Eötvös Loránd University

Faculty of Science

Semigroup algebras of finite representation type

Süle Márk Manó

Bachelor thesis

mathematics BSc

Supervisor:

Dr. István Ágoston

Budapest, 2025

Contents

1.0 Acknowledgement

I want to express my gratitude to my supervisor, István Ágoston, for proposing this topic and guiding me throughout the writing of my thesis. I am grateful for the countless consultations and conversations, not only about mathematics, but also on everyday topics. I thank him for shaping the way I think about algebra. I believe that the insights I learned from him will be truly valuable to me.

2.0 Prerequisites

2.1 Semigroups

Basic notions

In this section, we list some basic definitions and elementary, well-known facts from semigroup theory. The discussion of the topic closely follows parts of the first chapter of [Ste16]. A set S equipped with an associative binary operation '$\cdot$' is called a **semigroup**. We usually denote the product of elements with juxtaposition or by putting an '$\cdot$' between them. From now on, S will denote a semigroup. If S has an identity element, we call it a **monoid**. There is not much difference between monoids and semigroups since if S does not have an identity element we can adjoin one externally by introducing a new symbol 1, and by considering the semigroup on $S \cup \{1\}$ where the product of elements of S stays the same and for any $s \in S$ we define $1s = s1 = s$, furthermore $1 \cdot 1 = 1$. The semigroup obtained this way is called the **semigroup with externally adjoined identity** and will be denoted as S^1.

An element z of S is called a **zero element** if for every $s \in S$ we have $sz = z = zs$. Similarly, if S does not contain a zero element, we can adjoin one externally as above. The semigroup obtained this way is called the **semigroup with externally adjoined** zero and will be denoted as S^0.

A **subsemigroup** of a semigroup is a subset that also forms a semigroup. A **subgroup** of a semigroup is any subsemigroup that is also a group. The **subsemigroup generated by a subset** $X \subseteq S$ is the smallest semigroup in S, with respect to inclusion, which contains X as a subset.

An **ideal** of S is a subset J such that $S^1JS^1 \subseteq J$. From the definition, it follows that any ideal of S is also a subsemigroup of S. An ideal generated by one element s is defined as the ideal S^1sS^1 and is called the **principal ideal** generated by s. Let us note that if S contains a zero element, and I is an ideal of S, then the zero element always belongs to I.

A significant role is played by the **idempotent elements** of S, that is, elements satisfying the equation $e^2 = e$. The set of idempotents contained in a subset $X \subseteq S$ is denoted by $E(X)$. For any $e \in E(S)$, eSe is a monoid with respect to the multiplication inherited from S, where e is the identity element. The set of elements of eSe which are invertible with respect to e (i.e. $x \in eSe$ for which $\exists y \in eSe$ such that $yx = xy = e$) form a subgroup of $eSe \subseteq S$, and are called the **maximal subgroups** of S. Note that these subgroups are indeed maximal, and every maximal subgroup is of this form.

There is a partial order on $E(S)$ defined by $e \leq f \Leftrightarrow ef = fe = e$, or equivalently, if $eSe \subseteq fSf$. An idempotent is called **primitive** if it is minimal with respect to the partial order restricted to non-zero idempotents. We shall use the notation $E_m(X)$ to denote the set of maximal elements of the previous partial order restricted to $X \subseteq S$.

Next, we mention a few important classes of semigroups. We start by discussing some basic properties of cyclic semigroups, i.e semigroups generated by a single element. Let S be a finite semigroup and let $s \in S$. Since S is finite, there exists a smallest integer $c > 0$, called the index of s, such that $s^c = s^{c+d}$ for some integer $d > 0$. The smallest such d is called the period of s. Clearly $s^c = s^{c+qd}$ for every $q \geq 0$. It can be proved that for any $s \in S$ with index c and period d we have $s^i = s^j$ $(i, j \in \mathbb{N})$ if and only if $i = j$ or $i, j \geq c$ and $i \equiv j$ (d). Then it follows that $\langle s \rangle = \{s^n : n \geq 1\}$ the **cyclic subsemigroup** generated by s is, in fact, identical to $\{s, s^2, \ldots s^{c+d-1}\}$ with the original product, where c is the index and d is the period of s, and that these elements are all different. Moreover, the subsemigroup $C = \{s^n : n \geq c\}$ of $\langle s \rangle$ is a cyclic group of order d, and the identity of C, denoted as s^ω, is the unique idempotent in $\langle s \rangle$, and moreover, $s^\omega = s^m$ where $m \geq c$ and $m \equiv 0$ (d). The proof of these facts is elementary and can be found in [Ste16] chapter 1.2. For us, the most important corollary of the above discussion is the following.

Corollary 2.1.1. *In a finite semigroup, any subsemigroup generated by an element contains an idempotent.*

Definition 2.1.2. *Let $0 \in S$ be a semigroup. S is a* ***null*** *or* ***zero semigroup*** *if and only if $S^2 = 0$.*

Definition 2.1.3. *Let S be a semigroup. S is a* ***regular semigroup*** *if and only if for every element $a \in S$ there exists at least one element $b \in S$ such that $a = aba$ and $b = bab$.*

An important subclass of regular semigroups is formed by inverse semigroups. They abstract the notion of partial symmetry analogous to how groups abstract the notion of symmetry.

Definition 2.1.4. *Let S be a semigroup. S is called an* ***inverse semigroup*** *if and only if for every element $m \in M$ there exists a unique element $m^* \in S$ such that $m = mm^*m$ and $m^* = m^*mm^*$.*

A "generic" example of an inverse semigroup is the monoid formed by the partial bijections of a set. Let X be a set and define

$$\mathcal{I}_X := \{\alpha : A \longrightarrow B : A, B \subseteq X, \alpha \text{ is a bijection}\}$$

Then it can be proved that the product

$$\alpha * \beta := (\beta \circ \alpha)\big|_{\alpha^{-1}(\mathrm{im}\alpha \cap \mathrm{dom}\beta)} \qquad \alpha, \beta \in \mathcal{I}_X$$

defines an inverse semigroup on $\mathcal{I}_X$. It is called the **symmetric inverse semigroup** on X. Furthermore, it can also be proved that every inverse semigroup can be embedded into a symmetric inverse monoid, in the same way that every group can be embedded into a symmetric group. The proof of this fact can be found in [CP61], Theorem 1.20.

Matrix type semigroups and (0-)simple semigroups

Let S be a semigroup, let I, M be two index sets and let $P = (p_{mi})_{m \in M, i \in I}$ be a generalized (having finitely many non-zero elements) $M \times I$ matrix with entries from S^0. Consider the set of generalized $I \times M$ matrices over S^0 with at most one non-zero entry and their product defined as

$A * B := APB$, with the usual matrix product on the right hand size.

This defines a semigroup called a **semigroup of matrix type** over S or **Rees matrix semigroup** over S with sandwich matrix P, and is denoted by $\mathcal{MS}^0(S^0, I, M, P)$. Since any non-zero element of $\mathcal{MS}^0(S^0, I, M, P)$ is uniquely determined by its non-zero entry and may be identified with (i, s, m) this construction can be treated as the semigroup on the set $I \times S^0 \times M$ with the following product:

$$(i, s, m)(j, t, n) := (i, sp_{mj}t, n) \quad i, j \in I, m, n \in M, s, t \in S^0$$

identifying elements of the form $(i, 0, m)$ with the zero element of $\mathcal{MS}^0(S^0, I, M, P)$. An important modification is the semigroup $\mathcal{MS}(S^0, I, M, P)$ with the same product but with matrices having exactly one nonzero entry. We will use the notation $\mathcal{MS}(S, I, M, P)$ when there is no need to distinguish between the cases of having a zero element or not. Our interest will lie in the case when S is a group, as we shall see, by the Rees Theorem2.1.10, these semigroups play an important role in the theory of semigroups. For more results on Rees matrix semigroups, the reader should consult [Okn90], beginning of chapter 1, where the above construction can also be found. We continue by investigating some simple semigroups (in the most literal sense), for which we need to recall some basic definitions.

A proper ideal M of a semigroup S is called **maximal** if there is no ideal in S (S-ideals for short) strictly between M and S. Analogously, we can define a minimal ideal of a semigroup. An ideal M is called **minimal** if and only if M does not contain any proper S-ideal and called a **0-minimal** if and only if $M \neq \{0\}$ and M does not properly contain any ideal of S except $\{0\}$.

A semigroup is called **simple** if and only if the only ideal of S is S itself and called **0-simple** if and only if $S^2 \neq 0$ and S contains no other ideal than $\{0\}$ and itself. As we shall see the purpose of the condition $S^2 \neq 0$ is only to omit the null semigroup on two elements. Moreover, this condition also implies $S^2 = S$, since S^2 is an ideal of S. The following results can be found in [CP61], chapter 2.5.

Lemma 2.1.5. *Let $S \neq 0$ be a semigroup with zero having $\{0\}$ and S as its only ideals. Then S is either 0-simple or the null semigroup on 2 elements.*

Proof. Either $S^2 = S$ or $S^2 = 0$. In the former case, S is 0-simple. In the latter case S is a null semigroup. Since $S \neq 0$ there must be a nonzero element $a \in S$, and $\{0, a\}$ is an ideal, implying that there cannot be more elements in S. □

Next, we derive a practical condition to check 0-simplicity.

Lemma 2.1.6. *Let $S \neq 0$ be a semigroup with zero. Then S is 0-simple if and only if $SaS = S$ for every nonzero $a \in S$.*

Proof. Assume that S is 0-simple. For every element $a \in S$ either $SaS = 0$ or $SaS = S$. Let $B \subseteq S$ be the set of all elements for which $SbS = 0$. Clearly B is an ideal of S, hence either $B = 0$ or $B = S$. From $B = S$ it would follow that $S^3 = 0$ which contradicts $S^2 = S, S \neq 0$. Therefore $B = 0$ and one implication is proved. For the other direction, choose a non-zero ideal A of S, and pick $0 \neq a \in A$. Then from the assumption $S = SaS \subseteq A \subseteq S$, proving the other direction. □

Theorem 2.1.7. *Let M be a 0-minimal ideal of a semigroup S with zero. Then either $M^2 = 0$ or M is a 0-simple subsemigroup of S.*

Proof. If $M^2 \neq 0$ then, as mentioned earlier, $M^2 = M$. Choose an $0 \neq a \in M$. Then S^1aS^1 is a nonzero ideal contained in M implying that $S^1aS^1 = M$. Therefore, $M = M^3 = MS^1aS^1M \subseteq MaM \subseteq M$, hence $MaM = M$ and M is 0-simple by Lemma 2.1.6. □

An important subclass of (0-)simple semigroups is formed by completely (0-) simple semigroups.

Definition 2.1.8. *A (0-)simple semigroup S is called **completely (0-)simple** if and only if it contains a primitive idempotent, that is, a minimal nonzero idempotent with respect to the natural partial order on idempotents.*

The structure of such semigroups is well understood. The following results will be used later in the text, but since their proofs do not align with the main stream of the thesis, we shall omit them. They can be found in [CP61] as Lemma 3.1, Theorem 3.3, Theorem 3.5 and Theorem 3.9.

Theorem 2.1.9. *The Rees matrix semigroup $\mathcal{MS}(G, I, M, P)$ is regular if and only if each row and each column of P contains a non-zero entry. Furthermore, a Rees-matrix semigroup is 0-simple if and only if it is regular, and in that case it is completely 0-simple.*

Theorem 2.1.10 (Rees)**.** *A semigroup is completely 0-simple if and only if it is isomorphic to a regular semigroup of matrix type over a group with zero.*

Theorem 2.1.11. *Let S be a semigroup with zero. Then the following conditions are equivalent:*
1) S is a completely 0-simple inverse semigroup
2) S is isomorphic with a (regular) semigroup of matrix type $\mathcal{MS}(G, I, I, E)$ over a group with zero G^0 and E the $I \times I$ identity matrix as sandwich matrix.

Congruences, factor semigroups

Let S be a semigroup and let $\equiv$ be an equivalence relation on S. $\equiv$ is called a **congruence** if and only if $a \equiv b$ implies $uav \equiv ubv$ for all $u, v \in S^1$. Since every equivalence relation on S is a subset of $S \times S$, there is a natural partial order between equivalence relations, and consequently between congruences. Let us denote the set of all congruences on S by $Cong(S)$. Then $Cong(S)$ is closed under taking intersections and unions over chains.

A partially ordered set $(L, \leq)$ is a **complete lattice** if and only if every subset A of L admits a greatest lower bound (infimum or meet) and least upper bound (supremum or join). Defining $\inf\mathcal{A} := \bigcap_{A \in \mathcal{A}} A$ and $\sup\mathcal{A} := \bigcap_{\bigcup \mathcal{A} \subseteq B \in Cong(S)} B$ for $\mathcal{A} \subseteq Cong(S)$, $Cong(S)$ becomes a complete lattice with the natural partial order.

The importance of congruences lies in the property that by taking the quotient or factor modulo a congruence $\equiv$, i.e. the set of equivalence classes, we get a structure of the same type, i.e. the equivalence classes of a semigroup congruence form a semigroup with the product:

$$[s_1]_{\equiv}[s_2]_{\equiv} := [s_1 s_2]_{\equiv}$$

Furthermore, the surjection $\pi : S \longrightarrow S/{\equiv}$ becomes a semigroup homomorphism. Next, we introduce two important cases of semigroup congruences. The proof that these examples do indeed give congruences is straightforward.

First, let S and T be semigroups, and let $\varphi : S \longrightarrow T$ be a homomorphism between them. Then the equivalence relation $a \equiv b \Leftrightarrow \varphi(a) = \varphi(b)$ defines a congruence on S. We will denote this congruence by **ker** φ, i.e. a kerφ $b \Leftrightarrow \varphi(a) = \varphi(b)$.

Next, let S be a semigroup and I an ideal in it. The equivalence relation $x \,\mathcal{I}\, y \Leftrightarrow x = y$ or $x, y \in I$ defines a congruence on S. The quotient by this congruence is called the **Rees quotient** on S by I. This quotient can also be identified with the following semigroups:

If $I = \emptyset$ then $S/I \cong S$. If $S = I \neq \emptyset$ then $S/I = \{I\}$, the semigroup on one element. Finally, in case $\emptyset \neq I \neq S$ let us define a semigroup structure on $(S\backslash I) \cup \{\overline{0}\}$, with $\overline{0}$ adjoined externally by:

$$s * t = \begin{cases} st & s, t, st \notin I \\ 0 & otherwise \end{cases}$$

Then, since I is an ideal, the following map is a surjective semigroup homomorphism:

$$\sigma : S \longrightarrow ((S\backslash I) \cup \{\overline{0}\}, *) \qquad\qquad s \longmapsto \begin{cases} s & \text{if } s \notin I \\ \overline{0} & \text{if } s \in I \end{cases}$$

for which the congruence ker σ coincides with $\mathcal{I}$, thus $S/\mathcal{I} \cong S/\text{ker } \sigma \cong$ $((S\backslash I) \cup \{\overline{0}\}, *)$.

The above constructions allow us to state theorems analogous to the isomorphism theorems for groups. The results are from [CP61] chapter 2.6. As an illustration, we prove the first theorem; the second theorem can be found as Theorem 2.37.

Theorem 2.1.12. *Let J be an ideal and T be a subsemigroup of a semigroup S. Then $J \cap T$ is an ideal of T, $J \cup T$ is a subsemigroup of S and $(J \cup T)/J \cong T/(J \cap T)$.*

Proof. $(J \cup T)^2 = (J^2 \cup JT \cup TJ) \cup T^2 \subseteq J \cup T$, thus $J \cup T$ is a subsemigroup of S. It is clear that $J \cap T$ is an ideal in T, and J is an ideal in $T \cup J$. Therefore, the quotients $(J \cup T)/J$ and $T/(J \cap T)$ are well defined. Let us denote their zero elements as 0 and $0'$, respectively. Then

$$(J \cup T)/J \cong ([(J \cup T)\backslash J] \cup \{0\}, *) = (T\backslash J \cup \{0\}, *)$$
$$T/(J \cap T) \cong ([T\backslash (J \cap T)] \cup \{0'\}, *) = (T\backslash J \cup \{0'\}, *)$$

which are clearly isomorphic. □

Theorem 2.1.13. *Let J be an ideal of a semigroup S and let $\pi : S \longrightarrow S/J$ be the natural homomorphism. Then π induces an inclusion-preserving bijection between the set of ideals of S containing J and the set of ideals of S/J. Furthermore, if A is an ideal of S containing J, then $(S/J)/(A/J) \cong S/A$.*

Corollary 2.1.14. *Let J and J' be ideals of a semigroup S with $J \subset J'$. Then J is a maximal ideal in J' if and only if J'/J is a minimal ideal of S/J. Moreover, from 2.1.7 J'/J is either a 0-simple semigroup or a null semigroup.*

Corollary 2.1.15. *Let J be an ideal of a semigroup S. J is a maximal ideal of S if and only if S/J has no proper nonzero ideals, according to Lemma 2.1.5, if and only if S/J is either 0-simple or null on two elements.*

Principal factors

The results of this section can be found in [CP61] Chapter 2.6. Let S be a semigroup and $s \in S$ an element of it. The ideal S^1sS^1 is called the **principal ideal** generated by s and is denoted by J_s; the subset of J_s consisting of the elements that do not generate J_s (as an ideal) is denoted by I_s. We note that any (0-)minimal ideal is a principal ideal by the following argument. Let J be a (0-)minimal ideal of S, and choose $a \in J$. S^1aS^1 is an ideal in J, therefore, it is (either) J (or $\{0\}$.) If there is an a for which $S^1aS^1 = J$, then we are done; otherwise $J = S^1JS^1 = 0$, which is a contradiction.

Proposition 2.1.16. *In case J_s is a minimal ideal (with respect to inclusion), then $I_s = \emptyset$; in any other case, I_s is an ideal of S.*

Proof. If $I_s \neq \emptyset$ then for any $a \in S$ and $b \in I_s$, $S^1(ab)S^1 = S^1a(bS^1) \subseteq S^1aS^1 \neq J_s$, but since J_s is an ideal, $ab \in J_s$, hence $S^1(ab)S^1 \subseteq J_s$, which means that ab belongs to I_s. If J_s is a minimal ideal, it cannot contain any proper subideals, but in the case $I_s \neq \emptyset$ we just showed that there exists a non-trivial subideal. □

Remark 2.1.17. *If $0 \neq S$ contains a zero element, then the condition $I_s \neq \emptyset$ is automatically fulfilled. It is clear that for any 0-minimal principal ideal $I_s = 0$, the converse is true by Lemma 2.1.6.*

The factor $S_s{:=}J_s/I_s$ is called the **principal factor** at s. We make the convention that the quotient with the empty set means just the semigroup itself. The set $\mathscr{J}_s := J_s \backslash I_s$ is called the **$\mathscr{J}$-class** of s. It is the same as the equivalence classes of the congruence $x \ \mathscr{J} \ y \Leftrightarrow S^1xS^1 = S^1yS^1$. The congruences $\mathcal{R}$ and $\mathcal{L}$ can be defined similarly, with elements being equivalent if they generate the same right and left principal ideals, respectively. The above congruences are known as **Green's relations** (with two additional ones, not mentioned explicitly) and they play a fundamental role in semigroup theory. We proceed to prove some basic results on principal factors.

Theorem 2.1.18. *Let S be a semigroup. Any principal factor of S is either 0-simple, simple or null.*

Proof. Let $a \in S$. The first claim is that I_a is a maximal ideal (of S) in J_a. Suppose that B is an ideal of S such that $I_a \subset B \subseteq J_a$, and choose an element $b \in B \backslash I_a$. Then $b \in J_a \backslash I_a = \mathscr{J}_a$ meaning that $J_b = J_a$. Since $J_b \subseteq B$, we get $B = J_a$. If $I_a = \emptyset$, then by the previous argument, J_a is a minimal ideal of S, hence a simple semigroup, and the quotient, by convention, is just itself. If $I_a \neq \emptyset$ then by Corollary 2.1.14 J_a/I_a must be 0-simple or a null semigroup. □

Definition 2.1.19. *Let S be a semigroup. S is called a* ***(completely) semisimple semigroup*** *if and only if every principal factor of S is (completely)* 0*-simple or (completely) simple.*

Definition 2.1.20. *Let S be a semigroup. A* ***principal series*** *of S is a chain of ideals of S (and the empty set)*

$$\emptyset = S_0 \subset S_1 \subset \cdots S_{r-1} \subset S_r = S$$

such that no ideal of S is strictly between consecutive terms. If we know a priori that S contains a zero element, then we will omit the empty set and replace it with the zero ideal.

It is clear that any finite semigroup admits a principal series. Moreover, the factors of such a series are precisely the $\mathscr{J}$-classes of S in some order.

Proposition 2.1.21. *Let S be a semigroup that admits a principal series. Then the factor semigroups S_k/S_{k-1} are isomorphic in some order to the principal factors of S. Furthermore, each difference $S_k \backslash S_{k-1}$ for $1 \leq k \leq r$ is a $\mathscr{J}$-class and every $\mathscr{J}$-class arises for exactly one choice of k.*

Proof. Choose a factor S_k/S_{k-1}, and let $a \in S_k \backslash S_{k-1}$. Then $S_{k-1} \cup J_a$ is an ideal of S between S_k and S_{k-1} strictly containing S_{k-1}. Hence, by the definition of the series $S_{k-1} \cup J_a = S_k$. For any $b \in I_a$ it must be the case that $b \in S_{k-1}$, otherwise we would have $S_{k-1} \cup J_b = S_k$ implying $a \in J_b$ which contradicts the choice of b. Equivalently, we have $I_a \subseteq S_{k-1}$. Furthermore, for any $c \in J_a \cap S_{k-1}$, since S_{k-1} is an ideal, $J_c \subseteq S_{k-1}$, which implies that $I_a = J_a \cap S_{k-1}$. From this and 2.1.12, the following isomorphisms follow:

$$J_a/I_a \cong J_a/(J_a \cap S_{k-1}) \cong (J_a \cup S_{k-1})/S_{k-1} \cong S_k/S_{k-1}$$

In addition, the following sets are equal:

$$\mathscr{J}_a = J_a \backslash I_a = (J_a \cup S_{k-1}) \backslash (I_a \cup S_{k-1}) = S_k \backslash S_{k-1}$$

Hence, for any $a' \in S_k \backslash S_{k-1}$, $J_a = J_{a'}$, implying that the principal factor J_a/I_a is independent of the choice of a in $S_k \backslash S_{k-1}$. On the other hand, for any $a \in S$ there exists a k $(1 \leq k \leq r)$ such that $a \in S_k$ but $a \notin S_{k-1}$, which implies that the correspondence $S_k/S_{k-1} \longrightarrow J_a/I_a$ is a bijection between the set of the factors of the principal series and the set of principal factors. (We have the analogous bijection between the $\mathscr{J}$-classes and the differences). $\square$

Corollary 2.1.22. *Any two principal series of S have isomorphic factors in some order.*

2.2 Representation theory

Most of the time, we assume that our algebras contain an identity element. Although this assumption could seem too strong in the case of semigroup algebras, as it turns out, adjoining an identity to the semigroup, thus making its algebra unital, does not change the property we will be interested in.

Semigroup algebras

The results of this section can be found in [Okn90] chapter 4. For any semigroup S and field K, we can construct the **semigroup algebra** KS (or $K[S]$) in the following way. Consider the $|S|$ dimensional free vector space over K with the elements of S forming a basis. The multiplication of KS is given by the multiplication of S for the elements of S, and since these elements form a basis, we can extend the multiplication linearly to linear combinations. One can easily check that this gives an associative algebra over K. Additionally, we will identify the elements $\mathbb{1}_K s$ with their counterparts in the semigroup.

There is an important modification of the above construction. If S contains a zero element, then we want to identify it with the zero of the vector space. It can be achieved by factoring out with the two-sided ideal $K0_S$. This new algebra K_0S is called the **contracted semigroup algebra** of S. Explicitly:

$$K_0S := KS/K0_S.$$

We can identify K_0S with the algebra having $S\backslash\{0\}$ as a basis and the multiplication of basis elements given by

$$s * t = \begin{cases} st & st \neq 0_S \\ 0 & st = 0_S \end{cases}$$

In view of this, it is clear that $K_0[S^0] \cong K[S^0]/K0_S \cong KS$.

We proceed to prove some useful isomorphisms for semigroup algebras.

Proposition 2.2.1. *Let S be a semigroup and let I be an ideal of S. Then the following isomorphism holds*

$$KS/KI \cong K_0(S/I)$$

Proof. Denote the factor homomorphism between S and S/I by σ. Using this, we can define the following algebra homomorphism:

$$\pi : KS \longrightarrow K_0(S/I)$$
$$\Sigma\lambda_i x_i \longmapsto \Sigma\lambda_i\sigma(x_i)$$

It is easy to see that it is indeed an algebra homomorphism. The kernel is precisely the ideal KI, hence we have the desired isomorphism. □

In order to clarify the relation between KS and K_0S, we recall the following:

Proposition 2.2.2. *If R is a ring with unity, then for any central idempotent e, the ring can be decomposed as*

$$R \cong eR \oplus (1-e)R$$

Proof. If e commutes with every element, then so does $(1-e)$, implying that the two components are two-sided ideals. Clearly, the summands generate the whole ring, and the fact that their intersection is zero can be seen from

$$er_1 = (1-e)r_2 \Rightarrow er_1 = e(er_1) = e(1-e)r_2 = 0.$$

□

If J is an ideal of an algebra R and J contains an identity e, then for any $x \in R$ we have: $ex = e(ex) = e(xe) = (xe)e = xe$ since $xe, ex \in J$. Hence e is a central idempotent and $R \cong eR \oplus (1-e)R$.

Corollary 2.2.3. *Let S be a semigroup and let I be an ideal of S such that KI is an algebra with identity e. Then*

$$KS \cong KI \times K_0[S/I].$$

Proof. Let e denote the identity of KI. Since I is an ideal of S, KI is also an ideal of KS and $KI = eKS$. From the previous discussion, e must be a central idempotent, hence $KS = KI \oplus (1-e)KS$. With the homomorphism

$$\varphi : KS \longrightarrow (1-e)KS, \quad \varphi(x) = (1-e)x$$

we get that $KS/KI \cong (1-e)KS$. Finally, from 2.2.1 $KS/KI \cong K_0[S/I]$. □

Corollary 2.2.4. *If S has a zero element 0_S, then $K0_S$ is an ideal of KS with identity, and it follows that $KS = K0_S \oplus K(1-0_S) \cong K \times K_0S$.*

Proposition 2.2.5. *Let $K \subseteq L$ be a field extension and S and T be semigroups. The following isomorphisms hold:*

$$LS \cong L \otimes_K KS \quad \textit{and} \quad KS \otimes_K KT \cong K[S \times T]$$

Furthermore if S and T have zero elements $0_S, 0_T$ then

$$L_0S \cong L \otimes_K K_0S \quad \textit{and} \quad K_0S \otimes_K K_0T \cong K_0[(S \times T)/I],$$

where $I = \{(s,t) : s = 0_S \text{ or } t = 0_T\}$.

The statement follows from the universal property of the tensor product, using (in the finite case, a dimension argument), and the structure of the semigroup algebra.

Representations and modules

Definition 2.2.6. *Let S be a semigroup, let K be a field, and let V be a K-vector space. A semigroup homomorphism $\varphi : S \longrightarrow End_K(V)$ is called a* ***representation*** *of S over K. We say that $dim_K V$ is the degree of the representation.*

Two representations $\varphi : S \longrightarrow End_K(V)$ and $\psi : S \longrightarrow End_K(W)$ are called equivalent if there exists a K-vector space isomorphism $T : V \longrightarrow W$ such that $T^{-1}\psi(s)T = \varphi(s)$ for every $s \in S$. By fixing a basis, one can also think of representations as maps to $M_n(K)$; then equivalent representations correspond to the same representations written in different bases. There is an equivalent way to view representations and their equivalences using modules. We assume familiarity with the usual definition of a module. We note that a module (over a ring R) can also be viewed as an abelian group M, with an R-action given on it by a ring homomorphism $\lambda : R \longrightarrow End_+(M)$. If $1 \in R$ then we assume $\lambda(1) = id_M$. In some cases, when we do not have an identity in R, we shall make use of certain idempotents which will act as so-called local identities, i.e. identities on some subset of M. We will also write rm for $\lambda(r)m$. The kernel of λ is denoted as $Ann_R(M)$.

If $\mathbf{1} \in A$ is an algebra over K, then we clearly have the ring homomorphism $K \overset{k\mathbf{1}}{\hookrightarrow} A \overset{\lambda}{\to} End_+(M)$ from K to $End_+(M)$, hence M also becomes a K-vector space. Furthermore, with this embedding, K is central in A which implies that $a(km) = (ak)m = (ka)m = k(am)\ \forall \in M, k \in K, a \in A$, therefore we can assume λ to map into $End_K(M)$.

The following proposition shows the connection between representations, modules and equivalences of representations. Its proof is straightforward.

Proposition 2.2.7. *Let K be a field, let $\varphi : S \longrightarrow End_K(V)$ be a representation of a semigroup S, and let $\psi : KS \longrightarrow End_+(V)$ define a module over KS. Then:*
1) φ uniquely extends to an algebra homomorphism $\Phi : KS \longrightarrow End_K(V)$.
2) ψ restricted to S gives a representation of S.

3) Two representation of S are equivalent if and only if the corresponding KS-modules are isomorphic.

Example 2.2.8. *Let K be a field, and take the semigroup $S = (\mathbb{N}, +)$. Then KS is isomorphic to the polynomial algebra $K[x]$. The elements $0, 1 \in \mathbb{N}$ generate the semigroup. To define a representation, it is enough to prescribe the image of $1 \in \mathbb{N}$ in $End_K(V)$, since 0 is the identity element of $\mathbb{N}$, hence its image is equal to the identity map. Then, from the previous proposition, we get the well-known example that the modules over $K[x]$ are in bijective correspondence with vector spaces with a linear transformation given on them. These modules will be denoted by V_α, where V is a vector space and $\alpha \in End_K(V)$.*

Next, we prove some general results on modules. The results of this section are from [EH18].

Lemma 2.2.9. *Let I be an ideal of an algebra A. Then the modules over A/I are in bijective correspondence with those A modules M for which $I \subseteq Ann_A(M)$. Moreover, if M and N are such modules, then a map $\varphi : M \longrightarrow N$ is an A-module homomorphism if and only if it is an A/I-module homomorphism.*

Proof. Let M be an A-module given by $\lambda : A \longrightarrow End_+(M)$ such that $I \subseteq ker(\lambda)$. Take two elements a and b of an equivalence class of the factor with I. We need to show that $\lambda(a) = \lambda(b)$. Since the elements are from the same equivalence class we know that there exists $i \in I$ such that $a = b + i$, but then since λ is (A-)linear and its kernel contains I we have $\lambda(a) = \lambda(b + i) = \lambda(b) + 0$.

Now let N be an A/I-module given by $\mu : A/I \longrightarrow End_+(N)$. This map composed with the natural factor map $\pi : A \longrightarrow A/I$ gives $\lambda := \mu \circ \pi : A \longrightarrow End_+(N)$ an A-module structure on N, and since $\pi(I) = 0$ we also have $\lambda(I) = 0$, which completes the first part of the proof. The proof of the second statement is straightforward. □

One way to study modules is to decompose them into the direct sum of simpler modules. In this case, the fundamental building blocks will be the indecomposable modules. The 'ultimate' goal is to decompose a module into the direct sum of indecomposable modules.

Definition 2.2.10. *Let R be a ring and let M be a non-zero R-module. M is called **indecomposable** if and only if it cannot be written as $M = M_1 \oplus M_2$ for non-zero submodules M_1 and M_2 of M.*

We note that in the case of modules over an algebra, finding a direct sum decomposition is equivalent to finding a direct sum decomposition as a vector space, in which the summands are invariant under the action of the algebra. Similarly, we can define the indecomposability of representations as a direct sum decomposition of the underlying vector spaces into invariant summands of the representation of the semigroup.

Direct sum decompositions are strongly tied to idempotents in $End_R(M)$. Namely, if $M = M_1 \oplus M_2$ then the projections $e_1 : M \longrightarrow M_1$ and $e_2 : M \longrightarrow M_2$ are non-zero idempotents in $End_R(M)$, such that $id_M = e_1 + e_2$ and $e_1 e_2 = e_2 e_1 = 0$. (Note that if the latter equalities are satisfied, then the idempotents are called orthogonal.) This works the other way around as well, if we have non-zero idempotents $e_1, e_2 \in End_M(R)$ such that $id_M = e_1 + e_2$ and $e_1 e_2 = e_2 e_1 = 0$, then by setting $M_1 := Im(e_1)$ and $M_2 := Im(e_2)$ we obtain a direct sum decomposition of M. Of course, the same logic carries over to decomposition with more than two constituents, in the case a decomposition to non-zero submodules $M = M_1 \oplus \ldots \oplus M_k$ is equivalent to the existence of a system of orthogonal non-zero idempotents such that $id_M = e_1 + \ldots + e_k$. We call such a system a complete set of orthogonal idempotents. This argument, with the remark that if e is idempotent then e and $(id_M - e)$ are orthogonal idempotents, gives an equivalent characterisation of indecomposability:

Lemma 2.2.11. *Let R be a ring and let M be a non-zero R-module. M is indecomposable if and only if the only idempotents in $End_R(M)$ are 0 and id_M.*

Next, we show an application of this lemma, which will be useful later on.

Lemma 2.2.12. *Let $A := K[x]/(f)$, where $f \in K[x]$ is a non-constant polynomial, and let V_α be a finite-dimensional cyclic A-module (generated by one element) such that the minimal polynomial of V_α is equal to g^t for some irreducible $g \in K[x]$. Then V_α is an indecomposable A-module.*

Proof. The map $T : V_\alpha \longrightarrow V_\alpha$, $v \mapsto \alpha(v)$ is an A-module homomorphism (K-linear and commutes with α, the action of x) that also has minimal polynomial g^t. Let $\phi : V_\alpha \longrightarrow V_\alpha$ be an arbitrary A-module homomorphism. Let us denote the generator of the module as w. Since the module is finite-dimensional there exists an $m \in \mathbb{N}$ such that $w, T(w), \ldots, T^m(w)$ form a K-basis of V_α, hence the image of w can be expressed as $\phi(w) = \sum_i a_i T^i(w)$ for some $a_i \in K$. Moreover any $v \in V_\alpha$ can be uniquely written in the form $v = \sum_j c_j T^j(w)$ for $c_j \in K$, and since ϕ is an A-module homomorphism it commutes with T, thus with T^j as well; implying that the following equation holds:

$$\phi(v) = \phi(\textstyle\sum_j c_j T^j(w)) = \sum_j c_j T^j(\phi(w)) = \sum_j c_j T^j(\sum_i a_i T^i(w)) =$$
$$\textstyle\sum_j \sum_i c_j a_i T^{i+j}(w) = \sum_i a_i T^i(\sum_j c_j T^j(w)) = \sum_i a_i T^i(v)$$

Therefore $\phi = \sum_i a_i T^i =: h(T)$ is a polynomial in T. Now suppose that $\phi^2 = \phi$. Equivalently, $h(T)(id - h(T)) = 0$ in the endomorphism ring. Since the minimal polynomial of T is g^t, it must divide $h(h-1)$, moreover, $K[x]$ is a unique factorisation domain, g is irreducible and h and $h-1$ are coprime, implying that either $g^t|h$ or $g^t|h-1$. In the first case, h must be the zero map, and in the second case, h must be the identity of V_α; thus, by the previous lemma, V_α is indecomposable. $\square$

Corollary 2.2.13. *If α has Jordan matrix $J_n(\lambda)$ (with the 1-s on the subdiagonal) with respect to some basis, then V_α is indecomposable.*

Proof. The minimal polynomial of $J_n(\alpha)$ is $(x-\lambda)^n$, hence $A = K[x]/((x-\lambda)^n)$. Let us denote the basis from the proposition as $w_1, \dots, w_n$. Then we have $\alpha(w_i) = \lambda w_i + w_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha(w_n) = \lambda w_n$, implying that Aw_1 contains $w_1, \dots, w_n$, and therefore V_α is a cyclic A-module generated by w_1. Since $(x-\lambda)$ is irreducible, the indecomposability of V_α follows from the previous lemma. $\square$

It is well known from linear algebra that (over an algebraically closed field) every matrix is similar to a block matrix with Jordan blocks corresponding to eigenvalues. Another way to interpret this result is that every module over the polynomial algebra of the field can be written as the direct sum of modules whose defining linear transformations are given by Jordan blocks $J_{n_i}(\lambda_i)$ for some $n_i \in \mathbb{N}$ and $\lambda_i \in K$. We have seen that these modules are indecomposable, hence, we obtained a direct sum decomposition of every module into indecomposable ones. Such a decomposition can be acquired for any algebra, but perhaps with 'wilder' indecomposables. (But more on that in the next chapter...) Let us continue with two theorems formalising this idea.

Theorem 2.2.14. *Let K be a field, let A be a K-algebra and let M be a non-zero finite-dimensional A-module. Then M can be written as the direct sum of finitely many indecomposable A-modules.*

Proof. We use induction on $dim_K M$. If $dim_K M = 1$, then M is a simple A-module, hence indecomposable. Now suppose that the theorem holds up to dimension $k-1$ and let M be an A-module with $dim_K M = k$. If M is indecomposable, then we are done; if not, then there exist submodules U, V, with strictly smaller dimensions, such that $M = U \oplus V$, for which the induction hypothesis is true, and the proof is complete. $\square$

Moreover, this decomposition is unique in the following way. (We shall omit the proof of the theorem, but it can be found in [EH18] as Theorem 7.18.)

Theorem 2.2.15 (Krull-Schmidt)**.** *Let K be a field, let A be a K-algebra and let M be a non-zero finite-dimensional A-module. Suppose that M has two direct sum decompositions to indecomposable A-submodules*

$$M = U_1 \oplus \ldots \oplus U_k \text{ and } M = V_1 \oplus \ldots \oplus V_l$$

then $k = l$ *and there exists a permutation* σ *such that* $U_i \cong V_{\sigma(i)}$ *for all* $i = 1, \ldots, r$.

Representation type

We have reached the main topic of the thesis. The results of this section are from [EH18], except tame and wild algebras, and the last four theorems. In the previous section, we have seen that modules over an algebra can be built up, essentially uniquely, from indecomposables. Next, the natural question to consider is if there is some way to systematically classify these indecomposables. We distinguish three different types of rings based on the number and structure of isomorphism classes of indecomposable modules. The simplest one, the type we will discuss in depth for group and semigroup algebras, is the following.

Definition 2.2.16. *Let* R *be a ring.* R *is of* ***finite representation type*** *if and only if there is a finite number of isomorphism classes of indecomposable* R*-modules.*

Example 2.2.17. *Let* R *be a semisimple ring. Then* R *has finite representation type.*

Proof. The proof uses well-known results on semisimple rings, which can be found, for example, in [EH18]. Take the direct sum decomposition of ${}_RR$ into (finitely many) simple modules: ${}_RR \cong S_1 \oplus \ldots \oplus S_k$. Every module is semisimple over R, hence it is enough to show that every simple module is isomorphic to one of the S_i-s. Any simple module is a quotient of ${}_RR$, hence it is also a direct summand of ${}_RR$. Then the Krull-Schmidt Theorem (which stays true for modules of finite composition length) implies that any simple module is isomorphic to one of the S_i-s. □

Otherwise, we say that the ring has **infinite representation type**. The class of infinite representation type rings can be further divided into tame and wild rings. Since their precise definition is technical, we shall only include the general ideas behind them (in the case of algebras). The definition of tame algebras is inspired by how the indecomposable modules over the polynomial algebra correspond to Jordan blocks, which can be parameterised by size and eigenvalue. From the module categories of tame algebras, we require a similar property, that indecomposable modules in every dimension can be 'parameterised' by a finite number of 'one-parameter' families (possibly missing finitely many indecomposable modules). Wild algebras are defined by the property that their module categories contain a copy of the module category of the free algebra on two generators $K\langle x, y\rangle$. In the latter, the module category of any algebra can be embedded, which roughly means that if we could understand the module category of a wild algebra, we could understand the module category of any algebra. (Which seems to be a hopeless task.) It is a famous dichotomy theorem due to Drozd that every finite-dimensional algebra, over an algebraically closed field, is either representation tame or representation wild. A reference to the theorem and the topic can be found in [SS07]. Let us continue with an example of a 'small' algebra with infinite representation type.

Lemma 2.2.18. *Let K be a field and let*

$$A := K[x, y]/(x^2, y^2, xy)$$

be the 3-dimensional commutative K-algebra. Then A has infinite representation type.

Proof. Any A-module V is determined by two K-linear maps $\alpha_X : V \longrightarrow V$ and $\alpha_Y : V \longrightarrow V$ satisfying the equations:

$$\alpha_X^2 = 0, \quad \alpha_Y^2 = 0, \quad \alpha_X \alpha_Y = \alpha_Y \alpha_X = 0$$

Define the $2n$-dimensional A-modules V_n specified by block matrices

$$\alpha_X := \begin{bmatrix} 0 & 0 \\ I_n & 0 \end{bmatrix}, \quad \alpha_Y := \begin{bmatrix} 0 & 0 \\ J_n & 0 \end{bmatrix}$$

where I_n and J_n stand for the n-dimensional identity and n-dimensional Jordan block $J_n(0)$, respectively. (It is easy to verify that these matrices satisfy the above equations, thus define A-modules.) It is only left to show that for every $n \in \mathbb{N}_+$ these modules are indecomposable since their dimensions differ, hence they are non-equivalent. By Lemma 2.2.11, it suffices to show that the only idempotents in $End_A(V_n)$ are the zero and identity maps. Let $\varphi \in End_A(V_n)$, in particular it is also a K-linear map, hence we can write in block matrix form

$$\varphi = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

with $n \times n$ block sizes (in the basis of the defining matrices of the module). φ is an A-module homomorphism iff it commutes with the matrices α_X and α_Y. From the former we get that $A_2 = 0$ and $A_1 = A_4$, and from the latter we know that $J_n A_1 = A_4 J_n = A_1 J_n$. In conclusion

$$\varphi = \begin{bmatrix} A_1 & 0 \\ A_3 & A_1 \end{bmatrix}, \text{ where } A_1 \text{ commutes with the Jordan matrix } J_n.$$

Suppose that $\varphi^2 = \varphi$, in particular $A_1^2 = A_1$. Let $A := K[x]/(f)$ be the algebra with $f(x) = (x-1)^n$ and let V_α be an A-module with the action of x given by J_n. Since A_1 commutes with the Jordan matrix, it defines an A-module endomorphism of V_α. Moreover, $A_1^2 = A_1$, hence by 2.2.11 and 2.2.13 A_1 is either the zero or the identity matrix. Either way, from $\varphi^2 = \varphi$ it follows that $A_3 = 0$, and hence φ is either zero or the identity. This means that V_n is indecomposable for every $n \in \mathbb{N}$. $\square$

Next, we collect some properties of representation finite algebras.

Proposition 2.2.19. *Let K be a field, ${}_KA$ an algebra and $I \neq A$ a two sided ideal of A. If the factor algebra A/I has infinite representation type, then A also has infinite representation type.*

Proof. Let M be an A-module such that $IM = 0$. By lemma 2.2.9 M can also be seen as an A/I-module with the action given by $(a + I)m = am$. From this it follows that for any such module the A and A/I-submodules are the same, and such modules are indecomposable over A and A/I at the same time. Moreover, from the second part of Lemma 2.2.9 it follows that such modules are isomorphic over A if and only if they are isomorphic over A/I. All together, this implies that the infinitely many non-isomorphic indecomposable A/I-modules give rise to infinitely many non-isomorphic indecomposable A-modules. $\square$

Corollary 2.2.20. *Consider the algebra $A := K[x, y]/(x^r, y^r)$, $(r \geq 2)$, and take I the ideal generated by the cosets of x^2, y^2 and xy. Then $A/I \cong K[x, y]/(x^2, y^2, xy)$, which has infinite representation type by 2.2.18, hence A also has infinite representation type.*

Proposition 2.2.21. *Let $A_1, \dots, A_n$ be K-algebras with identity elements e_i respectively. Then $A := A_1 \times \dots \times A_n$ has finite representation type if and only if A_i has finite representation type for every $1 \leq i \leq n$.*

Proof. Let M be a non-zero A-module. Then by setting $\varepsilon_i := (0, \dots, 0, e_i, 0, \dots, 0)$ $(1 \leq i \leq n)$, the $\varepsilon_i M$-s become submodules, and we have the decomposition
$M = \varepsilon_1 M \oplus \dots \oplus \varepsilon_n M$. Suppose that M is an indecomposable A-module. Then there exists a k, $(1 \leq k \leq n)$ such that $M = \varepsilon_k M$ and for every $i \neq k : \varepsilon_i M = 0$.

In this case $\varepsilon_k M$ is also an indecomposable A-module, furthermore let $I := (\varepsilon_1, \dots, \varepsilon_{k-1}, \varepsilon_{k+1}, \dots, \varepsilon_n)$ then $A_k \cong A/I$ and $I\varepsilon_k M = 0$, hence $e_k M$ can also be viewed as an A_k-module. Using the argument in the previous lemma, we know that $\varepsilon_K M$ is simultaneously indecomposable over A and A_k, hence it is indecomposable as an A_k module as well. Furthermore, as above, such modules are isomorphic over A and A_k at the same time. By the proposition, there are finitely many indecomposable A_k modules up to isomorphism for every k, and there are finitely many options for k; hence A has finite representation type.

Every A_i in the direct sum is a factor algebra of A by the canonical projection, as above. If any A_i had infinite representation type, then so would A; thus, if A has finite representation type, then every A_i also has finite representation type. $\square$

Proposition 2.2.22. *Let K be a field and let $1 \in A$ be a finite-dimensional K-algebra. If A has finite representation type, then A has finitely many ideals.*

Proof. Take an arbitrary ideal I of A. Take a decomposition of A/I over A to indecomposable A-modules M_i $(1 \leq i \leq k)$: ${}_A A/I = \bigoplus_{i=1}^k M_i$. Similar to the previous propositions, this decomposition is also valid over A/I: ${}_{A/I} A/I = \bigoplus_{i=1}^k M_i$. Since A/I has an identity element $Ann_{A/I}(A/I) = 0$ which implies that $Ann_A(A/I) = I$. Furthermore, the former annihilator can also be written as $I = Ann_A(A/I) = \bigcap_{i=1}^k Ann_A(M_i)$, hence every ideal can be written as the intersection of annihilators of indecomposable modules. Since there is only a finite number of such modules (up to isomorphism), there can only be a finite number of ideals. (Annihilators of isomorphic modules coincide.) □

Let us list some additional results about finite representation type. For example, it is a famous result that every finite-dimensional algebra of finite representation type admits a multiplicative basis, meaning that there exists a (K-)basis of the algebra such that the product of any two elements from the basis is either a basis element or 0. This theorem is due to R. Bautista, P. Gabriel, A. V. Roiter and L. Salmerón [Bau+85]. Another way to interpret this result is that every finite-dimensional algebra of finite representation type can be considered as a semigroup algebra. It is also worth noting that, in some special cases, it is not a hopeless task to classify algebras of finite representation type. Gabriel acquired a classification in the case of quiver algebras, which play a decisive role for algebras over an algebraically closed field. A proof can be found, for example, in [EH18].

3.0 Semigroup algebras of finite representation type

Although the representation type of semigroup algebras is in general less understood than their group algebra counterpart, in the case of a few important classes a definitive answer has been acquired. A recurring theme is the connection between the representation type of a semigroup algebra and the representation types of its maximal subgroups. In this chapter, equivalent (internal) conditions will be given in the case of commutative and inverse semigroups for their algebras to have finite representation type. For more results on other types of semigroups, the reader should consult [Pon93].

If the field used is clear from the context, we will sometimes refer to the representation type of the semigroup algebra as the representation type of the semigroup. Our first observation is that we only have to deal with finite semigroups. If the semigroup algebra KS has finite representation type, then Lemma 2.2.22 tells us that KS must have finitely many ideals. In particular, KS has to be Artinian. But then, according to Zelmanov's Theorem [Zel77], if a semigroup algebra KS is Artinian, then S must be finite.

Furthermore, as a corollary of 2.2.4 and 2.2.21 KS and K_0S are of finite representation type at the same time, so when it is more convenient, we can switch to the contracted semigroup algebra without changing the representation type.

Another assumption we can make without changing the representation type is that the semigroup contains a zero or an identity element. If S does not contain a zero element, then, as we have mentioned at the beginning of the section on semigroup algebras, $K_0[S^0] \cong KS$, and the contracted semigroup algebra has the same representation type as its original semigroup algebra. Similarly, in the case when S does not contain an identity element, then from every KS-module $\lambda : KS \longrightarrow End_K(V)$ we obtain a $K[S^1]$-module by defining the image of $1 \in S^1$ as id_V. By switching to matrix representations of S (which is essentially the same), it can easily be seen that such pairs of representations are equivalent and indecomposable at the same time since the identity map is invariant under conjugation (basis-change transformations) and is compatible with any block matrix decomposition. It turns out that essentially we have covered every $K[S^1]$-module. We have not dealt with the case when the image of the identity of S^1 is mapped to a non-identity idempotent $e \in End_K(V)$. In this case, we have the vector space decomposition $V = eV \oplus (id_v - e)V$. Clearly for any $s \in S^1$ $s(id_V - e)V = 0$, and if for some $s \in S; v, w \in V$ we have $sev = (id_V - e)w$, then $esv = sev = (id_V - e)w \in eV \cap (id_V - e)V = 0$. Where the second equality holds since e is the image of the identity of $K[S^1]$, hence it commutes with the image of any element of the algebra. Thus, eV and $(id_V - e)V$ are $K[S^1]$-invariant subspaces, hence the decomposition is also valid for V viewed as a $K[S^1]$-module. Moreover, $(id_V - e)V$ further decomposes into $K[S^1]$-modules on which the action of the algebra is zero (every element is mapped to zero). Since the action is zero, any decomposition as an Abelian group is valid as a module decomposition, with the restriction that $(id_V - e)V$ is also a vector space. Therefore, the isomorphism types of indecomposables in this decomposition come from Z_p or $\mathbb{Q}$ (as an abelian group), depending on the characteristic of K. Therefore, we obtained a direct sum decomposition into a module, for which $1 \in S$ acts as an identity, and a number of copies of either Z_p or $\mathbb{Q}$. Then it is clear that KS and $K[S^1]$ are of finite representation type at the same time.

Now we understand what happens when we interfere with the semigroup, but what about the base field? As it is mentioned in [Pon93], the semigroup algebra of a semigroup S over a field K is of finite representation type if and only if the semigroup algebra of S over a field L is of finite representation type, where $K \subseteq L$ is a field extension of finite degree or it is the algebraic closure of K. We close the introduction of the chapter with an argument that unifies some previous results.

Lemma 3.0.1. *Suppose I is an ideal in a semigroup S. If S/I, the factor semigroup, has infinitely many non-equivalent r,epresentations then so does S.*

Proof. The assumption of the proposition is equivalent to $K[S/I]$ having infinite representation type. As we have mentioned at the beginning of the chapter this is further equivalent to $K_0[S/I]$ having infinite representation type. According to 2.2.1 $K_0[S/I]$ is isomorphic to KS/KI, hence KS has a factor algebra with infinite represents,tion type. Finally from proposition **??** this implies the infinite representation type of S. □

3.1 Group algebras

In the case of group algebras, there are two main theorems characterising finite representation type. The problem can be separated into two cases: First, if we consider group algebras over a field whose characteristic does not divide the order of the group, the key is Maschke's theorem, which tells us that these algebras are semisimple; therefore, by Proposition 2.2.17 are of finite representation type. In the second case, we have to deal with so-called modular representations, meaning that p, the characteristic of the field, divides the order of the group. As we shall see, the answer will solely depend on the p-Sylow subgroup(s) of the group, by Higman's theorem. In the following we shall present the necessary tools to deal with the modular case, then at the end of the chapter we state the two branches in one theorem, giving a definitive answer for a group algebra to be of finite representation type. The results on group algebras are from [EH18] Chapter 8.2.

p-groups

To understand general modular representations we will first investigate the simplest case, when the order of the group is a prime power. As we will see, the group algebra will be of finite representation type only for the cyclic p-groups. Every other group will have a 'bad' factor with infinite representation type, therefore being of infinite representation type itself. Let us begin with a definition.

Definition 3.1.1. *Let G be a finite group and let p be a prime number. G is called a **p-group** if and only if $|G| = p^a$ for some $a \in \mathbb{N}$.*

An important example of a p-group are the p-Sylow subgroups of a general finite group.

Definition 3.1.2. *Let p be a prime and let H be a subgroup of a finite group G with $|G| = p^\alpha m$, where $p \nmid m$. H is called a **p-Sylow subgroup** of G if and only if $|H| = p^\alpha$.*

It is well known that such a subgroup exist and all such subgroups are isomorphic, therefore sometimes we will only refer to them as the p-Sylow subgroup of G without further distinction between them. Next, we prove some isomorphism, which will show the representation types of the modular group algebras of C_{p^a} and $C_p \times C_p$.

Proposition 3.1.3. *Let $p > 0$ be a prime, C_{p^a} be the cyclic group of order p^a, where $a \in \mathbb{N}_+$, and K be a field. Then for the group algebra we have the following isomorphism: $KG \cong K[x_1]/((x_1^{p^a} - 1))$. Furthermore if $charK = p$, then $KG \cong K[x]/(x^{p^a})$*

Proof. Let $g \in G$ be a generator of the group. Define the surjective evaluation algebra homomorphism:

$$\phi : K[x_1] \longrightarrow KG$$
$$\sum a_i x_1^i \longmapsto \sum a_i g^i$$

We have $(x_1^{p^a}-1) \subseteq ker\phi$ because g^{p^a} is the unit of G, and therefore of KG. Since $K[x_1]/ker\phi \cong KG$, we know that the left hand side is at most p^a dimensional and we get that $ker\phi = (x_1^{p^a}-1)$ and the proposed isomorphism. In the modular case, by the binomial theorem $x_1^{p^a}-1 = (x_1-1)^{p^a}$, and with substituting $x = x_1 - 1$ we get the proposed isomorphism. □

Proposition 3.1.4. *If K is a field then the algebra $K[x]/(x^n)$ has finite representation type.*

Proof. A module over the polynomial ring is equivalent to a vector space with the action of x given on it by a linear transformation. This is also true for any factor algebra $K[x]$ with the action satysfing the polynomial. Then the action of x is nilpotent implying that there exists a basis such that the matrix of x is the direct sum of Jordan blocks. For any indecomposable module there can be only one block in their matrix form, i.e. the matrix describing the action of x is similar to $J_n(0)$. □

Corollary 3.1.5. *The two propositions imply that in characteristic $p > 0$ the algebra of any cyclic p-group is of finite representation type.*

Proposition 3.1.6. *Let $p > 0$ be a prime, $C_p \times C_p$ be the direct product of two cyclic groups of order p. Then we have the following isomorphism:*

$$K[C_p \times C_p] \cong K[x_1, x_2]/(x_1^p - 1, x_2^p - 1)$$

Futhermore if $charK = p$, then: $K[C_p \times C_p] \cong K[x, y]/(x^p, y^p)$

Proof. Choose generators a and b in the two copies of C_p. By defining the following map on a basis of $K[x_1, x_2]$ and extending it linearly we get a surjective algebra homomorphism to $K[C_p \times C_p]$

$$x_1 \longmapsto (g_1, 1),\ x_2 \longmapsto (g_2, b).$$

Since in the cyclic group $g_i^p = 1$ the ideal $(x_1^p - 1, x_2^p - 1)$ is part of the kernel of the map. Furthermore $K[C_p \times C_p]$ is p^2 dimensional, and the elements $x_1^i x_2^j$ $(0 \leq i, j \leq p-1)$ form a basis in $K[x]/(x_1^p - 1, x_2^p - 1)$, hence from the homomorphism theorem the two algebras are isomorphic.
The second isomorphism follows from the binomial formula in characteristic p and by substituting $x_1 - 1$ for x and $x_2 - 1$ for y. This substitution yields a well defined isomorphism. □

Corollary 3.1.7. *By Corollary 2.2.20 $K[x, y]/(x^p, y^p)$ is of infinite representation type, and therefore $K[C_p \times C_p]$ as well.*

Lemma 3.1.8. *Let p be a prime and let G be a p-group. If G is not cyclic, then it has a factor group isomorphic to $C_p \times C_p$.*

Proof. If G is abelian, then by the fundamental theorem of finite abelian groups the desired quotient follows. Now we proceed with the case when G is non-abelian. It is well known that conjugacy classes partition the group, hence we have the conjugacy class equation

$$|G| = |Z(G)| + \sum_{\substack{G_x conj.c. \\ |G_x| \geq 2}} |G_x|$$

where $Z(G)$ is the center of the group (equivalently the set of elements with conjugacy classes of one element). It is also known that class sizes divide the order of the group; therefore, in the case of p-groups the size of any class with at least two elements must be divisible by p. Since G is not abelian $Z(G) \neq G$, hence there exists an element with conjugacy class of at least size two. Then from the equation above, by divisibility, it follows that $|Z(G)|$ must be divisible by p.

$Z(G)$ is always a normal subgroup (union of conjugacy classes and subgroup), and from the above the factor $G/Z(G)$ is a p-group with strictly less elements. Indirectly suppose that the factor is cyclic. This would mean that the factor is generated by the coset $xZ(G)$ for some $x \notin Z(G)$. Therefore every element would belong to a coset $x^r Z(G)$ for some r. Then for any $g_1, g_1 \in G : g_1 = x^{r_1} z_1$ and $g_2 = x^{r_2} z_2$ for some $r_1, r_2 \in \mathbb{N}$ and $z_1, z_2 \in Z(G)$. But this would imply that every element commutes, which is a contradiction.

From here we can proceed by induction on $n \geq 2$, the power of p. (For $n = 0, 1$ the groups are cyclic.) If $n = 2$ then G must be commutative, otherwise the center would have p elements, by Lagrange's theorem, hence the factor would be cyclic. In this case from the classification of finite abelian groups G must be $C_p \times C_p$ itself, therefore the claim follows with $N = \{1\}$. If $n > 2$ then the factor $G/Z(G)$ is a non-cyclic p-group with fewer elements, hence by the induction hypotesis and the bijection between appropriate normal subgroups, there exists a normal subgroup $Z(G) \subseteq N \lhd G$ such that $(G/Z(G)/(N/Z(G)) \cong C_p \times C_p$. From the second isomorphism theorem $(G/Z(G)/(N/Z(G)) \cong G/N$ and the lemma follows. □

Theorem 3.1.9. *Let p be a prime and G be a p-group. Furthermore, let K be a field with $char(K) = p$. Then KG is of finite representation type if and only if G is cyclic.*

Proof. We have already seen in corollary 3.1.5 that for cyclic p-groups KG has finite representation type. For the other direction we define the surjective algebra homomorphism $\phi : KG \longrightarrow K(G/N)$ defined on the basis formed by the elements of G as $\phi(g_1) = g_1 N$ and extending it linearly. It is clear that this map is surjective and since $(g_1 g_2)N = (g_1 N)(g_2 N)$ it is an algebra homomorphism. But $K(G/N) \cong K(C_p \times C_p)$ which has infinite representation type by corollary 3.1.7. Finally from 2.2.19 it follows that KG also has infinite representation type. □

Connection between representations of subgroups and the entire group

To determine the representation type of a modular group algebra, we require new tools to relate the modules over the algebra of a subgroup and the entire group algebra. Let us denote the group by G, a subgroup of G by H, and let K be a field. One direction is easy to see:

Let M be a module over KG. If we restrict the set of scalars to the subalgebra KH, we get a KH module. We are going to denote this KH-module as $Res_{KH}M$. (We will only write $ResM$ when it is clear which subgroup we are referring to.)

In the other direction, we need a bit more complicated construction. Take an arbitrary KH module W. Since both KG and W are vector spaces over K we can take their tensor products over K. This becomes a KG module by defining

$$x \cdot (g \otimes_K w) := (xg) \otimes_K w \quad \text{for } x, g \in G,\ w \in W \text{ and extending it linearly.}$$

Consider the K-subspace

$$\mathscr{H} = span_K\{gh \otimes w - g \otimes hw : g \in G, h \in H, w \in W\}$$

Moreover, it is a KG-submodule of $KG \otimes_K W$. The factor module

$$KG \otimes_H W := (KG \otimes_K W)/\mathscr{H}$$

is called the **KG-module induced from the KH-module** W. The action of KG on the induced module is the same as above, but additionally, by the choice of $\mathscr{H}$, we take into account the action of KH on the original module, namely,

$$gh \otimes_H w = g \otimes_H hw \qquad \text{for all } g \in G, h \in H, w \in W$$

where, for clarity, $g \otimes_H w$ denotes the coset $g \otimes_K w + \mathscr{H}$. It can be proven that the map $\iota : W \longrightarrow KG \otimes_H W$, $w \mapsto 1 \otimes_H w$ is an injective KH-module homomorphism, hence if we identify W with its image $\iota(W)$ the KG-action on the induced module extends the KH-action on W. An explicit basis can also be constructed for the induced module. Let T be a set of representatives of left H-cosets, that is, $G = \bigcup_{t \in T} tH$. without loss of generality we can assume that $1 \in T$. Let W be a finite-dimensional KH-module with K-basis $\{w_1, \dots w_n\}$. Then a K-basis of $KH \otimes_H W$ is given by

$$\{t \otimes_H w_i : t \in T, i = 1, \dots n\}$$

The proof of this fact can be found in the appendix of [EH18].

Lemma 3.1.10. *Let K be field, G a finite group and H a subgroup of G. Then we have the following three properties of the restriction and induction operations:*
1) If ${}_{KG}M$ and ${}_{KG}N$ are finite-dimensional KG modules then

$$Res(M \bigoplus N) = ResM \bigoplus ResN$$

2) If W is a finite-dimensional KH-module then W is isomorphic to a direct summand of $Res(KG \otimes_H W)$
3) If M is a finite-dimensional KG-module then via the map

$$\mu : KG \otimes_H ResM \longrightarrow M,\ g \otimes_H m \longmapsto gm$$

(extended linearly) it is a factor module of $KG \otimes_H ResM$.

Proof. 1) Straightforward.
2) As we have seen $KG \otimes_H W$ has a K-basis $\{t \otimes_H w_i : t \in T, i = 1, \dots n\}$ where $1 \in T$ is a system of representatives of left H-cosets and $\{w_1, \dots w_n\}$ is a K-basis of W. In addition let $W_t := span\{t \otimes_H w_i : i = 1, \dots n\}$. For any element $h \in H$ and $t \in T$ $ht \in G$ belongs to exactly one left H-coset, i.e. there exist unique elements $s \in G, \overline{h} \in H$ such that $ht = s\overline{h}$. The action of KG is defined with multiplication from the left on the first part of the tensors, so the action of the restriction to KH acts the same way, thus from the previous argument

$$h(t \otimes_H w_i) = ht \otimes_H w_i = s\overline{h} \otimes_H w_i = s \otimes_H \overline{h}w_i$$

From this it is clear that W_1 is a KH-submodule of $Res(KG \otimes_H W)$. Moreover for any $h \in H \backslash \{1\}$ and $t \in T \backslash \{1\}$ we must have $s \neq 1$ (the s from the previous argument), otherwise we would have $ht = 1\overline{h} = \overline{h}$, thus $t \in H$, a contradiction. This shows that $(\sum_{t \in T \backslash \{1\}} W_t)$ is also a submodule of $Res(KG \otimes_H W)$. Since these submodules are spanned by elements of a basis of $KG \otimes_H W$ this gives us the decomposition

$$Res(KG \otimes_H W) = W_1 \bigoplus (\textstyle\sum_{t \in T \backslash \{1\}} W_t)$$

By extending linearly the map defined on a basis of W

$$\varphi : W \longrightarrow W_1,\ w_i \longmapsto 1 \otimes_H w_i$$

we obtain an isomorphism of KH-modules.
3) Let $\{m_i : i = 1, \dots n\}$ be a K-basis of M, and by definition, $\{g_j : g_j \in G\}$ is a K-basis of KG. First we want to prove that the map

$$\mu : KG \otimes_K ResM \longrightarrow M,\ g_j \otimes_H m_i \longmapsto g_j m_i \text{ (extended linearly)}$$

is a KG-module homomorphism. Since the map is defined on a basis of the tensor product it extends uniquely to a K-linear map, furthermore

$$\mu(g(f \otimes_K m)) = \mu(gf \otimes_K m) = gfm = g\mu(f \otimes_K m)$$

giving us that μ is a KG-module homomorphism. It is clear that the map is surjective. In order to show that μ is also a homomorphism from the factor $KG \otimes_H ResM = (KG \otimes_K ResM)/\mathscr{H}$ we need to show that $\mu(\mathscr{H}) = 0$. Using the definition of $\mathscr{H}$ (above)

$$\mu(gh \otimes_K w - g \otimes_K hw) = ghw - ghw = 0.$$

proving the last point of the lemma. □

Next, a small lemma before an important theorem. The poof can be found in [EH18], Lemma 2.30.

Lemma 3.1.11. *Let K be a field, let A be a K-algebra, and let M, N, N' be non-zero A-modules. Suppose that $j : N \longrightarrow M$ and $\pi : M \longrightarrow N'$ are A-module homomorphism, such that $\pi \circ j : N \longrightarrow N'$ is an isomorphism. Then j is injective, π is surjective and $M = im(j) \oplus ker(\pi)$*

Theorem 3.1.12. *Let K be a field and let H be a subgroup of a finite group G. Then the following hold:*
1) If KG has finite representation type then KH also has finite representation type.
2) Suppose further that the index $|G : H|$ is invertible in K then the following holds:
i) Every finite-dimensional KG-module M is isomorphic to a a direct summand of the induced module $KG \otimes_H ResM$.
ii) If KH has finite representation type then KG has finite representation type too.

Remark 3.1.13. *The condition of invertibility cannot be ommited for statement ii). Take a field K with $char(K) = p > 0$ and consider C_p as a subgroup of $C_p \times C_p$. As it was proved above, in spite of the former having finite representation type over K the latter has infinite representation type over K.*

Proof. 1) Let $M_1, \ldots, M_r$ be representatives of the isomorphism classes of finite-dimensional indecomposable modules over KG. Their restrictions can be written as direct sums of a finite number of finite-dimensional indecomposable KH-modules; putting them together we obtain a list of finitely many finite-dimensional indecomposable KH-modules. We want to show that every finite-dimensional indecomposable KH-module is on this list up to isomorphism.

Let W be a finite-dimensional indecomposable KH-module, then $KG \otimes_H W$ is a KG-module, hence

$$KG \otimes_H W \cong \bigoplus_{i=1}^{t} M_i^W \quad \text{where each } M_i^W \text{ is from the set } \{M_1, \ldots, M_r\}.$$

From the first part of the previous lemma the restriction of direct sums is the direct sum of restrictions, hence

$$Res(KG \otimes_H W) \cong \bigoplus_{i=1}^{t} Res(M_i^W)$$

which further decomposes to the direct sum of indecomposable KH-modules from the above list. From the second part of the previous lemma W is a direct summand of $Res(KG \otimes_H W)$, and since W is indecomposable, the Krull-Schmidt Theorem 2.2.15 implies that there exists a KH-module from the list such that W is isomorphic to

it.
2) Let $1 \in T = \{g_1, g_2 \ldots, g_r\}$ be a system of representatives of left H-cosets in G, as in the previous lemma.
i) First consider the map

$$\sigma : M \longrightarrow KG \otimes_H Res(M),\ m \longmapsto \sum_{i=1}^{r} g_i \otimes_H g_i^{-1} m$$

From the K-bilinearity of the tensor product it follows that σ is K-linear. Next, we show that σ does not depend on the coset representatives chosen, which will be used deduce that σ is a KG-module homomorphism. Take T, defined above, then every system of representatives of H-cosets has the form $g_1h_1, g_2h_2, \ldots, g_rh_r$ for some $h_1, \ldots, h_r \in H$. With this new system the image of an $m \in M$ reads as

$$\sum_{i=1}^{r} g_ih_i \otimes_H h_i^{-1}g_i^{-1}m = \sum_{i=1}^{r} g_ih_ih_i^{-1} \otimes_H g_i^{-1}m = \sum_{i=1}^{r} g_i \otimes_H g_i^{-1}m$$

by using the definition of the induced module. For any $g \in G$

$$\sigma(gm) = \sum_{i=1}^{r} g_i \otimes_H g_i^{-1}gm = \sum_{i=1}^{r} g_i \otimes_H (g^{-1}g_i)^{-1}m = *$$

We obtain a new set of representatives by setting $\overline{g}_i := g^{-1}g_i$. Since it does not matter which representatives we choose the equation continues the following way

$$* = \sum_{i=1}^{r} g_i \otimes_H (g^{-1}g_i)^{-1}m = \sum_{i=1}^{r} g\overline{g}_i \otimes_H \overline{g}_i^{-1}m = g(\sum_{i=1}^{r} \overline{g}_i \otimes_H \overline{g}_i^{-1}m) = g\sigma(m)$$

concluding that σ is a KG-module homomorphism.

Take the map $\mu : KG \otimes_H Res(M) \longrightarrow M$ defined in the previous lemma. The composition $\mu \circ \sigma : M \longrightarrow M$ looks the following way for all $m \in M$:

$$(\mu \circ \sigma)(m) = \mu(\sum_{i=1}^{r} g_i \otimes_H g_i^{-1}m) = \sum_{i=1}^{r} g_ig_i^{-1}m = |G : H|m$$

Now by the invertibility of the index of H we can set $\kappa := \frac{1}{|G:H|}\sigma$, which gives $\mu \circ \kappa = id_M$, thus the previous Lemma implies that $KG \otimes_H Res(M) \cong im(\kappa) \bigoplus \ker(\mu)$. In addition κ must be injective, since $\mu \circ \kappa = id_M$, therefore $KG \otimes_H Res(M) \cong M \bigoplus \ker(\mu)$, and i) is proved.
ii) Let $W_1, \ldots W_r$ be representatives of the finitely many isomorphism classes of finite-dimensional indecomposable KH-modules. The KG-modules $KG \otimes_H W_s$ decompose into finitely many finite dimensional indecomposables over KG. Alltogether we obtain finitely many indecomposable KG-modules from this process. It is enough to show that every finite-dimensional indecomposable KG-module is isomorphic to a direct summand of $KG \otimes_H W_s$ for some $1 \leq s \leq r$, since then it will also be isomorphic to an indecomposable direct summand of which, up to isomorphism, there are only finitely many. Let M be a finite-dimensional indecomposable KG-modules. Then the restriction of M to a KH-module can be written as

$$Res(M) \cong W_1^{\oplus a_1} \oplus \ldots \oplus W_r^{\oplus a_r}$$

Distributivity of tensor products over direct sums stay true for $\otimes_H$ as well. Let V and W be two KH-modules. Then

$$\mathscr{H} = span_K\{gh\otimes(v+w) - g\otimes h(v+w)\} = span_K\{gh\otimes v + gh\otimes w - g\otimes hv - g\otimes hw\} = span_K\{(gh\otimes v - g\otimes hv)\} + span_K\{(gh\otimes w - g\otimes hw)\} =: \mathscr{H}_1 + \mathscr{H}_2$$

and the intersection of $\mathscr{H}_1$ and $\mathscr{H}_2$ is clearly zero, thus $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$. Hence we have

$$KG\otimes_H (V\oplus W) = [KG\otimes(V\oplus W)]/\mathscr{H} \cong [(KG\otimes V)\oplus(KG\otimes W)]/(\mathscr{H}_1\oplus\mathscr{H}_2) \cong (KG\otimes V)/\mathscr{H}_1 \oplus (KG\otimes W)/\mathscr{H}_2 \cong KG\otimes_H V \oplus KG\otimes_H W$$

Using this distributivity the following holds

$$KG\otimes_H Res(M) \cong (KG\otimes_H W_1)^{\oplus a_1} \oplus \ldots \oplus (KG\otimes_H W_r)^{\oplus a_r}$$

By part i) we know that M is isomorphic to a direct summand of $KG\otimes_H Res(M)$, hence by the Krull-Schmidt Theorem 2.2.15 it is isomorphic to some indecomposable direct summond of one of $KG\otimes_H W_i$ for some $1\leq i\leq r$. □

The only thing left before the main theorem of the chapter is to state Maschke's Theorem. Since it is a well-known theorem, we shall omit its proof.

Theorem 3.1.14 (Maschke). *Let K be a field and let G be a finite group. If $char(K)\nmid |G|$ (e.g. $char(K)=0$), then the group algebra KG is semisimple.*

Theorem 3.1.15 (Higman). *Let K be a field and let G be a finite group. The group algebra KG is of finite representation type if and only if either $char(K) = 0$ or $char(K) = p > 0$ and the p-Sylow subgroup of G is cyclic.*

Proof. First, suppose that KG is of finite representation type. In case $char(K) = 0$ there is nothing to prove. Suppose that $char(K) = p > 0$, and let H be a p-Sylow subgroup of G. Since KG has finite representation type, by theorem 3.1.12 KH must have finite representation type as well. Moreover, KH is a p-group, hence by 3.1.9, can only be cyclic.

For the other direction first suppose that $char(K) = 0$. Then Maschke's theorem implies that KG is semisimple, therefore, by 2.2.17, it is of finite representation type.

The only case left to prove is when $char(K) = p > 0$ and H, the p-Sylow subgroup of G, is cyclic. Then $|G : H|$ is invertible in K, and Remark 3.1.5 implies that KH is of finite representation type. Now, from the second part of Theorem 3.1.12 it follows that KG has finite representation type. □

3.2 Commutative semigroup algebras

The results on commutative semigroup algebras are due to Ponizovskii, the chapter is based on his paper [Pon70]. From now on S will denote a commutative finite semigroup. As we have mentioned before, without loss of generality, we can assume that S contains an identity and a zero element. First we will construct the so called normal series, an ideal series, of S, and as it will turn out, the representation type of the whole semigroup will only depend on conditions involving the factors of this series.

Normal series for commutative semigroups

We start by constructing an ideal series of S, called the **normal series**. We construct the series

$$0 = S_0 \subset S_1 \subset \ldots \subset S_n = S$$

from right to left recursively, in the following way:

The first ideal in our series is S, the whole semigroup. We define S_{n-1} as the union of the ideals of all the idempotents besides 1 in S. Now suppose that S_{k+1} has already been constructed and pick a maximal idempotent e_{k+1} with respect to the natural partial order in it. S_k will be the union of the ideals of all idempotents except e_{k+1} in S_{k+1}. Using the notation of 2.1 we can write the series in the following way:

$$S_n = S, \quad S_{n-1} = \bigcup_{e \in E(S) \setminus \{1\}} Se, \, \ldots, \, S_k = \bigcup_{e \in E(S_{k+1}) \setminus \{e_{k+1}\}} Se, \quad S_0 = 0$$

Next, we prove some important properties of the series.

Proposition 3.2.1. *For every k $(1 \leq k \leq n)$ the following hold:*
1) $S_k \backslash S_{k-1} \subseteq Se_k$
2) $G_{e_k} \subseteq S_k \backslash S_{k-1}$, in particular $e_k \in S_k \backslash S_{k-1}$
3) $\forall r \in S_k \backslash (S_{k-1} \cup G_{e_k})\ \exists n_r : r^{n_r} \in S_{k-1}$

Proof. 1) For $k = n$ it is trivial, because 1 is an identity for every element. For $k < n$ it is clear from the definition.
2) For any $x \in G_{e_k}$ we have $x \in Se_k$, therefore $x \in S_k$. If x is contained in Se for some idempotent e in S_k, then we know that $x = ye$ with $y \in S$. Multiplying this equation with the inverse we get $e_k = x_{e_k}^{-1} x = (x_{e_k}^{-1} y)e$, but then $e \in G_{e_k}$. In a group the only idempotent is the identity of the group implying $e = e_k$.
3) Take $\langle r \rangle$, the subsemigroup generated by r, by Corollary 2.1.1 it contains an idempotent. If it was e_k, then we would either have $r = e_k$ or $r \in G_{e_k}$, since

$r^{n_r-1}r = rr^{n_r-1} = e_k$. (Note that the converse is also true, no element in G_{e_k} can have a power in S_{k-1}, otherwise we would have $e_k \in S_{k-1}$.) $\square$

Corollary 3.2.2. *From the first statement we have that e_k is an identity for every element in the difference $S_k \backslash S_{k-1}$.*

As a consequence of the previous proposition, the difference of the ideals can be written as $S_k \backslash S_{k-1} = G_k \cup R_k$, where G_k is the maximal subgroup associated to e_k, and e_k acts as an identity for every element in the difference. Since S_{k-1} is an ideal in S we can take the Rees factor with it. Let σ_{k-1} denote the homomorphism from S onto the quotient. Then we also deduced the structure of the factor semigroup, namely $\overline{S}_k := \sigma_{k-1}(S_k) = \overline{G}_k \sqcup \overline{R}_k$, where $\overline{G}_k = \sigma_{k-1}(G_k)$ and $\overline{R}_k = \sigma_{k-1}(R_k)$. $\overline{G}_k \cong G_k$, and $\overline{R}_k$ is a nilpotent ideal of the factor (it is an ideal because the original exponent sending an element to zero also sends its product with an arbitrary element to zero). Since e_k acts as an identity for the whole difference, its image $\overline{e}_k$ does as well in the factor. These types of semigroups play the most significant role in the theory, so we give them a name.

Definition 3.2.3. *A commutative semigroup E is called **elementary** if and only if $E = G \sqcup N$, where G is a subgroup of E with unit e, which acts as an identity for the whole semigroup; $0 \in N$ is a nilpotent ideal of E, and the 0 of N is the zero element of the whole semigroup.*

Our final interest is in the set $L_k = R_k \backslash R_k^2$. The reason is that we will prove that the group G_k acts on it via (left) multiplication.

Proposition 3.2.4. *G_k acts on L_k.*

Proof. In $\overline{S}_k$, the image of L_k is $\overline{R}_k \backslash \overline{R}_k^2$. $\overline{R}_k$ being an ideal of $\overline{S}_k$ implies that $\overline{R}_k^2$ is an ideal as well. The unit $\overline{e}_k$ acts as an identity for $\overline{R}^2$, therefore the elements, that are invertible with respect to $\overline{e}_k$, act as bijections on the ideal. Hence we have the following implications:

$$\overline{g} \in \overline{G}_k, \overline{r} \in \overline{R}_k, \overline{gr} \in \overline{R}_k^2 \Rightarrow \overline{g}_{\overline{e}_k}^{(-1)} \overline{gr} \in \overline{g}_{\overline{e}_k}^{(-1)} \overline{R}^2 \Rightarrow \overline{e}_k \overline{r} = \overline{r} \in \overline{g}_{\overline{e}_k}^{(-1)} \overline{R}^2 = \overline{R}^2$$

Therefore,

$$\overline{g} \in \overline{G}_k, \overline{r} \in \overline{R}_k \backslash \overline{R}_k^2 \Rightarrow \overline{gr} \in \overline{R}_k \backslash \overline{R}_k^2.$$

From this it follows that

$$g \in G_k, l \in L_k \Rightarrow gl \in L_k.$$

$\square$

Now we turn to the representations of the above semigroups. It turns out that as far as finite representation type is concerned we only have to understand the representations of the elementary factor semigroups in the normal series. Let $\overline{\phi}_k$ be an indecomposable representation of $\overline{S}_k$ over some field K. From this we construct a representation of S with the following map:

$$\phi(s) := \overline{\phi}_k(\sigma_{k-1}(se_k)) \tag{3.1}$$

By the definition of S_k we know that $se_k \in S_k$, so the expression above makes sense, and it is easy to verify that it gives a homomorphism, and therefore a representation of S over K.

Proposition 3.2.5. *ϕ is an indecomposable representation of S over K.*

Proof. From 3.2.2 we know that for any $s \in S_k \backslash S_{k-1}$: $se_k = s$. Therefore $\phi(s) = \overline{\phi}_k(\sigma_{k-1}(s))$. Any non zero element of $\overline{S}_k$ has the form $\sigma_{k-1}(s)$ for some $s \in S_k \backslash S_{k-1}$, so we have $\phi(S_k \backslash S_{k-1}) = \overline{\phi}_k(\overline{S}_k \backslash \overline{0})$. Therefore $\phi(S_k \backslash S_{k-1})$ is an indecomposable set of matrices, but then ϕ is an indecomposable representation of S. □

Proposition 3.2.6. *Let ϕ be an indecomposable representation of S over K. Then there exists a $\overline{S}_k$ from the normal series such that 3.1 holds.*

Proof. Since $S_0 = \{0\}$, there exists $k \geq 1$ such that $\phi(S_{k-1}) = 0$, but $\phi(S_k) \neq 0$. Take ${}_{KS}M$ the left KS module corresponding to ϕ. Then we know that

$$S_{k-1}M = 0 \quad \text{and} \quad e_k M \neq 0$$

The latter is true, because if e_k annihilated the module then every element of Se_k would annihilate it too, which would mean $\phi(S_k) = 0$. Take the following two subsets of M:

$$M' = \{m \in M : e_k m = m\} \qquad\qquad M'' = \{m \in M : e_k m = 0\}.$$

Both of these are closed with respect to addition, and because S is commutative also for multiplication with ring elements, hence they form submodules of M. Furthermore $M = M' \bigoplus M''$, because $M' \cap M'' = \{0\}$ and every $m \in M$ can be written as $m = e_k m + (1 - e_k)m \in M' + M''$. Now since M is indecomposable and $\{0\} \neq e_k M \subseteq M'$ we deduce that for every $m \in M$ $e_k m = m$.

Let $\overline{S} = \sigma_{k-1}(S)$, $\overline{s} = \sigma_{k-1}(s)$, $m \in M$. Since we know that $S_{k-1}M = 0$ and $e_k M \neq 0$, by defining

$$\overline{s}m := sm$$

we get a $K\overline{S}$-module structure on M. From this we get a $K\overline{S}_k$ module structure as well. This affords a representation $\overline{\phi}_k$, of $\overline{S}_k$ over K. From the fact that multiplication with e_k acts as an identity for the module and the definition we see that:

$$sm = s(e_k m) = (se_k)m = \sigma_{k-1}(se_k)m \qquad \forall m \in M, s \in S$$

But $\sigma_{k-1}(se_k) \in \overline{S}_k$, thus $\phi(s) = \overline{\phi}_k(\sigma_{k-1}(se_k))$. The equation above shows that the set of matrices ϕ and $\overline{\phi}_k$ coincide. Since the former is indecomposable so is the latter and we are done with the proof. □

It follows that S is of finite representation type if and only if all of its normal factors (i.e. factors of the normal series) are.

Representations of elementary semigroups

Here, E will denote an elementary semigroup, G and R will stand for its unique subgroup and nilpotent ideal, respectively. Equivalent conditions will be formulated for an elementary semigroup to have finite representation type both in the standard and modular cases.

Proposition 3.2.7. *Let $E = G \cup R$ be an elementary semigroup with, $|G| = n$, and let K be a field, containing the n-th roots of unity (e.g. algebraically closed), whose characteristic does not divide the order of G. Suppose that ϕ is an indecomposable representation of E over K. Then $\phi(G)$ consists only of scalar matrices and $\phi(R)$ is an indecomposable representation of R.*

Proof. Let $n \in \mathbb{N}$ be the order of the group, and pick a $g \in G$. Since $g^n = 1$, it is also true for $\phi(g)$, and therefore the minimal polynomial $m_{\phi(G)}$ of $\phi(g)$ divides $x^n - 1$. From K containing the roots of unity it follows that $m_{\phi(G)}$ splits over K, and since $\mathrm{char}(K) \nmid n$ these roots are distinct, implying that $\phi(g)$ is diagonalizable. By collecting the same scalars next to each other, for every $g \in G$ we can suppose that the image of g looks like the following block diagonal matrix:

$$\phi(g) = \begin{bmatrix} \lambda_{g_1} E_{g_1} & 0 & \cdots & 0 \\ 0 & \lambda_{g_2} E_{g_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{g_m} E_{g_m} \end{bmatrix} \tag{3.2}$$

where $\lambda_{g_i} \in K, i \neq j \Rightarrow \lambda_{g_i} \neq \lambda_{g_j}$ and E_{g_i} denotes the appropriate unit matrix. Consequently we want to prove that m=1. Suppose that it is not true and choose an $r \in R$ and write the matrix in block matrix form with the same block sizes as before:

$$\phi(r) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{p2} & \cdots & A_{mm} \end{bmatrix} \tag{3.3}$$

Since S is commutative we know that

$$(\lambda_{g_i} E_i)A_{ij} = A_{ij}(\lambda_{g_j} E_j)$$

Therefore $i \neq j \Rightarrow A_{ij} = 0$, which would imply that $\phi(R)$ is decomposable. Then, since the elements of $\phi(G)$ consist of scalar matrices with the same block sizes, we would get a decomposition for the whole semigroup, which contradicts the assumption. By the same logic, if $\phi(R)$ were decomposable, the (now known to be) scalar matrices of G would decompose the same way and this would give us a decomposition of the representation of E. □

Theorem 3.2.8. *Let $E = G \cup R$ be an elementary semigroup and K a field with characteristic not dividing the order of G. Then E is of finite representation type if and only if one of the following holds:*
1) $R = \{0\}$ (i.e. $E = G \cup \{0\}$)
2) $R \backslash R^2 \neq \emptyset$ and G acts transitively on $R \backslash R^2$.

Proof. 1) In the first case suppose that $R \backslash R^2 = \emptyset$. Equivalently $R = R^2$ which, by the nilpotency of R, means that $R = \{0\}$. Consequently we are dealing with the representation type of a group with an externally adjoined 0. Then by 2.2.1

$$KE \cong K(G \cup 0) \cong K \times K_0(G \cup 0) \cong K \times KG.$$

Therefore, by 3.1.5 KE has finite representation type if and only if KG has. But by Maschke's Theorem, KG is semisimple, and as such, by 2.2.17, is of finite representation type.

2) Now we deal with the case $R \backslash R^2 \neq \emptyset$. We can partition E by taking the 'cosets' with respect to G. Explicitly:

$$E = G \sqcup Gr_2 \sqcup ... \sqcup Gr_m$$

It is easy to check that, as in the case of groups, we have $Gr_i \cap Gr_j \neq \emptyset \Rightarrow Gr_i = Gr_j$ and clearly we get every element of E. Since for any $u \in E$ we know that $Ga = Gb \Rightarrow (Ga)u = (Gb)u \Rightarrow G(au) = G(bu)$ this decomposition also defines a congruence on the semigroup by the equivalence relation:

$$a \; \mathcal{G} \; b \Longleftrightarrow Ga = Gb \quad a, b \in E$$

Because G acts on $R\backslash R^2$ the cosets also interact nicely with this set, namely: $Gr_i \cap (R\backslash R^2) \neq \emptyset \Rightarrow Gr_i \subset (R\backslash R^2)$.

First, suppose that G does not act transitively on $R\backslash R^2$. Then there are at least two '*G-cosets*' in $R\backslash R^2$; we can assume them to be Gr_1 and Gr_2. Now we would like to prove that the following set is an ideal:

$$I := R\backslash(Gr_1 \sqcup Gr_2)$$

We see that $I \subset R$ and $I \supseteq R\backslash(R\backslash R^2) = R^2$, hence $R^2 \subseteq I \subset R$. In particular this implies that I is closed with respect to multiplication with elements from R. Let $g \in G$ and $r \in R$, then:

$$gr \in Gr_1 \sqcup Gr_2 \Rightarrow r = g^{-1}gr \in Gr_1 \sqcup Gr_2$$

which implies that I is also closed with respect to multiplication with group elements and therefore an ideal.
Let us denote the join of the ideal congruence of I and $\mathcal{G}$ in the congruence lattice as τ. In the factor E/τ I is 'compressed' to zero and the elements of *G-cosets* are 'united', this gives us a four element semigroup

$$E/\tau = \overline{e} \cup \overline{r}_1 \cup \overline{r}_2 \cup \overline{0}$$

With multiplication:

$$\overline{e} \quad \text{is the identity of } E/\tau \qquad \overline{0} \quad \text{is the zero of } E/\tau \qquad \overline{r}_1^2 = \overline{r}_2^2 = \overline{r}_1\overline{r}_2 = \overline{0}$$

By Lemma 2.2.18 the three element semigroup with zero multiplication has infinitely many nonequivalent indecomposable representations over any field. For any of these by adjoining the identity matrix we obtain an indecomposable representation of E/τ. (Since the identity matrix is compatible with any decomposition, if by adjoining it the representation became decomposable, then it would have been decomposable originally as well.) Then, by 3.0.1, these representations composed with the factor map, give infinitely many nonequivalent indecomposable representations of E.

Now suppose that G acts transitively on $R\backslash R^2$. Then we know that the set consists of only one '*G-coset*' for example Gr^*. Since $R\backslash R^2$ is a generating set for R we know that every element of R has the form:

$$(r^*)^n g \qquad \text{for some } n \in \mathbb{N}, g \in G$$

As we have mentioned in the introduction of the chapter, representation type is invariant under finite extensions of the base field, hence we can suppose the base field to contain the roots of unity equal to the order of G, since otherwise we could adjoin them. This means we are allowed to use Proposition 3.2.7, which tells us that for any indecomposable representation ϕ of E: $\phi(G)$ consist only of scalar matrices and $\phi(R)$ is an indecomposable representation of R. Since scalar matrices are compatible with any decomposition the indecomposability of R and the form of elements in R implies that $\phi(r^*)$ is an indecomposable matrix. Hence $\phi(\langle r^*\rangle)$ is an indecomposable representation of the subsemigroup generated by r^*. Furthermore, since $\phi(G)$ consists only of scalar matrices we can recover the representation of it just by storing a one dimensional representation, namely which scalars we used. Then the dimension of the vector space of ϕ fully determines $\phi(G)$. This works vice verse, the pair (δ, ψ), where δ is a one dimensional representation of G and ψ is an idecomposable representation of $\langle r^*\rangle$ uniquely defines (because of the form of elements) an indecomposable representation of E. Let us call such pairs equivalent if $\delta \sim \delta'$ and $\psi \sim \psi'$ as representations. We see that equivalent pairs describe equivalent indecomposable representations of E, therefore the number of equivalence classes of indecomposable representations of E is at most the number of equivalence classes of such pairs. Here the number of one-dimensional representations of G is finite (they map G into the set of roots of $x^n - 1$, where $|G| = n$). On the other hand r^* is nilpotent (say, $(r^*)^k = 0$), and since ψ is an indecomposable representation, $\psi(r^*)$ is similar to a nilpotent Jordan block of size at most $k \times k$. This gives finitely many equivalence classes of pairs (δ, ψ). $\square$

Now we turn to the modular case. Here we will use the Kronecker-product of matrices, defined the following way:

Definition 3.2.9. *Let K be a field and let $A \in K^{m\times n}, B \in K^{p\times q}$. Then the **Kronecker-product** $A \times B$ is defined as*

$$A \times B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{p2}B & \cdots & a_{mm}B \end{bmatrix} \in K^{pm\times qn}.$$

Remark 3.2.10. *It can be verified that the Kronecker-product of matrices with appropriate sizes is associative, distributive on both sides and $(A \times B)(C \times D) = AC \times BD$.*

In the next theorem we will also use the following notations: I_n is the n-dimensional unit matrix, J_n denotes the n-dimensional nilpotent Jordan block, and

$$\Delta = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \delta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

(Note that $\Delta = I_2 + \delta = I_2 + J_2$)

Lemma 3.2.11. *If $S \in M_{2m}(K)$ commutes with $I_m \times \Delta$, then*

$$S = S_0 \times I_2 + S_1 \times \delta$$

where $S_0, S_1 \in M_m(K)$. Furthermore if S is invertible then S_0 is invertible as well.

Proof. Write S as a block matrix partitioned by 2×2 square matrices S_{ij}. Since S commutes with $I_m \times \Delta$ we know that $\Delta S_{ij} = S_{ij}\Delta$. By substituting $\Delta = I_2 + \delta$ we get $\delta S_{ij} = S_{ij}\delta$. Multypling these matrices shows that every block can be written as:

$$S_{ij} = \begin{bmatrix} s_{ij}^{11} & 0 \\ s_{ij}^{21} & s_{ij}^{11} \end{bmatrix} = s_{ij}^{11} I_2 + s_{ij}^{21}\delta$$

Defining $S_0 := (s_{ij}^{11})$ and $S_1 := (s_{ij}^{21})$ gives us the desired form of S.

Now suppose that S is invertible. Then S^{-1} also commutes with $I_m \times \Delta$, which by the first part implies $S^{-1} = T_0 \times I_2 + T_1 \times \delta$. From the properties of the Kronecker-product and $\delta^2 = 0$ we have

$$I_{2m} = SS^{-1} = S_0T_0 \times I_2 + (S_0T_1 + S_1T_0) \times \delta$$

Hence it follows that $S_0T_1 + S_1T_0 = 0$ and $S_0T_0 = I_m$, which gives us a (right) inverse of S_0. (Left inverse can be shown the same way.) ☐

Lemma 3.2.12. *Let K be a field with characteristic $p > 0$, $E = G \cup R$ an elementary semigroup with maximal subgroup isomorphic to a cyclic p-group. In addition suppose that $R \neq 0$. Then E is of infinite representation type.*

Proof. Let us denote the generator of the cyclic *p-group* as g. We are going to define the following $2m$-dimensional representations of E:

$$D_m : E \longrightarrow M_{2m}(K)$$

$$D_m(g^k) := I_m \times \Delta^k, \quad D_m(r) := J_m \times \delta \quad (r \in R \backslash R^2), \quad D_m(R^2) := 0$$

By the properties of the Kronecker-product it can be checked that it is indeed a semigroup homomorphism. (Note that if $r \in R\backslash R^2$, $g \in G$ then $gr = rg \in R\backslash R^2$, and $\delta\Delta = \Delta\delta = \delta$.) We prove that this is an indecomplosable representation for every m. Fix an m and suppose that this is not an indecomposable representation. Then there exists an invertible matrix T, such that:

$$T^{-1}D_mT = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \tag{3.4}$$

In particular

$$T^{-1}D_m(g)T = \begin{bmatrix} M_1(g) & 0 \\ 0 & M_2(g) \end{bmatrix}$$

By definition $D_m(g) = I_m \times \Delta = I_m \times (I_m + \delta) = I_{2m} + (I_m \times \delta)$, from which it follows that:

$$T^{-1}(I_m \times \delta)T = \begin{bmatrix} M_1(g) - I' & 0 \\ 0 & M_2(g) - I'' \end{bmatrix}$$

for some identity matrices I', I''. Since $(I_m \times \delta)^2 = 0$ we know that $(M_1(g) - I')^2 = 0$ and $(M_2(g) - I'')^2 = 0$. Every nilpotent matrix, over an arbitrary field, is similar to a Jordan matrix, furthermore in our case the block sizes are at most 2, since the squares of the matrices are zero. Consequently there exist invertible matrices U_1, U_2 such that:

$$U_1^{-1}(M_1(g) - I')U_1 \text{ and } U_2^{-1}(M_2(g) - I'')U_2$$

are Jordan matrices. Define

$$U := \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}$$

It is clearly invertible, furthermore

$$U^{-1}T^{-1}D_m(g)TU = \begin{bmatrix} U_1^{-1}M_1(g)U_1 & 0 \\ 0 & U_2^{-1}M_2(g)U_2 \end{bmatrix}$$

is a Jordan matrix similar to $D_m(g)$, so its Jordan blocks are all Δ. It follows that

$$U^{-1}T^{-1}D_m(g)TU = \begin{bmatrix} D_r(g) & 0 \\ 0 & D_{m-r}(g) \end{bmatrix} = D_m(g)$$

Equivalently we have that $S := TU$ is invertible and commutes with $D_m(g) = I_m \times \Delta$, which by 3.2.11 implies that

$$S = S_0 \times I_2 + S_1 \times \delta$$

with the appropriate matrices and S_0 is also invertible. Choose an $r \in R$, from the commutativity of E we see that

$$(S^{-1}D_m(r)S)(S^{-1}D_m(g)S) = (S^{-1}D_m(g)S)(S^{-1}D_m(r)S),$$

but since $S^{-1}D_m(g)S = D_m(g)$ the equation implies that $S^{-1}D_m(r)S$ commutes with $D_m(g)$, therefore by 3.2.11 we know that

$$S^{-1}D_m(r)S = A \times I_2 + B \times \delta \tag{3.5}$$

with appropriate matrices A and B. This yields $D_m(r)S = S(A \times I_2 + B \times \delta)$. By substituting $D_m(r) = J_m \times \delta$ and $S = S_0 \times I_2 + S_1 \times \delta$

$$(J_m \times \delta)(S_0 \times I_2 + S_1 \times \delta) = (S_0 \times I_2 + S_1 \times \delta)(A \times I_2 + B \times \delta)$$

Using $\delta^2 = 0$ we get

$$J_m S_0 \times \delta = S_0 A \times I_2 + S_1 A \times \delta + S_0 B \times \delta$$

Multiplying this equation with $I_m \times \delta$ yields $0 = S_0 A \times \delta$, which means that $S_0 A = 0$. Since S_0 is invertible we must have $A = 0$. Then $J_m S_0 \times \delta = S_0 B \times \delta$ implying $J_m S_0 = S_0 B$. Because S_0 has an inverse

$$S_0^{-1} J_m S_0 = B \tag{3.6}$$

But by equation 3.4 and by definition U being a diagonal block matrix with the same block sizes

$$S^{-1} D_m(r) S = U^{-1} T^{-1} D_m(r) T U = \begin{bmatrix} U_1^{-1} M_1(r) U_1 & 0 \\ 0 & U_2^{-1} M_2(r) U_2 \end{bmatrix}$$

Comparing this with 3.5 and remembering that $A = 0$

$$\begin{bmatrix} U_1^{-1} M_1(r) U_1 & 0 \\ 0 & U_2^{-1} M_2(r) U_2 \end{bmatrix} = B \times \delta$$

But this can only happen if

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \tag{3.7}$$

Now we have our contradiction by 3.7 and 3.6, since J_m is a Jordan square matrix, hence indecomposable. For $m \neq n$ D_m and D_n cannot be equivalent since the dimensions of their vector spaces differ. So we obtained infinitely many non-equivalent indecomposable representations of E over K.

□

Lemma 3.2.13. *Let $E = G \cup R$ be an elementary semigroup and K a field with characteristic $p > 0$. If KE is of finite representation type, then the p-Sylow subgroup of G is cyclic.*

Proof. If the *p-Sylow* subgroup of G is not cyclic we are going to show infinitely many nonequivalent indecomposable modules over KE. The factor with R is isomorphic to $G \cup \{0\}$ the group with externally adjoined zero. Then by 2.2.1 we have

$$KE/KR \cong K_0(E/R) \cong K_0(G \cup 0) \cong KG$$

By Higman's theorem if the *p-Sylow* subgroup of G is not cyclic then KG has an infinite number of nonequivalent indecomposable modules, and since KG is a factor of KE by 2.2.19 KE has infinitely many as well. □

Theorem 3.2.14. *Let $E = G \cup R$ be an elementary semigroup and K a field with characteristic $p > 0$. Suppose that p divides the order of G. Then E is of finite representation type if and only if the following conditions hold simoultaneously:*
1)$R = 0$ (i.e. E is a group with an externally adjoined zero)
2)the p-Sylow subgroup of G is cyclic.

Proof. If $R = 0$ holds, we have the following isomorphisms 2.2.4:

$$KE \cong K \times K_0E \cong K \times KG$$

Which means that KE is of finite representation type $\Longleftrightarrow$ $K \times KG$ is of finite representation type $\Longleftrightarrow$ KG is of finite representation type $\Longleftrightarrow$ the *p-Sylow* subgroup of G is cyclic, where the second equivalence holds because of 2.2.21 and the final one is due to Higman's Theorem.

As a consequence of the previous lemma it is only left to show that if $R \neq 0$ then E is of infinite representation type. First we can make a few simplifications by taking factors of E. Then according to remark 3.0.1 if a factor semigroup of E has infinite representation type then so does E. If $R^2 \neq 0$ then factoring with the ideal gives us an elementary semigroup in which $\overline{R}$ is the nilpotent ideal and $\overline{R^2}$ is the zero element. Let P be a *p-Sylow* subgroup of G. Since G is Abelian with $p||G|$, there exists a non-trivial cyclic p-subgroup P and another subgroup N such that

$$PN = G \text{ and } P \cap N = 1.$$

If $N \neq e$ then instead of G we can consider E/N, by taking the quotient with the congruence of *N-cosets*, defined the same way as in 3.2.8. The factor is also an elementary semigroup with non-zero maximal subgroup $G/N \cong P$, as it was shown in 3.2.8 that a coset is either disjoint from $R \backslash R^2$ or is contained in it. Finally, Lemma 3.2.12 implies that this semigroup is of infinite representation type. □

Summing up the previous results we state the general theorem for an arbitrary commutative semigroup algebra:

Theorem 3.2.15. *Let K be a field with characteristic p, and let S be a commutative semigroup admitting a normal series $(S_k)_{k=0}^{n}$, with elementary quotient semigroups $E_k = G_k \cup R_k$. KS is of finite representation type if and only if for every k*
1) if $p \nmid |G_k|$, then and either $E_k = G_k \cup \{0\}$ or $R \backslash R^2 \neq \emptyset$ and G acts transitively on $R \backslash R^2$.
2) if $p \mid |G_k|$, then $E_k = G_k \cup \{0\}$ and the p-Sylow subgroup of G_k is cyclic.

3.3 Inverse semigroup algebras

In this section we determine which inverse semigroup algebras are of finite representation type. Actually, combined with a result from [Pon93] we can determine the representation type of an even broader class of semigroups, but since in that case the result is not elegent (opposed to the case of inverse semigroups), we shall only discuss the result informally.

The result will have a strong connection with completetely simple semigroups. By 2.1.10, we have seen that these semigroup coincide with Rees matrix semigroups over a group with zero. Due to this characterization, we begin with discussing the tool to handle the algebras of these semigroups. This section is mostly based on [Okn90], the beginning and end of chapter 5.

Algebras of matrix type

Let K be a field, and let R be an associative K-algebra. Fix two nonempty index sets I, M and $P = (p_{mi})_{m \in M, i \in I}$ a generalized (having finitely many non-zero elements) $M \times I$ matrix with $p_{mi} \in R$. Consider $A = (a_{im})$ and $B = (b_{im})$ two generalized $I \times M$ matrices with finitely many non-zero entries over R, and define operations between them as

$$A + B = ((a_{im} + b_{im})_{im}), \text{ i.e. element-wise addition}$$
$$A * B = APB, \text{ with the usual product of matrices}$$
$$\lambda A = (\lambda a_{im}), \text{ for } \lambda \in K.$$

The set of generalised $I \times M$ matrices with finitely many non-zero entries over R becomes an associative K-algebra with this product and is called the **algebra of matrix type** over R. It is denoted as $\mathcal{M}(R, I, M, P)$ or $\hat{R}$ for brevity. An important example and the motivation for this construction comes from the following observation.

Lemma 3.3.1. *Let $S = \mathcal{MS}(G, I, M, P)$ be a semigroup of matrix type. Then*

$$K_0[\mathcal{MS}(G, I, M, P)] \cong \mathcal{M}(KG, I, M, P)$$

Proof. In case S does not have a zero element we will mean the original semigroup algebra under the notation $K_0 S$. The map

$$\varphi : \mathcal{MS}(G, I, M, P) \longrightarrow \mathcal{M}(KG, I, M, P)$$
$$\varphi((i, g, m)) := (a_{jn}) = \begin{cases} g & (j, n) = (i, m) \\ 0 & otherwise \end{cases}$$

defines a semigroup homomorphism from S to the multiplicative semigroup of the algebra. Since the images of the non-zero elements of S generate the vector space of the algebra and are linearly independent they form a basis of the algebra. Hence extending φ linearly to a homomorphism of K-algebras gives the desired isomorphism $K_0[\mathcal{MS}(G, I, M, P)] \cong \mathcal{M}(KG, I, M, P)$. □

A generalized matrix over R with only one non-zero element r at the i-th row and m-th column is denoted as (r, i, m), furthermore $\hat{R}^{(m)}_{(i)} := \{(r, i, m) : r \in R\}$. For nonempty sets $J \subseteq I$ and $N \subseteq M$ we define $\hat{R}^{(N)}_{(J)} := \sum_{i \in J} \sum_{m \in N} \hat{R}^{(m)}_{(i)}$.

Let R, R' be two K-algebras and let $\varphi : R \longrightarrow R'$ be a homomorphism between them. The induced homomorphism $\hat{\varphi} : \mathcal{M}(R, I, M, P) \longrightarrow \mathcal{M}(R', I, M, \varphi(P))$, where $\varphi(P) = ((\varphi(p_{im}))_{im})$ is defined as $\hat{\varphi}((a_{im})) := ((\varphi(a_{im}))_{im})$.

Lemma 3.3.2. *Let R be a finite-dimensional K-algebra and let P be a $m \times n$ matrix over R. If $m > n$ then there exists a non-zero $n \times m$ matrix X over R such that $XP = 0$. If $n > m$ then there exists a non-zero $n \times m$ matrix Y over R such that $PY = 0$.*

For the proof of the Lemma, see [CP61], Theorem 5.11.

Proposition 3.3.3. *Let R be an algebra with a non-zero finite-dimensional homomorphic image. Then the following conditions are equivalent:*
1) The algebra $\hat{R}$ has an identity.
2)I, M are finite sets, $|I| = |M|$ and P is an invertible matrix in $M_{|I|}(R)$.
Moreover if 1) or 2) holds then $\hat{R} \cong M_{|I|}(R)$

Proof. First assume that $\hat{R}$ has an identity. We shall denote it with E. Then there exist finite sets J, N such that $E \in \hat{R}^{(N)}_{(J)}$. Fix an $i \in I$, from E being a left identity it follows that for any $m \in M$

$$(1, i, m) = E * (1, i, m) = EP(1, i, m) \in \hat{R}^{(M)}_{(J)}$$

which means that $i \in J$, implying $I = J$ is a finite set. Furthermore, the i-th column of EP is zero except at the entry (i, i) which is 1, otherwise there either would be a non-zero element elsewhere or a number other then 1 at the entry (i, m) in $(1, i, m)$. Hence EP is the $I \times I$ identity matrix. From E being a right identity it follows that $M = N$ is a finite set and PE is the $M \times M$ identity matrix. From here we only need to prove $|I| = |M|$ to establish 2), since then E is the inverse of P in the algebra $M_{|I|}(R)$. Let $\varphi : R \longrightarrow R'$ be a homomorphism to a finite-dimensional algebra R'. We can suppose that φ is surjective. Let $\hat{\varphi} : \hat{R} \longrightarrow \mathcal{M}(R', I, M, \varphi(P)) = \hat{R'}$ be the induced homomorphism of φ. I and M are finite sets, hence $\hat{R'}$ is a finite-dimensional algebra as well, and $\varphi(E)$ acts as an identity for it implying that for

every $X \in \hat{R}'$, $X = \varphi(E)\varphi(P)X = X\varphi(P)\varphi(E)$. If $I > M$ were the case, then applying the previous lemma to P would imply he existence of a non-zero X such that $X\varphi(P) = 0$, further implying $X\varphi(P)\varphi(E) = 0$, contradicting the the derived equation. Hence $M \geq I$, and by the same logic and the second part of the lemma it follows that $I \geq M$ implying $|I| = |M|$, and proving 2).
If 2) holds, then P is invertible, hence the inverse of P is an identity for the algebra and $\psi : \hat{R} \longrightarrow M_{|I|}(R)$, $\psi(X) := XP$ is an isomorphism of algebras. □

Corollary 3.3.4. *Let $S = \mathcal{MS}(G, I, M, P)$ be a finite semigroup of matrix type. Then the following conditions are equivalent:*
1) The algebra K_0S has an identity
2) I and M are finite sets, $|I| = |M|$ and P is an invertible matrix in $M_{|I|}(KG)$.
Moreover if 1) or 2) holds, then $K_0S \cong M_{|I|}(KG)$ and S is a completely 0-simple semigroup.

Proof. According to Lemma 3.3.1 $K_0S \cong \mathcal{M}(KG, I, M, P)$, and since KG is finite dimensional, the previous proposition applies to the former algebra, proving the equivalence. Moreover, the invertibility of P implies that it has no zero columns or rows, so by theorem 2.1.9 S is completely 0-simple. □

Strongly p-semisimple semigroups

Here, we define the wider class of semigroups mentioned in the introduction of the section. Their definition is based on a common property they share, but to decide whether a semigroup possesses it we are not aware of any accessible, explicit condition.

Definition 3.3.5. *Let p be zero or prime and let F be the prime field of characteristic p. A semigroup S is* ***strongly p-semisimple*** *if and only if all the principal factor algebras $F_0[S_t], t \in S$ have identity elements.*

Remark 3.3.6. *If $char(K) = p$ then $K_0[S_t] \cong K \otimes_{F_0} F_0[S_t]$ has an identity if and only if $F_0[S_t]$ has one.*

Remark 3.3.7. *If S is strongly p-semisimple then S cannot have null semigroups as principal factors, since algebras of null semigroups cannot contain an identity. Therefore, S is a semisimple semigroup.*

Theorem 3.3.8. *Let S be a finite non-zero strongly p-semisimple semigroup and let K be a field with $char(K) = p$. Then*

$$K_0S \cong K_0[T_1] \times \ldots \times K_0[T_r]$$

for non-zero principal factors T_i of S. Furthermore if S is completely semisimple then

$$K_0S \cong M_{n_1}(K[G_1]) \times \ldots \times M_{n_r}(K[G_r])$$

where G_i are maximal subgroups of S.

Proof. We only prove it for the case when S has a zero element, the proof is the same if S does not have one with forgetting about the $0-$ prefix. Any finite semigroup admits a principal series. The proof goes by induction on the number of constituents of the series. It is clear if the series consists solely of 0 and the semigroup. Suppose we have $0 \subset S_1 \subset S$ as the series. It was noted in the section on principal factors that a 0-minimal ideal is always principal. The first non-zero and nonempty term S_1 in the series must be a 0-minimal ideal of S, hence $S_1 = J_a$ for some $a \in S$. According to 2.1.16 $I_a = 0$, hence the principal factor at a is $T_1 := S_1/\{0\} \cong S_1$. From the conditions it is an algebra with identity hence by 2.2.3 $K_0S \cong K_0[T_1] \times K_0[S/S_1]$. Now suppose that the hypothesis is true for series with lengths at most k, and let S be a semigroup such that it has principal series of length $k+1$, denoted as $(S_k)_{k=0}^r$. Applying the previous case we get $K_0S \cong K_0[S_1] \times K_0[S/S_1]$, and from the bijection between ideals of the original semigroup and the factor a principal series of S/S_1 is

$$0 \subset S_2/S_1 \subset \cdots \subset S/S_1$$

which has less constituents, therefore the induction hypothesis applies and

$$K_0[S/S_1] \cong \prod K_0[(S_{i+1}/S_1)/(S_i/S_1)]$$

Furthermore, from the isomorphism theorem 2.1.13, we get that

$$(S_{i+1}/S_1)/(S_i/S_1) \cong S_{i+1}/S_i$$

which are precisely the principal factors of the original semigroup, and the proposition is proved. Moreover, if the principal factors are completely 0-simple by the previous corollary we have $K[T_i] \cong M_{n_i}(K[G_i])$. □

The theorem also stays true with the condition "finite" replaced with "having finitely many principal factors", but since only the algebras of finite semigroups can have finite representation type, we can constrain ourselves to finite semigroups. Furthermore, for finite semigroups semisimplicity, and complete semisimplicity coincide. Hence we deduced that for every finite strongly p-semisimple semigroup S

$$K_0S \cong M_{n_1}(K[G_1]) \times \ldots \times M_{n_r}(K[G_r])$$

reducing the problem of finite representation type to the group case.

Corollary 3.3.9. *Let K be a field with $char(K) = p$, and let S be a finite strongly p-semisimple semigroup. Then KS is of finite representation type if and only if all of its maximal subgroups are.*

Proof. K_0S and KS are of finite representation type at the same time, and the representation type of direct products of algebras is determined by the representation type of its constituents 2.2.21. Moreover, it is well known, from Morita theory that any ring R is Morita-equivalent to $M_n(R)$, hence they are of finite representation type at the same time. A discussion of these concepts can be found in [AF92]. □

Proposition 3.3.10. *Let S be a nonzero finite inverse semigroup. For any p, prime or zero, S is strongly p-semisimple.*

Proof. First, we show that S is semisimple. Since any principal factor is either (0-) simple or null we only have to rule out the possibility of a null principal factor. Choose an $a \in S$, since any inverse semigroup is regular we have $a^* \in S^1aS^1$ and $a \in S^1a^*S^1$, implying that $a, a^* \in J_a \backslash I_a$. Moreover, since the pseudo-inverse was unique in the original semigroup it is also unique in the factor,thus any principal factor is also an inverse semigroup, and a non-zero regular semigroup cannot be null. This proves that a finite inverse semigroup is semisimple. Finally, by theorem 2.1.11 and corollary 3.3.4, the algebras of the principal factors have identities. □

Theorem 3.3.11. *Let K be a field, and let S be an inverse semigroup. The semigroup algebra KS is of finite represenation type if and only if the group algebras of the maximal subgroups of S are of finite representation type.*

This result on inverse semigroups of finite representation type is due to Ponizovskii, and is mentioned in [Pon93]. (In the survey the author states this theorem without any particular reference.)

Remark 3.3.12. *The above discussion also gives an equivalent condition for the algebra of a finite inverse semigroup to be semisimple.*

References

[AF92] Frank W. Anderson and Kent R. Fuller. *Rings and Categories of Modules*. 2nd ed. Graduate Texts in Mathematics. Springer New York, NY, 1992. DOI: https://doi.org/10.1007/978-1-4612-4418-9.

[Bau+85] Raymundo Bautista et al. "Representation-finite algebras and multiplicative bases". In: *Inventiones mathematicae* 81 (1985), pp. 217–285. DOI: 10.1007/BF01389052.

[CP61] A.H. Clifford and G.B. Preston. *The Algebraic Theory of Semigroups (Vol. I)*. American Mathematical Society, 1961.

[EH18] Karin Erdmann and Thorsten Holm. *Algebras and representation theory*. Springer Undergraduate Mathematics Series. Springer Cham, 2018.

[Okn90] J Okninski. *Semigroup Algebras (1st ed.)* CRC Press, 1990. DOI: https://doi.org/10.1201/9781003066521.

[Pon70] I.S. Ponizovskii. "Matrix representations of finite commutative semigroups". In: *Sib Math J* 11 (1970), pp. 816–822. DOI: https://doi.org/10.1007/BF00967841.

[Pon93] J.S. Ponizovskii. "Semigroup algebras of finite representation type". In: *Semigroup Forum* 46 (1993), pp. 1–6. DOI: https://doi.org/10.1007/BF02573536.

[SS07] Daniel Simson and Andrzej Skowronski. "Tame and wild representation type of algebras". In: *Elements of the Representation Theory of Associative Algebras*. London Mathematical Society Student Texts. Cambridge University Press, 2007, pp. 309–358.

[Ste16] Benjamin Steinberg. *Representation theory of finite monoids*. Universitext. Springer Cham, 2016.

[Zel77] Efim Isaakovich Zel'manov. "Semigroup algebras with identities". In: *Siberian Mathematical Journal* 18.4 (1977), pp. 557–565.

Mesterséges Intelligencia használatának nyilatkozata

Alulírott Süle Márk Manó nyilatkozom, hogy szakdolgozatom elkészítése során az alább felsorolt feladatok elvégzésére a megadott MI alapú eszközöket alkalmaztam:

Feladat	**Felhasznált eszköz**	**Felhasználás helye**	**Megjegyzés**
Nyelvhelyesség ellenőrzése	Writefull, Grammarly	Teljes dolgozat	-
Hivatkozások generálása, 'Mesterséges Intelligencia használatának nyilatkozata' generálása	ChatGPT	Bibliográfia, MI nyilatkozat	BibTeX generálása

A felsoroltakon túl más MI alapú eszközt nem használtam.