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# Line transversals of families of convex sets

Bachelor thesis

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# Chapter 1

# Introduction

Combinatorial geometry is a branch of mathematics that studies discrete geometric structures through the lens of combinatorics. This relatively young mathematical discipline ties together geometry and combinatorics in elegant ways. Some of the results in this field are surprisingly useful not only in mathematics but also in computer science, robotics, scene analysis, and computer-aided design.

One of the fundamental results in combinatorial geometry came from Eduard Helly. In 1913, Helly famously proved that if $F_1, F_2, \ldots, F_n$ are convex sets in $\mathbb{R}^d$ such that every $d+1$ sets chosen from this collection have a common point, then every set in the collection has a common point. This is known as Helly's theorem, and this result opened up a whole new area of research known as "Helly-type theorems". These theorems explore the conditions under which local intersection properties of families of sets imply a global intersection property. Over time, there have been a large number of variations and generalizations of Helly's theorem. This thesis explores some of these generalizations, and moves in the direction where instead of intersecting all sets with a single point, we intersect the sets with a $k$-dimensional affine subspace. This is called a *geometric transversal* with a $k$-flat. We mostly look at cases of $k = 1$, in which the intersecting subspace is a line.

In the first part of the thesis, I prove Helly's original theorem, and discuss slight variations such as the colorful Helly theorem or the fractional Helly theorem. The next chapter examines some of the results concerning line transversals in different spaces. Finally, in the last chapter, I present my results, where I improve McGinnis and Zerbib's results regarding families of compact connected sets where every three sets in the family admit a line transversal.

Most of the notation and terminology in this paper is quite standard. If $\mathscr{F}$ is a family of sets (i.e., a set of sets), then $\bigcap \mathscr{F}$ means the intersection of the sets in $\mathscr{F}$. Similarly, $\bigcup \mathscr{F}$ means the union of the sets in $\mathscr{F}$. Additionally, let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

# Chapter 2

# Basic results

## 2.1 Original Helly's theorem

One of the most fundamental theorems in discrete geometry was discovered by Eduard Helly in 1913. However, he did not publish his results until 1923, by which time other proofs had surfaced. The theorem is the following:

**Theorem 2.1** (Helly's theorem). *Let $\mathscr{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If every $d+1$ sets have a nonempty intersection, then the whole family has a nonempty intersection.*

The simplest proof of Helly's theorem uses a lemma by J. Radon regarding a partitioning of points in $\mathbb{R}^d$:

**Lemma 2.2** (Radon's theorem). *Any set of $d+2$ points in $\mathbb{R}^d$ can be partitioned into two nonempty sets $A_1, A_2$ such that $conv(A_1) \cap conv(A_2) \neq \emptyset$.*

*Proof of Radon's theorem:* Consider any set $X = \{x_1, x_2, \ldots, x_{d+2}\} \subset \mathbb{R}^d$. With $d+2$ points, there exists $d+2$ coefficients $\{a_1, \ldots, a_{d+2}\}$ which fulfill the following equations:

$$\sum_{i=1}^{d+2} a_i x_i = \underline{0}\ , \ \ \sum_{i=1}^{d+2} a_i = 0 \tag{2.1}$$

The first equation is a linear equation on each coordinate of our points, therefore this is a system of linear equations where we have $d+2$ unknowns, and $d+1$ equations. Thus, there must be a solution $\{a_1, \ldots, a_{d+2}\}$ where not all of the $a_i$ are zero.
Now let us take the points of the nonnegative coefficients $A_1 := \{x_i \mid a_i \geq 0\}$ and negative coefficients $A_2 := \{x_i \mid a_i < 0\}$. Here $A_1$ and $A_2$ are non-empty, and they form a correct partition, because we can find a common point in their convex hulls:
We know that $\sum\limits_{x_i \in A_1} a_i x_i = \sum\limits_{x_i \in A_2} -a_i x_i$ due to the first equation in (2.1). Let $C := \sum\limits_{x_i \in A_1} a_i = \sum\limits_{x_i \in A_2} -a_i$. With this, we can now find the common point $p$:

$$p = \sum_{x_i \in A_1} \frac{a_i}{C} x_i = \sum_{x_i \in A_2} \frac{-a_i}{C} x_i \tag{2.2}$$

Here, the left side is a convex combination of the points in $A_1$, and the right side is a convex combination of the points in $A_2$, therefore their convex hulls intersect. $\square$

*Proof of Helly's theorem:* The proof will be by induction on the number of sets.
**Base case:** $d+2$ sets.
We have d+2 convex sets: $F_1, \ldots, F_{d+2}$. By our assumption, every $d+1$ sets have a nonempty intersection, therefore we can choose points $p_1, \ldots, p_{d+2} \in \mathbb{R}^2$ in the intersections: $\forall i \in [d+2] \quad p_i \in \bigcap_{j \neq i} F_j$.
Now, we can use Radon's theorem on these points. Using the theorem, we can partition $p_1, \ldots, p_{d+2}$ into two nonempty sets $A_1$ and $A_2$, such that $conv(A_1) \cap conv(A_2)$ is nonempty. Hence, we can choose a point $x \in conv(A_1) \cap conv(A_2)$. We claim that $x \in \bigcap F_i$.
Consider any $i \in [d+2]$. Here, $p_i$ is in either $A_1$ or $A_2$. If $p_i \in A_1$, then we know that for every $p_j \in A_2$, $p_j \in F_i$. Knowing that $F_i$ is convex, we can conclude that $conv(A_2) \subseteq F_i$, therefore $x \in F_i$. Similarly, if $p_i \in A_2$ then $conv(A_1) \subset F_i$, therefore $x \in F_i$. With this, the base case is proven.
**Inductive step:** we have $n > d+2$ sets.
Here, we can assume that the statement holds for $n-1$ sets. Using the base case, we also know that every $d+2$ sets have a nonempty intersection. Knowing these, we can replace $F_1, F_2$ with $F_1 \cap F_2$. In this new family of sets, we can see that every d+1 have a nonempty intersection. We now have $n-1$ sets, and we can use the inductive hypothesis to find a common point $x$. This $x$ will then be in every original set. This completes the proof. $\square$

## 2.2 Generalizations and variations

Having established the original Helly's Theorem, it is natural to explore its broader implications and extensions. Over time, mathematicians have discovered a variety of powerful generalizations that retain the core spirit of Helly's result while relaxing certain assumptions or introducing new combinatorial structures. One of the notable generalizations is the colorful Helly theorem by Lovász in [10]:

**Theorem 2.3** (colorful Helly theorem [10])**.** *Let $\mathscr{F}_1, \ldots, \mathscr{F}_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$. If every d+1 sets, chosen from different families, have a nonempty intersection, then there exists $1 \leq i \leq d+1$ such that the sets in $\mathscr{F}_i$ have a nonempty intersection.*

This theorem generalizes Helly's theorem, because if we choose all $d+1$ sets to be the same $\mathscr{F}$ family, then it yields Helly's theorem.
Next, research started to gravitate towards the question of whether similar to Helly's or stronger statements can be said about specific types of objects instead of the wide range of convex sets. This led to the introduction of the term *Helly number*.

**Definition 2.4.** *For a space $X$, and a collection of allowed sets $\mathscr{A}$ in $X$, the Helly number $h(X, \mathscr{A}) \in \mathbb{N}$ of $\mathscr{A}$ is defined as the smallest integer $h$ such that for any finite subfamily $\mathscr{F} \subset \mathscr{A}$, if every $h$ sets in $\mathscr{F}$ have a nonempty intersection, then all sets in $\mathscr{F}$ have a nonempty intersection. If no such $h$ exists, then $h(X, \mathscr{A}) := \infty$.*

Due to Theorem 2.1, and with the additional lemma that $d+1$ cannot be lowered, we know that the Helly number of convex sets in $\mathbb{R}^d$ is $d+1$. With $d = 1$, we get that the Helly number of intervals on a line is 2. Using this, we can also calculate the Helly number of axis-parallel boxes is $\mathbb{R}^d$: If two boxes intersect, clearly their projections to each axis also intersect. Therefore, if every two boxes intersect in $\mathscr{F}$, then we can find a point on each axis which is inside all of their projections. These points on the axes then define a point in $\mathbb{R}^d$ which must be inside all boxes. Hence, the Helly number of axis-parallel boxes in $\mathbb{R}^d$ is 2. Another example of a known Helly number is the Helly number of $d$-dimensional (hollow) spheres in $\mathbb{R}^d$. In [8] Maehera showed that this is equal to $d+2$. An additional well-known Helly number is for convex lattice sets. Convex lattice sets are sets in the form of $C \cap \mathbb{Z}^d$ where $C$ is a convex set in $\mathbb{R}^d$. Doignon in [9] proved that the Helly number of these sets in $\mathbb{R}^d$ is $2^d$.
Lovász's generalization of Helly's theorem, Theorem 2.3, inspired researchers to ask the question of whether similar generalizations can be asserted in the cases of the above specific sets.

**Definition 2.5.** *For a space $X$, and a collection of allowed sets $\mathscr{A}$ in $X$, the colorful Helly number $\eta(X, \mathscr{A}) \in \mathbb{N}$ of $\mathscr{A}$ is defined as the smallest integer $\eta$ for which the following statement holds: Let $\mathscr{F}_1, \mathscr{F}_2, \ldots, F_\eta \subset \mathscr{A}$ be finite subfamilies. If for every choice $F_1 \in \mathscr{F}_1, \ldots, F_\eta \in \mathscr{F}_\eta$, the intersection is nonempty: $\bigcap\limits_{i=1}^{\eta} F_i \neq \emptyset$, then there exists $i \in [\eta]$ such that $\bigcap \mathscr{F}_i \neq \emptyset$. If no such $\eta$ exists, then $\eta(X, \mathscr{A}) := \infty$.*

It is easy to see that $h(X, \mathscr{A}) \leq \eta(X, \mathscr{A})$, since the colorful theorem implies the non-colorful one, by choosing the families to be the same sets. In [10] Pohoata proved that for $d$-dimensional spheres, $h = \eta$. He also showed that the same is true for Hamming balls of fixed radius, and hypersurfaces of bounded degree. However, it is not always true that $h(X, \mathscr{A}) = \eta(X, \mathscr{A})$. In the case of axis-parallel boxes, we saw above that $h(X, \mathscr{A}) = 2$. On the other hand, it is easy to prove that $\eta(X, \mathscr{A}) = d+1$: From Lovász's theorem 2.3, we know that $\eta(X, \mathscr{A}) \leq d+1$. $\eta(X, \mathscr{A}) \geq d+1$ follows from the following construction: Consider a hypercube in $\mathbb{R}^d$, and let the $d$ families ("colors") of boxes be the $d$ pairs of opposite faces of the hypercube. Clearly, choosing one side from each pair of opposite sides will have a nonempty intersection (one of the corners), however none of the families have a common point. Hence, the colorful Helly number cannot be $d$ or lower: $\eta(X, \mathscr{A}) \geq d+1$.

In all the above cases, we looked at examples where we specified the examined sets and thus arrived at either a stronger or a weaker statements. However, what would happen, if we weakened the starting conditions? This is a direction that also received a great deal of attention over the years. The basic principle is the following: What can we say about $\mathscr{F}$, if we only know that out of all choices of $k$ sets, only a fraction of them have a common point? This question was first studied in 1979 by Katchalski and Liu, and they were able to prove the following for convex sets:

**Theorem 2.6** (Katchalski and Liu [12])**.** *Let $\mathscr{F}$ be a family of $n$ convex sets in $\mathbb{R}^d$. For every $\alpha > 0$, there exists $\beta = \beta(\alpha, d) > 0$, such that if out of every choice of $d+1$ sets in $\mathscr{F}$, at least $\alpha\binom{n}{d+1}$ of them have a nonempty intersection, then there exists a point which is contained in at least $\beta n$ sets.*

This theorem is known as the fractional Helly theorem. Later Kalai [13] found the current best possible function for $\beta$ is $(1-(1-\alpha)^{1/d+1})$. Note that here, if $\alpha \longrightarrow 1$, then $\beta \longrightarrow 1$, which means if "almost all" $d+1$ sets have a nonempty intersection, then "almost all" sets in $\mathscr{F}$ have a nonempty intersection.

# Chapter 3

# Line transversals

## 3.1 Line transversals in $\mathbb{R}^2$

A different way to weaken the local conditions in Helly's theorem, is by changing the intersecting object. Helly's condition can be rephrased the following way: For every $d+1$ sets in $\mathscr{F}$, there is a point which intersects them. Here, the condition can be weakened if instead of a point, we choose some larger set. Typically, this intersecting set is a $k$-dimensional affine subspace. In this chapter, we look at cases of $k = 1$, that is, we intersect by a line. If a line $\ell$ intersects every member of a family $\mathscr{F}$, then $\ell$ is called a *line transversal* of $\mathscr{F}$. A family $\mathscr{F}$ has the property $T(k)$ if every $k$ sets in $\mathscr{F}$ admit a line transversal. What can we say about a family globally, if we know it has the $T(k)$ property for some $k$?
Unfortunately, if $\mathscr{F}$ is a family of convex sets, it is not true that the $T(k)$ property for some $k$ implies that the family has a line transversal:

**Claim 3.1.** *For any $n$, there is a family $\mathscr{F}$ of $n$ convex sets in $\mathbb{R}^2$, such that every subfamily of $n-1$ sets in $\mathscr{F}$ have a line transversal, but $\mathscr{F}$ does not have a line transversal.*

*Proof.* Choose a point $P$, and divide the plane into $n$ equal pieces, by $n$ half-lines starting from $P$ equally spaced apart clockwise: half-lines $\ell_1, \ldots, \ell_n$. The convex sets will be the following: Let $C_i$ be the smaller, open set bounded by the half-lines $\ell_i$ and $\ell_{i+\lfloor n/2 \rfloor -1}$; which, in the case of even $n$, is one piece less than a half-plane, and in the case of odd $n$, it is one and a half piece less. Here, if $i + \lfloor n/2 \rfloor - 1 > n$, then it is taken modulo $n$. The family $\mathscr{F} = \{C_1, \ldots, C_n\}$ does not have a line transversal: take any line $\ell$ in the plane. Choose a direction of $\ell$, and this direction will be between the direction of $\ell_i$ and $\ell_{i+1}$ for some $i$. Then it is easy to see that $\ell$ cannot intersect the set $C_{i+1}$ and the set ending at $\ell_i$ ($C_{i-(\lfloor n/2 \rfloor -1)}$) at the same time. However, for any $C_i$, we can intersect every set except $C_i$: Take any line which is perpendicular to the angle bisector of $C_i$, and does not intersect $C_i$. $\square$

With slight modification of the above construction, we can also see that the sets can be compact in addition to being convex.

However, having the $T(k)$ property is still incredibly useful. While the whole family cannot be pierced by a single line (a line pierces a set if it intersects the set), some fraction of it can still be pierced, similarly to the fractional Helly's theorem:

**Theorem 3.2** (Katchalski and Liu [14]). *For any constant $0 \leq c < 1$ there exists $k = k(c)$ such that if a family of $n$ compact connected sets in $\mathbb{R}^2$ have the $T(k)$ property, then there exists a line which pierces at least $cn$ sets, if $n$ is sufficiently large.*

To prove this, we need a similar theorem for *symmetric twins*. A symmetric twin is a subset of a circle which consists of two closed arcs symmetric about the center of the circle. The whole circle is also considered a symmetric twin. The theorem is the following:

**Theorem 3.3** (Katchalski and Liu [14]). *Let $\mathscr{F}$ be a family of $n$ symmetric twins on the same circle. Let $k$ be an integer $1 < k < n$. If every $k$ sets in the $\mathscr{F}$ have a nonempty intersection, then there exists a subfamily of $\mathscr{F}$ of size at least $n\frac{k-2}{k+1}$ with a nonempty intersection.*

Note that this theorem is similar to Theorem 3.2 in the sense that for every $0 \leq c < 1$ we can choose $k$ such that $\frac{k-2}{k+1} > c$ given that $n$ is sufficiently large.
The appearance of symmetric twins seems surprising at first, however, the connection to Theorem 3.2 is quite simple: Consider two compact connected sets, and consider all directions for which there exists a common transversal of the two sets with this direction. These directions can be plotted on a circle around the origo, and these points will clearly form a symmetric twin. We will use Theorem 3.3 to find a good point which will be the direction of our piercing line.

*Proof of Theorem 3.3:* We may assume that $k$ is the largest number such that every $k$ sets in $\mathscr{F}$ have a nonempty intersection, since $\frac{k-2}{k+1}$ is an increasing function of $k$. Hence, there exists a subfamily $\mathscr{B} = \{B_1, \ldots, B_{k+1}\}$ with an empty intersection. Now let $a_i$ and $a_{i+k+1}$ be antipodal points in $\bigcap(\mathscr{B} - B_i)$. Note that we can choose $a_i$ and $a_{i+k+1}$ to be antipodal since the intersection of symmetric twins must be symmetric about the center of the circle. Now we relabel the points and sets such that $\{a_1, \ldots, a_{2k+2}\}$ are in clockwise order around the circle. $[a_i, a_j]$ will denote the closed arc starting from $a_i$ and ending at $a_j$ in clockwise direction. Further into the proof, any index bigger than $2k+2$ is taken modulo $2k+2$.
Since $B_i$ is a symmetric twin, and we know that $a_i, a_{i+k+1} \notin B_i$ and all other points are in $B_i$, we can conclude:

$$[a_{i+1}, a_{i+k}] \cup [a_{i+k+2}, a_{i-1}] \subset B_i \tag{3.1}$$

With this in mind, we can make the following observation:

$$\bigcap(\mathscr{B} - \{B_i, B_{i+1}\}) \subset [a_{i-1}, a_{i+2}] \cup [a_{i+k}, a_{i+k+3}] \tag{3.2}$$

Proof: Let us assume that there exists $x \in \bigcap(\mathscr{B} - \{B_i, B_{i+1}\})$ which is outside of the given region. Then, by equation (3.1), we know that $x$ must be in both $B_i$ and $B_{i+1}$. Thus, $x \in \bigcap \mathscr{B}$ which is a contradiction since $\bigcap \mathscr{B} = \emptyset$.

Now for any $F \in (\mathscr{F} - \mathscr{B})$, we know that $\{F\} \cup (\mathscr{B} - \{B_i, B_{i+1}\})$ must have a nonempty intersection, since it is a family of size $k$. Hence, $F$ must intersect $[a_{i-1}, a_{i+2}] \cup [a_{i+k}, a_{i+k+3}]$, and since $F$ is a symmetric twin, this means $F$ must also intersect only $[a_{i-1}, a_{i+2}]$ for every $1 \leq i \leq k+1$. From this, it follows that $F$ must contain every $a_i$, $1 \leq i \leq 2k+2$ with the possible exception of at most 6 points: $a_j, a_{j+1}, a_{j+2}$ and their opposite points for some $j$. Since this is true for every $F \in (\mathscr{F} - \mathscr{B})$, one of the points must belong to at least

$$\frac{2k+2-6}{2k+2}|\mathscr{F} - \mathscr{B}| = \frac{k-2}{k+1}(n-k-1) \tag{3.3}$$

sets of $(\mathscr{F} - \mathscr{B})$. This point also belongs to $k$ members of $\mathscr{B}$. This point is then the common point of $\frac{k-2}{k+1}(n-k-1) + k > \frac{k-2}{k+1}n$ sets, thus the proof is concluded. □

*Proof of Theorem 3.2:* Let $C$ be a fixed circle on the plane with center $O$. Now let us define the set $A_{ij}$ on $C$ for every $i \neq j$:

$x \in A_{ij}$ if the line passing through $x$ and $O$ is parallel to some common transversal of the sets $F_i$ and $F_j$. Here $A_{ij}$ will be a symmetric twin. Let $\mathscr{A} := \{A_{ij} \mid 1 \leq i, j \leq n,\ i \neq j\}$ denote the collection of these symmetric twins. Here, $|\mathscr{A}| = \binom{n}{2}$.

The value of $k$ will be determined later, however let us assume every $k$ members of $\mathscr{F}$ have a common transversal. Then, every $\lfloor k/2 \rfloor$ members of $\mathscr{A}$ must have a nonempty intersection, since they correspond to at most $k$ sets in $\mathscr{F}$ and thus have a common transversal. Due to theorem 3.3, there must be a point $x$ on $C$ which is contained in at least $\frac{\lfloor k/2 \rfloor - 2}{\lfloor k/2 \rfloor + 1}\binom{n}{2}$ symmetric twins. Now let us draw a line $\ell$ which is perpendicular to the line formed by $x$ and $O$. For each $F_i \in \mathscr{F}$, let $G_i$ be the projection of $F_i$ to $\ell$. Let $\mathscr{G}$ be the family of these projections. We know that a projection of a compact connected set is a closed segment, therefore $\mathscr{G}$ is a family of $n$ segments on a line. Out of the $\binom{n}{2}$ pairs of segments $\frac{\lfloor k/2 \rfloor - 2}{\lfloor k/2 \rfloor + 1}\binom{n}{2}$ pairs must have an intersection because having a common transversal perpendicular to $\ell$ is equivalent to their projections intersecting. With this, we can use a result of Abbott and Katchalski about segments on a line:

**Theorem 3.4** (Abbott and Katchalski [16]). *Let $\mathscr{G}$ be a family of $n$ closed intervals on a line, where $n$ is sufficiently large. Let $\alpha$ be any constant, $0 < \alpha < 1$. If at least $\alpha\binom{n}{2}$ pairs of intervals intersect, then there exists a point $x$ on the line which is in at least $(\sqrt{1 - \sqrt{1-\alpha}})n$ intervals.*

Let us define the function $f(k) := \sqrt{1-\sqrt{1-\frac{\lfloor k/2\rfloor-2}{\lfloor k/2\rfloor+1}}}$. Applying the above theorem, there exists a point $y$ on $\ell$ such that $y$ is in at least $f(k)n$ intervals of $\mathscr{G}$. Then, let $t$ be the line perpendicular to $\ell$ which intersects $\ell$ at point $y$. Here, $\mathscr{G}$ being the projection of the sets, $t$ will be the common transversal of at least $f(k)n$ sets in $\mathscr{F}$. With this, we can now finally determine the value of $k$: $k$ should be chosen such that $f(k) > c$. This can obviously be done, since

$$\lim_{k\to\infty}\left[\sqrt{1-\sqrt{1-\frac{\lfloor k/2\rfloor-2}{\lfloor k/2\rfloor+1}}}\right]=1 \tag{3.4}$$

With this, the proof is concluded. □

## 3.2 Line transversals in $\mathbb{R}^3$

In the previous section, we only investigated line transversals in 2 dimensions. Can a similar theorem be formulated in higher dimensions? The answer is no. Even in just one higher dimension, the $T(k)$ property becomes too weak to imply a global property of similar strength. For any $k$, there does not exist a $c > 0$ such that having the $T(k)$ property would imply that there is a line intersecting at least $c|\mathscr{F}|$ sets. This is a consequence of the following theorem:

**Theorem 3.5** (Alon, Kalai, Matousek, Meshulam [15]). *For every $k, m \in \mathbb{N}$ there exists a family of $m$ compact convex sets $\mathscr{F}$ in $\mathbb{R}^3$ such that every $k$ sets in $\mathscr{F}$ admit a line transversal, but no $k+4$ sets have a line transversal.*

The proof will use the following lemma:

**Lemma 3.6.** *Let $S_1, \ldots, S_m$ be subsets of $[n]$ for some $n, m \in \mathbb{N}$. There are convex sets $C_1, \ldots, C_m$ in $\mathbb{R}^3$ such that if some $\{S_i : i \in I\}$ have a common element, then the corresponding convex sets $\{C_i : i \in I\}$ have a line transversal, and whenever some $\{C_i : i \in I\}$ have a line transversal, then we can remove at most 3 elements from the index set $I$ to obtain $I'$, for which $\{S_i : i \in I'\}$ have a common element.*

*Proof of Theorem 3.5:* Using Lemma 3.6, we choose the sets $S_1, \ldots, S_m$ such that every $k$ of them intersect, but no $k+1$. This can be done the following way: index each subset of $[m]$ with size $k$ from 1 to $\binom{m}{k}$. Then, let $S_i$ be the indices for which the corresponding set contains $i$.
After this, using Lemma 3.6, the corresponding sets $C_1, \ldots, C_m$ will be a sufficient construction: Every $k$ of them will have a line transversal, but no $k+4$ can have a line transversal, since the reduced index set would still have at least $k+1$ elements, but no $k+1$ sets in $\{S_1. \ldots, S_m\}$ intersect. □

*Proof of Lemma 3.6:* The construction relies on the hyperbolic paraboloid defined by the equation $z = xy$, a surface that has been frequently employed in problems involving lines in $\mathbb{R}^3$.

Let $\Sigma$ denote the surface given by $z = xy$. For $i \in [n]$, define the line $\ell_i$ by the equations $x = \frac{i}{n}$, $z = \frac{i}{n}y$. It is straightforward to verify that each $\ell_i$ is part of the surface $\Sigma$. Now, consider a sequence of small positive numbers $0 < \varepsilon_1 \ll \varepsilon_2 \ll \cdots \ll \varepsilon_m \ll 1$ such that each $\varepsilon_i$ is significantly smaller than $\varepsilon_{i+1}$, and $\varepsilon_m$ is really small in terms of $n$ and $m$. Let $\Pi_j$ be the plane with equation $y = \frac{j}{m} + \varepsilon_j x$ which is the $xz$ plane shifted in the $y$ axis by $\frac{j}{m}$, and tilted by a very little amount. This way, its intersection with $\Sigma$ is a parabolic arc which is almost flat: The intersection, within $\Pi_j$, has the equation $z = \frac{j}{m}x + \varepsilon_j x^2$ which is a parabola very close to a line. Let $P_{ij}$ be the intersection of $\ell_i$ and $\Pi_j$. Now we define our convex sets: Let $C_j = conv(\{P_{ij} : i \in S_j\})$. Each $C_j$ forms a narrow convex polygon situated just above the surface $\Sigma$, and it can be calculated that the maximum vertical distance of a point in $C_j$ from $\Sigma$ is less than $\varepsilon_j$. Figure 1 is an illustration with $S_j = \{1, 3, 4\}$.

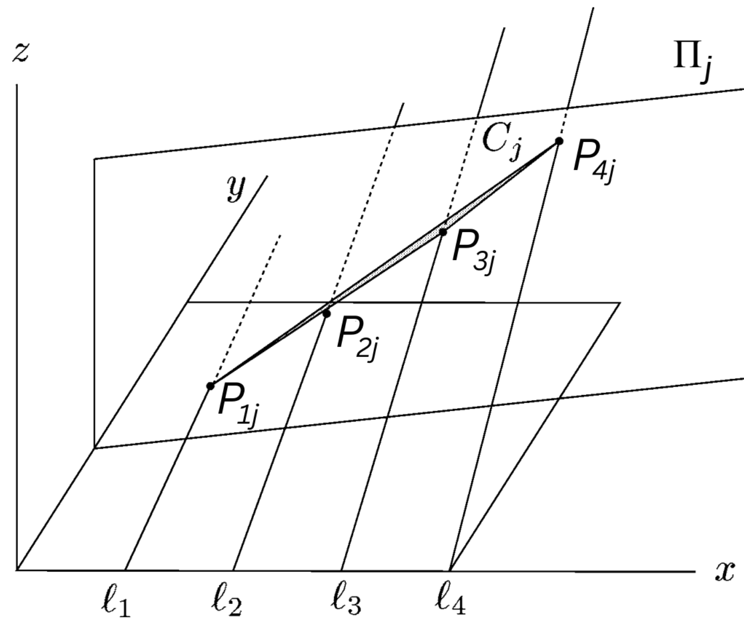



Figure 1: Source: Alon, Kalai, Matousek, Meshulam [15]

We see that whenever some sets $\{S_j : j \in I\}$ have a common element $i$, then the line $\ell_i$ will be a common transversal of the sets $\{C_j : j \in I\}$. Thus, the first claim of the lemma is fulfilled. Next, we need to show that whenever some sets $\{C_j : j \in I\}$ have a line transversal, then by removing at most 3 elements from $I$ we get a collection $\{C_j : j \in I'\}$ for which one of the $\ell_i$ lines is a transversal. To prove this, we divide each $C_j$ into a *high* region and a *low* region: a point is in the low region of $C_j$, if its vertical distance from $\Sigma$ is less than $\frac{\varepsilon_j}{100n^2}$. The rest of the polygon is in the high region. With calculation, it can be shown that the low region will consist of small very clearly distinct pieces cut off near $C_j$'s connecting points to $\Sigma$. An illustration of this can be seen in figure 2, where the low region is indicated by the black part.

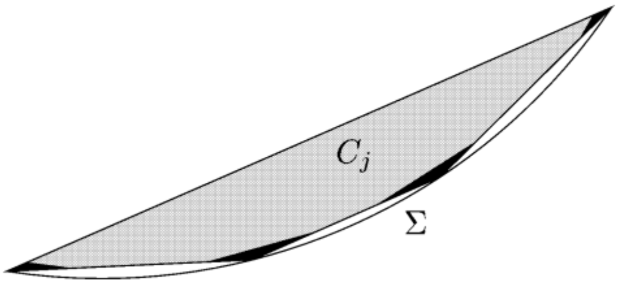



Figure 2: Source: Alon, Kalai, Matousek, Meshulam [15]

To finish the lemma, we have the following two claims:

**Claim 3.7.** *If a line $\lambda$ meets at least two $C_j$ in the low regions, then either the sets met by $\lambda$ in the low regions are also met by one of the $\ell_i$, or $\lambda$ only meets two sets.*

**Claim 3.8.** *A line $\lambda$ can meet at most 3 $C_j$ in the high regions.*

*Proof of Claim 3.7:* Let us assume that $\lambda$ meets two low region pieces not met by some $\ell_i$. This means that these pieces are near points $P_{i_1j_1}$ and $P_{i_2j_2}$ where $i_1 \neq i_2$. Therefore, their $x$ coordinates differ by at least $\frac{1}{n}$. This means that $\lambda$ goes "diagonally", and is not closely parallel to the $yz$ plane nor the $xz$ plane. Hence, $\lambda$ is not nearly parallel to the surface $\Sigma$, and since the sets $C_j$ are very close to $\Sigma$, if the $\varepsilon_j$ are small enough then $\lambda$ can only meet at most two of the sets.

*Proof of Claim 3.8:* Let us assume that $\lambda$ meets 4 sets in the high regions: $C_{j_1}, C_{j_2}, C_{j_3}, C_{j_4}$. Let us parametrize $\lambda$ by the $y$ coordinate, and let $d_\lambda(y)$ be the vertical distance of $\lambda$ from $\Sigma$ in terms of $y$. Since $\Sigma$ is a hyperbolic paraboloid, $d_\lambda(y)$ will be a quadratic polynomial. Let $y_1, y_2, y_3, y_4$ be the $y$ coordinate of the intersection points between $\lambda$ and $C_{j_1}, C_{j_2}, C_{j_3}, C_{j_4}$. We know that $y_k$ here is approximately equal to $\frac{j_k}{m}$. We also know that the vertical distance at these points cannot be high:

$$d_\lambda(y_k) < \varepsilon_{j_k} \quad k = 1, 2, 3, 4 \tag{3.5}$$

Moreover, since we intersect at the high regions, the vertical distance also cannot be too low:

$$d_\lambda(y_k) > \frac{\varepsilon_{j_k}}{100n^2} \quad k = 1, 2, 3, 4 \tag{3.6}$$

A quadratic polynomial however, cannot meet these conditions at the same time: at $y_1, y_2, y_3$ we know that the polynomial is upper bounded by $\varepsilon_{j_3}$. On the other hand, at $y_4$ the polynomial should reach at least $\frac{\varepsilon_{j_4}}{100n^2}$. A $2nd$ degree curve hitting 3 very low values, namely between 0 and $\varepsilon_{j_3}$, at 3 different points at least $\frac{1}{2m}$ apart, clearly cannot reach $\frac{\varepsilon_{j_4}}{100n^2}$ at $y_4$, if $\varepsilon_{j_3}$ is sufficiently smaller than $\varepsilon_{j_4}$ (in terms of $n$ and $m$). This is a contradiction.

Using Claim 3.7 and Claim 3.8, the proof of the lemma is now finished. $\square$

# Chapter 4

# My results

## 4.1 Line transversals of sets with T(3) property

While the $T(k)$ property does not imply a line transversal, we showed above, that in $\mathbb{R}^2$ it does imply that some nonzero fraction of the sets do have a line transversal. Along with such quantitative results, people have started to investigate the question of how many lines do we need to pierce every set? If $\mathcal{F}$ is a family of sets, we say thet $\mathcal{F}$ is *pierced* by $k$ lines, if there exists $k$ lines whose union intersect all sets in $\mathcal{F}$. The *line-piercing* number of $\mathcal{F}$ is the lowest such $k$.
The line-piercing number of sets in the plane has been a frequently studied topic since the 1960s. In particular, bounding the line-piercing number of compact convex sets proved to be an interesting problem. In 1964 Eckhoff [6] proved that if a family of compact convex sets has the $T(4)$ property, then it can be pierced by 2 lines. In 1974 [5] he gave an example of sets satisfying the $T(3)$ property which cannot be pierced by 2 lines.
The sets investigated in these results are assumed to be convex, but the results also apply to connected sets. This follows from the fact that if $F$ is a connected set in $\mathbb{R}^2$, then a line $\ell$ intersects $F$ if and only if $\ell$ intersects $conv(F)$.
For a while, it was unknown whether the $T(3)$ property implies a finite line-piercing number. However, in 1975 Kramer [7] proved that compact convex sets satisfying the $T(3)$ property can be pierced by 5 lines. After 18 years, in 1993 Eckhoff [4] was able to show that such families can be pierced by 4 lines. Finally, in 2021 McGinnis and Zerbib [1] proved that they can be pierced by only 3 lines. By their result, the line-piercing number of such families was finally resolved:

**Theorem 4.1** (McGinnis and Zerbib [1])**.** *Let $\mathcal{F}$ be a family of compact connected sets in $\mathbb{R}^2$. If every three sets in $\mathcal{F}$ admit a line transversal, then $\mathcal{F}$ is pierced by 3 lines.*

In fact, McGinnis and Zerbib also proved a generalized, "colorful" version of this result. In their paper they proved the theorem with 6 colors. Let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

**Theorem 4.2** (McGinnis and Zerbib [1]). *Let $\mathcal{F}_1, \dots, \mathcal{F}_6$ be families of compact connected sets in $\mathbb{R}^2$. If every three sets $F_1 \in \mathcal{F}_{i_1},\ F_2 \in \mathcal{F}_{i_2},\ F_3 \in \mathcal{F}_{i_3}, \quad 1 \leq i_1 < i_2 < i_3 \leq 6$ admit a line transversal, then there exists $i \in [6]$ such that $\mathcal{F}_i$ can be pierced by 3 lines.*

Similarly to the colorful Helly's theorem, this theorem generalizes Theorem 4.1, because if we choose all families to be the same family $\mathcal{F}$, it yields Theorem 4.1.
It can also be easily seen that if the statement is true for $k$ colors (families), then it is also true for all $l \geq k$ colors, by applying the $k$-color theorem to the first $k$ families. Therefore, lowering the number of colors makes a stronger statement.

Here I will prove that Theorem 4.2 is also true for concurrent lines and with 5 families, if the families are bounded. This theorem then can be extended to unbounded families if we allow the piercing lines to be parallel. Overall therefore, I also improve Theorem 4.2 to 5 colors.
My proof is a modification of McGinnis and Zerbib's marvelous method for proving Theorem 4.2.

**Theorem 4.3** (Main result). *Let $\mathcal{F}_1, \dots, \mathcal{F}_5$ be bounded families of compact connected sets in $\mathbb{R}^2$. If every three sets $F_1 \in \mathcal{F}_{i_1},\ F_2 \in \mathcal{F}_{i_2},\ F_3 \in \mathcal{F}_{i_3}, \quad 1 \leq i_1 < i_2 < i_3 \leq 5$ admit a line transversal, then there exists $i \in [5]$ such that $\mathcal{F}_i$ can be pierced by 3 concurrent lines.*

**Corollary 4.4.** *Let $\mathcal{F}$ be a bounded family of compact connected sets in $\mathbb{R}^2$. If every three sets in $\mathcal{F}$ admit a line transversal, then $\mathcal{F}$ can be pierced by 3 concurrent lines.*

The main tool used in the proof will be the colorful KKM theorem (Knaster–Kuratowski–Mazurkiewicz lemma [2], generalized by David Gale [3]).

**Theorem 4.5** (KKM theorem [2]). *Let $\Delta^{n-1}$ be an $n-1$ dimensional simplex with $n$ vertices $\{v_1, \dots, v_n\}$. Let $A_1, \dots, A_n$ be open sets of $\Delta^{n-1}$ such that for every face $\sigma$ of $\Delta^{n-1}$, we have $\sigma \subset \bigcup_{v_i \in \sigma} A_i$. Then $\bigcap A_i \neq \emptyset$. Namely, there is a point $x \in \Delta^{n-1}$ colored by all colors.*

In this case, we say that $A_1, \dots, A_n$ form a *KKM-covering* of $\Delta^{n-1}$.

**Theorem 4.6** (colorful KKM theorem [3]). *Let $\Delta^{n-1}$ be an $n-1$ dimensional simplex. Let $A_1^i, \dots, A_n^i$, $i \in [n]$ be open sets of $\Delta^{n-1}$ such that for every $i \in [n]$, $A_1^i, \dots, A_n^i$ form a KKM-covering of $\Delta^{n-1}$. Then there exists $\pi \in S_n$ permutation, such that $\bigcap_{i=1}^n A_i^{\pi(i)} \neq \emptyset$.*

*Proof of Theorem 4.3:* First, I will prove a simplified version of this result, where the sets have a non-empty interior of size $\delta$ for some $\delta > 0$. Formally, there exists $\delta > 0$ such that each $F \in \mathcal{F}_i$ contain some translation of $B_\delta$ as a subset, where $B_\delta$ is the disk with radius

$\delta$ centered at the origin. This condition applies in the special case where we have finite families of sets with non-empty interior, along with the condition of boundedness. Later, we will extend this proof to allow all compact connected sets.
We will also indirectly assume that no family can be pierced by 3 lines, and will arrive at a contradiction.
Using that the families are bounded, we may scale the plane such that every set in every family is contained inside a circle with unit circumference $U$. To use the KKM theorem, we will associate every point of a simplex with a set of 3 lines on the plane. Let $\Delta^4$ be the 4 dimensional simplex defined as follows: $\Delta^4 := \{(x_1, \ldots, x_5) \in \mathbb{R}^5 \ \mid \ x_i \geq 0, \ \sum x_i = 1\}$ which is the convex hull of the canonical basis vectors $e_1, \ldots, e_5$ of $\mathbb{R}^5$. Next, for every point $x$ $(x_1, \ldots, x_5) \in \Delta^4$, we will associate 5 points on the circle the following way: from a previously fixed point $P$ of $U$, we consecutively measure up the coordinates of $x$ along the curcumference of the circle creating 5 points $f_1(x), \ldots, f_5(x)$. Note that since the coordinates add up to 1, $f_5(x)$ will coincide with $P$ as seen in Figure 3.

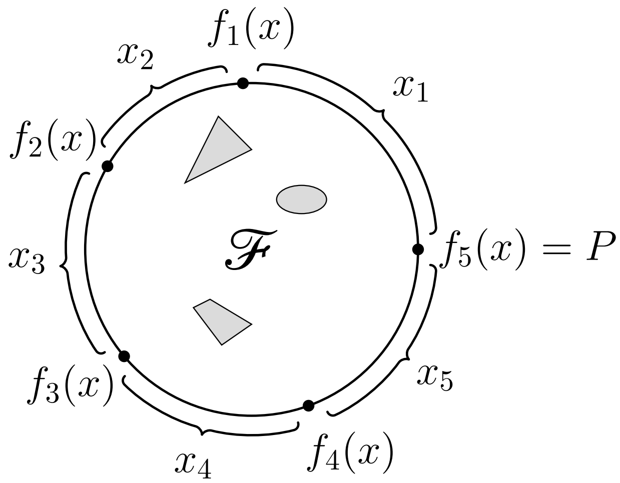



Figure 3: The coordinates of a point $x \in \Delta^4$ measured up along the circumference of $U$.

Let $\overline{AB}$ denote the line defined by points $A$ and $B$. With the 5 points on the circle, we can define 3 lines: $\ell_1 := \overline{f_1(x)f_4(x)}$, $\ell_2 := \overline{f_2(x)f_5(x)}$, and $\ell_3 := \overline{f_3(x)M}$ where $M$ is the intersection of $\ell_1$ and $\ell_2$. The definition of $\ell_3$ is the main difference between this proof and McGinnis and Zerbib's. In their paper, they used a 6th point on $U$ to define $\ell_3$.
Here however, $\ell_3$ is not defined when $\ell_1$ and $\ell_2$ coincide. We can avoid this, and other edge cases, by considering a slightly smaller simplex where each coordinate must be at least $\varepsilon$. Specifically, let $\Delta^4_\varepsilon := \{(x_1, \ldots, x_5) \in \mathbb{R}^5 \ \mid \ x_i \geq \epsilon, \ \sum x_i = 1\}$. This way, if $i \neq j$, then $f_i(x) \neq f_j(x)$. From now on, we will only consider $\Delta^4_\varepsilon$. The value of $\varepsilon$ will be determined later.

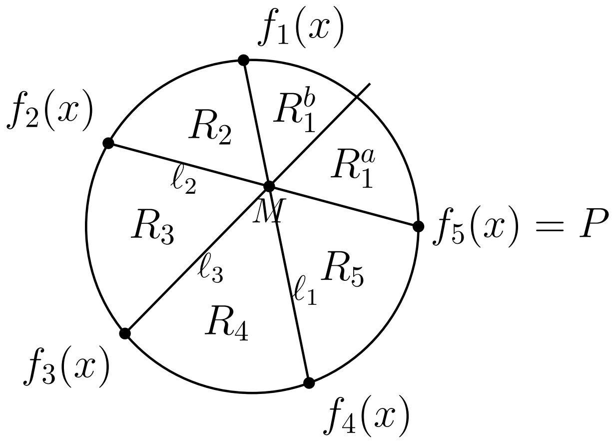



Figure 4: A point $x \in \Delta_\varepsilon^4$ defines 5 points on $U$, the 3 defined lines of which split the disk bounded by $U$ into 6 regions.

Name each open region bounded by the lines and $U$ as $R_1^a, R_1^b, R_2, R_3, R_4, R_5$ as seen in Figure 3. Of course, the regions depend on $x$.

Now for every $1 \leq j \leq 5$ we will define the colors corresponding to the family $\mathscr{F}_j$. For $i = 2, 3, 4, 5$, let $A_i^j$ be the set of $x \in \Delta_\varepsilon^4$ points, for which exists $F \in \mathscr{F}_j$ such that $F \subset R_i$. The case of $i = 1$ is special: let $A_1^j$ be the set of $x \in \Delta_\varepsilon^4$ points, for which exists $F \in \mathscr{F}_j$ such that $F$ is the subset of either $R_1^a$ or $R_1^b$, this means $F$ is in one of the regions divided by $\ell_3$.

**Claim 4.7.** *For each $j$, $A_1^j, \ldots, A_5^j$ form a KKM cover of $\Delta_\varepsilon^4$.*

*Openness:*

Since each $F \in \mathscr{F}_j$ is closed and each region is open, if $R_i$ contains a set $F$, then for a small ball around $x$, $R_i$ will still contain $F$, therefore all $A_i^j$ are open sets of $\Delta_\varepsilon^4$. Note that the regions $R_i$ change with $x$, and the set $F$ remains the in same position.

*Condition on the faces of $\Delta_\varepsilon^4$:*

We will prove this by first showing that each face only has colors which are associated with the vertices of the face, then we will show that every point on the simplex is colored. Firstly we need to check that if the $i$th coordinate of $x$ is $\varepsilon$, that is, we are on a face of $\Delta_\varepsilon^4$ not containing the $i$th vertex, then it cannot be colored by the $i$th color, formally $(x_i = \varepsilon) \Rightarrow (x \notin A_i^j)$. To prove this, we need to set $\varepsilon$ small enough that if $x_i = \varepsilon$, then $R_i$ cannot contain any set $F \in \mathscr{F}_j$. Here, we will use the fact that there exists $\delta > 0$ such that each $F \in \mathscr{F}_j$ contains some translation of $B_\delta$ as a subset. Let $\varepsilon := \frac{\delta}{4\pi}$. With this $\varepsilon$, if $x_i = \varepsilon$, then a simple calculation shows that the length of the arc bounding $R_i$ must be less than $\delta$. Knowing this, using the fact that the bounding lines meet at $M$ inside $U$, we can conclude that $R_i$ cannot contain any set, therefore $(x_i = \varepsilon) \Rightarrow (x \notin A_i^j)$ holds.

Secondly, we have to show that, $\Delta^4_\varepsilon = \bigcup_{i=1}^5 A_i^j$ for each $j$. This is true due to the indirect assumption that none of the families can be pierced by 3 lines. Suppose there was $x \in \Delta^4_\varepsilon$ such that $x \notin \bigcup_{i=1}^5 A_i^j$, then there is no $F \in \mathscr{F}_j$ which is in one of the 6 regions. Because every $F \in \mathscr{F}_j$ is connected, this means the union of $\ell_1, \ell_2, \ell_3$ pierce every $F \in \mathscr{F}_j$ which contradicts the assumption. Knowing $(x_i = \varepsilon) \Rightarrow (x \notin A_i^j)$ and $\Delta^4_\varepsilon = \bigcup_{i=1}^5 A_i^j$ we can conclude that the conditions of Theorem 4.6 hold. □

Thus, by Theorem 4.6, there exists some permutation $\pi \in S_5$ and a point $p \in \Delta^4_\varepsilon$ such that $p \in \bigcap_{i=1}^5 A_i^{\pi(j)}$. Therefore, for point $p$, each of the open regions $R_i$ contain a set $F_i \in \mathscr{F}_{\pi(i)}$, $i = 2, 3, 4, 5$. For $i = 1$ it means either $R_1^a$ or $R_1^b$ contain a set $F_1 \in \mathscr{F}_{\pi(1)}$. If $F_1 \subset R_1^a$, then the sets $F_1, F_2, F_4$ do not admit a line transversal. If $F_1 \subset R_1^b$, then the sets $F_1, F_3, F_5$ do not admit a line transversal. In both cases, we arrive at a contradiction. Therefore, the assumption that no $\mathscr{F}_j$ family can be pierced by 3 lines was incorrect, and there is a point $p \in \Delta^4_\varepsilon$ which is not colored by any $A_i^j$ for some $j$. The defined lines $\ell_1, \ell_2, \ell_3$ for $p$ then pierce the family $\mathscr{F}_j$. We can also see that these lines intersect at point $M$. □

With this, we can now generalize the proof for families of any compact connected sets: Let $\mathscr{F}(\delta)$ be the thickened version of $\mathscr{F}$ by $\delta$. More formally, $\mathscr{F}(\delta) := \{F + B_\delta \mid F \in \mathscr{F}\}$ for some $\delta \geq 0$, where $B_\delta$ is the closed disk with radius $\delta$ centered at the origin, and $F + B_\delta$ is the Minkowski sum of $F$ and $B_\delta$.

**Lemma 4.8.** *Let $\mathscr{F}$ be a bounded family of compact sets in the plane. If for all $\delta > 0$, $\mathscr{F}(\delta)$ is pierced by $n$ concurrent lines such that the intersection of said lines is always inside a fixed compact region, then $\mathscr{F}$ is pierced by $n$ concurrent lines.*

*Proof.* For each $\delta$ we choose $n+1$ points on the piercing lines the following way: let $p_0^\delta$ be the intersection of the lines, and let $p_1^\delta, \dots, p_n^\delta$ be points on each line such that the distance of the points from $p_0^\delta$ is 1 unit. Now let $\delta_i$ be a series which converges to 0. Then, since each point is inside a fixed compact region, there exists a subseries $\delta_{i_0}$ for which $p_0^{\delta_{i_0}}$ converges to some $p_0$. Next, let $\delta_{i_1}$ be a subseries of $\delta_{i_0}$ such that $p_1^{\delta_{i_1}}$ converges to some $p_1$. Continuing this for all $n+1$ points, we get the series $\delta_{i_n}$, and the limit points $p_0, \dots, p_n$. With these points we define the limit lines $\ell_1, \dots, \ell_n$. Let us assume that these lines do not pierce every set in $\mathscr{F}$. Specifically, no lines intersect some $F \in \mathscr{F}$. Since $F$ is closed and bounded, there exists $\epsilon$ such that $F(\epsilon)$ is still not pierced. If $F(\epsilon)$ is not pierced by the limit lines, then there exists an index $I$ where the lines associated with $\delta_{i_n}$ do not pierce $F$ if $i_n > I$. This is a contradiction, since if $\delta_{i_n} < \epsilon$, then the lines associated with $\delta_{i_n}$ must intersect $F(\epsilon)$. □

We can see in the above almost finished proof of Theorem 4.3, that the intersection $M$ is always inside the unit disk, therefore we can use Lemma 4.8 in the context of 5 families to prove Theorem 4.3 : If for every $\delta > 0$ one of the families $\mathscr{F}_i(\delta)$ $1 \leq i \leq 5$ is pierced by 3 concurrent lines , then it follows that it is true for one of the families $\mathscr{F}_i$ that for every $\delta > 0$, $\mathscr{F}_i(\delta)$ is pierced by 3 concurrent lines. With this, we can use Lemma 4.8 for this family. Hence, $\mathscr{F}_i$ is pierced by 3 concurrent lines. With this, the proof of Theorem 4.3 is concluded. □

We can now extend Theorem 4.3 to unbounded families of compact connected sets:

**Lemma 4.9.** *Let $\mathscr{F}$ be a family of compact sets on the plane. If every bounded subfamily of $\mathscr{F}$ is pierced by $n$ concurrent lines, then $\mathscr{F}$ is pierced by $n$ concurrent or parallel lines.*

*Proof.* The proof is by induction. The base case of $n = 0$ is trivial. Inductive step from $n-1$ to $n$: Assume that we know that the Lemma is true for every $k < n$. Then, if every bounded subfamily of $\mathscr{F}$ is pierced by $n-1$ lines, then the Lemma is trivially true. Hence, we can assume that there exists a bounded subfamily $\mathscr{F}_0$ which is not pierced by $n-1$ lines. Therefore, we know that for every bounded subfamily of $\mathscr{F}$, the $n$ piercing lines also each intersect the the convex hull of $\mathscr{F}_0$.
Now take an infinite series of larger and larger disks in the plane $D_1, D_2, \ldots$ with the same midpoint, while their radiuses approach infinity. Each of them cover a bounded subfamily of $\mathscr{F}$, let us call them $\mathscr{F}_1, \mathscr{F}_2, \ldots$. Each of these families can be pierced by $n$ lines. Since these lines always intersect the convex hull around $\mathscr{F}_0$, we can take a sufficiently large compact region around $\mathscr{F}_0$, and select 2 points with unit distance on each line, such that all points are inside this fixed region. Similarly to the proof of Lemma 4.8, since these points are in a compact region, we can take a subseries of the disks $D_1, D_2, \ldots$ such that each of these selected points on the piercing lines converge. The limit of these lines define the limit lines. These $n$ limit lines cannot be anything other than parallel or concurrent, because then there would be an index in the series after which the piercing lines are not concurrent. Since each set in $\mathscr{F}$ is pierced from an index onward, and each set is closed, they must be pierced by the limit lines. □

Using Lemma 4.9 in the context of 5 families, the extension can now be formulated as follows:

**Theorem 4.10.** *Let $\mathscr{F}_1, \ldots, \mathscr{F}_5$ be families of compact connected sets in $\mathbb{R}^2$. If every three sets $F_1 \in \mathscr{F}_{i_1}$, $F_2 \in \mathscr{F}_{i_2}$, $F_3 \in \mathscr{F}_{i_3}$, $1 \leq i_1 < i_2 < i_3 \leq 5$ admit a line transversal, then there exists $i \in [5]$ such that $\mathscr{F}_i$ can be pierced by 3 concurrent or parallel lines.*

**Corollary 4.11.** *Let $\mathscr{F}$ be a family of compact connected sets in $\mathbb{R}^2$. If every three sets in $\mathscr{F}$ admit a line transversal, then $\mathscr{F}$ can be pierced by 3 concurrent or parallel lines.*

Similarly to McGinnis and Zerbib's paper [1], the proof of Theorem 4.3 imply a slightly stronger result: if the families are bounded, due to the fact that $f_5(x)$ is fixed

every time, we can choose the position of $f_5(x)$. Therefore, we can choose any point $Q$ for which one of the piercing lines will go through, as long as $Q$ is outside the convex hull of the sets.
Moreover, by choosing $Q$ further and further in one direction, a simple convergence argument shows, that the direction of one of the piercing lines can be chosen in advance. All the results in this paper can be demonstrated in Eckhoff's [5] example of sets which cannot be pierced by 2 lines. His example can indeed be pierced by 3 concurrent lines, one of which can be horizontal.

With the number of colors brought down from 6 to 5, the question still remains whether it can be lowered to 3. I conjecture it to be true:

**Conjecture 4.12.** *Let $\mathscr{F}_1, \ldots, \mathscr{F}_3$ be families of compact connected sets in $\mathbb{R}^2$. If every three sets $F_1 \in \mathscr{F}_1,\ F_2 \in \mathscr{F}_2,\ F_3 \in \mathscr{F}_3$ admit a line transversal, then there exists $i \in [3]$ such that $\mathscr{F}_i$ can be pierced by 3 lines.*

Moreover, the concurrent version of the above conjecture:

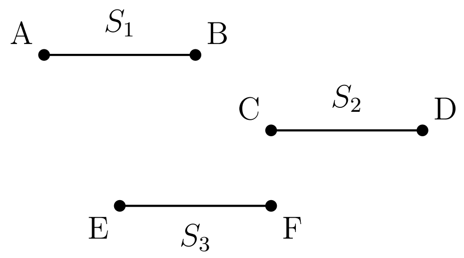
**Conjecture 4.13.** *Let $\mathscr{F}_1, \ldots, \mathscr{F}_3$ be bounded families of compact connected sets in $\mathbb{R}^2$. If every three sets $F_1 \in \mathscr{F}_1,\ F_2 \in \mathscr{F}_2,\ F_3 \in \mathscr{F}_3$ admit a line transversal, then there exists $i \in [3]$ such that $\mathscr{F}_i$ can be pierced by 3 concurrent lines.*

## 4.2 Line transversals of axis-parallel segments with T(3) property

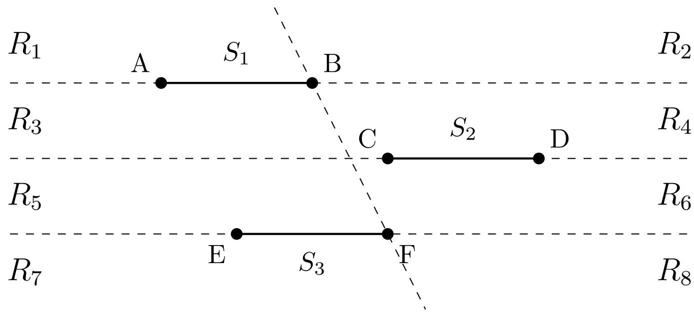
While we could not prove Theorem 4.2 for less than 5 colors, we can look at some special cases. One special case of 4 colors would be the following: Let $\mathscr{F}$ be a family of horizontal segments, and let $\mathscr{G}$ be a family of vertical segments. If the 4 families are $\mathscr{F}, \mathscr{F}, \mathscr{G}, \mathscr{G}$, then if every 3 segments, chosen from 3 different families admit a line transversal, then one of the families can be pierced by 3 lines. This theorem is indeed true, in fact, I was able to prove a stronger statement:

**Theorem 4.14.** *Let $\mathscr{F}$ be a family of horizontal segments, and let $\mathscr{G}$ be a family of vertical segments in $\mathbb{R}^2$. If every three segments containing vertical and horizontal segments have a line transversal, then the following is true: If $\mathscr{F}$ do not admit a line transversal, then $\mathscr{G}$ is pierced by 3 lines.*

*Proof.* It is a well-known result by Santaló [17], that if a family of horizontal segments cannot be pierced by a line, then there exists 3 segments in the family which cannot be pierced by a line. Assuming that $\mathscr{F}$ cannot be pierced by 1 line, let $AB$, $CD$, $EF$ be these 3 segments. The only way these segments can be arranged can be seen in figure 1: The top and bottom segments can be separated from the middle segment by a line.




Now we draw 4 lines: The extension of the 3 segments, and the $BF$ line.




With these 4 lines, the plane is split into 8 regions named $R_1, \ldots, R_8$. Now these 4 lines must pierce every set in $\mathscr{G}$:
The segments in $\mathscr{G}$ must have a common transversal with every pair in $\{S_1, S_2, S_3\}$. A set $G \in \mathscr{G}$ cannot be inside the regions $R_4$ and $R_6$, because $\{S_1, S_3, G\}$ would not have a common transversal. $G$ also cannot be inside regions $R_2, R_5, R_7$, because $\{S_1, S_2, G\}$ would not have a common transversal. Finally, $G$ cannot be inside regions $R_1, R_3, R_8$, because $\{S_2, S_3, G\}$ would not have a common transversal.
However, one of these lines can always be erased such that $\mathscr{G}$ is still pierced: Assume that we cannot erase any of the lines. Then, for each line, there must be a vertical segment in $\mathscr{G}$ which is only intersected by that line. Let $\ell_1$ and $\ell_2$ be the names of the extensions of $AB$ and $EF$. Then, there is $G_1, G_2 \in \mathscr{G}$ such that $G_1$ is only pierced by $\ell_1$, and $G_2$ is only pierced by $\ell_2$. Here, $G_1$ cannot be to the left of point $B$, because then $\{S_2, S_3, G_1\}$ would not have a common transversal. Similarly, $G_2$ cannot be to the left of point $F$, because then $\{S_1, S_2, G_2\}$ would not have a common transversal. Therefore, we can assume that both segments are on the right side of the $BF$ line. With this, either $\{S_1, G_1, G_2\}$ or $\{S_3, G_1, G_2\}$ will not have a common transversal depending on the position of $G_1$ and $G_2$. Therefore, we have arrived at a contradiction, and one of the lines can indeed be erased. □

Using this theorem for both $\mathscr{F}$ and $\mathscr{G}$, we get the following consequence:

**Corollary 4.15.** *Let $\mathscr{F}$ be a family of horizontal segments, and let $\mathscr{G}$ be a family of vertical segments in $\mathbb{R}^2$. If every three segments containing vertical and horizontal segments have a line transversal, then either one of $\mathscr{F}$ and $\mathscr{G}$ admit a line transversal, or both of them can be pierced by 3 lines.*

# Nyilatkozat

Alulírott Csizmadia Miklós Dániel nyilatkozom, hogy szakdolgozatom elkészítése során az alább felsorolt feladatok elvégzésére a megadott MI alapú eszközöket alkalmaztam:

| **Feladat** | **Felhasznált eszköz** | **Felhasználás helye** | **Megjegyzés** |
|---|---|---|---|
| Ábra készítés | ChatGPT-4o | 3. és 4. ábra | |
| Nyelvhelyesség ellenőrzése | ChatGPT-4o | Teljes dolgozat | |

A felsoroltakon túl más MI alapú eszközt nem használtam.

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