Eötvös Loránd Tudományegyetem
Természettudományi Kar

Kozári Dominik

Extremely amenable groups and Fraïssé limits

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Supervisor:
Máté András Pálfy

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Contents

1 Introduction

A very natural question in dynamics is the following: given a topological group G that acts continuously on a compact space X, when can we find a fixed point of this action? Let us pose an even more stronger question, when is is it true that *every* action of a group G on any compact space X has a fixed point? Such groups are called extremely amenable groups. In [3] this question was solved for a wild variety of groups, namely for the Polish non-Archimedean groups.

It turns out that Polish non-Archimedean groups can be realized as the automorphism groups of countable ultrahomogeneous structures and every such structure can be realized as a so-called Fraïsse limit of a Fraïsse class.

The first part of the thesis will introduce the reader in the world of Fraïsse limits and we give some examples of Fraïsse limits.

The second part of the thesis gives the proof the Kechris, Pestov, Todorcevic correspondence that roughly states that an automorphism group of a Fraïsse limit is extremely amenable if and only if the Fraïsse class it satisfies some Ramsey property.

2 Preliminaries

Notation 2.1. *For groups A and B, if A is a subgroup of B we write $A \leq B$*

Notation 2.2. *Suppose G is a group and $H \leq G$. Then G acts on the left-cosets of H in the usual way, that is $g.hH = ghH$ for all $g, h \in G$.*

Notation 2.3. *The symmetric group on some set X will be denoted by S_X.*

Notation 2.4. *The action of a group G on a set X is denoted by $G \curvearrowright X$. Furthermore, the action of an element $g \in G$ on an element $x \in X$ will be written as: $a(g, x) = g.x$.*

Definition 2.5. *If G is a group acting on a set X, then G also acts on X^n diagonally as follows: $g.(x_1, \ldots, x_n) = (g.x_1, \ldots, g.x_n)$. We refer to this action as the diagonal action of G on X^n.*

Notation 2.6. *Consider the group S_X for some countable set X, and suppose $F \subseteq X$. The pointwise stabilizer of F in S_X will be denoted by*

$$H_{(F)} = \{g \in S_X \mid g.x = x \text{ for all } x \in F\},$$

and the setwise stabilizer of F in S_X will be denoted by

$$H_F = \{g \in S_X \mid g.F = F\}.$$

Specifically, if $F = \{x\}$ we write

$$H_x = \{g \in S_X \mid g.x = x\}.$$

Notation 2.7. *Similarly, suppose $G \leq S_X$ for some countable set X and suppose $F \subseteq X$. The pointwise stabilizer of F in G will be denoted by*

$$G_{(F)} = \{g \in G \mid g.x = x \text{ for all } x \in F\},$$

and the setwise stabilizer of F in G will be denoted by

$$G_F = \{g \in G \mid g.F = F\}.$$

Specifically, if $F = \{x\}$ we write

$$G_x = \{g \in G \mid g.x = x\}.$$

Notation 2.8. *Using the previous two statements, we can write*

$$\begin{gathered} G_{(F)} = H_{(F)} \cap G \\ G_F = H_F \cap G \\ G_x = H_x \cap G \end{gathered}$$

Notation 2.9. *If X is a topological space and $G \subseteq X$, then the closure of G in X is denoted by $\overline{G}$.*

Notation 2.10. *Suppose A and B are $\mathcal{L}$-structures. If A is a substructure of B we will write $A \preceq B$.*

Notation 2.11. *We denote graphs in the usual way, for a graph $G = (V_G, E_G)$, the set of vertices of G is V_G, and the set of edges of G is E_G.*

We will assume that the reader is familiar with the following theorems. The interested reader can find the proofs of these theorems in [1], [4]:

Theorem 2.12. *The product of countably many separarable topological spaces is separable.*

Theorem 2.13. *The product of countably many completely metrizable topological spaces is complete metrizable.*

Theorem 2.14. *Tychonoff's theorem, which states that the product of compact topological spaces is compact.*

Theorem 2.15. *If a topological space X is compact and Hausdorff, then X is normal.*

Theorem 2.16. *(Uryhson's lemma) If X is a normal, then any two disjoint closed subset $Z_1, Z_2 \subseteq X$ can be separated by a continuous function, that is, there exists a continuous function $f : X \to [0,1]$ such that $f|_{Z_1} \equiv 0$ and $f|_{Z_2} \equiv 1$.*

Definition 2.17. *A topological space X is said to be Polish, if it is separable and completely metrizable.*

Definition 2.18. *A topological space X satisfies the countable chain condition if the are at most countably many disjoint open sets in X.*

Theorem 2.19. *A Polish space always satisfies the countable chain condition.*

Theorem 2.20. *If a topological space X is Polish, then a subspace Y of X is Polish if and only if Y is G_δ.*

Theorem 2.21. *The group S_ω is a topological group with the pointwise convergence topology, where basic open sets are of the form $gH_{(F)}$, for $g \in S_\omega$ and $F \in \omega$ finite, i.e., the left-cosets of pointwise stabilizers of finite subsets of ω.*

Theorem 2.22. *The group S_ω is a G_δ subspace of the Polish space ω^ω, therefore S_ω is Polish.*

Theorem 2.23. *A subgroup H of a Polish group G is Polish if and only if the subgroup H is closed in G.*

Theorem 2.24. *A homomorphism φ between topological groups G and H is continuous if and only if φ is continuous at 1_G.*

3 Automorphism groups of countable structures

Lemma 3.1. *Suppose X is a countable set and consider the topological group S_X. Let F be some subset of X^n for some $n \in \omega$. Then the setwise stabilizer of F is closed in S_X. Written formally:*

$$H_F = \{g \in S_X \mid g.F = F\}$$

is a closed set in S_X.

Proof. Fix an arbitrary $F \subseteq X^n$ for some n. The complement of H_F is $H_F^C = \{g \in S_X \mid g.F \nsubseteq F \text{ or } g.F \nsupseteq F\}$. Let $A := \{g \in S_X \mid g.F \nsubseteq F\}$ and let $g \in A$. This means there is some $x \in F$, such that $g.x \notin F$. The set gH_x is a basic open neighborhood of g. for all $g' \in gH_x$ $g'.x = g.x \notin F$, therefore $g'.F \nsubseteq F$. This shows that A is open. Similarly, we can show that $B := \{g \in S_X \mid g.Y \nsupseteq Y\}$ is open, and $H_F^C = A \cup B$ is also open, so H_F is closed. □

Lemma 3.2. *Suppose X is a countable set and $G \leq S_X$, then the closure of G is the set of all elements in S_X that fix all G-orbits on X^n for all n. Written formally:*

$$\overline{G} = \{g \in S_X \mid \forall n\ g.O = O \textit{ for all } G\textit{-orbits } O \textit{ on } X^n\} \tag{1}$$

Proof. Let G' be the group on the right-hand side of (1). The set G' is the intersection of the sets $H_O = \{g \in S_X \mid g.O = O\}$ for all G-orbits O on X^n for all n. By Lemma 3.1, All of these sets are closed, therefore their intersection is closed. This means G' is a closed set which clearly contains G, so it must contain $\overline{G}$.

On the other hand, suppose $g \in G'$, and B is an open neighborhood of g. We need to show that $B \cap G \neq \emptyset$. We can find a basic open neighborhood of g of the form $gH_{(F)}$ with $gH_{(F)} \subseteq B$, where $F \subseteq X$ is finite, and $H_{(F)} = \{g \in S_X \mid g.x = x \text{ for every } x \in F\}$. Enumerate F as a tuple x in $X^{|F|}$. Since g fixes all G-orbits, on all X^n for all n, $g.x$ is in the same G-orbit as x. This means there is some $h \in G$ for which $g.x = h.x$, therefore $h \in gH_{(F)} \subseteq B$. It follows that $g \in \overline{G}$ and $\overline{G} \supseteq G'$, so we have $\overline{G} = G'$. □

We are now ready to prove the following theorem, which characterizes closed subgroups of S_ω. Notice that the following theorem combines the fields of group theory, first-order logic, and topology.

Theorem 3.3. *Let G be a topological group, then the following are equivalent:*

(1) G is isomorphic (as a topological group) to a closed subgroup of S_ω,

(2) G is of the form $Aut(A)$, where A is a countable structure,

(3) G is a non-Archimedean and Polish group.

Proof. (1) ⇒ (2): We need to find a structure A, with some language $\mathcal{L}$, such that $G = Aut(A)$. Let the universe of A be ω. We construct the language $\mathcal{L}$ using the orbits of G on ω^n for all n, adding a different n-ary relation symbol for every orbit: for $n \geq 1$ let k_n be the number of G-orbits on ω^n, let $(O_{n,i})_{1\leq i\leq k_n}$ denote the G-orbits on ω^n, and let $(R_{n,i})_{1\leq i\leq k_n}$ be n-ary relation symbols. We want $R_{n,i}$ to be interpreted as which orbit the tuple $(k_1, k_2, \dots, k_n) \in \omega^n$ belongs to, that is, $R^A_{n,i} = O_{n,i} \subseteq \omega^n$. This gives us a structure A, which is usually referred to as the canonical structure of G. Now we must show $G = Aut(A)$. The set $Aut(A)$ is the set of permutations of ω that preserve every relation in $\mathcal{L}$, which is the set of permutations of ω that fix all G-orbits on ω^n for all n. By Lemma 3.2, we have $Aut(A) = \overline{G}$ and since G is closed $Aut(A) = \overline{G} = G$.

(2) ⇒ (1): First, suppose G is finite. By Cayley's theorem, G is isomorphic to a subgroup of S_G and therefore G is isomorphic to a subgroup of S_ω. Since G is finite and S_ω is Hausdorff, G is closed, so we are done. Now, assume G is infinite. Clearly A must be countably infinite, so we can say that the universe of A is ω. The set $Aut(A)$ is the set of permutations of ω which preserve all relations and functions of A. For a relation $R^A \subseteq \omega^n$ with arity n to be preserved by a permutation g is means that

$$(a_1, \dots, a_n) \in R^A \text{ iff } (g.a_1, \dots, g.a_n) \in R^A.$$

This means g preserves R^A if and only if g stabilizes the set R^A. For a function f^A with arity n, we can consider the relation $R_f \subseteq \omega^{n+1}$ for which $f^A(a_1, \dots, a_n) = a_{n+1}$ iff $(a_1, \dots, a_{n+1}) \in R_f$. An element g preserves f^A if and only if it stabilizes the set R_f. Therefore $Aut(A)$ is the set of permutations of ω, which stabilize certain subsets of ω^n for various n. By Lemma 3.1, this is a closed subgroup.

(1) ⇒ (3): The group S_ω is Polish, and is a closed subgroup of S_ω, therefore G is Polish. If F is a finite subset of ω, then $G_{(F)}$ is a subgroup and an open neighborhood of the identity 1_G. Therefore $\{G_{(F_i)} \mid F_i = \{0, 1, \dots, i\},\ i \in \omega\}$ is a system of subgroups which forms an open neighborhood basis at the identity, thus G is non-Archimedean.

(3) ⇒ (1): Since G is non-Archimedean, we can find a system of subgroups which forms an open neighborhood basis at the identity 1_G. We may suppose this basis is of the form $\{N_i \mid i \in \omega\}$, where $N_i \supseteq N_{i+1}$ for all i. The group G is separable, so it satisfies the countable chain condition. It follows that $|G : N_i|$ is countable for all i, therefore the set

$$H := \{gN_i \mid g \in G, i \in \omega\}$$

is countable. We would like to show that G is isomorphic, as a topological group, to a subgroup of S_H. Consider the function

$$\phi : G \to S_H$$
$$\phi(g)(hN_i) = hgN_i.$$

For any $g \in G$ the image $\phi(g)$ is an element of S_H, because multiplication by g from the right is an automorphism of G, therefore for a fixed N_i, the function

$\phi(g)$ is a permutation of $\{hN_i \mid h \in G\}$. We can also see that ϕ is a group homomorphism, since

$$\phi(gh)(kN_i) = kghN_i = \phi(h)(kgN_i) = \phi(g)\phi(h)(kN_i).$$

Suppose $g, h \in G$, $g \neq h$. This means $g^{-1}h \neq 1_G$, so there is some i such that $g^{-1}h \notin N_i$. For any $k \in G$, we have

$$\begin{aligned} g^{-1}hN_i \neq N_i &\Rightarrow hN_i \neq gN_i \Rightarrow khN_i \neq kgN_i \\ &\Rightarrow \phi(h)(kN_i) \neq \phi(g)(kN_i) \Rightarrow \phi(h) \neq \phi(g), \end{aligned}$$

thus ϕ is injective.

To show that ϕ is continuous it is enough to show it is continuous at 1_G. Suppose $\forall n \in \omega$ $g_n \in G$ and $g_n \to 1_G$. We must show that $\phi(g_n) \to \phi(1_G) = 1_{S_H}$. Let B be an open neighborhood of 1_{S_H}. We can find a basic open set $B' \subseteq B$ of the form $B' = \{h \in S_H \mid h.x = x \text{ for all } x \text{ in } F\}$, the pointwise stabilizer of F, where $F = \{h_1N_{i_1}, \ldots, h_kN_{i_k}\} \subseteq H$ finite. Let $M = \bigcap_{j=1}^{k} N_{i_j}$, which is an open neighborhood of 1_G. Therefore only finitely many $g_n \notin M$, which means there are only finitely many n for which $\phi(g_n)(M) \neq M$. Every other $\phi(g_n)$ fixes M, and therefore fixes F pointwise, thus $\phi(g_n) \to 1_{S_H}$.

We will show that ϕ^{-1} is continuous in a similar way. Suppose $\phi(g_n) \to 1_{S_H}$ and B is an open neighborhood of 1_G. There is some $i \in \omega$ such that $N_i \subseteq B$. Let $B' = \{h \in S_H \mid hN_i = N_i\}$, which is a basic open neighborhood of 1_{S_H}, therefore there are only finitely many n for which $\phi(g_n) \notin B'$. Thus there are only finitely many n for which $\phi(g_n)(N_i) \neq N_i$, which means only finitely many $g_n \notin N_i$, so ϕ^{-1} is continuous.

This shows that $\phi : G \to \phi(G)$ is a topological group isomorphism with $\phi(G) \leq S_H$. Since H is countable, S_H is isomorphic to a subgroup of S_ω, therefore G is isomorphic to a subgroup of S_ω. Furthermore, since G is Polish, $\phi(G)$ is a Polish subgroup of S_ω, therefore $\phi(G)$ is closed. □

Mentioned in the previous proof, we now define the notion of a canonical structure.

Definition 3.4. *Suppose $G \leq S_\omega$ is closed. The canonical structure of G, denoted by A_G, is the structure with universe ω and language*

$$\mathcal{L} = \{R_O \mid O \text{ is a } G\text{-orbit on } \omega^n \text{ for some } n = 1, 2, \ldots\},$$

where $R_O^{A_G} = O \subseteq R^n$ for all $n = 1, 2, \ldots$ and all G-orbits O on ω^n. As seen in the proof of Theorem 3.3 (1) $\Rightarrow$ (2), $G = Aut(A_G)$.

Theorem 3.5. *Suppose $G \leq S_\omega$ is closed and A_G is the canonical structure of G. If we have finitely generated substructures $A \subseteq A_G$ and $B \subseteq A_G$ and an isomorphism $f : A \to B$, then f extends to an automorphism $g : A_G \to A_G$, meaning $g|_A = f$.*

Proof. Notice that since $\mathcal{L}$ is a relational language, the structure generated by a subset $H \subseteq A_G$ is H. List the elements of A and B as $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. We can assume that $f(a_k) = b_k$ for all $1 \leq k \leq n$. Since f is an isomorphism, f preserves relations, therefore

$$(a_1, \ldots, a_n) \in R_O^{A_G} \iff (f(a_1), \ldots, f(a_n)) \in R_O^{A_G}$$

for every G-orbit O on ω^n for all n. This means $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are in the same G-orbit, so there exists some $g \in G$ for which $g.a_k = b_k$ for all $1 \leq k \leq n$. This means g extends f, and clearly g is an automorphism, since $G = Aut(A_G)$. □

4 Fraïssé limits

We define the nice property that appeared in Theorem 3.5.

Definition 4.1. *An $\mathcal{L}$-structure M is said to be ultrahomogeneous, if for any two finitely generated substructures $A, B \preceq M$ with an isomorphism $\phi : A \to B$, there exists an automorphism $\psi \in Aut(M)$ which is an extension of ϕ.*

Remark. *With this definition, the canonical structure A_G of some closed $G \leq S_\omega$ is said to be ultrahomogeneous.*

Definition 4.2. *For an $\mathcal{L}$-structure M, we denote by $Age(M)$ the class of $\mathcal{L}$-structures isomorphic to a finitely generated substructure of M.*

Definition 4.3. *(Extension Property, EP) Suppose M is an $\mathcal{L}$-structure. We say that M has the Extension Property if for any finitely generated $A \preceq M$ and an embedding $f : A \to B$ with $B \in Age(M)$, there is an embedding $g : B \to M$ for which $g(f(a)) = a$ for all a in A.*

An equivalent definition is the following: Suppose M is an $\mathcal{L}$-structure. We say that M has the Extension Property if for any $A, B \in Age(M)$ with $A \preceq B$ and an embedding $f : A \to M$, there is an embedding $g : B \to M$ which extends f.

Lemma 4.4. *If M is a countable ultrahomogeneous structure, then M has the Extension Property.*

Proof. Suppose $A \preceq M$ is finitely generated and $f : A \to B$ is an embedding with $B \in Age(M)$. First, suppose $B \preceq M$. Since f is an embedding, $f : A \to f(A)$ is an isomorphism between finitely generated substructures of M. By ultrahomogeneity of M, this extends to an automorphism $h : M \to M$. Therefore $g = h^{-1}|_B : B \to M$ is an embedding with $g(f(a)) = a$ for all a in A.

Now, for an arbitrary $B \in Age(M)$ we can find a finitely generated $\widehat{B} \preceq M$ and an isomorphism $\phi : B \to \widehat{B}$. the function $\widehat{f} := \phi \circ f : A \to \widehat{B}$ is an embedding, therefore, by the previous argument, there exists an embedding $g : \widehat{B} \to M$ with $g(\widehat{f}(a)) = a$ for all $a \in A$. This gives us an embedding $g \circ \phi : B \to M$ with $(g \circ \phi)(f(a)) = g(\widehat{f}(a)) = a$ for all $a \in A$, which proves that M satisfies EP. □

Lemma 4.5. *Suppose M and H are countable structures with the Extension Property where $Age(M) = Age(H)$, and suppose $f : M_0 \to H_0$ is an isomorphism between the finitely generated substructures $M_0 \preceq M$ and $H_0 \preceq H$. Then there is an isomorphism $g : M \to H$, which is an extension of f.*

Proof. Suppose we have M, H, f, M_0, H_0 as above. We will construct an isomorphism $\Phi : M \to H$ using a back-and-forth argument. We will construct a sequence of partial isomorphisms $(\Phi_k)_{k \in \omega}$ by induction on k, where each partial isomorphism extends the previous one. First, we list the elements of M and H so we have $M = \{m_0, m_1, \dots\}$ and $H = \{h_0, h_1, \dots\}$. Suppose

$\Phi_k : M_k \to H_k$ is an isomorphism with $M_k \preceq M, H_k \preceq H$ both finitely generated, $\{m_0, \dots, m_{k-1}\} \subseteq M_k$ and $\{h_0, \dots, h_{k-1}\} \subseteq H_k$.

If $m_k \in M_k$, then let $\widehat{M_k} := M_k$ and $\widehat{H_k} := H_k$ and $\widehat{\Phi_k} := \Phi_k$. If $m_k \notin M_k$, then let $\widehat{M_k}$ be the structure generated by $M_k \cup \{m_k\}$, which is clearly finitely generated. The function $\Phi_k^{-1} : H_k \to \widehat{M_k}$ is an embedding. By the Extension Property, there is an embedding $\widehat{\Phi_k} : \widehat{M_k} \to H$ with $\widehat{\Phi_k}(\Phi_k(h)) = h$ for all $h \in H_k$, therefore $\widehat{\Phi_k}$ is an extension of Φ_k. The range of $\widehat{\Phi_k}$ is clearly finitely generated, and will be denoted by $\widehat{H_k}$. We achieved that $m_k \in dom(\widehat{\Phi_k})$

We do a similar step to get Φ_{k+1} in a way such that $h_k \in ran(\Phi_{k+1})$. If $h_k \in \widehat{H_k}$, then let $M_{k+1} := \widehat{M_k}$ and $H_{k+1} := \widehat{H_k}$ and $\Phi_{k+1} := \widehat{\Phi_k}$. If $h_k \notin \widehat{H_k}$, then let H_{k+1} be the structure generated by $\widehat{H_k} \cup \{h_k\}$, which is finitely generated. The function $\widehat{\Phi_k} : \widehat{M_k} \to H_{k+1}$ is an embedding. By the Extension Property, there is an embedding $\Psi : H_{k+1} \to M$ with $\Psi(\widehat{\Phi_k}(m)) = m$ for all $m \in \widehat{M_k}$, therefore Ψ^{-1} is an extension of $\widehat{\Phi_k}$. The range of Ψ is finitely generated, and will be denoted by M_{k+1}. Now we let $\Phi_{k+1} = \Psi^{-1} : M_{k+1} \to H_{k+1}$. This means Φ_{k+1} is an isomorphism between M_{k+1} and H_{k+1}, where M_{k+1} and H_{k+1} are both finitely generated, $\{m_0, \dots, m_k\} \subseteq M$ and $\{h_0, \dots, h_k\} \subseteq H$, and Φ_{k+1} extends Φ_k. This proves the induction step.

For the base case, we let $\Phi_0 := f$, since $f : M_0 \to H_0$ is an isomorphism with $M_0 \preceq M$ and $H_0 \preceq H$ both finitely generated, and we have no restriction on the domain and range of f. In the sequence of functions $\Phi_0, \Phi_1, \dots$ each function extends the previous one, so the union is well-defined and we let $\Phi := \bigcup_{k \in \omega} \Phi_k$. Clearly Φ is an isomorphism, $dom(\Phi)$ must include all elements of M, and $ran(\Phi)$ must include all elements of H, therefore $\Phi : M \to H$ is an isomorphism that extends f. □

Corollary 4.6. *As a special case of the previous lemma, suppose $M = H$ is a countable structure with the Extension Property. The lemma states that given finitely generated substructures $M_1 \preceq M$ and $M_2 \preceq M$ and an isomorphism $f : M_1 \to M_2$, we can extend f to an automorphism $g : M \to M$, therefore M is ultrahomogeneous. This fact, along with Lemma 4.4 gives us the following corollary:*

Corollary 4.7. *For a countable structure M, the following are equivalent:*

(1) M is ultrahomogeneous,

(2) M has the Extension Property.

Definition 4.8. *A <u>Fraïssé class/amalgamation class</u> is a nonempty class $\mathcal{K}$ of finitely generated, countable $\mathcal{L}$-structures of countably many isomorphism types, for which the following are true:*

1. *(Isomorphism Property, IP) $\mathcal{K}$ is closed under isomorphisms,*
2. *(Hereditary Property, HP) $\mathcal{K}$ is closed under finitely generated substructures,*

3. *(Joint Embedding Property, JEP) For any $A_1, A_2 \in \mathcal{K}$, there exists some $B \in \mathcal{K}$ and embeddings $f_1 : A_1 \to B$ and $f_2 : A_2 \to B$,*

4. *(Amalgamation Property, AP) For any $A, B_1, B_2 \in \mathcal{K}$ and embeddings $f_1 : A \to B_1$ and $f_2 : A \to B_2$, there exists some $C \in \mathcal{K}$ and embeddings $g_1 : B_1 \to C$ and $g_2 : B_2 \to C$ such that $g_1(f_1(a)) = g_2(f_2(a))$ for all $a \in A$.*

We will now state the famous theorem of Fraïssé.

Theorem 4.9. *We state the following:*

(1) If M is an ultrahomogeneous countable structure, then $Age(M)$ is a Fraïssé class,

(2) If $\mathcal{K}$ is a Fraïssé class, then there is an ultrahomogeneous countable structure M for which $Age(M) = \mathcal{K}$,

(3) If $\mathcal{K}$ is a Fraïssé class, and M is a countable structure with the Extension property where $Age(M) = \mathcal{K}$, then M is determined up to isomorphism.

Corollary 4.10. *This means that for a Fraïssé class $\mathcal{K}$, there exists an ultrahomogeneous countable structure M for which $Age(M) = \mathcal{K}$, and M is determined up to isomorphism. We call this structure M the* <u>*Fraïssé limit*</u> *of $\mathcal{K}$.*

Proof. (1) Since M is countable, it has countably many finitely generated substructures, therefore $Age(M)$ contains countably many isomorphism types. By the definition of $Age(M)$ it is easy to see that $Age(M)$ consists of finitely generated structures, and satisfies IP and HP.

To prove that $Age(M)$ satisfies JEP suppose $A, B \in Age(M)$. Our goal is to find a structure $C \in Age(M)$ and embeddings $f_A : A \to C$ amd $f_B : B \to C$. Since $A, B \in Age(M)$, the structure A is isomorphic to some substructure $\widehat{A} \preceq M$, generated by some finite set $\{a_1, \ldots, a_k\} \subseteq M$. Similarly, B is isomorphic to $\widehat{B} \preceq M$ generated by some finite set $\{b_1, \ldots, b_l\} \subseteq M$. Therefore, let $\phi_A : A \to \widehat{A}$ and $\phi_B : B \to \widehat{B}$ be isomorphisms. Let C be the structure generated by the finite set $\{a_1, \ldots, a_k, b_1, \ldots, b_l\} \subseteq M$. Clearly $C \in Age(M)$ and $f_A = id|_{\widehat{A}} : \widehat{A} \to C$ is an embedding and $f_B = id|_{\widehat{B}} : \widehat{B} \to C$ is an embedding. Therefore $f_A \circ \phi_A : A \to C$ and $f_B \circ \phi_B : B \to C$ are embeddings, thus $Age(M)$ satisfies JEP.

To prove that $Age(M)$ satisfies AP suppose $A_0, A_1, A_2 \in Age(M)$ and $f_1 : A_0 \to A_1$ and $f_2 : A_0 \to A_2$ are embeddings. Our goal is to find a structure $C \in Age(M)$ and embeddings $g_1 : A_1 \to C$ and $g_2 : A_2 \to C$ with $g_1(f_1(a)) = g_2(f_2(a))$ for all $a \in A_0$.

First, assume $A_0, A_1, A_2 \preceq M$. Since f_1 is an embedding, $f_1 : A_0 \to f_1(A_0)$ is an isomorphism between finitely generated substructures of M. By ultrahomogeneity of M, this extends to an automorphism $h_1 : M \to M$. Similarly, we can extend f_2 to an automorphism $h_2 : M \to M$. Let $C := A_1 \cup h_1 \circ h_2^{-1}(A_2)$, which is a finitely generated substructure of M. Let $g_1 := id_{A_1} : A_1 \to C$

and $g_2 := h_1 \circ h_2^{-1} : A_2 \to C$, which are embeddings. For all $a \in A_0$ we have $g_2 \circ f_2(a) = h_1 \circ h_2^{-1} \circ f_2(a) = h_1(a) = f_1(a) = g_1 \circ f_1(a)$.

Now suppose we have $A_0, A_1, A_2 \in Age(M)$ arbitrary. There exist finitely generated $\widehat{A_0}, \widehat{A_1}, \widehat{A_2} \preceq M$, and isomorphisms $\phi_0 : A_0 \to \widehat{A_0}$, $\phi_1 : A_1 \to \widehat{A_1}$, $\phi_2 : A_2 \to \widehat{A_2}$. We can construct embeddings $\widehat{f_1} = \phi_1 \circ f_1 \circ \phi_0^{-1} : \widehat{A_0} \to \widehat{A_1}$ and $\widehat{f_2} = \phi_2 \circ f_1 \circ \phi_0^{-1} : \widehat{A_0} \to \widehat{A_2}$. By the previous case, we know there is some $C \preceq M$ and embeddings $g_1 : \widehat{A_1} \to C$ and $g_2 : \widehat{A_2} \to C$ with $g_1 \circ \widehat{f_1}(a) = g_2 \circ \widehat{f_2}(a)$ for all $a \in \widehat{A_0}$. Let $h_1 := g_1 \circ \phi_1 : A_1 \to C$ and $h_2 := g_2 \circ \phi_2 : A_2 \to C$, which are embeddings. For all $a \in A_0$ we have $h_2 \circ f_2(a) = g_2 \circ \phi_2 \circ f_2(a) = g_2 \circ \phi_2 \circ f_2 \circ \phi_0^{-1} \circ \phi_0(a) = g_2 \circ \widehat{f_2} \circ \phi_0(a) = g_1 \circ \widehat{f_1} \circ \phi_0(a) = h_1 \circ f_1(a)$, therefore $Age(M)$ satisfies AP.

(2) For the construction of M, which will be the Fraïssé limit of $\mathcal{K}$, we will use a standard bookkeeping argument. First, we will construct a sequence of structures $(A_k)_{k\in\omega}$ for which the following are true:

(a) $A_k \in \mathcal{K}$ for all $k \in \omega$,

(b) $A_k \preceq A_{k+1}$ for all $k \in \omega$,

(c) Every $A \in \mathcal{K}$ can be embedded into some A_k,

(d) Suppose $A, B \in \mathcal{K}$, and we have embeddings $f_1 : A \to A_n$ and $f_2 : A \to B$ for some $n \in \omega$. Then there is some $m \in \omega$ with $m > n$ and an embedding $g : B \to A_m$ with $f_1(a) = g(f_2(a))$ for all $a \in A$.

Since $\mathcal{K}$ has countably many isomorphism types, we can pick a structure from each isomorphism type and list them, so we let $\mathbb{T} := \{T_1, T_2, \dots\}$ be the set of representatives of the isomorphism types. We will construct the sequence $(A_k)_{k\in\omega}$ by induction on k. For the base case, $A_0 := T_0$.

Suppose we have $A_k \in \mathcal{K}$ for some odd number $k \in \omega$. By the Joint Embedding Property of $\mathcal{K}$, there is a structure $\widehat{A_{k+1}} \in \mathcal{K}$ and embeddings $f_1 : A_k \to \widehat{A_{k+1}}$ and $f_2 : T_{(k+1)/2} \to \widehat{A_{k+1}}$. Formally, this does not mean $A_k \preceq \widehat{A_{k+1}}$, but notice that since A_k can be embedded into $\widehat{A_{k+1}}$ we can build a structure isomorphic to $\widehat{A_{k+1}}$ by adding elements to the universe of A_k and defining the relations and functions on these new elements according to $\widehat{A_{k+1}}$. This gives us a structure A_{k+1} for which $A_k \preceq A_{k+1}$, and $T_{(k+1)/2}$ can be embedded into A_{k+1}.

Given $A, B \in \mathcal{K}$, there are only countably many ways to embed A into B, since A is finitely generated and an embedding is already determined by the images of the generating elements. This means that there are only countably many 6-tuples of the form (A, n, B, f, C, g), where $A, B, C \in \mathbb{T}$, $n \in \omega$, $f : A \to B$ is an embedding, and $g : A \to C$ is an embedding. We can construct a countable sequence $\mathbb{S} = (A_i, n_i, B_i, f_i, C_i, g_i)_{i\in\omega}$ in which every such 6-tuple appears infinitely many times.

Now suppose we have $A_k \in \mathcal{K}$ for an even number $k \in \omega$. Take the ith element of the sequence $\mathbb{S}$ where $i = k/2 + 1$, and denote it by $\mathbb{S}_i =$

$(A_i, n_i, B_i, f_i, C_i, g_i)$. If $k < n_i$ then we simply let $A_{k+1} := A_k$. If $k \geq n_i$, but A_{n_i} does not have isomorphism type C, then we let $A_{k+1} := A_k$ also. If $k \geq n_i$ and there is an isomorphism $\phi : C \to A_{n_i}$, then we let $g_i' := \phi \circ g_i : A \to A_{n_i}$, which is an embedding. We will construct A_{k+1} in a way such that it satisfies property (d) for the structures A_i and B_i and the embeddings $f_i : A_i \to B_i$ and $g_i' : A_i \to A_{n_i}$. By the Amalgamation Property of $\mathcal{K}$ there is some $\widehat{A_{k+1}} \in \mathcal{K}$ and embeddings $h_1 : A_{n_i} \to \widehat{A_{k+1}}$ and $h_2 : B_i \to \widehat{A_{k+1}}$ with $h_1(g_i'(a)) = h_2(f_i(a))$ for all $a \in A_i$. Again, formally we do not have $A_k \preceq \widehat{A_{k+1}}$, but by the same argument as before we can construct A_{k+1} isomorphic to $\widehat{A_{k+1}}$, for which $A_k \preceq A_{k+1}$ and $h_1 : A_{n_i} \to A_{k+1}$ is the identity. This gives us a structure A_{k+1} with $A_k \preceq A_{k+1}$ and an embedding $h_2 : B \to A_{k+1}$ with $g_i'(a) = h_2(f_i(a))$ for all $a \in A_i$, which is what we wanted.

We have now constructed the sequence of structures $(A_k)_{k\in\omega}$ and we want to prove this sequence satisfies properties (a)-(d). Clearly we have $A_k \in \mathcal{K}$ and $A_k \preceq A_{k+1}$ for all $k \in \omega$. Suppose we have an arbitrary $A \in \mathcal{K}$ with isomorphism type T_n for some $n \in \omega$. By our construction, T_n can be embedded into A_{2n}, therefore A can also be embedded into A_{2n}, which means our sequence satisfies property (c).

Suppose we have $A, B \in \mathcal{K}$, and embeddings $f_1 : A \to A_n$ and $f_2 : A \to B$ for some $n \in \omega$. We let $C := A_n$, and we suppose A, B, C have isomorphism types $T_{i_A}, T_{i_B}, T_{i_C}$ respectively. During our construction, we encountered the 6-tuple $(T_{i_A}, n, T_{i_B}, f_2', T_{i_C}, f_1')$ infinitely many times, where $f_1' = f_1$ and $f_2' = f_2$ up to isomorphism. This means we encountered the 6-tuple $(T_{i_A}, n, T_{i_B}, f_2', T_{i_C}, f_1')$ on some step number k, where k is an even number $k \in \omega$ and $k \geq n$. We constructed A_{k+1} in a way such that there exists an embedding $g' : B \to A_{k+1}$ with $f_1'(a) = g'(f_2'(a))$ for all $a \in A$. Formally, we do not have an embedding $g : B \to A_{k+1}$, but up to isomorphism $g' : B \to A_{k+1}$ satisfies $f_1(a) = g(f_2(a))$ for all $a \in A$, so our sequence satisfies property (d).

Since $A_k \preceq A_{k+1}$ for all $k \in \omega$, the union is well-defined and we let $M := \bigcup_{k\in\omega} A_k$, which is clearly countable. To show that $Age(M) = \mathcal{K}$, suppose $A \in \mathcal{K}$. We know A can be embedded into some $A_k \preceq M$ with an embedding $f : A \to A_k$. Since A is finitely generated, $f(A)$ is also finitely generated. This means that A is isomorphic to a finitely generated substructure of M, therefore $A \in Age(M)$, proving $\mathcal{K}$ is a subset of $Age(M)$.

On the other hand, suppose $A \in Age(M)$. The structure A is isomorphic to some finitely generated $B \preceq M$. The finitely many elements that generate B must all appear in some A_k, therefore B is a finitely generated substructure of some $A_k \in \mathcal{K}$. This means A is isomorphic to a finitely generated substructure of an element of $\mathcal{K}$, and since $\mathcal{K}$ satisfies HP and IP, $A \in \mathcal{K}$. This proves $Age(M)$ is a subset of $\mathcal{K}$, so we have $Age(M) = \mathcal{K}$.

To show that M is ultrahomogeneous, we will show that it has the Extension Property. Suppose $A \preceq M$ is finitely generated and $f_1 : A \to B$ is an embedding for some $B \in Age(M)$. The finitely many elements that generate A all appear in some A_k, so we have $A \preceq A_k$. In other words, the identity $f_2 = id|_A : A \to A_k$ is an embedding. Clearly $A \in Age(M)$, and since $Age(M) = \mathcal{K}$, we have $A, B \in \mathcal{K}$. By property (d), for some $m > n$ there exists an embedding $g : B \to A_m$ with $g(f_1(a)) = f_2(a) = a$ for all $a \in A$, which proves that M satisfies EP. Therefore, by Corollary 4.7, M is ultrahomogeneous, thus we have successfully constructed the Fraïssé limit of $\mathcal{K}$.

(3) Suppose M and H are both countable structures with the Extension Property and $Age(M) = Age(H) = \mathcal{K}$. We want to show that M is isomorphic to H. Let $M_0 \preceq M$ be the structure generated by the empty set in M, which is clearly finitely generated. Since $M_0 \in Age(M) = Age(H)$, there is an isomorphism $f : M_0 \to H_0$ for some finitely generated $H_0 \preceq H$. By Lemma 4.5 there is an extension of f which is an isomorphism $g : M \to H$. This proves that M and H are isomorphic and that M is determined up to isomorphism.

□

4.1 Examples

4.1.1 The set of rationals with the standard ordering

One of the most basic examples of a Fraïssé limit is the Fraïssé limit of the class of finite ordered sets.

Theorem 4.11. *We state the following:*

(1) The class of finite ordered sets, denoted by $\mathcal{K}$, is a Fraïssé class,

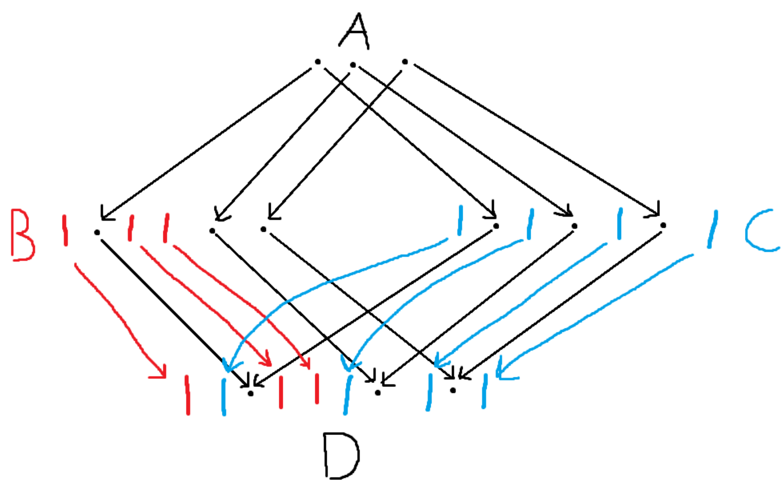
(2) The Fraïssé limit of $\mathcal{K}$ is the set of rational numbers with the standard ordering, denoted by $(\mathbb{Q}, <)$.

Proof. (1): Clearly every structure in $\mathcal{K}$ is countable and finitely generated, and $\mathcal{K}$ satisfies IP and HP. The class $\mathcal{K}$ contains only countably many isomorphism types, since for every finite n, there exists a unique ordered set on n elements, up to isomorphism. For any two finite ordered sets, the one with smaller or equal size can be embedded into the other one, thus $\mathcal{K}$ satisfies JEP.

To prove that $\mathcal{K}$ satisfies AP, suppose we have finite ordered sets A, B, C and embeddings $f_1 : A \to B$ and $f_2 : A \to C$. We need to find a finite ordered set D and embeddings $g_1 : B \to D$ and $g_2 : C \to D$, for which $g_1(f_1(a)) = g_2(f_2(a))$ for all $a \in A$. We can assume that $A \preceq B$ and $A \preceq C$, the functions f_1 and f_2 are the identity, and the universe of B and the universe of C intersect precisely in the universe of A. Let D be $B \cup C$, and let the functions $g_1 : B \to D$ and $g_2 : C \to D$ be the identity. Say $A = \{a_1, \dots, a_n\}$, then the n elements of A define $n + 1$ intervals of the forms:

$$\{x < a_1\},\ \{a_k < x < a_{k+1}\},\ \{a_n < x\},$$

some of which may be empty. We define the ordering on D the following way: two elements of the set $B \subseteq D$ will have the same relation as they had in the structure B, and two elements of the set $C \subseteq D$ will have the same relation as they had in the structure C. If an element of the set $B \subseteq D$ and an element of the set $C \subseteq D$ are in different intervals, the one in the "lower" interval will be defined to be smaller. If an element of the set $B \subseteq D$ and an element of the set $C \subseteq D$ are in the same interval, the element from B will always be defined to be smaller. This gives us an ordering on the set D for which g_1, g_2 are embeddings and $g_1(f_1(a)) = g_2(f_2(a))$ for all $a \in A$ because f_1, f_2, g_1, g_2 are all the identity. This proves that $\mathcal{K}$ satisfies AP, thus $\mathcal{K}$ is a Fraïssé class.



(2): Since $(\mathbb{Q}, <)$ is a relational structure, the structure generated by a finite subset $F \subseteq \mathbb{Q}$ is simply F. It is easy to see that any structure isomorphic to a finite substructure of $(\mathbb{Q}, <)$ is a finite ordered set, and that every finite ordered set is isomorphic to a finite substructure of $(\mathbb{Q}, <)$, therefore $Age((\mathbb{Q}, <)) = \mathcal{K}$.

To show that $(\mathbb{Q}, <)$ is ultrahomogeneous, suppose we have two finite substructures $\{p_1, \ldots, p_n\} \preceq \mathbb{Q}$ and $\{q_1, \ldots, q_n\} \preceq \mathbb{Q}$ which are isomorphic. We can assume $p_1 < \ldots p_n$ and $q_1 < \cdots < q_n$. We must find an automorphism of $(\mathbb{Q}, <)$, in other words, an order-preserving function $f : \mathbb{Q} \to \mathbb{Q}$, for which $f(p_i) = q_i$ for $i = 1, \ldots, n$. This is an easy task, as we can define $f(p_i) := q_i$ for $i = 1, \ldots, n$ and linearly interpolate everywhere else.

Since $(\mathbb{Q}, <)$ is countable, ultrahomogeneous, and $Age((\mathbb{Q}, <)) = \mathcal{K}$, it follows from Theorem 4.9 (3) that $(\mathbb{Q}, <)$ is the Fraïssé limit of $\mathcal{K}$, up to isomorphism. □

It is important to mention here that countability is crucial in determining the Fraïssé limit up to isomorphism, since the real numbers with the standard ordering, denoted by $(\mathbb{R}, <)$, is also an ultrahomogeneous structure with $Age((\mathbb{R}, <)) = \mathcal{K}$, but $(\mathbb{R}, <)$ is clearly not isomorphic to $(\mathbb{Q}, <)$.

4.1.2 The random graph

The Random graph is a well-known infinite graph with multiple equivalent definitions. The following theorem gives us three equivalent definitions, combining the fields of first-order logic, graph theory, and probability theory.

Theorem 4.12. *We state the following:*

(1) The class of finite graphs, denoted by $\mathcal{K}$, is a Fraïssé class. We will let G be the Fraïssé limit of $\mathcal{K}$,

(2) Suppose H is a graph with $V_H = \omega$ with the property that for any two sets $A \subseteq \omega$ and $B \subseteq \omega$ with $A \cap B = \emptyset$, there is some $v \in \omega$ where $v \notin A \cup B$ and v is connected to all vertices in A, and v is not connected to any vertices in B. Then H is isomorphic to G,

(3) Suppose we create a graph R with $V_R = \omega$ by connecting every pair of vertices with probability $1/2$ independently. Then R has the property in (2) with probability 1, and therefore R is isomorphic to G with probability 1.

Proof. (1): Similarly to the class of finite ordered sets, every structure in $\mathcal{K}$ is countable and finitely generated, $\mathcal{K}$ contains only countably many isomorphism types, and $\mathcal{K}$ satisfies IP and HP. For any two graphs A and B, which we can assume to be disjoint, A and B can be embedded into the disjoint union of A and B by the identity function, therefore $\mathcal{K}$ satisfies JEP.

We will prove that $\mathcal{K}$ satisfies AP similarly to how we proved that the class of finite ordered sets satisfies AP. Suppose we have finite graphs $A = (V_A, E_A)$, $B = (V_B, E_B)$, $C = (V_C, E_C)$ and embeddings $f_1 : A \to B$ and $f_2 : A \to C$. We need to find a finite graph D and embeddings $g_1 : B \to D$ and $g_2 : C \to D$, for which $g_1(f_1(a)) = g_2(f_2(a))$ for all $a \in A$. We can assume that $A \preceq B$ and $A \preceq C$, the functions f_1 and f_2 are the identity, and $V_B \cap V_C = V_A$. The universe of the structure D will be $V_D := V_B \cup V_C$, the edges of D will be $E_D = E_B \cup E_C$ and the embeddings $g_1 : B \to D$ and $g_2 : C \to D$ will be the identity. Clearly D is a finite graph and we have $g_1(f_1(a)) = g_2(f_2(a))$ for all $a \in V_A$ since f_1, f_2, g_1, g_2 are all the identity. This proves $\mathcal{K}$ satisfies AP, thus $\mathcal{K}$ is a Fraïssé class.

(2): Suppose that H is a graph on ω with the property in (2), and G is the Fraïssé limit of $\mathcal{K}$ as above. We would like to show that H is isomorphic to G. Since H is clearly countable, by Theorem 4.9 (3), it is enough to show that H has the Extension Property and $Age(H) = \mathcal{K}$.

Since the language of H is relational, the structure generated by a finite subset $F \subseteq H$ is simply F. To prove that H has the Extension Property, suppose we have $A, B \in Age(H)$ with $A \preceq B$ and an embedding $f_0 : A \to H_0 \preceq H$. Clearly A and B are both finite graphs, so we will let $V_A = \{a_1, \dots, a_m\}$ and $V_B = \{a_1, \dots, a_m, b_1, \dots, b_n\}$. We must find an embedding $g : B \to H$ that extends f_0 and we will do so by induction on n. Suppose we have an isomorphism $f_k : A_k \to H_k$ between the graphs A_k and H_k for some $1 \leq k \leq n$, where $V_{A_k} = V_A \cup \{b_1, \dots, b_k\}$ and $V_{H_k} = \{h_1, \dots, h_{m+k}\}$. We define the set of vertices in A_k which are connected to b_{k+1}, and the set of vertices in A_k which are not connected to b_{k+1} as follows:

$$A_k^{(1)} = \{v \in A_k \mid (v, b_{k+1}) \in E_B\} \text{ and } A_k^{(2)} = \{v \in A_k \mid (v, b_{k+1}) \notin E_B\}.$$

Notice that $A_k^{(1)}$ and $A_k^{(2)}$ are disjoint, and since f_k is an isomorphism, $f(A_k^{(1)})$ and $f(A_k^{(2)})$ are disjoint as well. By the property in (2), we can find a vertex $h_{m+k+1} \in \omega = V_H$, which is connected to all vertices in $f_k(A_k^{(1)})$ and not connected to any vertices in $f_k(A_k^{(2)})$. This gives us an isomorphism $f_{k+1} : A_{k+1} \to H_{k+1}$, where $V_{A_{k+1}} = V_A \cup \{b_1, \ldots, b_{k+1}\}$ and $V_{H_{k+1}} = \{h_1, \ldots, h_{m+k+1}\}$ and $\phi_{k+1}(a_{k+1}) = h_{m+k+1}$. The isomorphism f_{k+1} extends f_k, so this proves the induction step. Therefore we can find an isomorphism $f_n : B \to H_n$ which extends f_0, so H satisfies EP.

To show that $Age(H) = \mathcal{K}$, it is easy to see that every finite substructure of H is a finite graph, so we have $Age(H) \subseteq \mathcal{K}$. Now suppose $A \in \mathcal{K}$ is a finite graph. Pick any two points $a \in A$ and $h \in H$. The function $f : \{a\} \to \{h\}$ between the one-vertex graphs $\{a\}$ and $\{h\}$ is clearly an isomorphism. By the Extension Property, we can extend f to an isomorphism $g : A \to H_0$, for some finite subgraph H_0 of H. This shows $\mathcal{K} \subseteq Age(H)$, so we have $Age(H) = \mathcal{K}$. By Theorem 4.9 (3), the graphs H and G are isomorphic.

(3): We want to prove that the graph R has the property in (2) with probability 1. Suppose we have finite sets $A \subseteq \omega$ and $B \subseteq \omega$ with $A \cap B = \emptyset$, $|A| = n$, $|B| = m$, and some vertex $v \notin A \cup B$. Denote the event that v is connected to all vertices in A and no vertices in B by $Q_{A,B,v}$. Since every pair of vertices is connected with probability $1/2$ independently, the probability of $Q_{A,B,v}$ is $\mathbb{P}(Q_{A,B,v}) = (1/2)^n(1/2)^m = 2^{-(n+m)} > 0$. Therefore, the complement of $Q_{A,B,v}$ has probability $\mathbb{P}(Q^C_{A,B,v}) = 1 - 2^{-(n+m)} < 1$. For every $v \neq v'$ with $v, v' \notin A \cup B$, the events $Q_{A,B,v}$ and $Q_{A,B,v'}$ are independent, thus the probability that the event $Q_{A,B,v}$ does not occur for any $v \notin A \cup B$ is

$$\mathbb{P}\Big(\bigcap_{v \notin A \cup B} Q^C_{A,B,v}\Big) = \prod_{v \notin A \cup B} \mathbb{P}(Q^C_{A,B,v}) = \lim_{i \to \infty} (1 - 2^{-(n+m)})^i = 0.$$

Therefore, there is a vertex $v \notin A \cup B$, which is connected to all vertices in A and not connected to any vertices in B with probability 1. This is true for all finite subsets $A \subseteq \omega$ and $B \subseteq \omega$ with $A \cap B = \emptyset$, so we have shown that R satisfies the property in (2) with probability 1. □

5 Extreme amenability - Kechris, Pestov, Todorcevic

5.1 The KPT correspondence

In this section we will be following section 4. of Kechris, Pestov, Todorcevic [3]

Definition 5.1. *Suppose G is a Hausdorff topological group and X is a compact Hausdorff space. A G-flow is a continuous action of G on X. For simplicity of notation, we will say X is a G-flow, instead of saying $G \curvearrowright X$ is a G-flow.*

Remark. *For the rest of this section, we will always assume that G is Hausdorff.*

Definition 5.2. *A topological group G is said to be extremely amenable if every G-flow has a fixed point, in other words, if X is a compact Hausdorff space and X is a G-flow, there is some $x \in X$ for which $g.x = x$ for all $g \in G$.*

Lemma 5.3. *Suppose G is a topological group, and X is a G-flow, then the following are equivalent:*

(1) The G-flow X has a fixed point,

(2) For every $n = 1, 2, \dots,$ continuous function $f : X \to \mathbb{R}^n$, $\varepsilon > 0$, and finite $F \subseteq G$, there is some $x \in X$ such that $|f(x) - f(g.x)| \leq \varepsilon$ for every $g \in F$, where $|\ |$ is the Euclidean norm.

Proof. (1) $\Rightarrow$ (2): Clearly, if $x \in X$ is a fixed point, $x = gx$ for every $g \in G$, therefore $|f(x) - f(g.x)| = 0$ for every function f.

(2) $\Rightarrow$ (1): First, for fixed n, f, ε, F as in (2), we define the set

$$H_{f,\varepsilon,F} := \{x \in X \mid |f(x) - f(g.x)| \leq \varepsilon \text{ for every } g \in F\} \subseteq X.$$

Notice that if $F = \{g\}$, the function $f' : X \to \mathbb{R}^n$, where $f'(x) = |f(x) - f(g.x)|$, is continuous. Therefore, $H_{f,\varepsilon,\{g\}}$ is the preimage of the closed set $\{x \in \mathbb{R}^n \mid |x| \leq \varepsilon\}$, so $H_{f,\varepsilon,\{g\}}$ is closed. The set $H_{f,\varepsilon,F}$ is the intersection of $H_{f,\varepsilon,\{g\}}$ for all $f \in F$, where F is finite, thus $H_{f,\varepsilon,F}$ is closed as well. Since X is compact, $H_{f,\varepsilon,F}$ is also compact. We would like to show that the intersection of $H_{f,\varepsilon,F}$ for all f, ε, F is nonempty. By a compactness argument, it is enough to show that the intesection of finitely many $\{H_{f_i,\varepsilon_i,F_i}\}_{1 \leq i \leq n}$ is nonempty. Suppose $f_i : \mathbb{R} \to \mathbb{R}^{m_i}$, then we let

$$\begin{aligned} f' &:= (f_1, \dots, f_n) : \mathbb{R} \to \mathbb{R}^{m_1 + \cdots + m_n} \\ \varepsilon' &:= min(\varepsilon_1, \dots, \varepsilon_n) \\ F' &:= F_1 \cup \cdots \cup F_n. \end{aligned}$$

By our assumption, $H_{f',\varepsilon',F'}$ is nonempty, therefore $\bigcap\limits_{i=1}^{n} H_{f_i,\varepsilon_i,F_i} \neq \emptyset$. This shows that the intersection of $H_{f,\varepsilon,F}$ for all f, ε, F is nonempty. But notice that an element $x \in \bigcap\limits_{f,\varepsilon,F} H_{f,\varepsilon,F}$ must be a fixed point.

If x was not a fixed point, there would be some $g \in G$ with $g.x \neq x$. Since X is compact and Hausdorff, X is T_4. By Urysohn's lemma there is a continuous function $f : X \to \mathbb{R}$ with $f(x) = 0$ and $f(g.x) = 1$. This means, for $F = \{g\}$ and $0 < \varepsilon < 1$, the set $H_{f,\varepsilon,F}$ is empty, since $|f(x) - f(g.x)| = 1 > \varepsilon$, which would be a contradiction. □

We will now prove the following technical lemma, as we will need it for the proof of Theorem 5.5.

Lemma 5.4. *Suppose G is something, X is a G-flow, and $f : X \to \mathbb{R}^n$ for some $n \in \omega$. Then there is an open neighborhood V of the identity 1_G, such that for every $v \in V$ and every $x \in X$ we have $|f(x) - f(v.x)| \leq \varepsilon$, where $|\ |$ refers to the Euclidian norm.*

Proof. First, fix some $x \in X$. Since f is continuous, we can find some W_x neighborhood of x, for which $|f(x) - f(w)| \leq \varepsilon/2$ for all $w \in W_x$. Since the action $a : G \times X \to X$ is continuous, we can find an open neighborhood U_x of x and an open neighborhood V_x of 1_G such that $a(V_x \times U_x) \subseteq W_x$. This means that for every $v \in V_x$ and every $y \in U_x$ we have $|f(x) - f(v.y)| \leq \varepsilon/2$. Now we will use a compactness argument. Since X is compact, it can be written as the union of U_x for finitely many x, so we say $X = \bigcup_{i=1}^{n} U_{x_i}$. We let $V = \bigcap_{i=1}^{n} V_{x_i}$, which is a neighborhood of 1_G. For any $v \in V$ and $x \in X$ where $x \in U_{x_i}$ for some x_i, we have

$$|f(x) - f(v.x)| \leq |f(x) - f(x_i)| + |f(x_i) - f(v.x)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

because $x, x_i \in U_{x_i}$ and $v \in V \subseteq V_i$, so we are done. □

Theorem 5.5. *Suppose $G \leq S_\omega$ is closed, then the following are equivalent:*

(1) The group G is extremely amenable,

(2) For any open subgroup $V \leq G$, any k-coloring $c : G/V \to \{1, \dots, k\}$ of the left-cosets of V, and any finite set of left-cosets $A \subseteq G/V$, there is an element $g \in G$, for which $g.A$ is monochromatic. In other words, there is a color $1 \leq i \leq k$ for which $c(g.a) = i$ for every $a \in A$, where G acts on G/V in the usual way $g.hV = ghV$.

Proof. (1) ⇒ (2): Suppose we have V, c, A as above. Consider the set $C = \{1, \dots, k\}^{G/V}$ of k-colorings of the left-cosets of V. The space C is the product of Hausdorff spaces, so C is Hausdorff. Furthermore, by Tychonoff's theorem, the space C is compact (with respect to the natural topology), as it is the product of compact spaces. Consider the action $(g.p)(x) = p((g^{-1}).x)$ for $g \in G$, $p \in C$, and $x \in G/V$, which is associative since

$$g.(h.p)(x) = (h.p)((g^{-1}).x) = p((h^{-1})(g^{-1}).x) = (gh).p(x),$$

where $h \in G$. This gives us that C is a G-flow, and the closed subset $X := \overline{G.c} \subseteq C$ is also a G-flow. Since G is extremely amenable, we can find a fixed point

$c^* \in X$. The coloring c^* must be constant, since $c^*(x) = g.c^*(x) = c^*((g^{-1}).x)$ for all $g \in G$, and G acts transitively on G/V, so we have $c^*(x) = i$ for all $x \in G/V$. Since A is finite and $c^* \in \overline{G.c}$ we can find some $g \in G$ for which $(g^{-1}).c|_A = c^*|_A$. Therefore, $c(g.a) = c^*(a) = i$ for all $a \in A$, so we are done.

(2) $\Rightarrow$ (1): Let X be a G-flow, we want to show that X has a fixed point. By Lemma 5.3, it is enough to show that if we have $n \in \omega$, $f : X \to \mathbb{R}^n$ continuous, $\varepsilon > 0$, and $F \subseteq G$ finite, then there exists an $x \in X$ for which $|f(x) - f(g.x)| \leq \varepsilon$ for all $g \in F$. Using Lemma 5.4, we can find an open neighborhood V' of the identity 1_G, for which $\forall v \in V'$, $\forall x \in X$, $\forall \varepsilon > 0$ we have $|f(x) - f(v.x)| \leq \varepsilon/3$. Recall Theorem 3.3, which gives us that there is a system of subgroups which form an open neighborhood basis at the identity 1_G, since G must be non-Archimedean. This means there is an open subgroup $V \subseteq V'$, for which $\forall v \in V$, $\forall x \in X$, $\forall \varepsilon > 0 : |f(x) - f(v.x)| \leq \varepsilon/3$ holds.

We will partition the set $f(X) \subseteq \mathbb{R}^n$ into finitely many subsets $A_1, \dots, A_k \subseteq \mathbb{R}^n$ of diameter at most $\varepsilon/3$. For a fixed $x' \in X$, let

$$U_i := \{g \in G \mid f(g.x') \in A_i\}, \text{ and } V_i := VU_i = \{Vu \in G/V \mid u \in U_i\}.$$

Since $\bigcup_{i=1}^{k} V_i = G/V$ we can define a coloring on all of G/V. , we can find a k-coloring c of the left-cosets of V such that $c^{-1}(i) \subseteq V_i$ for all $1 \leq i \leq k$. Let $\widehat{F}$ be the finite set $F \cup \{1_G\}$. It is easy to see that property (2) is equivalent to the analogous statement for the space of right-cosets of V, where G acts in the usual way $g.Vh = Vhg^{-1}$. For V, c, $\widehat{F}V \subseteq V/G$, we can use this statement about right-cosets to get that there is a color $1 \leq i \leq k$ and some $g \in G$ such that $c((g^{-1}).V\widehat{F}) = i$ for all $a \in \widehat{F}$. This means that $V\widehat{F}g \subseteq V_i$, and since $1_G \in V$, we have that $\widehat{F}g \subseteq V_i$.

Now we will show that $x := gx'$ works, so we fix some $h \in \widehat{F}$, and we want to show that $|f(x) - f(hx)| \leq \varepsilon$ for all $h \in \widehat{F}$. Since $hg \subseteq V_i = VU_i$, we can write $hg = v_1' u_i$ for some $v_1' \in V$. Since V is a subgroup, $(v_1')^{-1} = v_1 \in V$, so we have $v_1 hg = u_i \in U_i$. Therefore,

$$f(v_1 hgx') \in A_i \ \Rightarrow \ f(v_1 hx) \in A_i.$$

Since $v_1 \in V$, we have

$$|f(v_1 hx) - f(hx)| \leq \varepsilon/3,$$

so $f(hx)$ is in the $\varepsilon/3$-neighborhood of A_i.

Similarly to before, since $1_G \in \widehat{F}$, we can write $g = v_2' u_i$ for some $v_2' \in V$. Since V is a subgroup, $(v_2')^{-1} = v_2 \in V$, so we have $v_2 g = u_i \in U_i$. Therefore,

$$f(v_2 gx') \in A_i \ \Rightarrow \ f(v_2 x) \in A_i.$$

Since $v_1 \in V$, we have

$$|f(v_2 x) - f(x)| \leq \varepsilon/3,$$

so $f(x)$ is in the $\varepsilon/3$-neighborhood of A_i.

Finally, since both $f(hx)$ and $f(x)$ are in the $\varepsilon/3$-neighborhood of A_i, we have

$$|f(x) - f(hx)| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$$

so we are done. □

Remark. *By the proof of (2) ⇒ (1) above, it is easy to see that we can restrict V to be from any given system of subgroups which from a neighborhood basis at 1_G. In particular, we can restrict V to be of the form $G_{(F)}$, the pointwise stabilizer of some finite $F \in \omega$, since the system of subgroups $\{G_{(F)} \mid F \subseteq \omega$ is finite$\}$ forms a neighborhood basis at 1_G. Furthermore, we can restrict F to be from any cofinal (under inclusion) set of finite subsets of ω.*

Definition 5.6. *Suppose $G \leq S_\omega$. A G-type σ is the G-orbit of some $F \subseteq \omega$, where F is finite and nonempty, formally $\sigma = G.F$*

Notation 5.7. *Suppose $G \leq S_\omega$ and σ and ρ are G-types. We write $\rho \leq \sigma$ if there is some $F \in \sigma$ and some $F' \in \rho$ for which $F' \subseteq F$.*
Equivalently, we write $\rho \leq \sigma$ if every $F' \subseteq \rho$ is a subset of some $F \subseteq \sigma$.
Equivalently, we write $\rho \leq \sigma$ if every $F \subseteq \sigma$ has some subset $F' \subseteq \rho$.

Consider a language $\mathcal{L} = \{(R_i)_{i\in I}, (f_j)_{j\in J}\}$. We will denote the space of all $\mathcal{L}$-structres with universe ω by $\mathcal{X}_\mathcal{L}$. Given that the relation R_i has arity n_i for all $i \in I$, and the function f_j has arity m_j for all $j \in J$, we have

$$\mathcal{X}_\mathcal{L} = \prod_{i\in I} 2^{\omega^{n_i}} \times \prod_{j\in J} \omega^{\omega^{m_j}}$$

If $\mathcal{L}$ is a relational language, then $J = \emptyset$, therefore $\mathcal{X}_\mathcal{L}$ is compact, since it is the product of compact spaces. The group S_ω acts canonically on $\mathcal{X}_\mathcal{L}$ in the following way: Given $A \in \mathcal{X}_\mathcal{L}$, we let $g.A$ be the the structure $B \in \mathcal{X}_\mathcal{L}$, such that

$$(a_1, \dots, a_{n_i}) \in R_i^B \iff ((g^{-1}).a_1, \dots, (g^{-1}).a_{n_i}) \in R_i^A,$$
$$f_j^B(a_1, \dots, a_{m_j}) = f_j^A((g^{-1}).a_1, \dots, (g^{-1}).a_{m_j}),$$

which means $g : A \to B$ is an isomorphism. This action of S_ω on $\mathcal{X}_\mathcal{L}$ is called the *logic action*, and we will quickly show why it is continuous:

It is enough to show that the logic action $a : S_\omega \times \mathcal{X}_\mathcal{L} \to \mathcal{X}_\mathcal{L}$ is continuous in $(1, A)$ for some $\mathcal{L}$-structure A. Fix a neighborhood W of A in $\mathcal{X}_\mathcal{L}$. We must find an open set in $S_\omega \times X$ whose image lies entirely within W. We can find a smaller, basic open neighborhood U of A, of the form

$$U = \{B \in \mathcal{X}_\mathcal{L} \mid F_i \in R_{n_i}^B \iff F_i \in R_{n_i}^A, \text{ where } F_i \text{ is a tuple and } i \in I \text{ finite}\},$$

i.e., the set of structures in $\mathcal{X}_\mathcal{L}$, which act the same as A on a finite collection of k-tuples for various k. Then, we can let F_0 be the set of all numbers that appear in these tuples. This gives us an open neighborhood of 1, namely $H_{(F_1)}$, the pointwise stabilizer of F_0. From here it is easy to see that $H_{(F_1)} \times U$ is an open set in $S_\omega \times X$ for which $a(H_{(F_1)} \times U) = U$, therefore a is continuous.

This means that if $\mathcal{L}$ is relational, the space $\mathcal{X}_\mathcal{L}$ is a G-flow for any $G \leq S_\omega$. Now, consider that our language $\mathcal{L}$ consists of a single binary relation symbol $\mathcal{L} = \{<\}$. We will denote by $LO \subseteq \mathcal{X}_\mathcal{L}$ the subspace of linear orderings on ω, where $<$ is interpreted as a linear order. Clearly LO is S_ω-invariant.

To see that LO is compact, it is enough to show that it is closed. Suppose A is an $\mathcal{L}$-structure with $A \notin LO$. This means $<^A$ is not a linear order, therefore it is either not irreflexive, not transitive, or it does not satisfy trichotomy. Then it is easy to see that there must be finitely subset of ω where A already fails to be a linear order. This means A has an open neighborhood in which every structure fails to be a linear order, thus the complement of LO is open.

Therefore, LO is subflow of $\mathcal{X}_\mathcal{L}$ for any $G \leq S_\omega$. This gives us the following definition:

Definition 5.8. *Let G be a subgroup of S_ω. We say that G preserves an ordering, if the G-flow LO has a fixed point, where LO is the space of linear orderings on ω. In other words, G preserves an ordering if there is a linear order $<^*$ on ω, for which $a <^* b \iff g.a <^* g.b$ for all $g \in G$.*

Theorem 5.9. *Suppose $G \leq S_\omega$ is closed. Then the following are equivalent:*

(1) G is extremely amenable,

(2) (a) For any $F \subseteq \omega$, where F is finite and nonempty, we have $G_{(F)} = G_F$, and (b) For any two G-types σ, ρ with $\rho \leq \sigma$, and a k-coloring $c : \rho \to \{1, \ldots, k\}$, there is some $F \in \sigma$ and a color $1 \leq i \leq k$ for which $c(F') = i$ for all $F' \subseteq F$, $F' \in \rho$,

(3) (c) G preserves an ordering, and (b) as in (2) above.

Proof. (3) ⇒ (2): Clearly, it is enough to prove that (c) implies (a). Suppose G preserves an ordering $<^*$, and suppose we have a nonempty finite set $F \subseteq \omega$. List the elements of F as $F = \{x_1, \ldots, x_n\}$, and we can assume that $x_1 <^* \cdots <^* x_n$. Since G preserves $<^*$, we have $g.x_1 <^* \cdots <^* g.x_n$ for all $g \in G$. Then, clearly, if some $g \in G$ stabilizes F setwise, it must also stabilize F pointwise, since it has the preserve the $<^*$-order of the elements, therefore $G_F = F_{(F)}$.
(1) ⇒ (3): Since G is extremely amenable, the G-flow LO has a fixed point. By definition, this means that G preserves an ordering, so we have shown (c).
To prove (b), suppose we have σ, ρ, and $c : \rho \to \{1, \ldots, k\}$ as in (2). We must find some $F \in \sigma$ for which $c(F') = i$ for every $F' \subseteq F$, $F' \in \rho$. Say ρ is of the form $\rho = G.F_0$, and consider the open subgroup $V = G_{(F_0)}$. Since (c) implies (a), $V = G_{(F_0)} = G_{F_0}$, this gives us that c is a k-coloring of G/V, as the elements of ρ correspond to left-cosets of $V = G_{F_0}$. Consider the finite set $A = \{F_0' \in \rho \mid F_0' \subseteq F^*\}$ for some $F^* \in \sigma$. We can apply Theorem 5.5 to V, c, A, therefore there is some $g \in G$ with $c(g.a) = i$ for all $a \in A$. Now we let $F = g.F^* \in \sigma$. For any $F' \subseteq F$, $F' \in \rho$, we have $(g^{-1}).F' \subseteq F^*$ and $(g^{-1}).F' \in \rho$, so $(g^{-1}).F' \in A$, therefore $c(g.(g^{-1}).F') = c(F') = i$, thus we

have shown (b).
(2) ⇒ (1): We will show (2) of Theorem 5.5, for V of the form $G_{(F)}$, where $F \subseteq \omega$ is finite. Consider the G-type $\rho = G.F$. Since $V = G_{(F)} = G_F$, the left-cosets of V can be identified with elements of ρ. Fix a k-coloring $c : \rho \to \{1, \ldots, k\}$ and some finite $A \subseteq \rho$. Now we let $\bigcup A = F_0$, and $\sigma = G.F_0$. By the property (b) there is some $1 \leq i \leq k$ and some $F^* \in \sigma$ such that $c(F') = i$ for every $F' \subseteq F^*$, $F' \in \rho$. If we write $F^* = g.F_0$ for some $g \in G$, we have $c(g.F') = i$ for all $F' \in A$, since $g.F' \subseteq F^*$ and $g.F' \in \rho$, so we have shown (2) of Theorem 5.5. □

For ease of notation, we introduce the following:

Notation 5.10. *Suppose $G \leq S_\omega$ and σ, ρ are G-types with $\sigma \leq \rho$. For $F \in \sigma$ we write*

$$\binom{F}{\rho} = \{F' \in \rho \mid F' \subseteq F\}$$

Notation 5.11. *Suppose $G \leq S_\omega$ and σ, ρ, τ are G-types with $\rho \leq \sigma \leq \tau$. For $k = 2, 3, \ldots,$ we write*

$$\tau \to (\sigma)^\rho_k,$$

if for every $F \in \tau$ and k-coloring $c : \binom{F}{\rho} \to \{1, \ldots, k\}$, there is some $F' \subseteq F$, $F' \in \sigma$ for which $\binom{F'}{\rho}$ is monochromatic, in other words $c(F_0) = i$ for all $F_0 \subseteq F'$, $F_0 \in \rho$. Note that this is equivalent to saying there is some $F \in \tau$ with this property.

Definition 5.12. *Suppose $G \leq S_\omega$. We say G has the Ramsey Property, if for all G-types σ and ρ with $\sigma \leq \rho$, and every $k = 2, 3, \ldots,$ there is a G-type τ with $\sigma \leq \tau$ for which $\tau \to (\sigma)^\rho_k$.*

Theorem 5.13. *Suppose $G \leq S_\omega$ closed. Then the following are equivalent:*

(1) G is extremely amenable,

(2) G preserves an ordering and G has the Ramsey Property.

Proof. (1) ⇒ (2): We have proven before that if G is extremely amenable, then it preserves an ordering, so we must show that G has the Ramsey Property. It can be shown that, by induction, we can restrict k to be 2 in Definition 5.12. By contradiction, assume that there are G-types σ and ρ with $\rho \leq \sigma$ for which does not exist a G-type τ with $\sigma \leq \tau$ and $\tau \to (\sigma)^\rho_k$. First, we will fix some $F_0 \in \sigma$. By our assumption, for every finite set E with $F_{\subseteq} E$, there must be some coloring $c_E : \binom{E}{\rho} \to \{1, 2\}$, for which there is no monochromatic set $F \in \binom{E}{\sigma}$. For some finite $F \subseteq \omega$, let C_F be the collection of sets $C_F := \{E \subseteq \omega \mid F \subseteq E\}$. The set $C := \{C_F \mid F \subseteq \omega \text{ finite}\}$ is a centered family of sets, since for any finitely many C_{F_i}, the set $\bigcup_{i=1}^{m} F_i$ is contained in the intersection $\bigcap_{i=1}^{m} C_{F_i}$. Therefore C can be extended to an ultrafilter U on the index set I of finite nonempty subsets of ω. The ultrafilter U will help us define a coloring in the following way: for some $D \in \rho$, either

$$\{E \subseteq \omega \mid D \cup F_0 \subseteq E,\ c_E(D) = 1\} \in U,$$

or

$$\{E \subseteq \omega \mid D \cup F_0 \subseteq E,\ c_E(D) = 2\} \in U.$$

Naturally, we let $c(D) = I \iff \{E \subseteq \omega \mid D \cup F_0 \subseteq E,\ c_E(D) = i\} \in U$, which gives us a 2-coloring $c : \rho \to \{1, 2\}$. By Theorem 5.9 (b), there is some $F \in \sigma$, which is monochromatic on $\binom{F}{\rho}$ with color i. For some $D \in \binom{F}{\rho}$, we know that $c(D) = i$, so we let $A_D = \{E \subseteq \omega \mid\ D \cup F \subseteq E,\ c_E(D) = i = c(D)\} \in U$. Since $\binom{F}{\rho}$ is finite, the set

$$\bigcap_{D \in \binom{F}{\rho}} A_D$$

is not empty, so we can pick some element of it, say E. Then we can see that $F_0 \subseteq E$, and for every $D \in \binom{F}{\rho}$, we have $c_E(D) = c(D) = i$, therefore $F \in \binom{E}{\sigma}$ is monochromatic for c_E, so we have arrived at a contradiction.

(2) $\Rightarrow$ (1): Since G preserves an ordering, by Theorem 5.9 it is enough to show that G satisfies (b) of (2) in Theorem 5.9. Clearly, since G has the Ramsey Property, this is true. □

Remark. *Suppose $G \leq S_\omega$. A set T of G-types is said to be cofinal if for any G-type ρ, there is some $\sigma \in T$ with $\rho \leq \sigma$.*

It can be shown that the previous theorem still holds if in Definition 5.12 we restrict the G-types to by in a given cofinal set of G-types.

We define the two pieces of notation in Notation 5.10 and Notation 5.11, as well as the Ramsey Property, for classes of $\mathcal{L}$-structures:

Notation 5.14. *Suppose $\mathcal{K}$ is a class of $\mathcal{L}$-structures and we have $A, B \in \mathcal{K}$ with $A \preceq B$. We write*

$$\binom{B}{A} = \{C \in \mathcal{K} \mid C \preceq B \text{ and } C \text{ is isomorphic to } A\}$$

Notation 5.15. *Suppose $\mathcal{K}$ is a class of $\mathcal{L}$-structures and we have $A, B, C \in \mathcal{K}$ with $A \preceq B \preceq C$. For $k = 2, 3, \dots,$ we write*

$$C \to (B)_k^A$$

if for every k-coloring $c : \binom{C}{A} \to \{1,\dots,k\}$, there is some $B' \in \binom{C}{B}$ for which $\binom{B'}{A}$ is monochromatic, in other words $c(A') = i$ for all $A' \preceq B'$, A' isomorphic to A.

Definition 5.16. *Suppose $\mathcal{K}$ is a class of $\mathcal{L}$-structures. We say $\mathcal{K}$ has the Ramsey Property, if for any two structures $A, B \in \mathcal{K}$ with $A \preceq B$, and every $k = 2, 3\dots$, there is some $C \in \mathcal{K}$ with $B \preceq C$ for which $C \to (B)^A_k$.*

The pieces of notation in Notation 5.14 and Notation 5.15, and consequently the Ramsey Property, are most interesting when $\mathcal{K}$ is a hereditary class of finite structures. In particular, when $\mathcal{K}$ is a Fraïssé class consisting of locally finite structures.

Definition 5.17. *Suppose $\mathcal{L}$ is a language with a binary relation symbol $<$. An order structure for $\mathcal{L}$, is an $\mathcal{L}$-structure in which the relation symbol $<$ is interpreted as a linear ordering.*
If $\mathcal{K}$ is a class of $\mathcal{L}$-structures, we say $\mathcal{K}$ is an order class if every structure in $\mathcal{K}$ is an order structure. In particular, if $\mathcal{K}$ is a Fraïssé class, and every structure in $\mathcal{K}$ is an order structure, we say $\mathcal{K}$ is a Fraïssé order class.

We now state the key theorem of Kechris, Pestoc, and Todorcevic:

Theorem 5.18. *Suppose $G \leq S_\omega$ is closed. The following are equivalent:*

(1) G is extremely amenable,

(2) G is of the form $G = Aut(M)$, where M is the Fraïssé limit of a Fraïssé order class with the Ramsey Property, and M is locally finite.

Proof. (1) $\Rightarrow$ (2): Let A_G be the canonical structure of G, which is ultrahomogeneous by Theorem 3.5. Since G is extremely amenable, G preserves a linear order $<^*$ on ω. Let $\mathcal{L}$ be the language we get by adding a binary relation symbol $<$ to the language on A_G. We will expand the structure A_G by intepreting $<$ as $<^*$, and we will denote the resulting $\mathcal{L}$-structure by A. Since G preserves $<^*$, we can see that $Aut(A) = G$ and that A is still ultrahomogeneous. Since the language of A is relational, A is locally finite, thefore $Age(A)$ is a locally finite Fraïssé order class. Since A is ultrahomogeneous, a G-type of A is precisely the set of all substructures of A isomorphic to some $A_0 \preceq A$, as an isomorphism between $A_0 \in Age(A)$ and $A_1 \in Age(A)$ extends to an automorphism $g \in G$. It is easy to see that G having the Ramsey Property is equivalent to $Age(A)$ having the Ramsey Property, so we are done by Theorem 5.13.

(2) $\Rightarrow$ (1): Since A is the Fraïssé limits of a locally finite Fraïssé order class, A is a locally finite order structure. This means that G preserves an ordering, since A is an order structure, and the G-types of finite substructures of A are cofinal in all the G-types, since A is locally finite. the G-type of a finite substructure of $A_0 \preceq A$ is the set of all substructures of A isomorphic to A_0. Similarly to

before, by Theorem 5.13, G has the Ramsey Property, and since G preserves an ordering, G is extremely amenable. □

We can rephrase the previous theorem in the following way:

Theorem 5.19. *Suppose $\mathcal{K}$ is a locally finite Fraïssé order class and M is the Fraïssé limit of $\mathcal{K}$. Then the following are equivalent:*

(1) $Aut(M)$ is extremely amenable,

(2) $\mathcal{K}$ has the Ramsey Property.

5.2 The extreme amenability of $Aut(\mathbb{Q}, <)$

Using Theorem 5.19, we can show that $Aut(\mathbb{Q}, <)$ is extremely amenable. By Theorem 4.12, we know that $(\mathbb{Q}, <)$ is the Fraïssé limit of the Fraïssé class of finite linear orderings. This class is clearly an order class and all of its elements are locally finite, so we just have to show is that the class of finite ordered sets satisfies the Ramsey Property. This is exactly the statement of the finite Ramsey theorem [5, Corollary 1.3], so we have proven that $Aut(\mathbb{Q}, <)$ is extremely amenable.

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Alulírott Kozári Dominik nyilatkozom, hogy szakdolgozatom elkészítése során nem használtam MI alapú eszközt. (ez angol legyen vagy magyar?)