2 by defining the p-adic fractal zeta function of the p-adic fractal string V. This is similar to the one-dimensional case, except the radii and the multiplicities are not unique. However we can assign to V the unique sequence $\{k_j\}_{j=1}^{\infty}$ if $V = \bigcup_j^* \bigcup_{i=1}^{*k_j} B_{p^{-j}}(x_{ji})$ and for each x_{ji} that's the maximum radius ball around it that's in the complement: $\forall 1 \leq i \leq k_j : B_{p^{-(j-1)}}(x_{ji}) \not\subset V$.

Definition 3.2.6. Assume that $V = \bigcup_j^* \bigcup_{i=1}^{*k_j} B_{p^{-j}}(x_{ji})$ is a decomposition into maximal balls as above. Then let

$$\zeta_V(s) = \sum_{j=1}^{\infty} k_j p^{-js}$$

the fractal zeta function of the p-adic fractal string V.

In Proposition 3.2.4 we proved that if $V = \mathbb{Z}_p^n \backslash V(I)$, then $k_j = p^n N(p^{j-1}) - N(p^j)$. And so

$$\zeta_{V(I)}(s) = \sum_{k=1}^{\infty} (p^n N(p^{k-1}) - N(p^k)) p^{-ks}$$

is the p-adic fractal zeta function of the variety of I.

3.3 Properties of the p-adic zeta function

Before going on to the most striking property of $\zeta_{V(I)}(s)$ and present the proof, we will now calculate the zeta function of an example.

Theorem 3.3.1. *If $I \lhd \mathbb{Z}[x_1, \ldots, x_n]$ is an ideal, then the series $\zeta_{V(I)}$ can be expressed as a rational function of $T = p^{-s}$.*

Example 3.3.2. $f(x, y) = x^2 + y^2 - 1$ we are looking for $([x]_k, [y]_k)$ pairs that satisfy the following knowing that $([x]_{k-1}, [y]_{k-1})$ satisfied the congruence $\mod p^{k-1}$:

$$\begin{aligned} [x]_k^2 + [y]_k^2 &\equiv 1 \mod p^k \\ ([x]_{k-1} + a_{k-1} p^{k-1})^2 + ([y]_{k-1} + b_{k-1} p^{k-1})^2 &\equiv 1 \mod p^k \\ [x]_{k-1}^2 + a_{k-1}[x]_{k-1} p^{k-1} + [y]_{k-1}^2 + b_{k-1}[y]_{k-1} p^{k-1} &\equiv 1 \mod p^k \end{aligned}$$

Since $[x]_{k-1}^2 + [y]_{k-1}^2 \equiv 1 \mod p^{k-1}$, we have $cp^{k-1} \equiv [x]_{k-1}^2 + [y]_{k-1}^2 - 1 \mod p^k$ for some $c \in \{0, \ldots, p-1\}$, which gives

$$c + a_{k-1}[x]_{k-1} + b_{k-1}[y]_{k-1} \equiv 0 \mod p$$

Since we reached a linear equality which gives us p solutions we can say that

$$N(p^k) = pN(p^{k-1}) = p^{k-1} N(p)$$

which means that for each k there are $p^2 N(p^{k-1}) - N(p^k)$ number of balls with p^{-k} radii. The geometric zeta function would be:

$$\begin{aligned} &\sum_{k=1}^{\infty} (p^2 N(p^{k-1}) - N(p^k)) p^{-ks} = N(p) \sum_{k=1}^{\infty} (p^k - p^{k-1}) p^{-ks} = \\ &\qquad = \frac{N(p)}{p}(p-1) \sum_{k=1}^{\infty} p^{k(1-s)} = N(p)(p-1) \frac{1}{1 - p^{1-s}} p^{-s} = N(p)(p-1) \frac{T}{1 - pT} \end{aligned}$$

which is a rational function of $T = p^{-s}$.

Definition 3.3.3. Let K be a field and $f \in K[x_1, \dots, x_n]$. The variety $V(f)$ is called non-singular if $\{x = (x_1, \dots, x_n) \mid f(x) = 0, \frac{\partial f}{\partial x_1}(x) = 0, \dots, \frac{\partial f}{\partial x_n}(x) = 0\} = \emptyset$.

The variety $V(I)$ is called non-singular over K if the vector space $\{(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)) | f \in I\}$ has the same dimension $\forall x \in V(I)$ over K.

When this maximum dimension is m we say that dimension of $V(I)$ is $n - m$.

This paper will only give a self-contained proof of Theorem 3.3.1 when $V(I)$ is non-singular over $\mathbb{Z}/p\mathbb{Z}$ [2]. In Chapter 5 we consider the general case, including singular varieties and show, using a hard theorem of Igusa that relies on Hironaka's resolution of singularities, that the fractal zeta function is rational even for singular varieties.

Our previous method (Example 3.3.2) fails in singular cases since the number of solutions $\mod p^k$ aren't distributed as evenly as for non-singular varieties. The number of solutions around a non-singular point $\mod p^k$ can be given as a geometric sequence, but around a singular point it varies and has to be calculated separately. The difficulties are well illustrated via the following example, which is the simplest singularity:

Example 3.3.4. $f(x, y) = xy$

$xy \equiv 0 \mod p$ if and only if at least one of them are dividable by p.

$xy \equiv [x]_2[y]_2 \equiv [x]_1[y]_1 + p(b_1[x]_1 + a_1[y]_1) \mod p^2$

If $[x]_1 \not\equiv 0 \mod p$ or $[y]_1 \not\equiv 0 \mod p$, thus (x, y) is not in the p^{-2} neighbourhood of the singularity, then this gives p solutions. If $[x]_1 \equiv [y]_1 \equiv 0 \mod p$, then it yields p^2 solutions.

Similarly for each m the p^{-m} neigbourhood of the singularity behaves differently and depending on the parity of m it brings 1 or p^2 results which complicates our previous method.

Luckily this can easily be resolved in the following way. Let us use the following notation:

$$M_k(p^l) = |\{([x]_k, [y]_k) | xy \equiv 0 \mod p^k, \gcd([x]_k, p^k) = p^l\}|$$

Since

$$\{([x]_k, [y]_k) | xy \equiv 0 \mod p^k\} = \cup_{l=0}^{*k} \{([x]_k, [y]_k) | xy \equiv 0 \mod p^k, \gcd([x]_k, p^k) = p^l\}$$

[2]If a variety is non-singular over $\mathbb{Z}/p\mathbb{Z}$, then it follows that it is non-singular over $\mathbb{Q}_p$, but the converse does not hold.

it is enough to find $M_k(p^l)$ to get $N(p^k)$, because

$$N(p^k) = \sum_{l=0}^{k} M_k(p^l).$$

If $0 \leq l < k$ and $\gcd([x]_k, p^k) = p^l$, then $p^{k-l}|y$, so

$$M_k(p^l) = (p^{k-l+1} - p^{k-l})p^l = (p-1)p^k.$$

Now $M_k(p^k) = p^k$, since $[x]_k$ has to be 0 and $[y]_k$ can be anything. Thus

$$N(p^k) = \sum_{l=0}^{k} M_k(p^l) = k(p-1)p^k + p^k.$$

Just like in the previous example, the p-adic fractal zeta function can be written as

$$\begin{aligned}\zeta_{V(f)}(s) &= \sum_{k=1}^{\infty}(p^2 N(p^{k-1}) - N(p^k))p^{-ks} = \\ &= \sum_{k=1}^{\infty}((k-1)(p-1)p^{k+1} + p^{k+1} - k(p-1)p^k - p^k)p^{-ks} = \\ &= (p-1)^2 \sum_{k=1}^{\infty}(k-1)(p^{1-s})^k = p^{2(1-s)}(p-1)^2 \sum_{k=1}^{\infty}(k-1)(p^{1-s})^{k-2}.\end{aligned}$$

Since

$$\sum_{k=1}^{\infty}(k-1)T^{k-2} = \left(\sum_{k=1}^{\infty} T^{k-1}\right)' = \left(\frac{1}{1-T}\right)' = \frac{1}{(1-T)^2}$$

we have

$$\zeta_V(f)(s) = p^2(p-1)^2 \frac{T^2}{1 - 2pT + p^2T^2}$$

a rational function in $T = p^{-s}$.

3.4 Proof of rationality of the zeta function

Now we turn to building up our proof for Theorem 3.3.1. We begin by introducing a lemma.

Lemma 3.4.1. *If $f \in \mathbb{Z}[x_1, \ldots, x_n]$ and its variety is non-singular over $\mathbb{Z}/p\mathbb{Z}$ then the number of solutions $\mod p^k$ satisfies*

$$N(p^k) = p^{(n-1)(k-1)} N(p)$$

Proof. Because of Lemma 3.2.5 we may assume that a given $[x]_{k-1}$ was a solution $\mod p^{k-1}$ and we are interested in how many $[x]_k$ solutions $\mod p^k$ are such that $[x]_k \equiv [x]_{k-1} \mod p^{k-1}$. For every i between 1 and n if $x_i = \sum_{j=0}^{\infty} b_{i,j} p^j$ then $[x_i]_k = [x_i]_{k-1} + b_{i,k-1} p^{k-1}$. Let us use the following notations:

$$\begin{aligned} \mathbf{x} &= (x_1, \ldots, x_n) \\ \forall k : \mathbf{b_k} &= (b_{1,k}, \ldots, b_{n,k}) \\ \nabla f(\mathbf{x}) &= (\frac{\partial f}{\partial x_1}(\mathbf{x}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{x})) \\ \forall k : \langle \nabla f(\mathbf{x}), \mathbf{b_k} \rangle &= \sum_{i=1}^{n} b_{i,k} \frac{\partial f}{\partial x_i}(\mathbf{x}) \end{aligned}$$

Then we get

$$f([\mathbf{x}]_k) \equiv f([\mathbf{x}]_{k-1} + \mathbf{b_{k-1}} p^{k-1}) \mod p^k$$

Notice that for any $x \in \mathbb{Z}_p$

$$[x]_k^d \equiv ([x]_{k-1} + bp^{k-1})^d \equiv [x]_{k-1}^d + d[x]_{k-1}^{d-1} bp^{k-1} \mod p^k$$

because any component which would give a higher degree would have a constant multiplier that is divisible by p^k.

And so

$$\begin{aligned} [x_1]_k^{d_1} \ldots [x_n]^{d_n} &\equiv [x_1]_{k-1}^{d_1} \ldots [x_n]_{k-1}^{d_n} + (\sum_{i=1}^{n} d_i b_{i,k} [x_i]_{k-1}^{d_i - 1} \Pi_{i \neq j} [x_j]_{k-1}^{d_j}) p^{k-1} \mod p^k \\ &\equiv [x_1]_{k-1}^{d_1} \ldots [x_n]_{k-1}^{d_n} + \langle \nabla([x_1]_{k-1}^{d_1} \ldots [x_n]_{k-1}^{d_n}), \mathbf{b_{k-1}} \rangle p^{k-1} \mod p^k \end{aligned}$$

Thus

$$f([\mathbf{x}]_k) \equiv f([\mathbf{x}]_{k-1}) + \langle \nabla f(\mathbf{x}), \mathbf{b_{k-1}} \rangle p^{k-1} \mod p^k$$

Just like in the example the components which aren't dependent on $b_{i,k-1}$ give

$$f([\mathbf{x}]_{k-1}) \equiv 0 \mod p^{k-1}$$

which means there is a $c = c([\mathbf{x}]_{k-1}) \in \{0, \ldots, p-1\}$ such that $f([\mathbf{x}]_{k-1}) \equiv cp^{k-1} \mod p^k$.

Thus $f([\mathbf{x}]_k) \equiv 0 \mod p^k$ if and only if

$$c + \langle \nabla f([\mathbf{x}]_{k-1}), \mathbf{b_{k-1}} \rangle \equiv c + \langle \nabla f([\mathbf{x}]_1), \mathbf{b_{k-1}} \rangle \equiv 0 \mod p.$$

Therefore

$$N(p^k) = \sum_{\substack{[\mathbf{x}]_{k-1} \bmod p^{k-1} \\ f([\mathbf{x}]_{k-1}) \equiv 0}} |\{\mathbf{b} \in (\mathbb{Z}/p\mathbb{Z})^n : \langle \nabla f([\mathbf{x}]_1), \mathbf{b} \rangle \equiv -c([\mathbf{x}]_{k-1}) \mod p\}.$$

Since f is non-singular, $\nabla f \neq 0$, and so:

$$N(p^k) = \sum_{\substack{[\mathbf{x}]_{k-1} \bmod p^{k-1} \\ f([\mathbf{x}]_{k-1}) \equiv 0}} |\{\mathbf{b} \in (\mathbb{Z}/p\mathbb{Z})^n : \langle \nabla f([\mathbf{x}]_{k-1}), \mathbf{b} \rangle \equiv -c([\mathbf{x}]_{k-1}) \mod p\} =$$
$$= |\{\mathbf{b} \in (\mathbb{Z}/p\mathbb{Z})^n : \langle \nabla f([\mathbf{x}]_{k-1}), \mathbf{b} \rangle \equiv 0 \mod p\}| N(p^{k-1}) = p^{n-1} N(p^{k-1}).$$

Using induction on k we get:

$$N(p^k) = p^{(n-1)(k-1)} N(p)$$

□

Lemma 3.4.2. *If $V(I)$ is non-singular over $\mathbb{Z}/p\mathbb{Z}$ and of dimension $n-m$, then the number of solutions $\mod p^k$ is the following:*

$$N(p^k) = p^{(n-m)(k-1)} N(p)$$

Proof. Let $f_1, \ldots, f_m \in I$ be polynomials so that in $[\mathbf{x}]_1$ their gradients span the vector space $\{(\frac{\partial f}{\partial x_1}(\mathbf{x}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{x})) | f \in I\}$, thus, they are linearly independent. Since $V(I)$ is non-singular and with algebraic dimension $n-m$, there exist such polynomials in I.

Like in our previous proof for each f_i we get a c_i so that

$$c_i p^{k-1} \equiv f_i([\mathbf{x}]_{k-1}) \mod p^k$$

and so we are looking for the number of $\mathbf{b} \in (\mathbb{Z}/p\mathbb{Z})^n$ which satisfy

$$\forall i : \langle \nabla f_i([\mathbf{x}]_{k-1}), \mathbf{b} \rangle \equiv \langle \nabla f_i([\mathbf{x}]_1), \mathbf{b} \rangle \equiv -c_i \mod p$$

So we are looking for the number of solutions $\mod p$ in $(\mathbb{Z}/p\mathbb{Z})^n$ for the equation $J\mathbf{x} = -\mathbf{c}$, where $J_{ij} = \frac{\partial f_i}{\partial x_j}([\mathbf{x}]_1)$ is the element in the ith row in the jth column, J is the Jacobian matrix of $F = (f_1, \ldots, f_m)^T$.

By the definition of $f_1, \dots, f_m$ the rank of J is m. Therefore, since $\dim(\operatorname{im} J) = m$,

$$J : (\mathbb{Z}/p\mathbb{Z})^n \to (\mathbb{Z}/p\mathbb{Z})^m$$

is surjective, and $\dim(\ker J) = n - m$ through the dimension theorem for vector spaces. Therefore, there are p^{n-m} solutions for $J\mathbf{x} = \mathbf{y}$ for any $\mathbf{y} \in (\mathbb{Z}/p\mathbb{Z})^m$. Thus

$$N(p^k) = |\{\mathbf{b} \in (\mathbb{Z}/p\mathbb{Z})^n : J\mathbf{b} \equiv -\mathbf{c} \mod p\}| N(p^{k-1}) = p^{n-m} N(p^{k-1}) = p^{(n-m)(k-1)} N(p).$$

□

We will now prove Theorem 3.3.1 in the non-singular case.

Proof. We may assume that $V(I)$ has dimension $n-m$. Through Proposition 3.2.4 there are $p^n N(p^{k-1}) - N(p^k)$ maximal balls of radius p^{-k} in the complement of $V(I)$. Using Lemma 3.4.2 we get:

$$\begin{aligned}
\zeta_{V(I)}(s) &= \sum_{k=1}^{\infty} (p^n N(p^{k-1}) - N(p^k)) p^{-ks} = N(p) \sum_{k=1}^{\infty} (p^{(n-m)(k-2)+n} - p^{(n-m)(k-1)}) p^{-ks} = \\
&= N(p)(p^{-(n-2m)} - p^{-(n-m)}) \sum_{k=1}^{\infty} (p^{n-m-s})^k = N(p)(p^{-(n-m)} - p^{-n}) \frac{p^{n-s}}{1 - p^{n-m-s}} = \\
&= N(p)(p^m - 1) \frac{T}{1 - p^{n-m}T}
\end{aligned}$$

□

Our results can be expressed in the terms of the generating function of the sequence $N(p^k)$. This is called the Poincaré series of $V(I)$ [13, 14]:

Definition 3.4.3.

$$P_{V(I)}(T) = \sum_{k=0}^{\infty} N(p^k) p^{-nk} T^k$$

We then have the following connection:

Proposition 3.4.4.

$$\zeta_{V(I)}(s) = (p^n T - 1) P_{V(I)}(p^n T) + 1$$

Note that by Lemma 3.4.2. The Poincaré series of a non-singular variety $V(I)$ is also a rational function.

Chapter 4

Dimension Concepts for p-adic Varieties and Fractal Zeta Functions

The dimension of an algebraic variety can be defined in many ways. One may take the dimension to be the maximum dimension of the tangent spaces at the variety's non-singular points like we did in Definition 3.3.3. Any ideal $I = (f_1, \ldots, f_m)$ generated by m independent polynomials over a field will have at least $n - m$ as its variety's dimension in the affine space $\mathbb{A}^n$.

Since we are in a space equipped with a metric, this gives us the opportunity to introduce and observe other concepts for the dimension of the variety. Recall from Chapter 2 that for one-dimensional real fractal strings, the abscissa of convergence of a fractal string's zeta function coincides with its Minkowski dimension. We will now prove that this still holds for a non-singular p-adic variety's zeta function after defining its Minkowski dimension. The proof will consist of proving for each of them separately that they agree with the algebraic dimension.

As before let $I \lhd \mathbb{Z}[x_1, \ldots, x_n]$.

4.1 The abscissa of convergence

Definition 4.1.1. The abscissa of convergence for the p-adic fractal zeta function $\zeta_{V(I)}(s) = \sum_{j=1}^{\infty} k_j p^{-js}$ is the following:

$$\sigma_{V(I)} = \inf\{\alpha \in \mathbb{R} : \sum_{j=1}^{\infty} k_j p^{-js} \text{converges if} \operatorname{Re} s > \alpha\}$$

Proposition 4.1.2. *The abscissa of convergence of the p-adic zeta function of $V(I)$ when non-singular over $\mathbb{Z}/p\mathbb{Z}$ and with algebraic dimension $n-m$, is $n-m$.*

Proof.

$$\zeta_{V(I)}(s) = N(p)(p^{-(n-2m)} - p^{-(n-m)}) \sum_{k=1}^{\infty} (p^{n-m-s})^k < \infty \iff$$

$$\iff |p^{n-m-s}| < 1 \iff \mathrm{Re}(n-m-s) < 0 \iff n-m < \mathrm{Re}\, s$$

□

4.2 The Minkowski dimension

The Minkowski dimension, also called the box dimension of a set in a metric space measures how well it can be covered by balls. We may use a different approach and define the Minkowski dimension using the set's Minkowski content. In the non-singular case both ways lead to the same result, the algebraic dimension.

Definition 4.2.1. The volume of the ε tube of the variety is

$$\mathrm{vol}(\varepsilon) = \lambda(\{x \in \mathbb{Z}_p^n \backslash V(I) | \exists y \in V(I) : |x-y|_p < \varepsilon\}).$$

Similarly to fractal strings, we can define the variety's Minkowski dimension using the tube's volume:

Definition 4.2.2. The Minkowski dimension of the variety $V(I)$ is

$$D_{V(I)} = \inf\{\alpha \geq 0 : \mathrm{vol}(\varepsilon) = O(\varepsilon^{n-\alpha}) \text{ as } \varepsilon \to 0^+\}.$$

Notice that for planecurves and surfaces in $\mathbb{R}^3$ this gives the expected results:
If γ is a smooth, simple, closed plane curve, then [16]

$$\mathrm{vol}(\varepsilon) = 2\mathrm{Length}(\gamma)\varepsilon = O(\varepsilon) = O(\varepsilon^{2-1})$$

thus, the dimension is 1 as expected.
If Σ is an oriented closed surface in $\mathbb{R}^3$, then Weyl's tube formula for a surface in $\mathbb{R}^3$ gives [16]

$$\mathrm{vol}(\varepsilon) = 2\mathrm{Area}(\Sigma)\varepsilon + \frac{4\pi}{3}\chi(\Sigma)\varepsilon^3 = O(\varepsilon) = O(\varepsilon^{3-2})$$

where $\chi(\Sigma)$ is the Euler characteristic of the surface, which makes the dimension to be 2.

Proposition 4.2.3. *The Minkowski dimension of the variety $V(I)$ when non-singular over $\mathbb{Z}/p\mathbb{Z}$ and with algebraic dimension $n-m$, is $n-m$.*

Proof. It is sufficient to show for $\varepsilon = p^{-l}$. If $x \in Z_p^n \backslash V(I)$ and $\forall y \in V(I) : |x-y|_p \geq p^{-k}$, then $B_{p^{-k}}(x) \subset \mathbb{Z}_p^n \backslash V(I)$. So to find $\mathrm{vol}(p^{-l})$, we need to add the measure of the balls with radii at most $p^{-(l+1)}$ in the representation of the fractal string:

$$\mathrm{vol}(p^{-l}) = \sum_{k=l+1}^{\infty} (p^n N(p^{k-1}) - N(p^k))p^{-nk}$$

From Lemma 3.4.2 we find that in this case $N(p^k) = p^{(n-m)(k-1)}N(p)$, and so

$$\begin{aligned} \mathrm{vol}(p^{-l}) = N(p) \sum_{k=l+1}^{\infty} (p^{(n-m)(k-2)+n} - p^{(n-m)(k-1)})p^{-nk} = \\ = N(p)(p^{-(n-2m)} - p^{-(n-m)}) \sum_{k=l+1}^{\infty} p^{-mk} = \\ = N(p)(p^{-(n-2m)} - p^{-(n-m)}) \cdot \frac{p^{-ml}}{p^m - 1} = \frac{N(p)}{p^{(n-m)}} p^{-ml} \end{aligned}$$

Thus

$$\log_{p^{-l}}(\mathrm{vol}(p^{-l})) = \frac{\log(N(p)) - (lm+n-m)\log(p)}{-l\log(p)}$$

And so

$$\lim_{l \to \infty} \log_{p^{-l}}(\mathrm{vol}(p^{-l})) = m$$

□

Or we may use the general definition for a set's Minkowski dimension:

Definition 4.2.4. Let $N(V(I), \varepsilon)$ denote the minimal number of sets with diameter at most ε that cover $V(I)$. The box dimension of the variety $V(I)$ is the limit

$$\dim_{\mathcal{M}}(V(I)) = \lim_{\varepsilon \to 0} \frac{\log N((V(I), \varepsilon))}{\log \frac{1}{\varepsilon}}$$

when it exists. [1]

Proposition 4.2.5. *The box dimension of the variety $V(I)$ when non-singular over $\mathbb{Z}/p\mathbb{Z}$, where I is with algebraic dimension $n-m$, is $n-m$.*

Proof. It is sufficient to show for $\varepsilon = p^{-k}$, because $N(V(I), \varepsilon) = N(V(I), p^{-k})$ for some k. However $N(V(I), p^{-k}) = N(p^k)$, because for each $x \in V(I)$ the balls with radius p^{-k} around the solutions of $\mod p^k$ give a minimal cover.

$$\dim_{\mathcal{M}}(V(I)) = \lim_{\varepsilon \to 0} \frac{\log(N(V(I), \varepsilon))}{\log \frac{1}{\varepsilon}} = \lim_{k \to \infty} \frac{\log(N(V(I), p^{-k}))}{k \log(p)} = \lim_{k \to \infty} \frac{\log(N(p^k))}{k \log(p)}$$

But since $V(I)$ is non-singular we know that $N(p^k) = p^{(n-m)(k-1)} N(p)$, thus,

$$\dim_{\mathcal{M}} V(I) = \lim_{k \to \infty} \frac{\log(p^{(n-m)(k-1)} N(p))}{k \log(p)} = \lim_{k \to \infty} ((n-m)\frac{k-1}{k} + \frac{\log(N(p))}{k \log(p)}) = n - m$$

□

Thus, we have proved the following:

Theorem 4.2.6. *The Minkowski dimension of a non-singular variety over $\mathbb{Z}/p\mathbb{Z}$ aligns with the abscissa of convergence of its p-adic zeta function.*

4.3 The Hausdorff dimension

As mentioned above, the Minkowski dimension and the Hausdorff dimension agree in the case of self-similiar sets that satisfy the open set condition in $\mathbb{R}^n$. We may introduce self-similarity in the following way on the p-adic numbers: $f : \mathbb{Q}_p^n \to \mathbb{Q}_p^n$ is a contraction if $\forall x, y \in \mathbb{Q}_p^n : |f(x) - f(y)|_p < |x - y|_p$. Then if $\mathcal{F} = \{f_1, \ldots, f_k\}$ is a finite set of contractions, there exists a unique non-empty compact set $K \subset \mathbb{Q}_p$ called the attractor [6] that satisfies

$$K = \bigcup_{i=1}^{k} f_i(K)$$

We say that K is a *p-adic self-similiar set.* If $\forall x, y \in \mathbb{Q}_p^n : |f(x) - f(y)|_p = r|x - y|_p$ for some $0 < r < 1$, then we call r the *contraction ratio.*

Definition 4.3.1. The diameter of $X \subset \mathbb{Q}_p^n$ is

$$diam(X) = \sup_{x,y \in \mathbb{Q}_p^n} |x - y|_p$$

Definition 4.3.2. When $\varepsilon > 0$ and $d \geq 0$

$$H^d_\infty(X) = \inf\{\sum_{i=0}^{\infty} diam_p(X_i)^d | X \subset \bigcup_{i=0}^{\infty} X_i\}$$

and this helps us to define the Hausdorff dimension of the set

$$\dim_{\mathcal{H}}(X) = \inf\{d \geq 0 | H^d_\infty(X) = 0\}$$

Lemma 4.3.3. *If the variety $V(I)$ is non-singular $\mathbb{Z}_p^n \setminus V(I)$ is a self-similiar set, moreover if the algebraic dimension is $n-m$, then $\mathbb{Z}_p^n \setminus V(I) = \bigcup_{i=1}^{*p^{n-m}} (x_i + p(\mathbb{Z}_p^n \setminus V(I)))$ for some $x_i \in \mathbb{Z}_p^n$.*

Proof. This follows from the proof we have given for Lemma 3.4.2. □

So $\mathbb{Z}_p^n \setminus V(I)$ is the attractor of $\mathcal{F} = \{f_i | \forall 1 \leq i \leq n-m : f_i(x) = x_i + px\}$. Thus $\mathbb{Z}_p^n \setminus V(I)$ is the disjoint union of p^{n-m} many $p(\mathbb{Z}_p^n \setminus V(I))$, which makes it easier to calculate the Hausdorff dimension of the fractal string.

Proposition 4.3.4. *When $V(I)$ is non-singular with algebraic dimension $n-m$, $\dim_{\mathcal{H}}(\mathbb{Z}_p^n \setminus V(I)) = n-m$.*

Proof. We will first show that $\dim_{\mathcal{H}}(\mathbb{Z}_p^n \setminus V(I)) \leq n-m$:

For this let $\mathbb{Z}_p^n \setminus V(I) \subset \bigcup_{j=0}^{\infty} X_j$ be an arbitrary cover. Then since $\mathbb{Z}_p^n \setminus V(I)$ is non-singular, Lemma 4.3.3 shows that it is self-similar:

$$\mathbb{Z}_p^n \setminus V(I) = \bigcup_{i=1}^{*p^{n-m}} (x_i + p(\mathbb{Z}_p^n \setminus V(I))) \subset \bigcup_{i=1}^{*p^{n-m}} (x_i + p(\bigcup_{j=0}^{\infty} X_j))$$

and thus the sets $X_{ij} = x_i + pX_j$ also form a cover.

$$\begin{aligned} H^d_\infty(X) &= \inf\{\sum_{i=0}^{\infty} diam_p(X_i)^d | X \subset \bigcup_{i=0}^{\infty} X_i\} \\ &\leq \inf\{\sum_{i=0}^{\infty} p^{n-m} \left(\frac{diam_p(X_i)}{p}\right)^d | X \subset \bigcup_{i=0}^{\infty} X_i\} = \frac{p^{n-m}}{p^d} H^d_\infty(\mathbb{Z}_p^n \setminus V(I)) \end{aligned}$$

We may iterate the above reasoning and through that we can prove inductively the following inequality for any positive integer k:

$$H^d_\infty(\mathbb{Z}_p^n \setminus V(I)) \leq \left(\frac{p^{n-m}}{p^d}\right)^k H^d_\infty(\mathbb{Z}_p^n \setminus V(I))$$

Since this stands for any positive integer k, taking the limit leads to

$$0 < H^d_\infty(\mathbb{Z}_p^n \setminus V(I)) \leq \lim_{k\to\infty}(\frac{p^{n-m}}{p^d})^k H^d_\infty(\mathbb{Z}_p^n \setminus V(I)) < \infty \implies \frac{p^{n-m}}{p^d} \geq 1 \iff d \leq n-m$$

Note that the sets $U_i = B_{\frac{1}{p}}(x_i)$ for $1 \leq i \leq p^{n-m}$ form a cover of $\mathbb{Z}_p^n \setminus V(I)$, thus for each

$$d \geq 0 : H^d_\infty(\mathbb{Z}_p^n \setminus V(I)) \leq \frac{p^{n-m}}{p^d} < \infty$$

.

This observation and the inequality of the limit show that for any

$$d > n-m : H^d_\infty(\mathbb{Z}_p^n \setminus V(I)) = 0$$

making $\dim_{\mathcal{H}}(\mathbb{Z}_p^n \setminus V(I)) \leq n-m$.

Now to prove the equality we show that $\dim_{\mathcal{H}}(\mathbb{Z}_p^n \setminus V(I)) \geq n-m$.

This is based on the fact that for any positive integer $k : p^{-kd} \leq p^{n-m}p^{-(k+1)d}$, when $d \leq n-m$. So in this case, we recieve a smaller value if we take a cover with bigger radii balls. Thus the above mentioned $\{U_i\}_{i=1}^{p^{n-m}}$ cover gives the infimum of the sums.

Therefore $\dim_{\mathcal{H}}(\mathbb{Z}_p^n \setminus V(I)) = n-m$. $\square$

Chapter 5

Connection to the Igusa Local Zeta Function

5.1 The Igusa local zeta function and the Poincaré series

The case of a singular variety's fractal zeta function can be dealt with if we consider its connections to the Poincaré series (Definition 3.4.3). This research was initiated by Jun-Ichi Igusa, in a more generalized context regarding the so called Igusa local zeta function [8]:

$$Z_{K,\phi}(s) = \int_{K^n} \phi(x)|f(x)|_K^s |dx|_K$$

for $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 0$ where K is a local field, $|.|_K$ is an absolute value on K,$f \in K[x_1, \dots, x_n]$ is a polynomial, $|dx|_K$ a Haar measure compatible with that absolute value and $\phi \in \mathcal{S}(K^n)$ is a compactly supported locally constant function[1].

If $K = \mathbb{Q}_p$ and $\phi = \chi_{\mathbb{Z}_p^n}$ the characteristic function of $\mathbb{Z}_p^n$ then we get the Igusa local zeta function for p-adic numbers:

$$Z_{V(f)}(s) = \int_{\mathbb{Z}_p^n} |f(x)|_p^s d\lambda(x)$$

A survey of the main results can be found in [13, 14].

Theorem 5.1.1. *$Z_{V(f)}(s)$ is a rational function in $T = p^{-s}$.*

[1] A so called Schwartz-function on K^n.

Igusa [8] showed this in 1974 using Hironaka's theorem [7] about the resolution of singularities.

It is also known that the above mentioned Poincaré series can be given as a rational function of the Igusa local zeta function. We will present a proof for this as seen below.

Proposition 5.1.2. *[8]*

$$Z_{V(f)}(s) = \frac{(p^{-s}-1)P_{V(f)}(p^{-s})+1}{p^{-s}}$$

Proof.

$$\begin{aligned} Z_{V(f)}(s) &= \int_{\mathbb{Z}_p^n} |f(x)|_p^s d\lambda(x) = \sum_{k=0}^{\infty} \int_{\{x\in\mathbb{Z}_p^n \mid |f(x)|_p=p^{-k}\}} p^{-ks} d\lambda(x) = \\ &= \sum_{k=0}^{\infty} p^{-ks} \int_{\{x\in\mathbb{Z}_p^n \mid |f(x)|_p=p^{-k}\}} d\lambda(x) = \sum_{k=0}^{\infty} p^{-ks} \lambda(\{x\in\mathbb{Z}_p^n \mid |f(x)|_p = p^{-k}\}) = \\ &= \sum_{k=0}^{\infty} p^{-ks}(\lambda(\{x\in\mathbb{Z}_p^n \mid |f(x)|_p \le p^{-k}\}) - \lambda(\{x\in\mathbb{Z}_p^n \mid |f(x)|_p \le p^{-(k+1)}\})) \end{aligned}$$

Now observe that:

$$\lambda(\{x\in\mathbb{Z}_p^n \mid |f(x)|_p \le p^{-k}\}) = \lambda(\{x\in\mathbb{Z}_p^n \mid f([x]_k) \equiv 0 \mod p^k\}) = p^{-nk}N(p^k)$$

Which means that:

$$\begin{aligned} Z_{V(f)}(s) &= \sum_{k=0}^{\infty} p^{-ks}(\lambda(\{x\in\mathbb{Z}_p^n \mid |f(x)|_p \le p^{-k}\}) - \lambda(\{x\in\mathbb{Z}_p^n \mid |f(x)|_p \le p^{-(k+1)}\})) = \\ &= \sum_{k=0}^{\infty} p^{-ks}(p^{-nk}N(p^k) - p^{-n(k+1)}N(p^{k+1})) = \\ &= \sum_{k=0}^{\infty} p^{-ks}p^{-nk}N(p^k) - \frac{1}{p^{-s}}\sum_{k=1}^{\infty} p^{-ks}p^{-nk}N(p^k) = \\ &= P_{V(f)}(p^{-s}) - \frac{1}{p^{-s}}(P_{V(f)}(p^{-s})-1) = \frac{(p^{-s}-1)P_{V(f)}(p^{-s})+1}{p^{-s}} \end{aligned}$$

□

Based on Proposition 5.1.2 we may deduce the following corollary.

Corollary 5.1.3.

$$P_{V(f)}(T) = \frac{1 - T\cdot Z_{V(f)}(s)}{1-T}$$

5.2 Closed formula of the p-adic fractal zeta function

Since the Igusa local zeta function can be given as a rational function in T, and the Poincaré series is a rational function of the Igusa local zeta function, and our fractal zeta function is a linear function of the Poincaré series, we can see that it also can be given as a rational function of T. The relationships between the three functions suggest the following theorem.

Theorem 5.2.1. *The p-adic zeta function of a p-adic affine variety given by a principal ideal can be given in a closed formula as a p-adic integral.*

Proof. From Proposition 3.4.4 and 5.1.2 we get

$$\zeta_{V(f)}(s) = p^{-(n-s)} Z_{V(f)}(-(n-s)) = p^{-(n-s)} \int_{\mathbb{Z}_p^n} |f(x)|_p^{-(n-s)} d\lambda(x)$$

so we have given our fractal zeta function in a closed formula. □

Notice that the Igusa local zeta function is only defined on varieties given by a principal ideal, but all varieties can be given in such a form:

Proposition 5.2.2. *If $I \lhd \mathbb{Z}_p[x_1, \ldots, x_n]$, then $\exists f \in \mathbb{Z}_p[x_1, \ldots, x_n]$, so that $V(I) = V(f)$.*

Remark 5.2.3. Here $V(I)$ refers to the set of points in $\mathbb{Z}_p^n$ as in Definition 3.2.3 and not the abstract algebraic variety in the sense of scheme theory.

Proof. Let I be the ideal generated by $f_1, \ldots, f_k \in \mathbb{Z}_p[x_1, \ldots, x_n]$, then it is sufficient to find a $g \in \mathbb{Z}_p[x_1, \ldots, x_k]$ that's only root is $(0, \ldots, 0) \in \mathbb{Z}_p^k$ since then $f = g(f_1, \ldots, f_k)$ satisfies the conditions given.

It is enough to show that there is an $h \in \mathbb{Z}_p[x_1, x_2]$ that on $\mathbb{Z}_p^2$ vanishes only at the point $(0, 0)$. Given such an h, the polynomial

$$g(x_1, \ldots, x_k) = h(x_1, h(x_2, \ldots h(x_{k-2}, h(x_{k-1}, x_k)) \ldots))$$

is a polynomial with integer coefficients whose only root is $(0, \ldots, 0) \in \mathbb{Z}_p^k$.

Let $h(x, y) = x^2 - py^2$. We will show that $h(x, y)$ meets the above conditions, i.e. if $x, y \in \mathbb{Z}_p$ then $h(x, y) = 0$ if and only x, y are both 0. It is enough to see that $y = 0$, because $h(x, y) = 0$ if and only if $x^2 = py^2$, thus if $y = 0$, then $x^2 = 0$, and so $x = 0$. To prove this, we will show that $\forall k\ p^k | y$ using induction on k:

For the base case assume first that $k = 1$: since $x^2 = py^2$, $p|x^2$, thus $p|x$, meaning $p^2|py^2$, which shows, that $p|y$.

Now assume that $p^k|y$: again since $x^2 = py^2$, $p^{2k+1}|x^2$, meaning $p^{k+1}|x$, which shows, that $p^{2k+2}|py^2$, and so $p^{k+1}|y$. $\square$

Therefore, since every p-adic affine variety can be described as a principal ideal's variety, Theorem 5.2.1 shows that the p-adic fractal zeta function of all p-adic affine varieties can be given as a p-adic integral.

Chapter 6

p-adic Projective Fractal Strings

The concepts introduced for p-adic affine varieties can be easily transferred to p-adic projective varieties as well. In this chapter we show how our base definitions extend to the p-adic projective space after briefly introducing the space itself.

6.1 The p-adic projective space

Definition 6.1.1.

$$\mathcal{S}^n = \{x \in \mathbb{Z}_p^{n+1} : |x|_p = 1\}$$

$$x, y \in \mathcal{S}^n : x \sim y \iff \exists \lambda \in \mathbb{Z}_p : \lambda x = y, |\lambda|_p = 1$$

$$\mathbb{P}\mathbb{Z}_p^n = \mathcal{S}^n / \sim$$

Remark 6.1.2. This $\mathbb{P}\mathbb{Z}_p^n$ introduced above is homeomorphic with the projective space $\mathbb{P}^n(\mathbb{Q}_p)$, the points of which are the lines that go through the origo in $\mathbb{Q}_p^{n+1}$. From here on, we consider this topological space instead of the algebraic construct, since we may define the following metric on $\mathbb{P}\mathbb{Z}_p^n$:

Definition 6.1.3. Let $d_p(x, y)$ denote the distance between the equivalence class of x and y on $\mathcal{S}^n$.

Proposition 6.1.4. *$\mathbb{P}\mathbb{Z}_p^n = \bigcup_{i=0}^{*n} \mathcal{U}_i$, where $\mathcal{U}_i = \{x \in \mathcal{S}^n : |x_1|_p < 1, \ldots, |x_{i-1}|_p < 1, |x_i|_p = 1\}$.*

These maps give a better insight as to how this metric actually behaves.

Lemma 6.1.5. *If $x \in \mathcal{U}_i, y \in \mathcal{U}_j$, then $d_p(x, y)$ is equal to*

1. $|x' - y'|_p$, *when* $i = j$ *and* $x \sim x', y \sim y', x'_i = y'_i = 1$.

2. *1, when* $i \neq j$.

Remark 6.1.6. If $x \in \mathcal{U}_i$, then $\exists x' \in \mathcal{U}_i : x \sim x', x'_i = 1$, since by the definition of $\mathcal{U}_i$ its ith coordinate is a unit, and thus x_i has a multiplicative inverse.

Proof. 1. It suffices to show that for any $\lambda \in \mathbb{Z}_p^\times : |x' - \lambda y'|_p \geq |x' - y'|_p$, because $\forall \lambda, \mu \in \mathbb{Z}_p^\times : |\mu x' - \lambda y'|_p = |\mu|_p |x' - \frac{\lambda}{\mu} y'|_p$.

If for $\lambda \in \mathbb{Z}_p^\times : |x' - \lambda y'|_p < |x' - y'|_p = p^{-k}$, then $\forall 1 \leq j \leq n+1 : |x'_j - \lambda y'_j|_p < p^{-k}$. Even for $j = i$, thus since $|1 - \lambda|_p < p^{-k}$, $\lambda \equiv 1 \mod p^{k+1}$.

If $|x'_j - y'_j|_p = p^{-k}$, then $x'_j \equiv y'_j \mod p^k$, but $x'_j \not\equiv y'_j \mod p^{k+1}$. Therefore $x'_j \not\equiv \lambda y'_j \mod p^{k+1}$, which means, that $|x'_j - \lambda y'_j|_p \geq p^{-k}$.

2. We may assume that $i < j$.

$d_p(x, y) = \min_{\lambda \in \mathbb{Z}_p^\times} |x - \lambda y|_p = \min_{\lambda \in \mathbb{Z}_p^\times} \max_{0 \leq k \leq n} |x_k - (\lambda y)_k|_p$ and since $\forall \lambda : |x_i|_p = 1, |\lambda y_i|_p < 1 \implies \forall \lambda : |x_i - \lambda y_i|_p = 1$ it has to be 1.

$\square$

Proposition 6.1.7. *If* $x \in \mathcal{U}_i$ *and* $dist(x, V(I)) = r$, *then* $B_r(x) \subset \mathcal{U}_i$.

Proof. This follows directly from the second point of the lemma. $\square$

6.2 The p-adic projective fractal zeta function

Definition 6.2.1. $V \subset \mathbb{P}\mathbb{Z}_p^n$ is a p-adic projective fractal string if it is the disjoint union of countably many balls.

Like for the affine case, this decomposition is not at all unique, but again we may assign to the projective fractal string the maximal ball decomposition where the radii are of the form p^{-j}.

Definition 6.2.2. Assume that $V = \bigcup_j^* \bigcup_{i=1}^{*k_j} B_{p^{-j}}(x_{ji})$ is a decomposition into maximal balls. Then let

$$\hat{\zeta}_V(s) = \sum_{j=1}^{\infty} k_j p^{-js}$$

be the fractal zeta function of the p-adic projective fractal string V.

Proposition 6.1.7 ensures that the maximal balls in the complement stay on their respective maps, which will make it easier to trace back the problems to our previous results.

Definition 6.2.3. Let $f(x) \in \mathbb{Z}_p[x_1, \ldots, x_{n+1}]$ be a homogeneous polynomial, and let $I \lhd \mathbb{Z}_p[x_1, \ldots, x_{n+1}]$ be a homogeneous ideal - an ideal for which if $f \in I$, then all of its homogeneous components are in I as well [3]. Then let us introduce the following notations:

$f_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) = f(px_1, \ldots, px_{i-1}, 1, x_{i+1}, \ldots, x_{n+1})$

$I_i = \{f_i : f \in I\}$

$N_i(p^k)$ denotes the number of points $\mod p^k$ of $V(I_i)$

Proposition 6.2.4. *I_i is an ideal.*

Proof. If $f_i, g_i \in I_i$ with $f, g \in I$, then $f_i + g_i \in I_i$, since $(f+g)_i = f_i + g_i$. Similarly for any $q \in \mathbb{Z}_p[x_1, \ldots, x_n] : qf_i \in I_i$, since if Q is the homogenous form of q, then $(Qf)_i = qf_i$. □

Definition 6.2.5. Let $I \lhd \mathbb{Z}_p[x_1, \ldots, x_{n+1}]$ be as above, then

$$\hat{V}(I) = \{x \in \mathbb{P}\mathbb{Z}_p^n : f(x) = 0, \forall f \in I\}$$

the topological space defined by the variety of I in the p-adic projective.

This object is well defined since if $x, y \in \mathcal{S}^n, f \in \mathbb{Z}_p[x_1, \ldots, x_{n+1}]$ is homogeneous, $f(x) = 0$ and $\lambda x = y$ for some $\lambda \in \mathbb{Z}_p$, then $f(y) = f(\lambda x) = \lambda^{n+1} f(x) = 0$. Thus, if $x \in \mathcal{S}^n$ is in the space, then its whole equivalence class is.

Proposition 6.2.6. *$\mathbb{P}\mathbb{Z}_p^n \setminus \hat{V}(I)$ is a p-adic projective fractal string.*

Proof. $\iota_i(x_1, \ldots, x_n) = (px_1, \ldots, px_{i-1}, 1, x_i, \ldots, x_n)$ gives a bijection between $\mathbb{Z}_p^n \setminus V(I_i)$ and $\mathcal{U}_i \setminus \hat{V}(I)$.

$\iota_i(B_{p^{-k}})(x) = \{y : |\iota_i(x)_j - y_j|_p = p^{-k}\ \forall 1 \leq j < i, |\iota_i(x)_j - y_j|_p < p^{-k}\ \forall j \geq i\} \cup B_{p^{-k}}(\iota_i(x))$

is the disjoint union of p^{i-1} number of balls with radius p^{-k}, moreover, if $B_{p^{-k}}$ is a maximal ball in $\mathbb{Z}_p^n V(I_i)$ then the image is the union of p^{i-1} maximal balls, because

$$\iota_i^{-1}(B_{p^{-k}})(x) = \{y : |\iota_i^{-1}(x)_j - y_j|_p < p^{-k}\ \forall j \geq i\} \cap B_{p^{-(k-1)}}(\iota_i^{-1}(x))$$

And since $\mathbb{Z}_p^n \setminus V(I_i)$ is a p-adic fractal string, $\mathbb{P}\mathbb{Z}_p^n \setminus \hat{V}(I)$ is the disjoint union of countably many balls. □

The functions ι_i give us the understanding we need to find the connection between the p-adic projective fractal zeta function $\hat{\zeta}_{\hat{V}(I)}(s)$ and the p-adic fractal zeta functions on the maps $\mathcal{U}_i$ corresponding with the variety.

Proposition 6.2.7.

$$\hat{\zeta}_{\hat{V}(I)}(s) = \sum_{i=1}^{n+1} p^{i-1}\zeta_{V(I_i)}(s) = \sum_{k=1}^{\infty} p^{-ks}(\sum_{i=1}^{n+1} p^{i-1}(p^n N_i(p^{k-1}) - N_i(p^k)))$$

Proof. If k_j is as before and k_{ij} is the number of maximal balls with radius p^{-j} on $\mathcal{U}_i$, then $k_{ij} = p^{i-1}k'_{ij}$, where k'_{ij} is the number of balls with radius p^{-j} in the maximal ball decomposition of $V(I_i)$. Thus using Proposition 3.2.4 we find that

$$\hat{\zeta}_{\hat{V}(I)}(s) = \sum_{j=1}^{\infty} k_j p^{-js} = \sum_{j=1}^{\infty}\sum_{i=1}^{n+1} k_{ij}p^{-js} = \sum_{i=1}^{n+1} p^{i-1}\sum_{j=1}^{\infty} k'_{ij}p^{-js} =$$
$$= \sum_{i=1}^{n+1} p^{i-1}\zeta_{V(I_i)}(s) = \sum_{k=1}^{\infty} p^{-ks}(\sum_{i=1}^{n+1} p^{i-1}(p^n N_i(p^{k-1}) - N_i(p^k))$$

□

Through this proposition, we may reap the results of our labour and prove the rationality of the function by giving a closed formula for it.

$V(I_i)$ is an affine variety, and so, as shown in Proposition 5.2.2, for each $1 \leq i \leq n+1$ there exists a $g_i \in \mathbb{Z}_p[x_1, \ldots, x_n]$, such that the $\mathbb{Z}_p$ points of the varieties $V(I_i)$ and $V(g_i)$ are identical or form the same set. In which case, the following holds.

Theorem 6.2.8. *If $V(I_i) = V(g_i)$, then*

$$\hat{\zeta}_{\hat{V}(I)}(s) = p^{-(n-s)}\sum_{i=1}^{n+1} p^{i-1}\int_{\mathbb{Z}_p^n} |g_i(x)|_p^{-(n-s)} d\lambda(x)$$

Proof. Since $V(I_i) \subset \mathbb{Z}_p^n$ is an affine variety we can apply Theorem 5.2.1:

$$\zeta_{V(I_i)}(s) = \zeta_{V(g_i)}(s) = p^{-(n-s)}\int_{\mathbb{Z}_p^n} |g_i(x)|_p^{-(n-s)} d\lambda(x)$$

and taking Proposition 6.2.7 into consideration we get that

$$\hat{\zeta}_{\hat{V}(I)}(s) = \sum_{i=1}^{n+1} p^{i-1}\zeta_{V(I_i)}(s) = \sum_{i=1}^{n+1} p^{i-1}p^{-(n-s)}\int_{\mathbb{Z}_p^n} |g_i(x)|_p^{-(n-s)} d\lambda(x)$$

□

Theorem 6.2.9. *$\hat{\zeta}_{\hat{V}(I)}(s)$ is rational in p^{-s}.*

Proof.

$$\hat{\zeta}_{\hat{V}(I)}(s) = p^{-(n-s)} \sum_{i=1}^{n+1} p^{i-1} \int_{\mathbb{Z}_p^n} |g_i(x)|_p^{-(n-s)} d\lambda(x) = p^{-(n-s)} \sum_{i=1}^{n+1} p^{i-1} Z_{V(g_i)}(-(n-s))$$

and $Z_{V(g_i)}(s)$ is rational in p^{-s} for all g_i. □

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Alulírott Robin Eszter Melinda nyilatkozom, hogy szakdolgozatom elkészítése során a megadott MI alapú eszközöket alkalmaztam:

Feladat	**Felhasznált eszköz**	**Felhasználás helye**	**Megjegyzés**
Nyelvhelyesség ellenőrzése	Writefull	Teljes dolgozat	
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A felsoroltakon túl más MI alapú eszközt nem használtam.