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Erdős–Szekeres–type theorems for monotone paths

Thesis

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1. Introduction

An *ordered graph* is a graph together with a total ordering on its vertices. Let $\mathcal{H}$ be an ordered k-uniform hypergraph. We say that $\mathcal{H}$ contains a *monotone path* of length n if there is an ordered sequence of vertices of length $n + k - 1$ for which $\mathcal{H}$ contains every k-tuple (i.e. edge) of consecutive vertices in the sequence. Let $N_k(q, n)$ be the smallest integer N so that every q-coloring of the edges of the ordered complete k-uniform hypergraph on N vertices contains a monochromatic monotone path of length n. The Ramsey-type problem of bounding $N_k(q, n)$ was studied by Fox, Pach, Sudakov and Suk [4]; however, it goes back (implicitly) to the seminal 1935 paper of Erdős and Szekeres [3].

In the first part of the thesis, we present two of the most well-known results in combinatorics, that appeared in the paper of Erdős and Szekeres mentioned above.

Then, in the second part, we present a construction by Moshkovitz and Shapira [5] that improves the bound of Fox et al. and, as a by-product, provides a new proof of the Erdős–Szekeres Theorem. Furthermore, we investigate how $N_k(q, n)$ changes if the coloring is restricted to stricter properties, and we present a construction by Balko [1] that addresses this case.

For the sake of clarity of presentation, we maintain some conventions throughout the thesis. For a positive integer n, we use $[n]$ to denote the set $\{1, \ldots, n\}$. For a set A, let $\binom{A}{k}$ be the set $\{S \subseteq A : |S| = k\}$. We write $v \leq w$ for two vectors $v, w \in [n]^d$ if $v_i \leq w_i$ for all coordinates. For $v, w \in [n]^d$, we say that v is *lexicographically smaller* than w if for the smallest $i \in [d]$ such that $v_i \neq w_i$ we have $v_i < w_i$. Let $\mathrm{tow}_h(x)$ be the tower function of height h, that is, $\mathrm{tow}_1(x) = x$ and $\mathrm{tow}_h(x) = 2^{\mathrm{tow}_{h-1}(x)}$ for every $h \geq 2$.

2. Erdős-Szekeres Lemma and Theorem

In the seminal 1935 paper of Erdős and Szekeres, in addition to establishing explicit bounds for graph and hypergraph Ramsey numbers, they also proved two fundamental results in combinatorics, which have become known as the Erdős-Szekeres Lemma and Theorem. In this section, we provide proof for both well-known results.

Let $f(a, b)$ be the smallest integer such that every sequence of $f(a, b)$ distinct real numbers contains either an increasing subsequence of length a or a decreasing subsequence of length b (the elements of the subsequences do not have to be adjacent in the given sequence). Then, the Erdős-Szekeres Lemma is the following:

Lemma 2.1 (Erdős-Szekeres Lemma)**.** $f(a, b) \leq (a - 1)(b - 1) + 1$.

Steele [7] collected seven proofs of the lemma; the one presented below is by Seidenberg [6].

Proof. Label the i-th number in the sequence with $(x_i, y_i) \in \mathbb{Z}^2$, where x_i and y_i are the lengths of the longest monotone increasing and decreasing subsequences ending with the i-th number, respectively.

Note that every label is distinct. If we suppose to the contrary that there exist indices $i < j$ such that $(x_i, y_i) = (x_j, y_j)$, then, by adding the j-th number to the longest increasing or decreasing subsequence ending at the i-th number − depending on whether the j-th number is greater or smaller than the i-th − the corresponding coordinate in the label would increase, contradicting our assumption.

If there is no increasing subsequence of length a or decreasing sequence of length b, then for all labels, we have $1 \leq x_i \leq a - 1$ and $1 \leq y_i \leq b - 1$. Therefore, there are at most $(a - 1)(b - 1)$ distinct labels, which means that we can have at most $(a - 1)(b - 1)$ distinct numbers without the required subsequences. □

Before continuing with the proof of the Erdős–Szekeres Theorem, we introduce a few definitions and a preliminary observation that will be used in the proof. A finite set of points in the plane is in *general position* if no three points are collinear and no two points share the same x-coordinate. A set of a points in general position ($a \geq 2$), labeled $p_1, \ldots, p_a$ with increasing x-coordinates is called an *a-cap* (respectively, *a-cup*) if

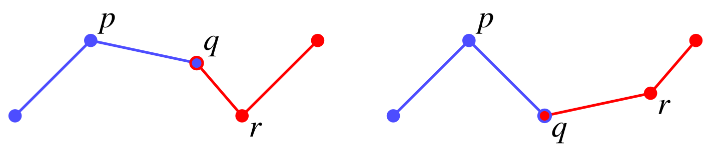


Figure 2.1: Connecting a-cap and b-cup can be extended.

the slopes of the segments $(p_1, p_2), (p_2, p_3), \ldots, (p_{a-1}, p_a)$ are decreasing (respectively, increasing).

Observation 2.2. *If the first point of a b-cup and the last point of an a-cap are the same, one of them can be extended by one point.*

Proof. Let q be the common point of the a-cap and the b-cup, also p and r be the point before and after q in the a-cap and the b-cup, respectively. The points p, q, r form a 3-cap or a 3-cup. In the first case r can be added to the a-cap, in the second case, p to the b-cup (Fig. 2.1). □

Let $g(a, b)$ be the smallest integer so that every set of $g(a, b)$ points in the plane in general position contains either an a-cap or a b-cup. The Erdős-Szekeres Theorem states the following:

Theorem 2.3 (Erdős-Szekeres Theorem)**.** $g(a, b) \leq \binom{a+b-4}{b-2} + 1$.

The following is the original proof by Erdős and Szekeres [3]; later in Section 3, we present an alternative proof of the theorem.

Proof. First, we will show that $g(a + 1, b + 1) \leq g(a, b + 1) + g(a + 1, b) - 1$. Suppose to the contrary that we have $g(a, b + 1) + g(a + 1, b) - 1$ points with no $(a + 1)$-cap nor $(b + 1)$-cup. Let S be the set of all endpoints of the a-caps.

If $|S| \geq g(a + 1, b)$, then, by definition, there must be either an $(a + 1)$-cap or a b-cup among the points. In the first case, we have an immediate contradiction. In the second case, we found a b-cup starting with the endpoint of an a-cap, so with Observation 2.2 we are done.

If $|S| < g(a+1, b)$, then, in the complement of S there are at least $g(a, b+1)$ points. So, there must be a $(b+1)$-cup or an a-cup. The first case contradicts the original assumption, the second with the definition of S, since we found an endpoint of an a-cap outside of S.

If $G(a, b) := \binom{a+b-4}{b-2} + 1$, then the recursion $G(a, b) = G(a, b-1) + G(a-1, b) - 1$ holds by Pascal's rule. To complete the proof by induction, we note that $g(2, b) = G(2, b)$ and $g(a, 2) = G(a, 2)$. Thus, the result follows. □

Note that the bounds on both $f(a, b)$ and $g(a, b)$ are tight. The papers from which we presented the proofs contain examples of tight examples.

3. Ramsey-number of paths in ordered graphs

Let $k \geq 2$ be an integer and $\mathcal{H} = (H, \prec)$ be an ordered k-uniform hypergraph. Recall that, we say $\mathcal{H}$ contains a monotone path of length n if there is an ordered sequence of $n + k - 1$ vertices $v_{i_1} \prec \cdots \prec v_{i_{n+k-1}}$ for which $\mathcal{H}$ contains every k-tuple (i.e. edge) $\{v_{i_j}, \ldots, v_{i_{j+k-1}}\}$ for all $j \in [n]$. We denote the ordered complete k-uniform hypergraph on n vertices by $\mathcal{K}_n^k$. A *coloring* $\mathfrak{C}$ of an ordered k-unirofm hypergraph $\mathcal{H}$ is a function that assigns to each edge of $\mathcal{H}$ an element from a fixed finite C set. If $|C| = q$, we say that $\mathfrak{C}$ is a *q-coloring*. A monotone path is *monocromathic* in $\mathfrak{C}$ if every edge of the path receives the same color.

Fox, Pach, Sudakov and Suk [4] introduced the number $N_k(q, n)$ that is the smallest integer N such that every q-coloring of the edges of $\mathcal{K}_N^k$ contains a monochromatic monotone path of length n. We can call this number the ordered multicolor Ramsey-number of monotone paths as its definition aligns with the original Ramsey-number. Observe that this framework of Fox et al. puts the Erdős-Szekeres Lemma and Theorem under one roof:

In the case of the Erdős-Szekres Lemma, for a given sequence of N distinct numbers $\{a_1, \ldots, a_N\}$, we define a complete ordered graph in the following way: Let the vertex set of the graph be the given set of numbers. The vertices are ordered according to their position in the sequence, that is, if $i < j$, then $a_i \prec a_j$ (this is the $\prec$ order on the vertices). For $a_i \prec a_j$, we color the edge $\{a_i, a_j\}$ blue if $a_i < a_j$ (this is the ordinary $<$ order on the numbers), and red otherwise. If $N \geq N_2(2, n)$, then by the definition of the number $N_2(2, n)$ there must be a monochromatic monotone path of length n in the coloring we defined above. That is, we have a subsequence of length $n + 1$ of the given numbers such that for each consecutive term in the subsequence the numbers grow if the color of the path is blue and decrease if its color is red. Therefore, we obtain $f(n+1, n+1) \leq N_2(2, n)$.

Similarly, in the case of the Erdős-Szekres Theorem, a given set of N points in general position in the plane $\{(x_1, y_1), \ldots, (x_N, y_N)\}$, we define a complete ordered 3-uniform hypergraph in the following way: Let the vertex set of the graph be the given point set. The vertices are ordered according to the x-coordinates of the corresponding points, that is, if $x_i < x_j$, then $(x_i, y_i) \prec (x_j, y_j)$. For $(x_i, y_i) \prec (x_j, y_j) \prec (x_k, y_k)$, we color the edge $\{(x_i, y_i), (x_j, y_j), (x_k, y_k)\}$ blue if the points $\{(x_i, y_i), (x_j, y_j), (x_k, y_k)\}$ form a 3-cap, and red if they form a 3-cup. If $N \geq N_3(2, n)$, then by the definition of the number $N_3(2, n)$ there must be a monochromatic monotone path of length n in the coloring we defined above. That is, we have a sequence of points of length $n + 2$ for which each consecutive triple of points forms a 3-cap if the color of the path is blue and a 3-cup if its color is red. Therefore, we obtain $g(n + 2, n + 2) \leq N_3(2, n)$.

Hence, we can prove both Erdős-Szekeres Lemma and Theorem by bounding $N_k(q, n)$. For $k = 3$, Fox et al. showed that

$$2^{(n/q)^{q-1}} \leq N_3(q, n) \leq n^{n^{q-1}}.$$

In this section, we first go through the constructions that Moshkovitz and Shapira [5] gave to prove the following better bounds on the number $N_k(q, n)$.

Theorem 3.1. *For every $k \geq 3$, $q \geq 2$, and sufficiently large n, we have*

$$tow_{k-1}(n^{q-1}/2\sqrt{q}) \leq N_k(q, n) \leq tow_{k-1}(2n^{q-1}).$$

Then, in Subsection 3.5, we examine how the number $N_k(q, n)$ changes when it is restricted to special types of colorings. We present the construction that Balko gave in [1], which resolves the problem of asymptotically bounding $N_k(q, n)$ restricted to those special colorings.

3.1 High-dimensional integer partitions

For the case $k = 3$ in Theorem 3.1, Moshkovitz and Shapira established a surprising connection between the problem of bounding $N_3(q, n)$ and the problem of enumerating high-dimensional integer partitions. In their proof, they used a subset of these partitions that they could identify with the vertex set of the graph that they construct for the lower bound or get for the upper bound. In this subsection, we define high-dimensional integer partitions and present some of their key properties.

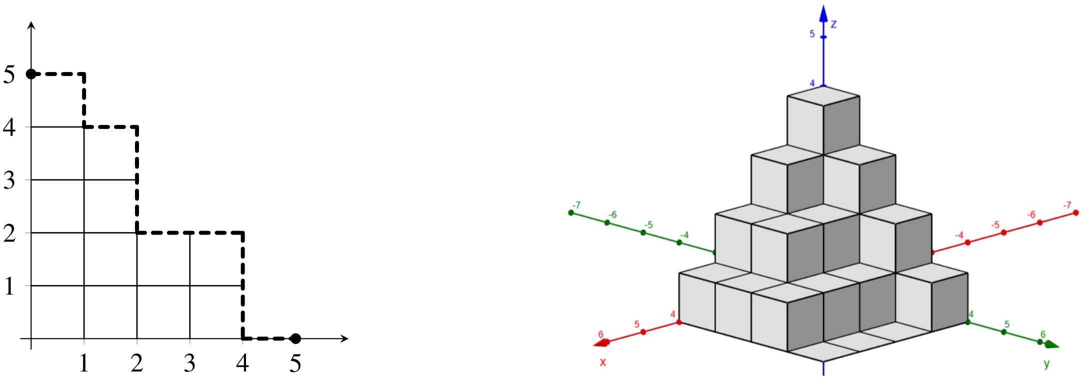
Definition 3.2. *A sequence of nonnegative integers $a_1, a_2, \ldots, a_m$ is called a line partition if $a_1 \geq a_2 \geq \cdots \geq a_m$.*

One can visualize a line partition in 2-dimensions as a sequence of stacks of height a_i each (see Fig. 3.1a).

Definition 3.3. *A matrix of nonnegative integers A is called a plane partition if $A_{i,j} \geq A_{i+1,j}$ and $A_{i,j} \geq A_{i,j+1}$ for all possible pairs of (i, j).*

One can visualize a plane partition in 3-dimensions as stacks of height $A_{i,j}$ at location (i, j) (see Fig. 3.1b).

Following this pattern, we can define higher-dimensional partitions in general:



(a) Line partition

(b) Plane partition

Figure 3.1: Partitions

Definition 3.4. *A d-dimensional (hyper)matrix of nonnegative integers A is called a d-dimensional partition if $A_{i_1,\dots,i_t,\dots i_d} \geq A_{i_1,\dots,i_t+1,\dots i_d}$ for all possible $i_1,\dots,i_d$ and $1 \leq t \leq d$.*

One can visualize a d-dimensional partition in $(d+1)$-dimensions as stacks of height $A_{i_1,\dots,i_d}$ at location $(i_1,\dots,i_d)$.

Note that Definition 3.4 means that for $\mathbf{i},\mathbf{j} \in [n]^d$ if $\mathbf{i} \leq \mathbf{j}$, then $A_\mathbf{i} \geq A_\mathbf{j}$; however, if we only know that $\mathbf{i}$ is lexicographically smaller than $\mathbf{j}$, then we do not know anything about the relation between $A_\mathbf{i}$ and $A_\mathbf{j}$.

To identify each vertex with a d-dimensional partition, we must define a total order on the partitions. We order the partitions lexicographically. For two d-dimensional partitions $A \neq B$, denote by $\delta(A,B)$ the lexicogripically smallest $(i_1,\dots,i_d) \in [n]^d$ where $A_{i_1,\dots,i_d} \neq B_{i_1,\dots,i_d}$. Then, we define the lexicographical order on the d-dimensional partitions as follows.

Definition 3.5. $A \prec B \iff A_{\delta(A,B)} < B_{\delta(A,B)}$.

Now we define the subset of partitions that will serve as the vertex set of the graph. Let $p_d(n)$ denote the set of d-dimensional partitions such that $1 \leq i_j \leq n$ for all $1 \leq j \leq d$ and $0 \leq A_{i_1,\dots,i_d} \leq n$ for all possible $i_1,\dots,i_d$. That is, if the partition is visualized in $(d+1)$-dimension, then it can fit in a $(d+1)$-dimensional (hyper)cube with edge length of n. Denote the cardinality of $p_d(n)$ with $P_d(n)$.

As it turns out, the number $P_d(n)$ can be estimated well.

Proposition 3.6. $P_1(n) = \binom{2n}{n}$.

Proof. Using the visualization above, a line partition in $p_d(n)$ can be represented as a lattice path from $(0,n)$ to $(n,0)$ where in each step the path moves either one unit to the right or one unit downward (see Fig. 3.1a). Since each such path consists of exactly n right steps and n down steps (in some order), the total number of such paths is $\binom{2n}{n}$, because that is the number of ways we can choose which n are downward from the $2n$ steps. □

Proposition 3.7. $P_d(n) \leq \binom{2n}{n}^{n^{d-1}}$.

Proof. Observe that in a d-dimensional partition A, if we fix an index $t \in [d]$ and fix all coordinates of $(i_1, \ldots, i_t, \ldots, i_d)$ except for i_t, then the sequence $(A_{i_1,\ldots,i_t,\ldots,i_d})$, where i_t ranges over $[n]$, is a line partition. In this way, the d-dimensional partition can be decomposed into n^{d-1} line partitions, as there are n^{d-1} ways to fix the remaining $d-1$ coordinates of the tuple $(i_1, \ldots, i_d)$. Therefore, we obtain the upper bound $P_d(n) \leq \binom{2n}{n}^{n^{d-1}}$ by independently choosing an arbitrary line partition for each of the n^{d-1} possible coordinate configurations. □

We simply mention the following lower bound on $P_d(n)$; the proof can be found in [5].

Theorem 3.8. *For every $d \geq 1$ and $n \geq 1$ we have*

$$P_d(n) \geq 2^{\frac{2}{3}n^d/\sqrt{d+1}}.$$

To prove the lower bound on the number $N_3(q, n)$, Guy Moshkovitz and Asaf Shapira establish the connection between the vertex set of the graph and the $p_d(n)$ subset of partitions through the following definition and observation:

Definition 3.9. *A set $S \subseteq [n]^d$ is a down-set if $s \in S$ implies $x \in S$ for every $x \leq s$.*

Observation 3.10. *The number of down-sets $S \subseteq [n]^d$ is $P_{d-1}(n)$.*

Proof. We construct a bijection between $p_{d-1}(n)$ and the down-sets in $[n]^d$.

Let $S \subseteq [n]^d$ be a down-set. For every $1 \leq i_1, \ldots, i_{d-1} \leq n$ define

$$A_{i_1,\ldots,i_{d-1}} := \max\{i \in [n] : (i_1, \ldots, i_{d-1}, i) \in S\}$$

with the convention that $A_{i_1,\ldots,i_{d-1}} = 0$ if no such i exists. We claim that the array A we get is a $(d-1)$-dimensional partition. Suppose the contrary, that there exists a tuple $(i_1, \ldots, i_t, \ldots, i_{d-1}) \in [n]^{d-1}$ such that

$$A_{i_1,\ldots,i_t,\ldots,i_{d-1}} < A_{i_1,\ldots,i_t+1,\ldots,i_{d-1}}.$$

By the definition of A we know that for the points

$$\mathbf{x} = (i_1, \ldots, i_t, \ldots, i_{d-1}, A_{i_1,\ldots,i_t,\ldots,i_{d-1}}) \text{ and } \mathbf{y} = (i_1, \ldots, i_t + 1, \ldots, i_{d-1}, A_{i_1,\ldots,i_t+1,\ldots,i_{d-1}})$$

we have $\mathbf{x}, \mathbf{y} \in S$ and $\mathbf{x} \leq \mathbf{y}$. Since S is a down-set, any point less than or equal to $\mathbf{y}$ must also lie in S. In particular, by our assumption, the point $(i_1, \ldots, i_t, \ldots, i_{d-1}, A_{i_1,\ldots,i_t+1,\ldots,i_{d-1}})$ is also in S. However, this is a contradiction with the choice of $A_{i_1,\ldots,i_t,\ldots,i_{d-1}}$.

Let $A \in p_{d-1}(n)$. For every $1 \leq i_1, \ldots, i_{d-1} \leq n$ define

$$S_{i_1,\ldots,i_{d-1}} := \left\{x \in [n]^d : x \leq (i_1, \ldots, i_{d-1}, A_{i_1,\ldots,i_{d-1}})\right\}$$

with the convention that $S_{i_1,\ldots,i_{d-1}} = \emptyset$ if $A_{i_1,\ldots,i_{d-1}} = 0$. We claim that the set S defined as $\bigcup_{1\le i_1,\ldots,i_{d-1}\le n} S_{i_1,\ldots,i_{d-1}}$ is a down-set in $[n]^d$. To prove this, let $s \in S$ and suppose $x \in [n]^d$ satisfies $x \le s$. By definition of S, there exists a tuple $i_1, \ldots, i_{d-1}$ such that

$$s \le (i_1, \ldots, i_{d-1}, A_{i_1,\ldots,i_{d-1}})$$

since the relation $\le$ between the ponts in $[n]^d$ is transitive, it follows that

$$x \le s \le (i_1, \ldots, i_{d-1}, A_{i_1,\ldots,i_{d-1}}),$$

and hence $x \in S_{i_1,\ldots,i_{d-1}} \subseteq S$. Therefore, S is a down-set. □

3.2 3-uniform hypergraphs

Now we are ready to show the following connection between $N_3(q,n)$ and the set $p_{q-1}(n)$.

Theorem 3.11. *For every $q \ge 2$ and $n \ge 2$ we have*

$$N_3(q,n) = P_{q-1}(n) + 1.$$

We begin by proving the case $q = 2$. Observe that, with Proposition 3.6, it provides an alternative proof of Theorem 2.3 in the case $a = b = n$. Then, we proceed to prove the theorem for general q in two steps: first by establishing the upper bound in Lemma 3.14, and then by proving the lower bound in Lemma 3.15.

Lemma 3.12. $N_3(2,n) \le P_1(n) + 1$.

Proof. Fix a blue-red coloring of the edges of K_N^3 without monochromatic monotone path of length n. We need to show that $N \le P_1(n)$. Let us define $C(u,v)$ for every $u < v$ pair of vertices as $(n_b + 1, n_r + 1)$, where n_b (respectively n_r) is the length of the longest blue (respectively red) monotone path ending with the vertices $\{u,v\}$. Note that by our assumption, that there is no monochromatic monotone path of length n, $C(u,v) \in [n]^2$. Define

$$D(v) := \left\{x \in [n]^2 : \exists u < v : x \le C(u,v)\right\}.$$

Notice that $D(v)$ is a downset in $[n]^2$, therefore using Observation 3.10, it is enough to prove that $D(v) \ne D(u)$ for every pair of vertices, since then, if we would have more than $P_1(n)$ vertecies the pigeonhole principle implies a contradiction with $D(v) \ne D(u)$.

So suppose to the contrary that $u < v$ and $D(u) = D(v)$. Then, $C(u,v) \in D(u)$, so by definition of $D(u)$ there exists a vertex such that $C(u,v) \le C(t,u)$. However, this is a contradiction, because the longest monotone path ending with (t,u) that has the same

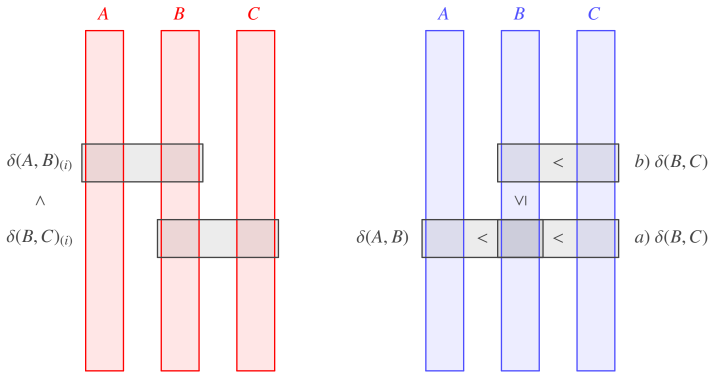


Figure 3.2: The coloring rule for $\{A, B, C\}$.

color as $\{t, u, v\}$ can be extended with $\{t, u, v\}$, so in the corresponding $i \in [2]$ coordinate $C(u, v)_i \nleq C(t, u)_i$. □

Lemma 3.13. $N_3(2, n) > P_1(n)$.

Proof. We need to construct a coloring of the edges of $K^3_{P_1(n)}$ that has no monochromatic path of length n. Identify each vertex with a distinct element of $p_1(n)$ and order them lexicographically. For given $A < B < C$ vertices, color the edge $\{A, B, C\}$ blue if $\delta(A, B) \geq \delta(B, C)$ and red otherwise (Fig. 3.2).

It is enough to show that for a monochromatic monotone path of length l ending with $B < C$ vertices, if its color is red, then $\delta(B, C) > l$; if it is blue, then $C_{\delta(B,C)} > l$. Since both $C_{\delta(B,C)}$ and $\delta(B, C)$ are, by definition, at most n, the previous claim follows that $n - 1 \geq l$.

In the first case, if its color is red, then, between every three consecutive vertices (an edge of the path), the δ index grows by one at least by the definition of the coloring (Fig. 3.2 left side); therefore, on the whole path (containing $l + 2$ vertices) it grows by l at least, so on the last two vertices δ must be greater than l.

In the second case, if its color is blue, then for every three consecutive vertices $\{A, B, C\}$ (an edge of the path) we have:

$$A_{\delta(A,B)} < B_{\delta(A,B)} \leq B_{\delta(B,C)} < C_{\delta(B,C)},$$

where the middle inequality comes from the definition of the elements of $p_1(n)$, and the other two from the definiton of the δ indices (Fig. 3.2 right side). Hence, the sequence of the corresponding coordinates increase by at least one between every two consecutive vertices, so, on the whole path by $l + 1$ at least. Therefore, the corresponding coordinate of the last vertex must be greater than $l + 1$. □

We can generalize the previous two lemmas for q-colorings:

Lemma 3.14. $N_3(q, n) \leq P_{q-1}(n) + 1$.

Proof. Fix a q-coloring of the edges of K_N^3 without a monochromatic monotone path of length n. We need to show that $N \le P_{q-1}(n)$. Let us define $C(u, v)$ for every $u < v$ pair of vertices as $(n_1 + 1, \dots, n_q + 1)$, where n_i is the length of the longest monotone path colored with the i-th color ending with the vertices $\{u, v\}$. Note that by our assumption, that there is no monochromatic monotone path of length n, $C(u, v) \in [n]^q$. Define

$$D(v) := \{x \in [n]^q : \exists u < v : x \le C(u, v)\}.$$

Notice that $D(v)$ is a downset in $[n]^q$, therefore using Observation 3.10, it is enough to prove that $D(v) \ne D(u)$ for every pair of vertices, since then, if we would have more than $P_1(n)$ vertecies the pigeonhole principle implies a contradiction with $D(v) \ne D(u)$.

So suppose the contrary that $u < v$ and $D(u) = D(v)$. Then $C(u, v) \in D(u)$, so by definition, there is a t vertex such that $C(u, v) \le C(t, u)$. However, this is a contradiction, because the longest monotone path ending with the vertices $\{t, u\}$ that has the same color as $\{t, u, v\}$ can be extended with $\{t, u, v\}$, so in the corresponding i-th ($i \in [q]$) coordinate $C(u, v)_i \not\le C(t, u)_i$. □

Lemma 3.15. *For every $q \ge 2$ and $n \ge 2$ we have*

$$N_3(q, n) > P_{q-1}(n).$$

Proof. Let $d = q - 1$. We need to construct a coloring of the edges of $K^3_{P_d(n)}$ that has no monochromatic path of length n. Identify each vertex with a distinct d-dimensional partition for which $1 \le i_j \le n$ for all $1 \le j \le d$ and $0 \le A_{i_1,\dots,i_d} \le n$ for all possible $i_1, \dots, i_d$. Order the vertices lexicographically. For given $A \lessdot B \lessdot C$ vertices, color the edge $\{A, B, C\}$ with the j-th color if $\delta(A, B)_j < \delta(B, C)_j$ (if there are several such j we choose the minimal), otherwise use the q-th color.

It is enough to show that for a monochromatic monotone path of length l ending with $B \lessdot C$ vertices, if its color is the j-th color, that is $1 \le j \le q - 1$, then $\delta(B, C)_j > l$; if it is the q-th color, then $C_{\delta(B,C)} > l$. Since $C_{\delta(B,C)}$ and $\delta(B, C)_j$ for each $j \in [d]$ are, by definition, at most n, from the previous claim follows that $n - 1 \ge l$.

If the color of the path is j that is $1 \le j \le q - 1$, then, between every three consecutive vertices of the graph (an edge of the path), the j-th coordinate of the δ index grows at least by one by the definition of the coloring (see Fig. 3.2 left side); therefore, on the whole path (containing $l + 2$ vertices) it grows by l at least, so on the last two vertices the j-th coordinate of δ must be greater than l.

If the color of the path is q, then for every three consecutive vertices of the path $\{A, B, C\}$ (an edge of the path) we have:

$$A_{\delta(A,B)} < B_{\delta(A,B)} \le B_{\delta(B,C)} < C_{\delta(B,C)}.$$

The middle inequality comes from the definition of the elements of $p_d(n)$ because we know that if the color is q, then $\delta(A, B) \geq \delta(B, C)$ (see def. 3.4 and the remark afterwards). The other two inequality hold by the definiton of the δ indices (Fig. 3.2 right side). Hence, the corresponding coordinate increases by at least one between every two consecutive vertices, so, on the whole path by $l + 1$ at least. Therefore, the corresponding coordinate of the last vertex must be greater than $l + 1$. □

From Theorem 3.11 and from the bounds in Proposition 3.7 and Theorem 3.8, we obtain:

Corollary 3.16. *For every $q \geq 2$ and $n \geq 2$ we have*

$$2^{\frac{2}{3}n^{q-1}/\sqrt{q}} \leq N_3(q, n) \leq 2^{2n^{q-1}}.$$

3.3 Higher-order line partitions

As the numbers $N_3(q, n)$ turn out to have a close relation to high-dimensional integer partitions in the framework of down-sets, the numbers $N_k(q, n)$ for general k are related to some kind of higher-order generalization of partitions. For an easier presentation, we consider the case $q = 2$, but we mention a possible generalization later.

Recall that with Observation 3.10 we get a bijection between the down-sets of $[n]^2$ and the elements of the set $p_1(n)$. We will generalize this in the following way. Let $\mathcal{P}^2(n)$ be the set $[n]^2$, and $\mathcal{P}^3(n)$ be the family of down-sets in $[n]^2 = \mathcal{P}^2(n)$ (so $\mathcal{P}^3(n)$ is in bijection with $p_1(n)$). For a set S in $\mathcal{P}^3(n)$ we know that if $x \in S$, then for all $x' \leq x$ we have $x' \in S$. For easier generalization, we think of each $x \in [n]^2$ as a down-set, so for $x, x' \in [n]^2$ if we have $x' \leq x$, then we can say $x' \subseteq x$.

Following this pattern, we define $\mathcal{P}^4(n)$ as the family of down-sets in $\mathcal{P}^3(n)$ (i.e. family of down-sets of line partitions in $[n]^2$). Observe that with the definition above of $x' \subseteq x$ for $x, x' \in [n]^2$, we can say that a line partition is a subset of an other.

In general, we inductively define $\mathcal{P}^k(n)$, whose members are down-sets in $\mathcal{P}^{k-1}(n)$ as follows.

Definition 3.17. *Let $\mathcal{P}^2(n) = [n]^2$ and suppose we have already defined $\mathcal{P}^{k-1}(n)$. A set $\mathcal{F} \subseteq \mathcal{P}^{k-1}(n)$ is in $\mathcal{P}^k(n)$ if $S \in \mathcal{F}$ implies $S' \in \mathcal{F}$ for any $S' \subseteq S$.*

We refer to these sets as higher-order line partitions; for a specific k, we call them order-k line partitions. Note that the elements of a set from $\mathcal{P}^k(n)$ are themselves elements in $\mathcal{P}^{k-1}(n)$. Also, if we denote the cardinality of $\mathcal{P}^k(n)$ by $\rho_k(n)$, then by the definition above we can easily get the following:

$$\rho_1(n) = n^2, \quad \rho_2(n) = \binom{2n}{n}, \quad \rho_k(n) \leq 2^{\rho_{k-1}(n)} \text{ for } k \geq 3.$$

To identify each vertex with a higher-order line partition, we must define a total order on the set of higher-order line partitions. For this purpose, we extend the lexicographical order to $\mathcal{P}^k(n)$. As the sets $\mathcal{P}^k(n)$ are defined recursively, we also define the extension of the lexicographical order recursively.

For $k = 2$, let $\lessdot$ be the standard lexicographical order on $[n]^2 = \mathcal{P}^2(n)$. Suppose that $\lessdot$ has already been defined on $\mathcal{P}^{k-1}(n)$. Then, for $\mathcal{P}^k(n)$, we say that an order-k partition A is lexicographically smaller than an order-k partition B, denoted $A \lessdot B$, if the lexicographical minimum (with respect to the previously defined lexicographical order on $\mathcal{P}^{k-1}(n)$) of the elements in the symmetric difference $A \triangle B$ is in B.

For two sets A and B from $\mathcal{P}^k(n)$ such that $A \not\supseteq B$, we define $\delta(A, B)$ as the lexicographically smallest element in $B \setminus A$.

Observation 3.18. *For any sequence of order-k line partitions $\mathcal{F}_1 \not\supseteq \mathcal{F}_2 \not\supseteq \cdots \not\supseteq \mathcal{F}_t$ we have $\delta(\mathcal{F}_1, \mathcal{F}_2) \not\supseteq \delta(\mathcal{F}_2, \mathcal{F}_3) \not\supseteq \cdots \not\supseteq \delta(\mathcal{F}_{t-1}, \mathcal{F}_t)$.*

Proof. It is enough to show that for $t = 3$ it holds, because then we can use it for every consecutive triples in $\mathcal{F}_1 \not\supseteq \mathcal{F}_2 \not\supseteq \cdots \not\supseteq \mathcal{F}_t$ and get the claim of the observation.

By definiton $\delta(\mathcal{F}_1, \mathcal{F}_2) \in \mathcal{F}_2$ and $\delta(\mathcal{F}_2, \mathcal{F}_3) \notin \mathcal{F}_2$. However, if $\delta(\mathcal{F}_1, \mathcal{F}_2) \supseteq \delta(\mathcal{F}_2, \mathcal{F}_3)$, then $\delta(\mathcal{F}_2, \mathcal{F}_3)$ would be in $\mathcal{F}_2$, since $\mathcal{F}_2$ is closed under taking subsets; therefore, we get $\delta(\mathcal{F}_1, \mathcal{F}_2) \not\supseteq \delta(\mathcal{F}_2, \mathcal{F}_3)$. □

Notice that this observation can be used itaretively, because the elements of an order-k line partition are order-$(k-1)$ line partitions and we get exactly the necessary strating conditions with $\delta(\mathcal{F}_1, \mathcal{F}_2) \not\supseteq \delta(\mathcal{F}_2, \mathcal{F}_3) \not\supseteq \cdots \not\supseteq \delta(\mathcal{F}_{t-1}, \mathcal{F}_t)$.

3.4 *k*-uniform hypergraphs

Now we can characterize $N_k(2, n)$ as follows:

Theorem 3.19. *For every $k \geq 2$ and $n \geq 2$ we have*

$$N_k(2, n) = \rho_k(n) + 1.$$

We prove Theorem 3.19 in two steps: first by establishing the upper bound in Lemma 3.20, and then by proving the lower bound in Lemma 3.21.

Lemma 3.20. $N_k(2, n) \leq \rho_k(n) + 1$.

Proof. Fix a blue-red coloring of the edges of K_N^k with no monochromatic monotone path of length n. Our goal is to show that $N \leq \rho_k(n)$. To do this, we need to establish a relationship between the vertices and the elements of $\mathcal{P}^k(n)$. For every set of $k-1$ vertices $x_1 < \cdots < x_{k-1}$, define $D(x_1, \ldots, x_{k-1}) := (n_b + 1, n_r + 1)$, where n_b (respectively, n_r) is the length of the longest blue (respectively, red) monotone path ending with the vertices

$\{x_1, \ldots, x_{k-1}\}$. By assumption, we have $D(x_1, \ldots, x_{k-1}) \in [n]^2$. Similarly to the proof of Lemma 3.14, we now define $D(v)$ for all vertices using the same idea, taking the downset, but this time $k-2$ times. For $r \in \{k-2, \ldots, 1\}$ and vertices $x_{k-r}, \ldots, x_{k-1}$, we recursively define:

$$D(x_{k-r}, \ldots, x_{k-1}) := \left\{S \in \mathcal{P}^{k-r}(n) : S \subseteq D(x, x_{k-r}, \ldots, x_{k-1}) \text{ for some } x < x_{k-r}\right\}.$$

Therefore, $D(v) \in \mathcal{P}^k(n)$, as desired, so it is enough to show that $D(u) \neq D(v)$ for every pair of $\{u, v\}$ vertices. To prove this, we use the following observation:

$$D(\mathbf{x}, x_{k-1}) \subseteq D(x_{k-r}, \mathbf{x}) \Rightarrow \exists x < x_{k-r} : D(x_{k-r}, \mathbf{x}, x_{k-1}) \subseteq D(x, x_{k-r}, \mathbf{x})$$

for any $2 \leq r \leq k-1$ and vertices $x_{k-r}, \ldots, x_{k-1}$, where $\mathbf{x}$ is the short form of $x_{k-r+1}, \ldots,$ x_{k-2}. By definition, $D(x_{k-r}, \mathbf{x}, x_{k-1})$ is in $D(\mathbf{x}, x_{k-1})$. Furthermore, by our assumption, it is also in $D(x_{k-r}\mathbf{x})$. Thus, using the definition of $D(x_{k-r}, \mathbf{x})$, there is an $x < x_{k-r}$ such that $D(x_{k-r}, \mathbf{x}, x_{k-1}) \subseteq D(x, x_{k-r}, \mathbf{x})$.

Now we are ready to prove that $D(u) \neq D(v)$ for every pair of vertices $\{u, v\}$. Suppose the contrary, that there exist vertices $u < v$ such that $D(u) = D(v)$. By applying our observation to $D(v) \subseteq D(u)$ there exists an $x_{k-2} < u$ such that $D(u, v) \subseteq D(x_{k-2}, u)$. We can continue iteratively using our observation $k-3$ more times, to get vertices $x_1 < \cdots < x_{k-3} < x_{k-2}$ such that $D(x_2, \ldots, x_{k-2}, u, v) \subseteq D(x_1, x_2, \ldots, x_{k-2}, u)$. However, consider the longest monotone path ending with the vertices $\{x_1, x_2, \ldots, x_{k-2}, u\}$ that has the same color as the edge $\{x_1, x_2, \ldots, x_{k-2}, u, v\}$ to it. This path can be extended by adding the edge $\{x_1, x_2, \ldots, x_{k-2}, u, v\}$. This contradicts the assumption that $D(x_2, \ldots, x_{k-2}, u, v) \nsubseteq D(x_1, x_2, \ldots, x_{k-2}, u)$, which completes the proof. □

Lemma 3.21. $N_k(2, n) > \rho_k(n)$.

Proof. We need to construct a blue-red coloring of the edges of $K^k_{\rho_k(n)}$ that has no monochromatic path of length n. To do so, identify each vertex with a distinct element of $\mathcal{P}^k(n)$ and order lexicographically. For the edge $\{\mathcal{F}_1, \ldots, \mathcal{F}_k\}$, we define its color using the previous observation as follows: the sets $\mathcal{F}_i$ are ordered lexicographically, so we have $\mathcal{F}_1 \lessdot \mathcal{F}_2 \lessdot \cdots \lessdot \mathcal{F}_k$. This implies, by definition, that $\mathcal{F}_1 \nsupseteq \mathcal{F}_2 \nsupseteq \cdots \nsupseteq \mathcal{F}_k$. By the previous observation, we obtain $\delta(\mathcal{F}_1, \mathcal{F}_2) \nsupseteq \delta(\mathcal{F}_2, \mathcal{F}_3) \nsupseteq \cdots \nsupseteq \delta(\mathcal{F}_{k-1}, \mathcal{F}_k)$. We can continue using the observation itaretively as we noticed it previously. By applying it i times, we obtain a $k-i$ long sequence of order $k-i$ line partitions. In particular, for $i = k-2$, we get a pair of order-2 line partitions X_1 and X_2 such that $X_1 \nsupseteq X_2$. Notice that $X_1, X_2 \in \mathcal{P}^2(n) = [n]^2$, so we can write (x_1, y_1) and (x_2, y_2) instead of X_1 and X_2, respectively. We color the edge $\{\mathcal{F}_1, \ldots, \mathcal{F}_k\}$ blue if $x_1 < x_2$; otherwise, we color it red. Notice that for the red edges we must have $y_1 < y_2$; otherwise, we would get $(x_1, y_1) \supseteq (x_2, y_2)$.

Suppose that we have a monochromatic monotone path of length l on the vertices

$\mathcal{F}_1 \lessdot \mathcal{F}_2 \lessdot \cdots \lessdot \mathcal{F}_{k+l-1}$. For each edge in the path $\{\mathcal{F}_i, \ldots, \mathcal{F}_{k+i-1}\}$ we have a pair from $[n]^2$. Notice that the l edges together determine $l+1$ distinct elements from $[n]^2$, and since the path is monochromatic, these elements strictly increase in either the first or second coordinate. Therefore, we get $l+1 \leq n$ because the coordinates cannot be greater than n, which completes the proof. □

We can generalize the approach above by using higher-order d-dimensional partitions instead of higher-order line partitions to obtain a characterization for any $q > 2$ colors as follows. Let $\mathcal{P}_d^2(n)$ be the set $[n]^d$. Then for $k \geq 3$ we define $\mathcal{P}_d^k(n)$ as the family of down-sets in $\mathcal{P}_d^{k-1}(n)$. Note that in this case, by Observation 3.10 we have $\mathcal{P}_d^3(n)$ in bijection with $p_{d-1}(n)$. If we denote the cardinality of $\mathcal{P}_d^k(n)$ by $\rho_{k,d}(n)$, then we get the following:

$$\rho_{2,d}(n) = n^d, \quad \rho_{3,d}(n) = P_{d-1}(n), \quad \rho_{k,d}(n) \leq 2^{\rho_{k-1,d}(n)} \text{ for } k \geq 3.$$

Using the same arguments as in the previous two lemmas, but now for higher-order d-dimensional partitions, we get the following characterization:

Theorem 3.22. *For every $k \geq 2$ and $n \geq 2$ we have*

$$N_k(q,n) = \rho_{k,q}(n) + 1.$$

Observe that the coloring defined in Lemma 3.15 coincides with the coloring defined in Lemma 3.21 for $k = 3$ (this is also true with the previous generalization for q-colorings in Lemma 3.21).

To establish the bounds on $N_k(q,n)$ in general for Theorem 3.1, we reduce the problem to the $k = 3$ case using the following Theorem 3.23 and Theorem 3.24 and then apply the bounds from Corollary 3.16.

Theorem 3.23. *For every $k \geq 3$, $q \geq 2$, and $n \geq 2$, we have*

$$N_k(q,n) \leq tow_{k-2}(N_3(q,n)).$$

Proof. It is enough to prove that for $k \geq 4$ the following holds:

$$N_k(q,n) \leq N_{k-2}(N_3(q,n) - 1, 2). \tag{3.1}$$

Indeed, if the previous is true, then we can use induction in the following way.

For $k = 3$, there is nothing to prove.

For $k = 4$:

$$N_4(q,n) \leq N_2(N_3(q,n) - 1, 2) \leq 2^{N_3(q,n)-1} + 1 \leq \mathrm{tow}_2(N_3(q,n)).$$

The first inequality comes from (3.1). For the second, we use Theorem 3.22 for $k = 2$ (i.e.

$N_2(q,n) \le n^q + 1$). The third inequality trivially holds.

For $k \ge 5$:

$$N_k(q,n) \le N_{k-2}(N_3(q,n)-1,2) \le \mathrm{tow}_{k-4}(N_3(q',2)) \le \mathrm{tow}_{k-3}(2\cdot 2^{q'-1}) \le \mathrm{tow}_{k-2}(N_3(q,n)),$$

where $q' = N_3(q,n) - 1$. The first inequality comes from 3.1. For the second, we use the induction hypotesis. The third follows from $N_3(q,n) \le 2^{2n^{q-1}}$, which is the corollary of Proposition 3.7 and Lemma 3.14.

Now, we prove 3.1. Fix a q-coloring of K_N^k that has no monochromatic monotone path of length n. We need to show that $N < N_{k-2}(N_3(q,n)-1,2)$. For vertices $v_1 < \cdots < v_{k-1}$, let us define $C(v_1,\dots,v_{k-1}) := (n_1+1,\dots,n_q+1)$, where n_i ($1 \le i \le q$) is the length of the longest path colored by i ending with the vertices $v_1,\dots,v_{k-1}$. Notice that, by our assumption, $C(v_1,\dots,v_{k-1})$ is in $[n]^q$. We continue by defining $D(v_2,\dots,v_{k-1})$ for vertices $v_2 < \cdots < v_{k-1}$ as the set $\{x \in [n]^q : \exists u < v_2 : x \le C(u,v_2,\dots,v_{k-1})\}$. Observe that $D(v_2,\dots,v_{k-1})$ is a down-set in $[n]^q$. Therefore, we obtain by Lemma 3.14 that:

$$\left|\left\{D(v_2,\dots,v_{k-1}) : v_2 < \cdots < v_{k-1} \text{ vertices of } K_N^k\right\}\right| \le N_3(q,n) - 1. \tag{3.2}$$

Now we consider the $(k-2)$-uniform hypergraph on the same vertex set as K_N^k and color the edge $\{v_2,\dots,v_{k-1}\}$ where $v_2 < \cdots < v_{k-1}$ with $D(v_2,\dots,v_{k-1})$. We claim that there is no monochromatic monotone path of length 2.

Suppose the contrary, that there are vertices $v_1 < \cdots < v_{k-1}$ for which the edges $\{v_1,\dots,v_{k-2}\}$ and $\{v_2,\dots,v_{k-1}\}$ have the same color, e.i. $D(v_1,\dots,v_{k-2}) = D(v_2,\dots,v_{k-1})$. Notice that $C(v_1,\dots,v_{k-1})$ is in $D(v_2,\dots,v_{k-1})$. By the definition of $D(v_1,\dots,v_{k-2})$ and our assumption, there exists a vertex $u < v_1$ such that $C(v_1,\dots,v_{k-1}) \le C(u,v_1,\dots,v_{k-2})$. However, this is a contradiction, because the longest monotone path ending with the vertices $\{u,v_1,\dots,v_{k-2}\}$ that has the same color as $\{u,v_1,\dots,v_{k-1}\}$ can be extended with $\{u,v_1,\dots,v_{k-1}\}$, so in the corresponding i-th ($i \in [q]$) coordinate we get $C(v_1,\dots,v_{k-1})_i \nleq C(u,v_1,\dots,v_{k-2})_i$.

We conclude that our coloring of K_N^{k-2} has no monochromatic monotone path of length 2. Therefore, $N < N_{k-2}(c,2)$ must hold, where c is the number of colors that we used. Since, by inequality (3.2), we have $c \le N_3(q,n) - 1$, and because the value of $N_k(q,n)$ increases as the number of colors increases, the proof is complete. □

We do not present the full proof of the following theorem, which establishes a recursive lower bound on $N_k(q,n)$; instead, we provide the part of the proof in Lemma 3.25, where we make use of the characterization given in Theorem 3.19. The complete proof can be found in [5].

Theorem 3.24. *There is an absolute constant n_0 so that for every $k \ge 3$, $q \ge 2$, and*

$n \geq n_0$, *we have*

$$N_k(q,n) \geq tow_{k-2}(N_3(q,n)/3n^q).$$

Lemma 3.25. *For every $k \geq 4$, $q \geq 2$, and $n \geq 2$, we have*

$$N_k(q,n) \geq 2^{N_{k-1}(q,n)/N_{k-2}(q,n)}.$$

Proof. By Theorem 3.22, it is enoungh to show that $\rho_{k,q}(n)+1 \geq 2^{\rho_{k-1,q}(n)+1/\rho_{k-2,q}(n)+1}$. Let L_i be the set $\left\{S \in \mathcal{P}_q^{k-1}(n) : |S| = i\right\}$ for $0 \leq i \leq \rho_{k-2,q}(n)$. Recall that the elements of $\mathcal{P}_q^{k-1}(n)$ are subsets of $\mathcal{P}_q^{k-2}(n)$ so the upper bound on i is well-defined. Denote $l_i := |L_i|$. We claim that every collection $\{S_1, \dots, S_t\} \subseteq L_i$ determines a distinct element of $\mathcal{P}_q^k(n)$ and thus $\rho_{k,q}(n) \geq 2^{l_i}$. To prove the claim, consider for any $\{S_1, \dots, S_t\} \subseteq L_i$ the set

$$\mathcal{F} = \left\{S \in \mathcal{P}_q^{k-1}(n) : \ \exists 1 \leq j \leq t \ \text{ such that } \ S \subseteq S_j\right\}.$$

This set indeed belongs to $\mathcal{P}_q^k(n)$ because the subset relation is transitive. Observe that elements in L_i cannot be subsets of one other (since all have the same size i), so $S_1, \dots, S_t$ are maximal in $\mathcal{F}$. Thus, for any other collection $\left\{S'_1, \dots, S'_{t'}\right\} \subseteq L_i$, the determined set F' differs from F, because the sets of maximal elements are different. This proves the claim.

Now, let $l_{\max}$ be the maximum among the l_i values. Then,

$$\rho_{k-1,q}(n) = \sum_{j=0}^{\rho_{k-2,q}(n)} l_j \leq l_{\max} \cdot (\rho_{k-2,q}(n) + 1),$$

since $\mathcal{P}_q^{k-1}(n) = \bigcup_{j=0}^{\rho_{k-2,q}(n)} L_j$. Moreover, the previous inequality is stirct, because $1 = l_0 < l_3$. Thus, we get

$$\frac{\rho_{k-1,q}(n) + 1}{\rho_{k-2,q}(n) + 1} \leq l_{\max},$$

This completes the proof. □

Now we can complete the proof of Theorem 3.1 using Theorem 3.23 and Theorem 3.24 with Corollary 3.16.

Proof of Theorem 3.1.

$$N_k(q,n) \overset{3.23}{\leq} \text{tow}_{k-2}(N_3(q,n)) \overset{3.16}{\leq} \text{tow}_{k-2}\left(2^{2n^{q-1}}\right) = \text{tow}_{k-1}\left(2n^{q-1}\right)$$

$$N_k(q,n) \overset{3.24}{\geq} \text{tow}_{k-2}\left(\frac{N_3(q,n)}{3n^q}\right) \overset{3.16}{\geq} \text{tow}_{k-2}\left(\frac{2^{\frac{2}{3}n^{q-1}/\sqrt{q}}}{3n^q}\right) \gtrapprox \text{tow}_{k-1}\left(\frac{n^{q-1}}{2\sqrt{q}}\right).$$ □

3.5 Transitive and k-monotone colorings

In this subsection, we investigate the problem of how the number $N_k(2,n)$ changes if the coloring is restricted to special colorings that satisfy stricter properties.

Definition 3.26. *A coloring* $\mathfrak{C}$ *of* $(K_n^k; \prec)$ *is called transitive if for every* $(k+1)$*-tuples of vertices* $\{v_1, \ldots, v_{k+1}\}$ *that satisfies* $v_1 \prec \cdots \prec v_{k+1}$ *and* $\mathfrak{C}(v_1, \ldots, v_k) = \mathfrak{C}(v_2, \ldots, v_{k+1})$ *it holds that all k-tuples from* $\binom{\{v_1, \ldots, v_{k+1}\}}{k}$ *have the same color in* $\mathfrak{C}$*.*

To motivate the definition above, observe that the colorings we defined in order to establish the connection between the number $N_k(q, n)$ and the Erdős-Szekeres Lemma and Theorem at the beginning of Section 3 are *transitive*.

Consider first the case of the Erdős-Szekeres Lemma. Suppose $v_1 \prec v_2 \prec v_3$ are vertices such that the edges $\{v_1, v_2\}$ and $\{v_2, v_3\}$ receive the same color. Then, by the transitivity of the natural ordering of the numbers, the edge $\{v_1, v_3\}$ must also receive the same color.

Similarly, consider the case of the Erdős-Szekeres Theorem. Suppose $v_1 \prec v_2 \prec v_3 \prec v_4$ are vertices such that the edges $\{v_1, v_2, v_3\}$ and $\{v_2, v_3, v_4\}$ receive the same color. Then, because $\{v_1, v_2, v_3, v_4\}$ forms either a 4-cup or a 4-cap, the edges $\{v_1, v_2, v_4\}$ and $\{v_1, v_3, v_4\}$ must also receive the same color.

Denote by $N_k^{\text{trans}}(q, n)$ the variant of $N_k(q, n)$ restricted to transitive colorings. The question of bounding the number $N_k^{\text{trans}}(q, n)$ was raised by Eliáš and Matušek in [2]. Clearly, we have $N_k^{\text{trans}}(q, n) \leq N_k(q, n)$ and with Theorem 3.1 we have:

Corollary 3.27. $N_k^{trans}(q, n) \leq tow_{k-1}(2n^{q-1})$.

Problem 1. *What is the growth rate of* $N_k^{trans}(q, n)$*?*

Moshkovitz and Shapira [5] note that it may be the case that $N_k^{\text{trans}}(q, n) = N_k(q, n)$, as this equality holds for their construction when $k = 2, 3$.

Remark 3.28. *For* $k = 2, 3$ *the coloring defined in Lemma 3.21 (and its generalization for Theorem 3.22) is transitive.*

Proof. For $k = 2$, the vertex set is $\mathcal{P}_q^2(n) = [n]^q$ and $\lessdot$ is the regular lexicographical order on $[n]^q$. Suppose that we have vertices $A \lessdot B \lessdot C$ such that $\mathfrak{C}(A, B) = \mathfrak{C}(B, C) = i \in [q]$. We have to show that $\mathfrak{C}(A, C) = i$. By our assumption and the definition of the coloring $\mathfrak{C}$ we know that $A_i < B_i$, $B_i < C_i$ and $A_j = B_j$, $B_j = C_j$ for $j < i$. Thus, $A_i < C_i$ and $A_j = C_j$ for $j < i$, i.e. $\mathfrak{C}(A, C) = i$.

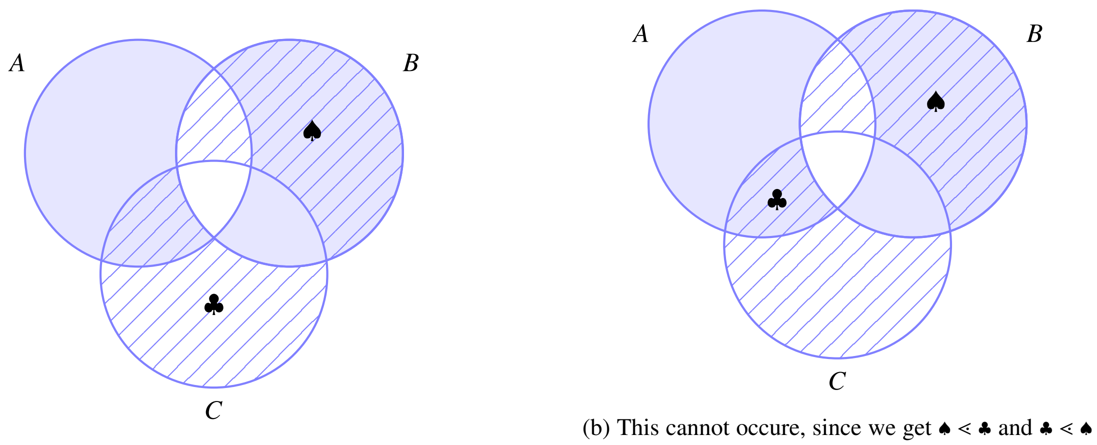
For $k = 3$, the vertex set is $\mathcal{P}_q^3(n) = \{S \subseteq [n]^q : S \text{ is a down-set in } [n]^q\}$, the order is $\lessdot$, and if we write minimal, we mean that it is minimal in the $\lessdot$ order. Suppose that we have vertices $A \lessdot B \lessdot C \lessdot D$ in $\mathcal{P}_q^3(n)$ such that $\mathfrak{C}(A, B, C) = \mathfrak{C}(B, C, D) = i \in [q]$. We have to show that $\mathfrak{C}(A, C, D) = \mathfrak{C}(A, B, D) = i$. We only prove it for the edge $\{A, C, D\}$ as the case of $\{A, B, D\}$ is symmetric.

Notice that we have to apply γ only once to obtain elements in $[n]^q$ and determine the coloring. Thus, we have to show

$$\gamma(A, C)_i < \gamma(C, D)_i \text{ and } \gamma(A, C)_j = \gamma(C, D)_j \text{ for } 1 \leq j < i. \tag{3.3}$$

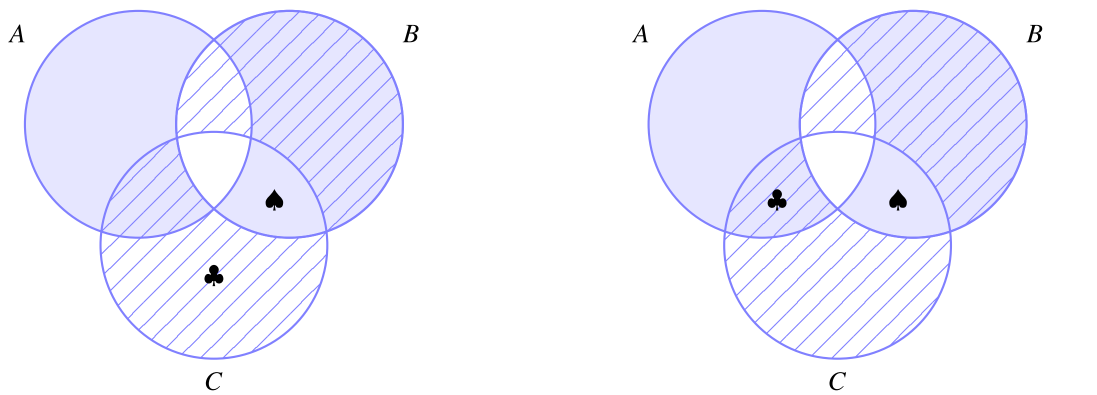
Therefore, we have to determine $\gamma(A, C)$. We claim that $\gamma(A, C)$ is the minimum of $\gamma(A, B)$ and $\gamma(B, C)$. From that we are done, since if $\gamma(A, C) = \gamma(B, C)$, then (3.3) trivially holds; otherwise, we can argue similarly to the case $k = 2$ and obtain (3.3) as well.

By the definition of the order $\lessdot$ and the function γ, $\gamma(A, B) \in B$ and $\gamma(B, C) \in C$ is the minimal element of $A \triangle B$ (filled area) and $B \triangle C$ (hatched area), respectively. Denoting $\gamma(A, B)$ as ♠ and $\gamma(B, C)$ as ♣, we can distinguish the following four cases based on the positions of ♠ and ♣:



(a) min{♠, ♣} = ♣

(b) This cannot occure, since we get ♠ $\lessdot$ ♣ and ♣ $\lessdot$ ♠ at the same time - contradiction



(c) min{♠, ♣}

(d) min{♠, ♣} = ♠

Figure 3.3: The four cases based on the location of ♠ = $\gamma(A, B)$ and ♣ = $\gamma(B, C)$.

Observe that, in the possible cases (Fig. 3.3 a, c, d), if we choose the minimum of $\gamma(A, B)$ and $\gamma(B, C)$, then it is the minimal element in $(A \cup B \cup C) \setminus (A \cap B \cap C)$ since the $\lessdot$ order is transitive. Furthermore, in all three cases, the minimum of $\gamma(A, B)$ and $\gamma(B, C)$ is in $C \setminus A$. Therefore, it follows that $\gamma(A, C)$ is the minimum of $\gamma(A, B)$ and $\gamma(B, C)$. □

As it turn out, for $k \geq 4$ the colorings defined in Lemma 3.21 (and its generalization for Theorem 3.22) are not transitive.

Let us consider the case $k = 4$ and $q = 2$, meaning we have vertices $A_1 \lessdot \cdots \lessdot A_5$ such that $\mathfrak{C}(A_1, \ldots, A_4) = \mathfrak{C}(A_2, \ldots, A_5) = c$. We aim to show that $\mathfrak{C}(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_5)$ is also c for $2 \leq i \leq 4$. To do this, we always have to determine $\gamma(A_{i-1}, A_{i+1})$. Consider the case $i = 2$. By the same argument used in Remark 3.28 for $k = 3$, we have $\gamma(A_1, A_3) = \min_{\lessdot}(\gamma(A_1, A_2), \gamma(A_2, A_3))$. If this minimum is $\gamma(A_2, A_3)$, then the sequence $\gamma(A_1, A_3), \gamma(A_3, A_4), \gamma(A_4, A_5)$ that determines $\mathfrak{C}(A_1, A_3, A_4, A_5)$ is the same as the sequence $\gamma(A_2, A_3), \gamma(A_3, A_4), \gamma(A_4, A_5)$, which determines $\mathfrak{C}(A_2, A_3, A_4, A_5)$. Therefore, we get that $\mathfrak{C}(A_1, A_3, A_4, A_5) = c$. However, if $\gamma(A_1, A_3) = \gamma(A_1, A_2)$, then we have to determine $\gamma(\gamma(A_1, A_2), \gamma(A_3, A_4))$. By the definition of γ, we know that $\gamma(\gamma(A_2, A_3), \gamma(A_3, A_4))$ is the minimum in $\gamma(A_3, A_4) \setminus \gamma(A_2, A_3)$, $\gamma(\gamma(A_1, A_2), \gamma(A_3, A_4))$ is the minimum in $\gamma(A_3, A_4) \setminus \gamma(A_1, A_2)$, and since $\gamma(A_2, A_3) \lessdot \gamma(A_1, A_2)$, it follows that $\gamma(\gamma(A_1, A_2), \gamma(A_2, A_3))$ is the minimum in $\gamma(A_1, A_2) \triangle \gamma(A_2, A_3)$. We use the following notations for simplicity:

$$\spadesuit := \gamma(\gamma(A_1, A_2), \gamma(A_2, A_3)), \quad \clubsuit := \gamma(\gamma(A_2, A_3), \gamma(A_3, A_4)), \quad \blacklozenge := \gamma(\gamma(A_1, A_2), \gamma(A_3, A_4)).$$

We can distinguish several cases based on the location of $\spadesuit$, $\clubsuit$, and $\blacklozenge$. Observe the case in Fig. 3.4.

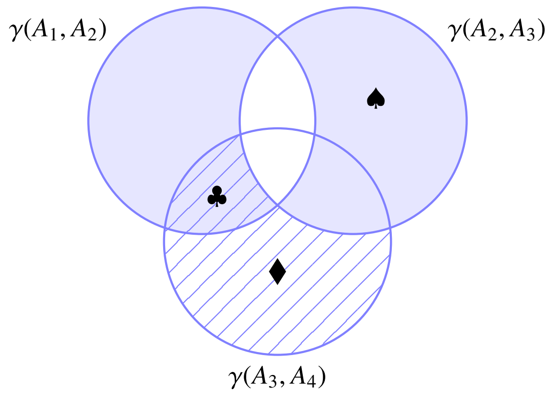


Figure 3.4: The problematic case.

Here $\clubsuit \lessdot \spadesuit \lessdot \blacklozenge$, that is, it can be the case that $\blacklozenge = \gamma(\gamma(A_1, A_2), \gamma(A_3, A_4))$ is even larger than $\gamma(\gamma(A_3, A_4), \gamma(A_4, A_5))$ in $\lessdot$, and that can contradict with transitive property. Observe the following example in Fig. 3.5.

Therefore, the question of deciding whether $N_k^{\text{trans}}(q, n) = N_k(q, n)$ remains an interesting open problem.

Balko, in [1], resolves Problem 1 by constructing a transitive coloring of $\mathcal{K}_N^k$ that contains no monochromatic monotone path of length $2n$, where $N \geq \text{tow}_{k-1}((1 - o(1))n)$. In fact, the coloring he constructs satisfies a property even stricter than transitivity.

For its definition, we use the folling notation: let B be a sequence with cardinality n, for $\{i_1, \ldots, i_r\} \subseteq [n]$ we use $B^{(i_1, \ldots, i_r)}$ to denote the subsequence of B that we obtain by deleting the i_j-th elements ($j \in [r]$) from B.

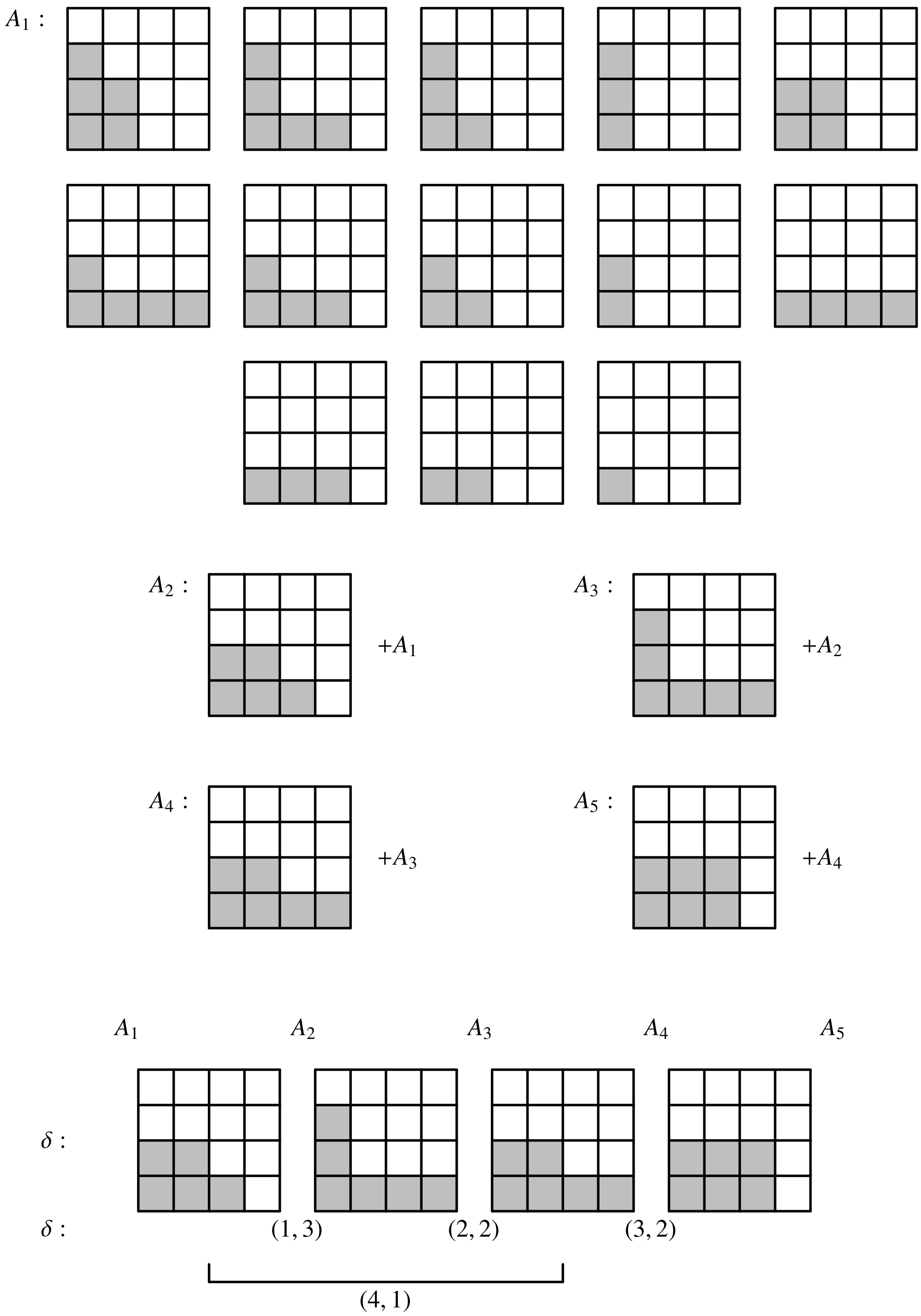


Figure 3.5: Observe that, here $\gamma(A_1, A_3) = \gamma(A_1, A_2)$, ♣ $= (1, 3)$, ♠ $= (2, 2)$, ♦ $= (4, 1)$, and $\gamma(\gamma(A_3, A_4), \gamma(A_4, A_5)) = (3, 2)$, furthermore, $\mathfrak{C}(A_1, A_2, A_3, A_4)$ and $\mathfrak{C}(A_2, A_3, A_4, A_5)$ are blue, since $(1, 3) \lessdot (2, 2)$ and $(2, 2) \lessdot (3, 2)$, however, $(4, 1) \not\lessdot (3, 2)$, that is, $\mathfrak{C}(A_1, A_2, A_3, A_4)$ is red - contradicting the transtitive property.

Definition 3.29. *A* 2*-coloring* $\mathfrak{C}$ *of* $(K_n^k; \prec)$ *that assigns* -1 *or* $+1$ *to every edge is called* k*-monotone if for every* $(k+1)$*-tuples of vertices* $S = \{v_1, \ldots, v_{k+1}\}$ *that satisfies* $v_1 \prec \cdots \prec v_{k+1}$ *we have* $\mathfrak{C}(S^{(1)}) \leq \cdots \leq \mathfrak{C}(S^{(k+1)})$ *or* $\mathfrak{C}(S^{(1)}) \geq \cdots \geq \mathfrak{C}(S^{(k+1)})$.

Remark 3.30. *Every* k*-monotone coloring is transitive.*

Denote with $N_k^{\text{mon}}(q,n)$ the variant of $N_k(q,n)$ restricted to k-monotone colorings. From the previous remark, we clearly have $N_k^{\text{mon}}(q,n) \leq N_k^{\text{trans}}(q,n)$. In the remaining part of the thesis, we present the construction of Balko [1] that gives us the following lower-bound on $N_k^{\text{mon}}(q,n)$.

Theorem 3.31. *For positive integers* k *and* n *with* $k \geq 3$*, we have*

$$N_k^{mon}(2, 2n) \geq tow_{k-1}((1 - o(1))n).$$

We will construct a coloring c_k of the edges of $\mathcal{K}_N^k$ such that it contains no monochromatic monotone path of length $2n$, where $N > \text{tow}_{k-1}((1 - o(1))n)$. First, we define the coloring and verify that it contains no monochromatic monotone path of length $2n$. Afterwards, we prove that the coloring c_k is a k-monotone coloring.

Let us start with a brief overview of the construction of the coloring c_k.

- For every $k \geq 3$ and positive integer n, we define a set $F_k(n)$ to serve as our vertex set, such that the elements in $F_k(n)$ are subsets of F_{k-1} and $|F_1(n)| = 2$, $|F_2(n)| = 2n$, and $|F_k(n)| = 2^{|F_{k-1}(n)|/2}$ for $k \geq 3$.
- We also partition $F_k(n)$ into two subsets: $F_k^+(n)$ and $F_k^-(n)$, and define a bijection σ_k between $F_k^+(n)$ and $F_k^-(n)$.
- Furthermore, we introduce an equivalence relation on $F_k(n)$ by declaring $A, B \in F_k(n)$ equivalent in $\equiv_k$, denoted $A \equiv_k B$ if $A = B$, $A = \sigma_k(B)$, or $\sigma_k(A) = B$.
- We define $<_k$, a linear order on the vertex set.
- Every element in $F_k^-(n)$ will precede the elements in $F_k^+(n)$.
- We also define $\lhd_k$, a linear order on the equivalence classes of the relation $\equiv_k$.
- The σ_k bijection will be $<_k$ order-reversing.
- Furthermore, if we regard the $\lhd_k$ order on $F_k^-(n)$ and $F_k^+(n)$, we will have $(F_k^-(n), \lhd_k) = (F_k^-(n), <_k)$ and $(F_k^+(n), \lhd_k) = (F_k^+(n), >_k)$ (see Fig. 3.6).
- We define a function γ that assigns to each pair of sets $A, B \in F_k(n)$ an element $\gamma(A, B) \in F_{k-1}(n)$, which is the first element (according to the order $\lhd_{k-1}$) at which A and B differ.

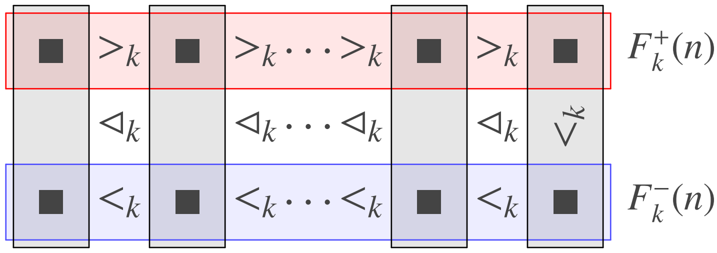


Figure 3.6: The $<_k$ and $\lhd_k$ orders - the vertical rectangles denote the equivalence classes of $\equiv_k$, and the black squares are the elements of $F_k(n)$.

- We obtain the color of the edge $\{A_1, \dots, A_k\}$, where $A_i \in F_k(n)$ and $A_1 <_k \cdots <_k A_k$, by applying γ to consecutive terms A_i and A_{i+1} iteratively.

For $k = 1$, we define $F_1^-(n) := \{-\}$, $F_1^+(n) := \{+\}$, $F_1(n) := \{-, +\}$, and $- <_1 +$. The bijection $\sigma_1 : F_1^-(n) \to F_1^+(n)$ simply let be $\sigma_1(F_1^-(n)) := F_1^+(n)$. Thus, $- \equiv_1 +$.

For $k = 2$, let

$$F_2^-(n) := \{(2n - i + 1, i) : \ i \in [n]\} \subseteq [2n]^2 \quad \text{and}$$
$$F_2^+(n) := \{(i, 2n - i + 1) : \ i \in [n]\} \subseteq [2n]^2.$$

Furthermore, $F_2(n) := F_2^-(n) \cup F_2^+(n)$ (note that this is indeed a disjoint union). We define the linear order $<_2$ on $F_2(n)$ as $(2n, 1) <_2 (2n-1, 2) <_2 \cdots <_2 (1, 2n)$. Let $\sigma_1 : F_1^-(n) \to F_1^+(n)$ be the one to one correspondence that maps $(2n - i + 1, i)$ to $(i, 2n - i + 1)$. The elements A and B are equivalent in $\equiv_2$ if $A = B$, $A = \sigma_1(B)$, or $B = \sigma_1(A)$. We now define a linear order $\lhd_2$ on the equivalence classes of $F_2(n)$ under $\equiv_2$. Identify the equivalence classes with the elements of F_2^- and let the $\lhd_2$ order on $(F_2(n))_{\equiv_2}$ be the $<_2$ order on $F_2^-(n)$. Slightly abusing the notation we can consider $\unlhd_2$ as a linear order on $F_2(n)$ if we let two equivalent elements equal in $\unlhd_2$.

Let $k \geq 3$ and assume that we have already constructed $F_{k-1}(n)$. We define $F_k(n)$ as the collection of sets such that every set in $F_k(n)$ contains exactly one set from each equivalence class of $F_{k-1}(n)$ over $\equiv_{k-1}$. Since there are two choices in each equivalence class and $F_{k-1}(n)$ has $|F_{k-1}(n)|/2$ equivalence classes, it follows that $|F_k(n)| = 2^{|F_{k-1}(n)|/2}$. Notice that the maximal and minimal elements of $F_{k-1}(n)$ with respect to $<_{k-1}$ are equivalent over $\equiv_{k-1}$. Thus, we can distinguish the sets in $F_k(n)$ based on whether the sets contain the maximum or the minimum (since exactly one of these conditions holds for every set). In this way, we obtain a partition of $F_k(n)$ into two disjoint subsets: $F_k^-(n)$ and $F_k^+(n)$, consisting of those sets that contain the minimal element and the maximal element of $F_{k-1}(n)$ in $<_{k-1}$, respectively. For two distinct sets A and B from $F_k(n)$, define $\gamma(A, B)$ to be the element of $B \cap E$, where E is the first equivalence class in $(F_{k-1}(n))_{\equiv_{k-1}}$ (with respect to the order $\lhd_{k-1}$) on which A and B differ. We define the linear order $<_k$ on $F_k(n)$

by letting $A <_k B$ if $\gamma(A, B) \in F^+_{k-1}(n)$. Notice that $\gamma(A, B) \in F^+_{k-1}(n)$ if and only if $\gamma(B, A) \in F^-_{k-1}(n)$ and thus $<_k$ is assymetric and connected. It remains to show that $<_k$ is also transitive. Suppose that we have A, B, C in $F_k(n)$ such that $A <_k B$ and $B <_k C$, we have to show that $A <_k C$. Let $N := |F_{k-1}(n)|/2$ and denote the equivalence classes of $F_{k-1}(n)$ with E_i, where $1 \le i \le N$ and $E_1 \lhd_{k-1} \cdots \lhd_{k-1} E_N$. There exist E_i and E_j, such that $\gamma(A, B) = B \cap E_i$ and $\gamma(B, C) = C \cap E_j$; furthermore, for every $l \in [N]$ such that $l < i$, we have $A \cap E_l = B \cap E_l$ (i.e. A and B differ in E_i first), and similarly for every $m \in [N]$ such that $m < j$, we have $B \cap E_m = C \cap E_m$ (i.e. B and C differ in E_j first). Note that $i \neq j$, otherwise $B \ni \gamma(A, B) = \gamma(B, C) \notin B$ is a contradtiction. If $j < i$, then $B \cap E_j = A \cap E_j$, thus $\gamma(A, C) = \gamma(B, C)$ and $A <_k C$. If $i < j$, then $B \cap E_i = C \cap E_i$, thus $\gamma(A, C) = \gamma(A, B)$ and $A <_k C$. Observe that if $A \in F^-_k(n)$ and $B \in F^+_k(n)$, then $A <_k B$, since the maximal and minimal element in $F_{k-1}(n)$ are equivalent in $\equiv_{k-1}$ and also the minimum in the $\lhd_{k-1}$ order, and A and B differ in these elements.

Let us define γ for $k = 2$ and show that the definition of the $<_2$ order is equivalent to the general definition of the order $<_k$ for $k = 2$. If $A = (a_1, a_2)$ and $B = (b_1, b_2)$ are distinct elements of $F_2(n)$, we define $\gamma(A, B) = -$ if $a_1 < b_1$, and similarly $\gamma(A, B) = +$ if $a_1 > b_1$ where $<$ is the standard ordering of $\mathbb{R}$. Thus, by the original definition of $<_2$, for distinct $A, B \in F_2(n)$, we have $A <_2 B$ if and only if $\gamma(A, B) \in F^+_2(n) = \{+\}$, which coincides with the general definition of the order $<_k$ for $k = 2$.

Now, let us move to the definition of the bijection $\sigma_k : F^-_k(n) \to F^+_k(n)$, for $k \ge 3$. For a set $\{A_1, \ldots, A_{|F_{k-1}(n)|/2}\} \in F_k(n)$, where each A_i belongs to the i-th equivalent class of $F_{k-1}(n)$ over $<_{k-1}$, define:

$$\sigma_k(\{A_1, \ldots, A_{|F_{k-1}(n)|/2}\}) := \{\sigma_{k-1}(A_1), \ldots, \sigma_{k-1}(A_{|F_{k-1}(n)|/2})\}\,.$$

Observe that σ_r is indeed a bijection. Two elements A, B are equivalent over $\equiv_k$ if $A = B$, $A = \sigma_k(B)$, or $B = \sigma_k(A)$. We identify each $A \in F^-_k(n)$ with $\sigma_k(A) \in F^-_k(n)$, and define the total order $\lhd_r$ on $(F_k(n))_{\equiv_k}$ as the ordering $<_r$ on $F^-_k(n)$. Slightly abusing the notation we can consider $\unlhd_k$ as a linear order on $F_k(n)$ if we let two equivalent elements equal in $\unlhd_k$.

For $r, k \ge 2$ and a sequence $A_1, \ldots, A_r$ in $F_k(n)$ in which each consecutive term is distinct, let $\Gamma(A_1, \ldots, A_r)$ be the sequence $\gamma(A_1, A_2), \ldots, \gamma(A_{r-1}, A_r)$. Observe that for $2 \le i \le r$ each term $\gamma(A_{i-1}, A_i)$ is in $F_{k-1}(n)$. By the definition of the γ function $\gamma(A_{i-1}, A_i) \in A_i$ and $\notin A_{i-1}$, thus, any two consecutive terms of the sequence $\gamma(A_1, A_2), \ldots, \gamma(A_{r-1}, A_r)$ are distinct (otherwise $A_i \ni \gamma(A_{i-1}, A_i) = \gamma(A_i, A_{i+1}) \notin A_i$ is a contradiction). Therefore, if $k \ge 3$, Γ can be applied on the sequence $\gamma(A_1, A_2), \ldots, \gamma(A_{r-1}, A_r)$ in $F_{k-1}(n)$. In general, applying Γ on the sequence $A_1, \ldots, A_r$ in $F_k(n)$ i times ($1 \le i \le \min(r-1, k-1)$) we obtain a sequence $\Gamma(\Gamma(\cdots\Gamma(A_1, \ldots, A_r))) =: \Gamma^i(A_1, \ldots, A_r)$ of $r - i$ elements in $F_{k-i}(n)$. For convinience, denote the sequence $A_1, \ldots, A_r$ with $\Gamma^0(A_1, \ldots, A_r)$.

Let $\mathcal{K}^k_{|F_k(n)|}$ be the k-uniform complete hypergraph on the vertex set $F_k(n)$ with the ordering $<_k$. For vertices $A_1 <_k \cdots <_k A_k$ color the edge $\{A_1, \ldots, A_k\}$ with $c_k(\{A_1, \ldots, A_k\}) := \Gamma^{k-1}(A_1, \ldots, A_k)$.

Lemma 3.32. *For all positive integers n and k with $k \geq 3$, there is no monochromatic monotone path of length $2n$ in $\mathcal{K}^k_{|F_k(n)|}$ colored with c_k.*

Proof. Let $\mathcal{P}$ be a monochromatic monotone path of length l in c_k. We aim to show that $l < 2n$. Let the vertices of $\mathcal{P}$ be $A_1 <_k \cdots <_k A_{l+k-1}$. Since $\mathcal{P}$ is monochromatic in c_k, we know that

$$\Gamma^{k-1}(A_1, \ldots, A_k) = \Gamma^{k-1}(A_2, \ldots, A_{k+1}) = \cdots = \Gamma^{k-1}(A_l, \ldots, A_{l+k-1}). \tag{3.4}$$

For each $1 \leq i \leq l$ we can rewrite the term in (3.4) as

$$\Gamma^{k-1}(A_i, \ldots, A_{i+k-1}) = \gamma\Big(\, \Gamma^{k-2}(A_i, \ldots, A_{i+k-2})\,,\ \Gamma^{k-2}(A_{i+1}, \ldots, A_{i+k-1})\,\Big).$$

Observe that the terms $\Gamma^{k-2}(A_i, \ldots, A_{i+k-2})$ are in $F_2(n)$, and each consecutive pair from (3.4) share a common term like that. Hence, we obtain a series $\left\{\Gamma^{k-2}(A_i, \ldots, A_{i+k-2})\right\}_{i=1}^{l+1}$ in $F_2(n)$. Furthermore, by the definition of the γ function for $k = 2$ and by the coloring of $\mathcal{P}$ we obtain that $\left\{\Gamma^{k-2}(A_i, \ldots, A_{i+k-2})\right\}_{i=1}^{l+1}$ is strictly increasing or decreasing in $<_2$ (i.e. the terms are distinct), thus we get $l + 1 \leq |F_2(n)| = 2n$ and that completes the proof. □

It remains to prove that the coloring is k-monotone, that is for every $(k+1)$-tuples of vertices $S = \{A_1, \ldots, A_{k+1}\} \subseteq F_k(n)$ that satisfies $A_1 <_k \cdots <_k A_{k+1}$ we have $c_k(S^{(1)}) \leq \cdots \leq c_k(S^{(k+1)})$ or $c_k(S^{(1)}) \geq \cdots \geq c_k(S^{(k+1)})$. Observe that to determine the color of $S^{(i)}$ for each $2 \leq i \leq k$, we always have to determine $\gamma(A_{i-1}, A_{i+1})$ when we apply Γ to $S^{(i)}$. The following lemma gives the answer for this problem.

Lemma 3.33. *For positive integers n and k with $k \geq 2$, let A, B, C be a sequence of distinct sets from $F_k(n)$. For $k \geq 3$, $\gamma(A, C) = min_{\lhd_{k-1}} \{\gamma(A, B), \gamma(B, C)\}$ if $\gamma(A, B) \not\equiv_{k-1} \gamma(B, C)$ and $\gamma(A, B), \gamma(B, C) \lhd_{k-1} \gamma(A, C)$ otherwise. For $k = 2$, $\gamma(A, C) \in \{\gamma(A, B), \gamma(B, C)\}$ if $\gamma(A, B) \neq \gamma(B, C)$ and $\gamma(A, C) = \gamma(A, B) = \gamma(B, C)$ otherwise.*

Proof. Let $k \geq 3$, $N := |F_{k-1}(n)|/2$ and denote the equivalence classes of $F_{k-1}(n)$ with E_i, where $1 \leq i \leq N$ and $E_1 \lhd_{k-1} \cdots \lhd_{k-1} E_N$. First, suppose that $\gamma(A, B) \not\equiv_{k-1} \gamma(B, C)$. There exist E_i and E_j ($i \neq j$ by our assumption), such that $\gamma(A, B) = B \cap E_i$ and $\gamma(B, C) = C \cap E_j$; furthermore, for every $l \in [N]$ such that $l < i$, we have $A \cap E_l = B \cap E_l$ (i.e. A and B differ in E_i first), and similarly for every $m \in [N]$ such that $m < j$, we have $B \cap E_m = C \cap E_m$ (i.e. B and C differ in E_j first). If $i < j$ (i.e. $\gamma(A, B) \lhd_{k-1} \gamma(B, C)$) the previous follows that $A \cap E_l = C \cap E_l$ for every $l \in [N]$ such that $l < i$ and $A \cap E_i \neq B \cap E_i = C \cap E_i$, which follows that $\gamma(A, C)$ is $\gamma(A, B)$. If $j < i$ (i.e. $\gamma(B, C) \lhd_{k-1} \gamma(A, B)$) it follows that $\gamma(A, C) = \gamma(B, C)$.

Now, suppose that $\gamma(A,B) \equiv_{k-1} \gamma(B,C)$, then $\gamma(A,B) \neq \gamma(B,C)$, since $\gamma(A,B) \in B$ but $\gamma(B,C) \notin B$. There exists E_i, such that $\gamma(A,B) = B \cap E_i$, $\gamma(B,C) = C \cap E_i$, and for $j \in [N]$, such that $j < i$, we have $A \cap E_j = B \cap E_j = C \cap E_j$. Therefore, $A \cap E_i = C \cap E_i$ and we get $\gamma(A,B), \gamma(B,C) \lhd_{k-1} \gamma(A,C)$.

For $k = 2$, by using Γ once we get elements in $\{+,-\}$. Thus, if $\gamma(A,B) \neq \gamma(B,C)$ we clearly have $\gamma(A,C) \in \{\gamma(A,B), \gamma(B,C)\} = \{+,-\}$. If $\gamma(A,B) = \gamma(B,C) = +$, we have $A <_2 B <_2 C$. And since the $<_2$ order is transitive, it follows that $A <_2 C$, that is $\gamma(A,C) = +$. If $\gamma(A,B) = \gamma(B,C) = -$, we have $A >_2 B >_2 C$, and so $\gamma(A,C) = -$. □

The following lemma is a technical one that we need for the proof of Lemma 3.35.

Lemma 3.34. *For positive integers n and k with $k \geq 2$, let A, B, A', B' be sets from $F_k(n)$ such that $A \neq B$.*

$$(i) \text{ Assume } A' \neq B. \text{ If } A \leq_k A', \text{ then } \gamma(A',B) \leq_{k-1} \gamma(A,B).$$
$$(ii) \text{ Assume } A \neq B'. \text{ If } B \leq_k B', \text{ then } \gamma(A,B) \leq_{k-1} \gamma(A,B').$$

Proof. We start with part (i). We can assume that $A <_k A'$, otherwise (i) clearly holds.

For $k = 2$, with obtaining γ once we get elements in $\{+,-\}$. Thus, we only have to check the case $\gamma(A,B) = -$, otherwise $\gamma(A',B) \leq_{k-1} \gamma(A,B)$ clearly holds. If $\gamma(A,B) = -$, we have $B <_2 A$, therefore $B <_2 A <_2 A'$, and since the $<_2$ order is transitive, we have $B <_2 A'$. Therefore, $\gamma(A',B) = -$ and then, $\gamma(A',B) \leq_{k-1} \gamma(A,B)$ holds.

Consider the case $k \geq 3$. There are three possibilities where to place B with respect to A and A':

1. If $A <_k B <_k A'$, then $\gamma(A,B)$ is in $F^+_{k-1}(n)$ and $\gamma(A',B)$ is in $F^-_{k-1}(n)$, therefore, by the $<_{k-1}$ order we have $\gamma(A',B) \leq_{k-1} \gamma(A,B)$.

2. If $B <_k A <_k A'$, then $\gamma(B,A)$ and $\gamma(A,A')$ are both in $F^+_{k-1}(n)$, and since $\gamma(B,A) \in A$ and $\gamma(A,A') \notin A$ we have $\gamma(B,A) \neq \gamma(A,A')$, thus $\gamma(B,A) \not\equiv_{k-1} \gamma(A,A')$. Therefore, by Lemma 3.33, we obtain that $\gamma(B,A')$ is the minimum of $\gamma(B,A)$ and $\gamma(A,A')$ in $\lhd_{k-1}$, in particular, we obtain that $\gamma(B,A') \unlhd_{r-1} \gamma(B,A)$. Observe that $\gamma(B,A')$ is also in $F^+_{k-1}(n)$, because $<_k$ is transitive (e.i. we have $B <_k A'$). Since $(F^+_{k-1}(n), \lhd_{k-1}) = (F^+_{k-1}(n), >_{k-1})$, we have $\gamma(B,A') \geq_{k-1} \gamma(B,A)$. Using the fact that $\gamma(X,Y) = \sigma_{k-1}(\gamma(Y,X))$ and σ_{k-1} is $\leq_{k-1}$ order-reversing, it follows that $\gamma(A',B) \leq_{r-1} \gamma(A,B)$.

3. If $A <_k A' <_k B$, then $\gamma(A,A')$ and $\gamma(A',B)$ are both in $F^+_{k-1}(n)$, and as before $\gamma(A,A') \not\equiv_{k-1} \gamma(A',B)$. Using Lemma 3.33 again, we obtain that $\gamma(A,B)$ is the minimum of $\gamma(A,A')$ and $\gamma(A',B)$, in particular $\gamma(A,B) \lhd_{k-1} \gamma(A',B)$. Since $(F^+_{k-1}(n), \lhd_{k-1}) = (F^+_{k-1}(n), >_{k-1})$, we have $\gamma(A,B) >_{r-1} \gamma(A',B)$.

Thus, part (i) is done.

$$\begin{array}{rcc} & i_1(A) & i_2(A) \\ & \downarrow & \downarrow \\ A: & = \cdots = \leq = & \cdots = \leq = \cdots = \end{array}$$

Figure 3.7: Definition of the $i_1(A)$ and $i_2(A)$ indices.

Observe that, except for the case $k = 2$, we see that $\gamma(A', B)$ and $\gamma(A, B)$ are always of type $+$. Thus, using part (i) for $k \geq 3$ and $B \leq_{k-1} B'$ we obtain $\gamma(B', A) \leq_{k-1} \gamma(B, A)$, where $\gamma(B', A)$ and $\gamma(B, A)$ are both of type $+$. Therefore, using $(F^+_{k-1}(n), \lhd_{k-1}) = (F^+_{k-1}(n), >_{k-1})$, we get $\gamma(B', A) \rhd_{k-1} \gamma(B, A)$. From which we obtain $\gamma(A, B') \rhd_{k-1} \gamma(A, B)$ since $\gamma(A, B) \equiv_{k-1} \gamma(B, A)$ and $\gamma(A, B') \equiv_{k-1} \gamma(B', A)$. Now we can use $(F^-_{k-1}(n), \lhd_{k-1}) = (F^-_{k-1}(n), <_{k-1})$, and it follows that $\gamma(A, B') \geq_{k-1} \gamma(A, B)$.

For $k = 2$, we only have to check the case $\gamma(A, B) = +$, otherwise $\gamma(A, B) \leq_{k-1} \gamma(A, B')$ clearly holds. If $\gamma(A, B) = +$, we have $A <_2 B$, therefore $A <_2 B <_2 B'$, and since the $<_2$ order is transitive, we have $A <_2 B'$. Therefore, $\gamma(A, B') = -$ and then $\gamma(A', B) \leq_{k-1} \gamma(A, B)$ holds. □

Note that $c_k(S^{(1)}), \ldots, c_k(S^{(k+1)})$ is the sequence $\Gamma^{k-1}(S^{(k+1)}), \ldots, \Gamma^{k-1}(S^{(1)})$ by the definition of the coloring. Therefore, to prove the monotone property of the coloring, we have to determine the relation between each consecutive term in $\Gamma^{k-1}(S^{(k+1)}), \ldots, \Gamma^{k-1}(S^{(1)})$. It turns out that we can obtain those relations recursively from the relations of consecutive terms in the sequences $\Gamma^{k-2}(S^{(k,k+1)}), \ldots, \Gamma^{k-1}(S^{(1,k+1)})$ and $\Gamma^{k-2}(S^{(1,k+1)}), \ldots, \Gamma^{k-1}(S^{(1,2)})$. In the last lemma we show this recursion by induction.

But first, we introduce some definitions that we need for the proof. For two sequences S_1 and S_2 denote the concatenation of S_1 and S_2 by $S_1 \cdot S_2$. We call a sequence of $\leq, \geq$, and $=$ symbols a *profile*. Let $O_l := (\leq, =, \leq, =, \ldots)$ and $E_l := (=, \geq, =, \geq, \ldots)$ be two profiles of length l. We call a P profile of length l *odd* or *even* if it can be obtained from O_l or E_l by changing some symbols $\leq$ or $\geq$ to $=$, respectively. For two profiles P_1 and P_2, we say that P_1 and P_2 have the *same parity* if both profiles are even or odd, otherwise, we say P_1 and P_2 have *distinct parity*. The *opposite profile* $\overline{P}$ of a profile P is the profile that is obtained from P by changing each term $\leq$ to $\geq$ and each term $\geq$ to $\leq$. Let the *profile of the* $A = A_1, \ldots, A_k$ *sequence* from $F_k(n)$ be the $k - 1$ long $p(A)$ sequence, such that whenever $A_j <_k A_{j+1}$, or $A_j >_k A_{j+1}$, or $A_j = A_{j+1}$, then the j-th term of $p(A)$ is $\leq$, or $\geq$, or $=$, respectively. Let $i_1(A)$ be the smallest $i \in [k]$ such that the i-th term of $p(A)$ is not $=$. Similarly, let $i_2(A)$ be the smallest $i \in [k]$ such that for every $i \leq j \leq k - 1$ the j-th term of $p(A)$ is $=$ (see Fig. 3.7). Note that from the previous definition it follows that $i_1(A) + 1 \leq i_2(A)$. In the following definition, the lower right j index denotes the j-th term of the profiles. For two profiles of length $k - 1$, $p(R_1)$ and $p(R_2)$ for which there is no $j \in [k - 1]$ such that $p(R_1)_j$ is $\leq$ and $p(R_2)_j$ is $\geq$ (or vice versa), we define the j-th term

$$H = (\Gamma^2(A_1,A_2,A_3), \Gamma^2(A_1,A_2,A_4), \Gamma^2(A_1,A_3,A_4), \Gamma^2(A_2,A_3,A_4))$$

$$\begin{array}{cccc} \Gamma^2(A_1,A_2,A_3) & \Gamma^2(A_1,A_2,A_4) & \Gamma^2(A_1,A_3,A_4) & \Gamma^2(A_2,A_3,A_4) \\ \gamma(A_1,A_2)\ \underline{\gamma(A_2,A_3)} & \gamma(A_1,A_2)\ \underline{\gamma(A_2,A_4)} & \gamma(A_1,A_3)\ \underline{\gamma(A_3,A_4)} & \gamma(A_2,A_3)\ \underline{\gamma(A_3,A_4)} \end{array}$$

$$H_1 = (\gamma(A_1,A_2), \gamma(A_1,A_3), \gamma(A_2,A_3)) \qquad H_2 = (\gamma(A_2,A_3), \gamma(A_2,A_4), \gamma(A_3,A_4))$$

Figure 3.8: Recursion for H if $k = 4$ and $S = \{A_1, A_2, A_3, A_4\}$.

($j \in [k-1]$) of the profile $p(R_1) \circ p(R_2)$ as follows:

$$(p(R_1) \circ p(R_2))_j := \begin{cases} = & \text{if } p(R_1)_j \text{ and } p(R_1)_j \text{ is } =, \\ \leq & \text{if } p(R_1)_j \text{ is } = \text{ and } p(R_1)_j \text{ is } \leq \text{ or vice versa,} \\ \geq & \text{if } p(R_1)_j \text{ is } = \text{ and } p(R_1)_j \text{ is } \geq \text{ or vice versa.} \end{cases} \tag{3.5}$$

Lemma 3.35. *For positive integers n, k, and s with $k \geq 3$ and $3 \leq s \leq k+1$, let $S := (A_1, \ldots, A_s)$ be a sequence of s set from $F_k(n)$ with $A_1 <_k \cdots <_k A_s$. Then the sequence $H := (\Gamma^{s-2}(S^{(s)}), \ldots, \Gamma^{s-2}(S^{(1)}))$ has either odd or even profile.*

Proof. We will use induction on the size of S. For this, we need to obtain the sequence H from smaller cases, to which we can apply the induction hypothesis. We will define two sequences, H_1 and H_2, for which the induction hypothesis applies, and from which we can reconstruct H and determine the profile of H from their profiles.

Before we define H_1 and H_2, observe that for a sequence $(A_1, \ldots, A_r)$ from $F_k(n)$ we have $\Gamma^{r-1}(A_1, \ldots, A_r) = \gamma(\Gamma^{r-2}(A_1, \ldots, A_{r-1}), \Gamma^{r-2}(A_2, \ldots, A_r))$. In particular,

$$\begin{aligned} \Gamma^{s-2}(S^{(s)}) &= \gamma(\Gamma^{s-3}(S^{(s-1,s)}), \Gamma^{s-3}(S^{(1,s)})), \\ \Gamma^{s-2}(S^{(1)}) &= \gamma(\Gamma^{s-3}(S^{(1,s)}), \Gamma^{s-3}(S^{(1,2)})), \\ \Gamma^{s-2}(S^{(i)}) &= \gamma(\Gamma^{s-3}(S^{(i,s)}), \Gamma^{s-3}(S^{(1,i)})) \text{ for } 1 < i < s. \end{aligned}$$

Now, define H_1 as the sequence $(\Gamma^{s-3}(S^{(s-1,s)}), \ldots, \Gamma^{s-3}(S^{(1,s)}))$ and H_2 as the sequence $(\Gamma^{s-3}(S^{(1,s)}), \ldots, \Gamma^{s-3}(S^{(1,2)}))$. Denote by G_1 the sequence $(\Gamma^{s-3}(S^{(s-1,s)}) \cdot H_1$ and by G_2 the sequence $H_2 \cdot (\Gamma^{s-3}(S^{(1,2)}))$. In this way, we can obtain the i-th term of H as $\gamma(X_i, Y_i)$, where X_i and Y_i are the i-th terms of G_1 and G_2, respectively (see Fig. 3.8).

For the base case $s = 3$, we have $H = (\gamma(A_1,A_2), \gamma(A_1,A_3), \gamma(A_2,A_3)), H_1 = (A_1, A_2)$, $H_2 = (A_2, A_3), G_1 = (A_1, A_1, A_2), G_2 = (A_2, A_3, A_3)$. Since $A_1 <_k A_2 <_k A_3$, it follows that $\gamma(A_1,A_2) \in A_2$ and $\gamma(A_2,A_3) \notin A_2$ both have type + so $\gamma(A_1,A_2) \not\equiv_k \gamma(A_2,A_3)$. Therefore, by Lemma 3.33 and $(F^+_{k-1}(n), \lhd_{k-1}) = (F^+_{k-1}(n), >_{k-1})$, we have

$$\begin{gathered} \gamma(A_1,A_2) = \gamma(A_1,A_3) >_{k-1} \gamma(A_2,A_3) \text{ if } \gamma(A_1,A_2) \lhd_{k-1} \gamma(A_2,A_3) \quad \text{and} \\ \gamma(A_1,A_2) <_{k-1} \gamma(A_1,A_3) = \gamma(A_2,A_3) \text{ if } \gamma(A_2,A_3) \lhd_{k-1} \gamma(A_1,A_2). \end{gathered}$$

Thus, we choose $p(H)$ to be the even profile $(=, \geq)$ if $\Gamma(S^{(3)}) \lhd_{k-1} \Gamma(S^{(1)})$ or the odd profile

$(\leq, =)$ if $\Gamma(S^{(1)}) \lhd_{k-1} \Gamma(S^{(3)})$. We also let $p(H_1) := (\leq)$, $p(H_2) := (\leq)$, $p(G_1) := (=, \leq)$, and $p(G_2) := (\leq, =)$. Observe that, if $\Gamma(S^{(3)}) \lhd_{k-1} \Gamma(S^{(1)})$, then $p(H)$ is $\overline{p(G_1)}$ and if $\Gamma(S^{(1)}) \lhd_{k-1} \Gamma(S^{(3)})$, then $p(H)$ is $p(G_2)$. Also, notice that $p(H_1)$ and $p(H_2)$ have the same parity and $i_1(G_1) = i_2(G_2)$.

For the induction step, we assume that $s \geq 4$. Briefly, we will define the sequences $H_{1,1}, H_{1,2}, H_{2,1}$, and $H_{2,2}$ to which we can apply the induction hypothesis to construct H_1 and H_2 similarly, as we obtained H from H_1 and H_2. Furthermore, we will define the profile $p(H)$ based on $p(H_1)$ or $p(H_2)$ and, as part of the induction step, we will prove that this is indeed the profile of H.

We define

$$H_{1,1} := (\Gamma^{s-4}(S^{(s-2,s-1,s)}), \ldots, \Gamma^{s-4}(S^{(1,s-1,s)})) \quad \text{and}$$
$$H_{1,2} := (\Gamma^{s-4}(S^{(1,s-1,s)}), \ldots, \Gamma^{s-4}(S^{(1,2,s)})).$$

By setting $G_{1,1} := (\Gamma^{s-4}(S^{(s-2,s-1,s)})) \cdot H_{1,1}$ and $G_{1,2} := H_{1,2} \cdot (\Gamma^{s-4}(S^{(1,2,s)}))$, we can obtain the i-th term of H_1 as $\gamma(X_i, Y_i)$, where X_i and Y_i are the i-th terms from $G_{1,1}$ and $G_{1,2}$, respectively. Similarly, define

$$H_{2,1} := H_{1,2} \quad \text{and}$$
$$H_{2,2} := (\Gamma^{s-4}(S^{(1,2,s)}), \ldots, \Gamma^{s-4}(S^{(1,2,3)})),$$

and let $G_{2,1} := (\Gamma^{s-4}(S^{(1,s-1,s)})) \cdot H_{2,1}$ and $G_{2,2} := H_{2,2} \cdot (\Gamma^{s-4}(S^{(1,2,3)}))$. Then, the i-th term of H_2 is again of the form $\gamma(X_i, Y_i)$, where X_i and Y_i are the i-th terms of $G_{2,1}$ and $G_{2,2}$, respectively.

Now we continue with the definition of $p(H)$, and we will prove later that $p(H)$ is, in fact, the profile of H.

1. If $p(H_1)$ and $p(H_2)$ have the same parity let $p(H)$ be the profile

   $$\begin{aligned} &\overline{p(G_1)} \quad \text{if } \Gamma^{s-2}(S^{(s)}) \lhd_{k-s+2} \Gamma^{s-2}(S^{(1)}) \text{ or} \\ &p(G_2) \quad \text{if } \Gamma^{s-2}(S^{(1)}) \lhd_{k-s+2} \Gamma^{s-2}(S^{(s)}). \end{aligned}$$

   We will also prove that $i_1(G_1) \geq i_2(G_2)$.

2. If $p(H_1)$ and $p(H_2)$ have distinct parity, we define $p(H)$ as the profile $\overline{p(G_1)} \circ p(G_2)$ and will also show that $i_1(G_1) \geq i_1(G_2)$ and $i_2(G_1) \geq i_2(G_2)$.

   Remark that if $p(H_1)$ and $p(H_2)$ have distinct parity, then there is no $i \in [s-1]$ such that the i-th term of $p(G_1)$ is $\leq$ and the i-th term of $p(G_2)$ is $\geq$ (or vice versa), thus, $\overline{p(G_1)} \circ p(G_2)$ is correctly defined in this case (definition 3.5). Also note that $\overline{p(G_1)} \circ p(G_2)$ and $p(H_1)$ have distinct parity.

Observe that for $k = 3$ the definition above of $p(H)$ and the additional claims on the indicies i_1, i_2 hold. Therefore, we are done with the base case. By the induction hypotesis, for every $j \in \{1,2\}$ we get:

1. If $p(H_{j,1})$ and $p(H_{j,2})$ have the same parity, we have $i_1(G_{j,1}) \geq i_2(G_{j,2})$ and $p(H_j)$ is the profile

$$\begin{array}{lll} \overline{p(G_{j,1})} & \text{if} & (\text{the first term of } H_j) \lhd_{k-s+3} (\text{the last term of } H_j) \\ p(G_{j,2}) & \text{if} & (\text{the last term of } H_j) \lhd_{k-s+3} (\text{the first term of } H_j). \end{array}$$

2. If $p(H_{j,1})$ and $p(H_{j,2})$ have distinct parity:

$$p(H_j) = \overline{p(G_{j,1})} \circ p(G_{j,2}) \quad \text{and} \quad i_1(G_{j,1}) \geq i_1(G_{j,2}),\ i_2(G_{j,1}) \geq i_2(G_{j,2}).$$

Notice that if the definition of $p(H)$ holds, then it has the same parity as $\overline{p(G_1)}$ or $p(G_2)$, in particular, the profile $p(H)$ of H is odd or even, which gives the statement of the lemma.

For the proof that $p(H)$ is a profile of H and our additional claims hold for the indices i_1 and i_2, assume first that $p(H_1)$ and $p(H_2)$ have the same parity.

We start with showing that $i_1(G_1) \geq i_2(G_2)$. First, consider the case when $p(H_1)$ and $p(H_2)$ are odd. We will distinguish some cases. Note that for $j \in \{1,2\}$ we have *(i)*: $i_j(G_{1,2}) = i_j(G_{2,1}) - 1$ since $H_{2,1} = H_{1,2}$ (i.e. $p(H_{2,1}) = p(H_{1,2})$) and in $p(G_{1,2})$ the profile $p(H_{1,2})$ is shifted one to the right; furthermore, by our induction hypothesis $p(H_{1,2})$ contains at least one term which is not $=$.

1. Assume that $p(H_1) \notin \{\overline{p(G_{1,1})}, p(G_{1,2}\}$ or $p(H_2) \notin \{\overline{p(G_{2,1})}, p(G_{2,2})\}$. Note that, by the definition of $p(H_j)$ for $j \in \{1,2\}$, if $p(H_j) \notin \{\overline{p(G_{j,1})}, p(G_{j,2}\}$, then $p(H_j) = \overline{p(G_{j,1})} \circ p(G_{j,2})$ and $p(H_{j,1})$ and $p(H_{j,2})$ have distinct parity. Also observe that the parity of $\overline{p(G_{j,1})} \circ p(G_{j,2})$ is the opposite of the parity of $p(H_{j,1})$ as we discussed after the definition of $p(H)$.

   (a) if $p(H_1) \notin \{\overline{p(G_{1,1})}, p(G_{1,2})\}$ and $p(H_2) \in \{\overline{p(G_{2,1})}, p(G_{2,2})\}$, then $p(H_1) = \overline{p(G_{1,1})} \circ p(G_{1,2})$ so $p(H_{1,1})$ is even. Therefore, $p(H_{1,2}) = p(H_{2,1})$ is odd. This follows that $\overline{p(G_{2,1})}$ is even, so it must be the case $p(H_2) = p(G_{2,2})$. Since $p(G_{2,2})$ and $p(H_{2,2})$ have the same parity by the definition of $G_{2,2}$, we get that $p(H_{2,2})$ is odd.

   Since $p(H_{1,1})$ and $p(H_{1,2})$ have distinct parity, by the induction hypotesis, we get $i_1(G_{1,1}) \geq i_1(G_{1,2})$ and $i_2(G_{1,1}) \geq i_2(G_{1,2})$.

   Since $p(H_{2,1})$ and $p(H_{2,2})$ have the same parity, we get *(ii)*: $i_1(G_{2,1}) \geq i_2(G_{2,2})$. From $p(H_1) = \overline{p(G_{1,1})} \circ p(G_{1,2})$ and $i_1(G_{1,1}) \geq i_1(G_{1,2})$, we get *(iii)*: $i_1(G_1) \geq i_1(G_{1,2}) + 1$, since in G_1 we have H_1 shifted one to the right (def of G_1) and i_1 is a minimum.

Recall that $p(H_2) = p(G_{2,2})$. It follows that (*iv*): $i_2(G_2) = i_2(G_{2,2})$, since in G_2 we have H_2 shifted to the right. Thus we obtain:

$$i_1(G_1) \overset{(iii)}{\geq} i_1(G_{1,2}) + 1 \overset{(i)}{=} i_1(G_{2,1}) \overset{(ii)}{\geq} i_2(G_{2,2}) \overset{(iv)}{=} i_2(G_2).$$

(b) if $p(H_1) \in \{\overline{p(G_{1,1})}, p(G_{1,2})\}$ and $p(H_2) \notin \{\overline{p(G_{2,1})}, p(G_{2,2})\}$, then $p(H_2) = \overline{p(G_{2,1})} \circ p(G_{2,2})$ so $p(H_{2,1})$ is even. Therefore, $p(H_{1,2}) = p(H_{2,1})$ is even and $p(H_{2,2})$ is odd. Since $p(H_{1,2})$ is even, thus $p(G_{1,2})$ is even, so it must be the case $p(H_1) = \overline{p(G_{1,1})}$. This follows, $p(H_{1,1})$ is even.

Since $p(H_{1,2})$ and $p(H_{2,2})$ have distinct parity, by the induction hypotesis, we get $i_1(G_{2,1}) \geq i_1(G_{2,2})$ and $i_2(G_{2,1}) \geq i_2(G_{2,2})$.

Since $p(H_{1,1})$ and $p(H_{1,2})$ have the same parity, we get (*ii*): $i_1(G_{1,1}) \geq i_2(G_{1,2})$. From $p(H_2) = \overline{p(G_{2,1})} \circ p(G_{2,2})$ and $i_2(G_{2,1}) \geq i_2(G_{2,2})$, we get (*iii*): $i_2(G_{2,1}) \geq i_2(G_2)$, since in G_2 we have H_2 shifted one to the left (def of G_2).

Recall that $p(H_1) = \overline{p(G_{1,1})}$. It follows that that (*iv*): $i_1(G_1) = i_1(G_{1,1}) + 1$, since in G_1 we have H_1 shifted to the right. Thus we obtain:

$$i_1(G_1) \overset{(iv)}{=} i_1(G_{1,1}) + 1 \overset{(ii)}{\geq} i_1(G_{1,2}) + 1 \overset{(i)}{=} i_2(G_{2,1}) \overset{(iii)}{\geq} i_2(G_2).$$

(c) if $p(H_1) \notin \{\overline{p(G_{1,1})}, p(G_{1,2})\}$ and $p(H_2) \notin \{\overline{p(G_{2,1})}, p(G_{2,2})\}$, then $p(H_{1,1})$ and $p(H_{2,1})$ is even. However, this is a contradiction because we also know that $p(H_{1,1})$ and $p(H_{1,2})$ have distinct parity, so $p(H_{1,2}) = p(H_{2,1})$ is odd.

2. Assume that $p(H_1) \in \{\overline{p(G_{1,1})}, p(G_{1,2})\}$ and $p(H_2) \in \{\overline{p(G_{2,1})}, p(G_{2,2})\}$.

(a) if $p(H_1) = \overline{p(G_{1,1})}$ and $p(H_2) = p(G_{2,2})$, then $p(H_{1,1})$ is even and $p(H_{2,2})$ is odd. We also have by definition of G_1 and G_2 that (*v*): $i_1(G_1) = i_1(G_{1,1}) + 1$ and (*vi*): $i_2(G_2) = i_2(G_{2,2})$. We have to deal with two cases, whether $p(H_{1,2}) = p(H_{2,1})$ is odd or even.

If $p(H_{1,2}) = p(H_{2,1})$ is odd, then $p(H_{1,1})$ and $p(H_{1,2})$ have distinct parity, and $p(H_{2,1})$ and $p(H_{2,2})$ have the same parity. It follows from the induction hypotesis, that (*vii*): $i_1(G_{1,1}) \geq i_1(G_{1,2}),\ i_2(G_{1,1}) \geq i_2(G_{1,2})$ and (*viii*): $i_1(G_{2,1}) \geq i_2(G_{2,2})$. Thus we obtain:

$$i_1(G_1) \overset{(v)}{=} i_1(G_{1,1}) + 1 \overset{(vii)}{\geq} i_1(G_{1,2}) + 1 \overset{(i)}{=} i_1(G_{2,1}) \overset{(viii)}{\geq} i_2(G_{2,2}) \overset{(vi)}{=} i_2(G_2).$$

If $p(H_{1,2}) = p(H_{2,1})$ is even, then $p(H_{1,1})$ and $p(H_{1,2})$ have the same parity, and $p(H_{2,1})$ and $p(H_{2,2})$ have distinct parity. It follows from the induction hypotesis, that (*viii*): $i_1(G_{1,1}) \geq i_2(G_{1,2})$ and (*vii*): $i_1(G_{2,1}) \geq i_1(G_{2,2}),\ i_2(G_{2,1}) \geq i_2(G_{2,2})$.

Thus we obtain:

$$i_1(G_1) \overset{(v)}{=} i_1(G_{1,1}) + 1 \overset{(vii)}{\geq} i_1(G_{1,2}) + 1 \overset{(i)}{=} i_1(G_{2,1}) \overset{(viii)}{\geq} i_2(G_{2,2}) \overset{(vi)}{=} i_2(G_2).$$

(b) if $p(H_1) = \overline{p(G_{1,1})}$ and $p(H_2) = \overline{p(G_{2,1})}$, then $p(H_{1,1})$ and $p(H_{2,1}) = p(H_{1,2})$ are even. We also have by definition of G_1 and G_2 that (v): $i_1(G_1) = i_1(G_{1,1}) + 1$ and (vi): $i_2(G_2) = i_2(G_{2,1})$. Since $p(H_{1,1})$ and $p(H_{1,2})$ have the same parity, by the induction hypotesis we get (vii): $i_1(G_{1,1}) \geq i_2(G_{1,2})$. Thus we obtain:

$$i_1(G_1) \overset{(v)}{=} i_1(G_{1,1}) + 1 \overset{(vii)}{\geq} i_2(G_{1,2}) + 1 \overset{(i)}{=} i_2(G_{2,1}) \overset{(vi)}{=} i_2(G_2).$$

(c) if $p(H_1) = p(G_{1,2})$ and $p(H_2) = p(G_{2,2})$, then $p(H_{1,2}) = p(H_{2,1})$ and $p(H_{2,2})$ are odd. We also have by definition of G_1 and G_2 that (v): $i_1(G_1) = i_1(G_{1,2}) + 1$ and (vi): $i_2(G_2) = i_2(G_{2,2})$. Since $p(H_{2,1})$ and $p(H_{2,2})$ have the same parity, by the induction hypotesis we get (vii): $i_1(G_{2,1}) \geq i_2(G_{2,2})$. Thus we obtain:

$$i_1(G_1) \overset{(v)}{=} i_1(G_{1,2}) + 1 \overset{(i)}{=} i_1(G_{2,1}) \overset{(vii)}{\geq} i_2(G_{2,2}) \overset{(vi)}{=} i_2(G_2).$$

(d) if $p(H_1) = p(G_{1,2})$ and $p(H_2) = \overline{p(G_{2,1})}$, then $p(H_{1,2})$ is odd and $p(H_{2,1})$ is even, which is a contradiction, since $p(H_{1,2}) = p(H_{2,1})$.

Note that in the cases above, we only rely on the facts that the parity of the profiles $p(H_1)$ and $p(H_2)$ is the same or different, we do not use the actual parity. Therefore, by symmetry, the inequality $i_1(G_1) \geq i_2(G_2)$ holds if $p(H_1)$ and $p(H_2)$ are even.

Now using $i_1(G_1) \geq i_2(G_2)$, we show that $p(H)$ is in fact the profile of H, and

$$\begin{aligned} p(H) &= \overline{p(G_1)} \quad \text{if } \Gamma^{s-2}(S^{(s)}) \lhd_{k-s+2} \Gamma^{s-2}(S^{(1)}) \text{ or} \\ p(H) &= p(G_2) \quad \text{if } \Gamma^{s-2}(S^{(1)}) \lhd_{k-s+2} \Gamma^{s-2}(S^{(s)}). \end{aligned}$$

Since $p(H_1)$ and $p(H_2)$ have the same parity we have, by the definition of H_1, H_2:

$$\begin{aligned} &\Gamma^{s-3}(S^{(s-1,s)}) <_{k-s+3} \Gamma^{s-3}(S^{(1,s)}) <_{k-s+3} \Gamma^{s-3}(S^{(1,2)}) \quad \text{or} \\ &\Gamma^{s-3}(S^{(s-1,s)}) >_{k-s+3} \Gamma^{s-3}(S^{(1,s)}) >_{k-s+3} \Gamma^{s-3}(S^{(1,2)}). \end{aligned}$$

This implies that the first term $\Gamma^{s-2}(S^{(s)}) = \gamma(\Gamma^{s-3}(S^{(s-1,s)}), \Gamma^{s-3}(S^{(1,s)}))$ of H and the last term $\Gamma^{s-2}(S^{(1)}) = \gamma(\Gamma^{s-3}(S^{(1,s)}), \Gamma^{s-3}(S^{(1,2)}))$ of H have the same type. Note that if $s \leq k$, then the previous two terms are not equvivalent, since $\Gamma^{s-2}(S^{(s)}) \in \Gamma^{s-3}(S^{(1,s)})$ while $\Gamma^{s-2}(S^{(1)}) \notin \Gamma^{s-3}(S^{(1,s)})$ (i.e. cannot be equal). Thus, we either have

$$\Gamma^{s-2}(S^{(s)}) \lhd_{k-s+2} \Gamma^{s-2}(S^{(1)}) \quad \text{or} \quad \Gamma^{s-2}(S^{(1)}) \lhd_{k-s+2} \Gamma^{s-2}(S^{(s)}).$$

1. If $\Gamma^{s-2}(S^{(s)}) \lhd_{k-s+2} \Gamma^{s-2}(S^{(1)})$, notice that by Lemma 3.33 for $A := \Gamma^{s-3}(S^{(s-1,s)})$, $B :=$

$$
\begin{array}{lcccc}
 & & & i_1(G_1) & i_2(G_1) \\
 & & & \downarrow & \downarrow \\
G_1: & = \cdots = = = & \cdots = = & \leq = \cdots = & \leq = \cdots = \\
G_2: & = \cdots = \leq = & \cdots = \leq = = & \cdots = = = & \cdots = \\
 & \uparrow & \uparrow & & \\
 & i_1(G_2) & i_2(G_2) & &
\end{array}
$$

Figure 3.9: $i_1(G_1) \geq i_2(G_2)$

$\Gamma^{s-3}(S^{(1,s)})$, and $C := \Gamma^{s-3}(S^{(1,2)})$ we obtain that

$$\gamma(\Gamma^{s-3}(S^{(s-1,s)}), \Gamma^{s-3}(S^{(1,2)})) = \Gamma^{s-2}(S^{(s)}) \quad \text{(i.e. 1-st term of } H\text{)}.$$

2. If $\Gamma^{s-2}(S^{(1)}) \triangleleft_{k-s+2} \Gamma^{s-2}(S^{(s)})$, using Lemma 3.33 on the same terms follows that

$$\gamma(\Gamma^{s-3}(S^{(s-1,s)}), \Gamma^{s-3}(S^{(1,2)})) = \Gamma^{s-2}(S^{(1)}) \quad \text{(i.e. } s\text{-th term of } H\text{)}.$$

Observe that if $s = k+1$, then the terms of H are from $\{+, -\} = F_1(n)$; therefore, applying Lemma 3.33 if $\Gamma^{s-2}(S^{(s)}) = \Gamma^{s-2}(S^{(1)})$, we get:

$$\Gamma^{s-2}(S^{(1)}) = \gamma(\Gamma^{s-3}(S^{(s-1,s)}), \Gamma^{s-3}(S^{(1,2)})) = \Gamma^{s-2}(S^{(s)}).$$

Assume that $\Gamma^{s-2}(S^{(s)}) \triangleleft_{k-s+2} \Gamma^{s-2}(S^{(1)})$. By the $i_1(G_1) \geq i_2(G_2)$ inequality (Fig. 3.9), we get that the $i_1(G_1)$-th term of H is $\gamma(\Gamma^{s-3}(S^{(s-1,s)}), \Gamma^{s-3}(S^{(1,2)})) = \Gamma^{s-2}(S^{(s)})$ (i.e. equal to the 1-st term of H). For every $i \leq i_1(G_1)$, the i-th term of H is $\gamma(\Gamma^{s-3}(S^{(s-1,s)}), Y_i)$, where Y_i is the i-th term of G_2. Since G_2 has an odd or even profile (i.e., it is monotone in $\leq_{k-s+3}$) we can apply Lemma 3.34 (*ii*) part. It follows that the first $i_1(G_1)$ terms of H are monotone in $\leq_{k-s+2}$. Notice that if we know the previous monotonicity and also that the 1-st and $i_1(G_1)$-th term are the same, it follows that the first $i_1(G_1)$ terms of H are equal to $\Gamma^{s-2}(S^{(s)})$. Using the $i_1(G_1) \geq i_2(G_2)$ inequality again, we get that for every $i > i_1(G_1)$, the i-th term of H is $\gamma(X_i, \Gamma^{s-3}(S^{(1,2)}))$, where X_i is the i-th term of G_1. Since G_1 also has an odd or even profile (i.e., it is monotone in $\leq_{k-s+3}$), we can apply Lemma 3.34 (*i*) part to the $\gamma(X_i, \Gamma^{s-3}(S^{(1,2)}))$ sequence. Consequently, we coclude that $p(H) = \overline{p(G_1)}$, and it is indeed a valid profile of H.

Finishing the proof, assume that $p(H_1)$ and $p(H_2)$ have distinct parity. First, consider the case when $p(H_1)$ is odd and $p(H_2)$ is even. Then, by the definition of G_1 and G_2, the pair consisting the i-th term of $p(G_1)$ and the i-th term of $p(G_2)$ can be $\{=, =\}, \{\leq, =\}, \{=, \geq\}$, or $\{\leq, \geq\}$. In each case, we apply Lemma 3.34 for X_i, X_{i+1} from G_1 and Y_i, Y_{i+1} from $G - 2$:

1. In the case $\{=, =\}$, we have $X_i = X_{i+1}$ and $Y_i = Y_{i+1}$. Thus, for the i-th term $\gamma(X_i, Y_i)$

and $(i+1)$-th term $\gamma(X_{i+1}, Y_{i+1})$ of H, we get:

$$\gamma(X_i, Y_i) = \gamma(X_{i+1}, Y_{i+1}).$$

2. In the case $\{\leq, =\}$, we have $X_i \leq_{k-s+3} X_{i+1}$ and $Y_i = Y_{i+1}$. Thus, for the i-th term $\gamma(X_i, Y_i)$ and $(i+1)$-th term $\gamma(X_{i+1}, Y_{i+1})$ of H, we get:

$$\gamma(X_i, Y_i) \geq_{k-s+2} \gamma(X_{i+1}, Y_{i+1}).$$

3. In the case $\{=, \geq\}$, we have $X_i = X_{i+1}$ and $Y_i \geq_{k-s+3} Y_{i+1}$. Thus, for the i-th term $\gamma(X_i, Y_i)$ and $(i+1)$-th term $\gamma(X_{i+1}, Y_{i+1})$ of H, we get:

$$\gamma(X_i, Y_i) \geq_{k-s+2} \gamma(X_{i+1}, Y_{i+1}).$$

4. In the case $\{\leq, \geq\}$, we have $X_i \leq_{k-s+3} X_{i+1}$ and $Y_i \geq_{k-s+3} Y_{i+1}$. Thus, first, for the i-th term $\gamma(X_i, Y_i)$ of H and $\gamma(X_{i+1}, Y_1)$, we get:

$$\gamma(X_i, Y_i) \geq_{k-s+2} \gamma(X_{i+1}, Y_1).$$

   Then, for $\gamma(X_{i+1}, Y_1)$ and the $(i+1)$-th term of H, we get:

$$\gamma(X_i, Y_i) \geq_{k-s+2} \gamma(X_{i+1}, Y_{i+1}).$$

   Therefore, we obtain $\gamma(X_i, Y_i) \geq_{k-s+2} \gamma(X_{i+1}, Y_{i+1})$, since $\geq_{k-s+2}$ is transitive.

These imply that $p(H) = \overline{p(G_1)} \circ p(G_2)$, and it is indeed a valid profile of H. Furthermore, the resulting profile $p(H)$ is even. In the other case, when $p(H_1)$ is even and $p(H_2)$ is odd, by simmetry, we again have $p(H) = \overline{p(G_1)} \circ p(G_2)$, and in this case, $p(H)$ is odd.

It remains to show that $i_1(G_1) \geq i_1(G_2)$ and $i_2(G_1) \geq i_2(G_2)$. We know by induction that $p(H_1)$ is in $\{\overline{p(G_{1,1})}, p(G_{1,2}), \overline{p(G_{1,1})} \circ p(G_{1,2})\}$ and $p(H_2)$ is in $\{\overline{p(G_{2,1})}, p(G_{2,2}), \overline{p(G_{2,1})} \circ p(G_{2,2})\}$. It follows that $i_j(H_1)$ is in $\{i_j(G_{1,1}), i_j(G_{1,2})\}$ and $i_j(H_2)$ is in $\{i_j(G_{2,1}), i_j(G_{2,2})\}$ for $j \in \{1, 2\}$. Now fix $i \in \{1, 2\}$.

If $p(H_{i,1})$ and $p(H_{i,2})$ have the same parity we have $i_1(G_{i,1}) \geq i_2(G_{i,2})$ by the induction hypotesis. Recall that, by the definition of i_1, i_2 indices we have $i_2(G_{j,k}) \geq i_1(G_{j,k})$ for $j, k \in \{1, 2\}$. Thus, we obtain: $i_2(G_{i,1}) \geq i_1(G_{i,1}) \geq i_2(G_{i,2}) \geq i_1(G_{i,2})$. Which follows that $i_2(G_{i,1}) \geq i_2(G_{i,2})$ and $i_2(G_{i,1}) \geq i_2(G_{i,2})$.

If $p(H_{i,1})$ and $p(H_{i,2})$ have distinct parity we have $i_1(G_{i,1}) \geq i_1(G_{i,2})$ and $i_2(G_{i,1}) \geq i_2(G_{i,2})$ by the induction hypotesis.

That is, we always get $i_1(G_{i,1}) \geq i_1(G_{i,2})$ and $i_2(G_{i,1}) \geq i_2(G_{i,2})$. Therefore, in all four cases, whether $p(H_{j,1})$ and $p(H_{j,2})$ have distinct or the same parity for $j \in \{1, 2\}$, we have $i_k(G_{1,1}) \geq i_k(G_{1,2})$ and $i_k(G_{1,1}) \geq i_k(G_{1,2})$ for $k \in \{1, 2\}$.

Alltogeather, we obtain that $i_k(H_1) \geq i_k(G_{1,2})$ and $i_k(H_2) \leq i_k(G_{2,1})$ for $k \in \{1, 2\}$. Note that, we also have for $j \in \{1, 2\}$ that $i_j(G_{1,2}) = i_j(G_{2,1}) - 1$ (as we already discussed it, see (*i*)). Thus, $i_k(H_2) \leq i_k(G_{2,1}) \leq i_k(H_1) + 1$. Since $i_j(G_1) = i_j(H_1) + 1$ and $i_j(G_2) = i_j(H_2)$ by the defintion of G_1 and G_2, $i_1(G_1) \geq i_1(G_2)$ and $i_2(G_1) \geq i_2(G_2)$ follows. □

Now, we can completet the proof of Theorem 3.31.

Proof of Theorem 3.31. We have to show that for a vertex set $S = \{A_1, \ldots, A_{k+1}\}$ in $F_k(n)$ that satisfies $A_1 <_k \cdots <_k A_{k+1}$ we have $\mathfrak{C}(S^{(1)}) \leq \cdots \leq \mathfrak{C}(S^{(k+1)})$ or $\mathfrak{C}(S^{(1)}) \geq \cdots \geq \mathfrak{C}(S^{(k+1)})$. Notice that the sequence $\mathfrak{C}(S^{(1)}), \ldots, \mathfrak{C}(S^{(k+1)})$ is $\Gamma^{k-1}(S^{(k+1)}), \ldots, \Gamma^{k-1}(S^{(1)})$ by the definition of the coloring. From Lemma 3.35 it follows that the previous sequence has an odd or even profile, in particular, it is monotone in $\leq_1$. □

Comparison with the construction by Moshkovitz and Shapira. The construction of Balko builds on similar approach and ideas as the one by Moshkovitz and Shapira. The key difference is that Moshkovitz and Shapira's coloring is defined over a larger vertex set, and it does not satisfy Lemma 3.35 since their construction is not transitive, as we showed for $k = 4$.

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Alulírott Vizsy Domonkos nyilatkozom, hogy szakdolgozatom elkészítése során az alább felsorolt feladatok elvégzésére a megadott MI alapú eszközöket alkalmaztam:

Feladat	**Felhasznált eszköz**	**Felhasználás helye**	**Megjegyzés**
Nyelvhelyesség ellenőrzése	Writefull GPT-4o	Teljes dolgozat	
LaTeX kód írás	GPT-4o	Teljes dolgozat 18., 36. Oldalak Figure 3.3, 3.4, 3.7 MI nyilatkozat	Fejezetszámozás \customproof parancs Táblázat

A felsoroltakon túl más MI alapú eszközt nem használtam.