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The Kakeya property

Bachelor thesis

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Budapest, 2025

Contents

1 Acknowledgement

First, I would like to express my gratitude to my supervisor, Tamás Keleti — not only for his help in the writing of this thesis, but more importantly for his attention and knowledge which were crucial to the research leading up to it.

I would also like to thank my friends for making my university years wonderful. As I cannot thank them each individually, I would like to highlight Zsigmond Fleiner, who inspired me with his talent and elevated my standards.

Last but not least, I would like to thank my family and my girlfriend for their deep belief in me throughout the years.

2 Notation

- Let $a \cdot b$ denote the standard dot product of a and b.
- For a vector v denote the projection onto it by

$$p_v(X) = \{x \cdot v \mid x \in X\}.$$

- $B(x, r)$ denotes the open ball centred at x with radius r. If the centre is understood to be the origin then $B(r) = B(0, r)$.
- Measure always refers to the Lebesgue measure, which we denote by λ.
- For sets A, B let $A + B$ denote their Minkowski sum.
- $\mathrm{Isom}^+(\mathbb{R}^n)$ is the set of orientation preserving isometries of $\mathbb{R}^n$ equipped with the natural topology. If $n = 2$ then a metric can be easily defined by writing each isometry as the composition of a rotation and a translation.
- A (continuous) *motion* is a continuous map $M : [0, 1] \to \mathrm{Isom}^+(\mathbb{R}^n)$.
- For a motion M and a set E the area swept by E during M is

$$\lambda(\{M_t(e))|t \in [0, 1], e \in E\}).$$

- For a translation α, by vector v, let $M^\alpha(t)$ be the translation by tv. Similarly for a rotation α, around P and by angle φ, let $M^\alpha(t)$ be the rotation around P by angle $t\varphi$.
- We refer to the Kakeya property as property (K) and the strong Kakeya property as (K^S). For definitions see Section 3.

3 Introduction

In 1917, Kakeya posed a question now known as the Kakeya needle problem: Among planar sets, in which a line segment of unit length can be fully rotated, which one has minimal area? This seemingly simple question turned out to have very deep implications and sparked a wide area in geometric measure theory. The first breakthrough came when Besicovitch [2] showed that there is no such minimum, the area of such a set can be arbitrarily small.

In the process, Besicovitch showed that there exists a compact planar set, which contains a unit line segment in every direction and has measure zero.

Definition 3.1. A set is a *Besicovitch set* if it contains a unit line segment in every direction.

The next natural question on how small Besicovitch sets can be is to ask if they must have Hausdorff dimension n. This is a very central open question to this day, with it being proven true for $n = 2$ by Davies in 1971 [11] and proven true by Wang and Zahl for $n = 3$ [5] just this year.

This thesis will mostly deal with problems similar to the original question of Kakeya. In Section 4 we will look at some constructions for the rotation of the unit line segment and we will construct a compact Besicovitch set of measure zero.

Section 5 and Section 6 deals with two properties first defined by Csörnyei, Héra, and Laczkovich in [8].

Definition 3.2. A set E in $\mathbb{R}^n$ has the *Kakeya property* (or shortly *property* (K)) if there exists a non-identity $\phi \in \mathrm{Isom}^+(\mathbb{R}^n)$ such that E can be moved to $\phi(E)$ in an arbitrarily small n-volume.

Definition 3.3. A set E in $\mathbb{R}^n$ has the *strong Kakeya property* (or shortly *property* (K^S)) if for every $\phi \in \mathrm{Isom}^+(\mathbb{R}^n)$ E can be moved to $\phi(E)$ in an arbitrarily small n-volume.

In Section 5 we will look at results regarding the characterization of planar sets with the Kakeya property. We will show that nice enough sets with the Kakeya property are concentric circles or lines (or their subsets).

In Section 6 we show that some sets have the strong Kakeya property, for example the union of a finite number of parallel line segments or a circle missing diametrically opposite arcs.

Section 7 and Section 8 exclusively contains results due to the author [9]. In Section 7 we discuss a result regarding the rotation of a square. In Section 8 we show how this theorem can be used to show that a wide family of sets in $\mathbb{R}^3$ have the strong Kakeya property.

4 Rotating a line segment

This section deals with different versions and proofs of the following theorem, first proved by Besicovitch in [2].

Theorem 4.1 (Besicovitch)**.** *For every ε there exists a continuous motion of the unit line segment during which it does a full rotation and sweeps an area less than ε.*

Observation 4.2 (Pál)**.** The first important observation is that we can translate the unit line segment in an arbitrarily small area. To do this we rotate the segment by a small angle and then we translate the segment in its own direction and when we reach the desired line we rotate the segment back to its original direction and translate it to the desired position (see Fig. 1). Such movements are called Pál joins.

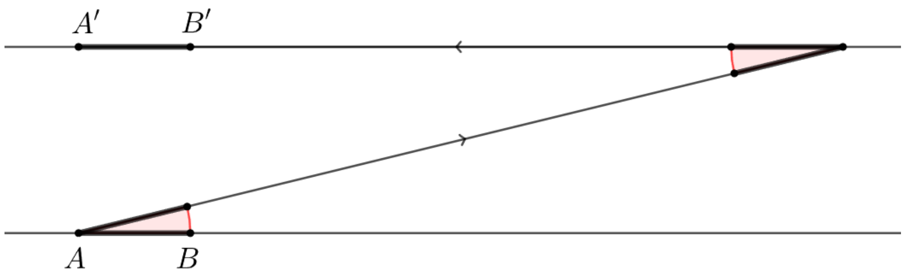

Figure 1: A Pál join from AB to $A'B'$, with the swept area marked red.

Observation 4.3 (Pál)**.** Let B be a set which is the finite union of triangles such that for every direction d, there is a triangle that contains a unit line segment in direction d. We can fully rotate a line segment in a way that it sweeps small area outside of B, since we can rotate the segment as much as we can inside a triangle and then move to a triangle, in which we can rotate it further, by a translation. By Observation 4.2 the segment can sweep an arbitrarily small area during the translations.

Therefore, if we construct a set of measure less than ε such that it is the finite union of triangles and contains a unit line segment in every direction than we have proved Theorem 4.1.

Now we present the full original proof.

Lemma 4.4. *Let ABC be any triangle. There exists a set of measure less than ε which is the finite union of triangles and contains a unit line segment in every direction between CA and CB.*

Proof. We can suppose that A and B lie on the x-axis. Let ω be the length of the segment AB and $a = \lambda(ABC)$, and let h be the second coordinate of C. For any given n we can draw rays from C such that they divide the segment AB into n equal parts. In whatever way we (horizontally) translate the n new triangles, their union retains the property that it contains a unit line segment in every direction between the directions of CA and CB. Therefore, it is enough to show that for every k there exists n such that

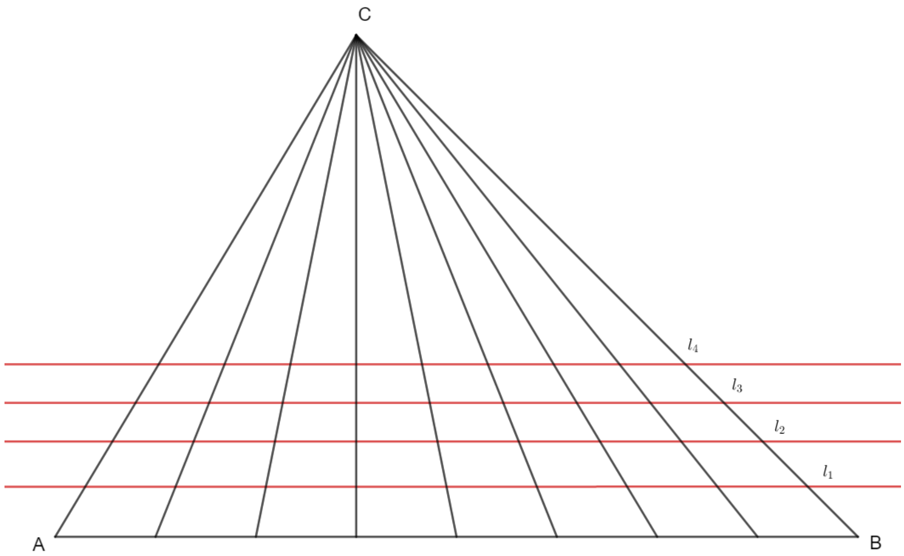

Figure 2: The case of n = 8 and L = 4.

if we cut ABC into n triangles as outlined then we can horizontally translate these triangles such that their union has area less then $\frac{a}{k}$.

For every $i \in \mathbb{N}$ draw line $l_i = \{y = h(1 - (\frac{2k-1}{2k})^i)\}$ (see Figure 2). Let L be such that $(\frac{2k-1}{2k})^{2L} < \frac{1}{2k}$. The reason for defining L this way is that the part of ABC above l_L has area less than $\frac{a}{2k}$, so we can ignore it (horizontal translations cannot increase the area above l_L).

We have broken up ABC into a triangle above l_L and L horizontal strips $S_1, \ldots, S_L$ each containing n trapezia. Note that S_i lies between l_{i-1} and l_i. Let $t_{i,j}$ denote the j-th trapezoid in the i-th strip. By similarity it is clear that the lower base of every trapezia in S_i has the same length, namely $a_i = \frac{\omega}{n}(\frac{2k-1}{2k})^i$.

Take m consecutive trapezia in i-th strip: $t_{i,j}, t_{i,j+1}, \ldots, t_{i,j+m-1}$. Take a trapezoid T in S_i such that its left side is parallel to the left side of $t_{i,j+m-1}$ and its right side is parallel to the right side of $t_{i,j}$. If we translate any of the m trapezia so that their lower base is in the lower base of T then the whole trapezoid will be in T. Therefore, we will take many consecutive trapezia and translate them into such a trapezoid (as we will see, this greater reduces their area). Naturally we need the lower base of T to be larger than a_i, denote their ratio by $r \geq 1$.

The top of T has length $ra_i + \frac{(m-2)a_i}{2k}$, this can be most easily seen by considering the fact that the rightmost point of $t_{i,j} \cap l_{i-1}$ is $(m-2)a_i$ from the leftmost point of $t_{i,j+m-1} \cap l_{i-1}$, while on l_i the same distance is $(m-2)a_{i+1} = (m-2)\frac{2k-1}{2k}a_i$, see Figure 3.

Therefore, the area of T is

$$\left(2r + \frac{(m-2)}{2k}\right) a_i \frac{h_i}{2},$$

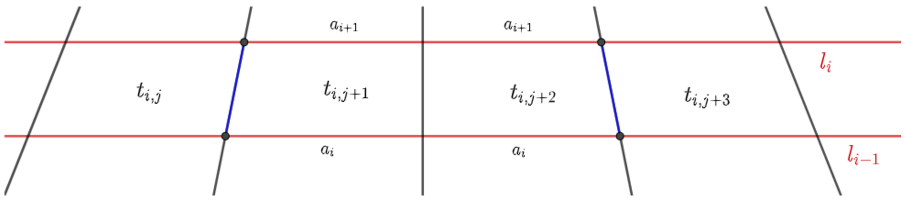

Figure 3: The case of m = 4.

where h_i is the height of S_i. While the sum of the areas of the m trapezia is

$$m\frac{4k-1}{2k}a_i\frac{h_i}{2}.$$

It is easy to see that for a fixed r, as m grows the ratio of these to numbers tends to $\frac{1}{4k-1}$. In fact, if

$$m > \frac{8k^2}{2k-1}r,$$

then the area of T will be less than $\frac{1}{2k}$ times the area of the trapezia.

The plan is the following: We will translate m_1 consecutive triangles (m_1 is to be chosen later), using the technique outlined above, so that their area in S_1 becomes small. This results in a set which is the union of triangles. "Glue" these triangles together, in the following steps these triangles will always be translated together. Group triangles so that each group contains m_1 consecutive triangles and perform this operation on each group. Now we get a few of these glued sets, which contain some consecutive triangles. In whatever way we horizontally translate these sets we cannot increase their area in S_1, so in the next step we only need to consider their area in S_2, and so on.

To achieve this, we will need to specify how many neighbouring sets we want to unite at every level, for the i-th level call this number m_i. Therefore, the number of triangles is $n = m_1 m_2 \dots m_L$ and in S_i we will construct sets containing $m_1 m_2 \dots m_i$ triangles.

Let $r_1 = 1$ and m_1 be greater than $\frac{8k^2}{2k-1}r_1$. In the first step group the triangles so that each group contains m_1 consecutive triangles. For each group translate every triangle in the group so that its base coincides with the base of the first triangle. We have constructed sets $A_1, A_2, \dots, A_{m_2 \dots m_L}$, now denote $T_i = A_i \cap S_i$. As we have seen, we now have

$$\lambda(\bigcup T_i) < \frac{1}{2k}\lambda((ABC) \cap S_1).$$

Since we do not have perfect control over A_i (and its intersection with l_1), we will need to increase the the interval into which we can pack the consecutive triangles. Let $r_2 = (r_1 + \frac{(m_1-2)}{2k})\frac{2k}{2k-1}$, this is the ratio between the length of the segment $A_i \cap l_1$ and a_2. Naturally let m_2 be such that $m_1 m_2 > \frac{8k^2}{k-1}r_2$ (we will group together $m_1 m_2$ consecutive triangles). Now group the sets A_i so that each group contains m_2 consecutive sets. In each group translate every element left so that the corresponding T_i trapezia have the same base on l_1. Now we get sets B_i containing $m_1 m_2$ triangles, and their intersections with l_1 is contained in a segment of length $r_2 a_1$. Therefore,

$$\lambda(\bigcup B_i \cap S_2) < \frac{1}{2k}\lambda((ABC) \cap S_2).$$

We can continue such steps, where the general formula in the i-th step is

$$r_i = \left(r_{i-1} + \frac{m_1 m_2 \dots m_{i-1} - 2}{2k}\right) \frac{2k}{2k-1},$$

$$\text{and } m_i \text{ is such that } m_1 m_2 \dots m_i > \frac{8k^2}{2k-1} r_i.$$

By induction, in the previous step we have constructed sets $R_1, R_2, \dots, R_{m_i m_{i+1} \dots m_L}$ for which

- We get R_j by translating $m_1 m_2 \dots m_{i-1}$ consecutive triangles.
- The ratio of the area of $\bigcup R_j$ and ABC under l_{i-1} is less than $\frac{1}{2k}$.
- $R_j \cap l_{i-1}$ is contained in a line segment of length $r_i a_i$, denote it by e_j.

Group R_j so that each group contains m_i consecutive elements. In each group translate every element left so that the corresponding e_j move to the same segment (leave the first element in place).

This way we have constructed $R'_1, R'_2, \dots, R'_{m_{i+1} m_{i+2} \dots m_L}$, all of which we got by translating $m_1 m_2 \dots m_i$ triangles. The intersection of all of these triangles with l_{i-1} is contained in a segment of length $r_i a_i$. As such their area is suitable by the choice of m_i. The last property follows as well.

After L steps we are left with a single set. This set has area less than $\frac{a}{2k}$ under the line l_L, the original triangle had area less than $\frac{a}{2k}$ above l_L therefore, this set has area less than $\frac{a}{k}$.

□

Remark 4.5. Note that in the above construction the base of every triangle stays in AB and we only use horizontal translations to the left.

Proof of Theorem 4.1. Draw a regular hexagon with an inscribed circle of radius 1 and partition it into 6 triangles using neighbouring vertices of the hexagon and the centre as vertices. By applying Lemma 4.4 to each triangle with $\frac{\varepsilon}{6}$ we get a set of measure less than ε which satisfies the conditions of Observation 4.3. □

Theorem 4.6. *There exists a Besicovitch set of Jordan measure zero.*

Proof relying on the details of the proof of Lemma 4.4. Similarly, we will only prove that for a triangle ABC of height 1 there is such a set containing a unit line segment in every direction between CA and CB. Let a be the area of ABC. Roughly speaking we will apply Lemma 4.4 to ABC, which results in a set of triangles. We will then apply the

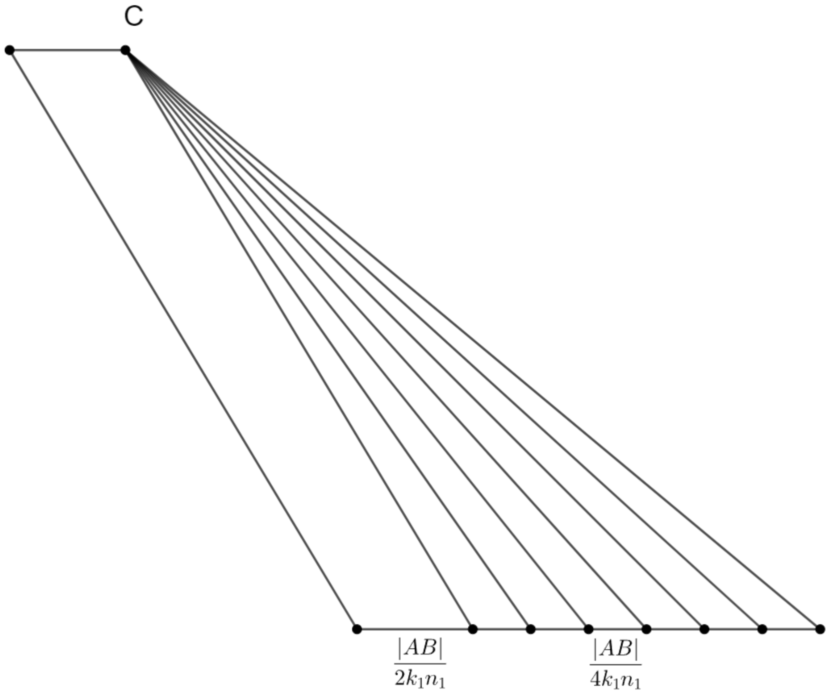

Figure 4: The operations on a triangle of T_1.

lemma again to these smaller triangles. Let D be the set of segments connecting a point of the segment AB to C. We can think about these operations as acting on D, always translating them left, while their lower endpoint remains in the segment AB (to avoid ambiguity we can assume that every segment which is the side of a triangle during an operation sticks to the triangle on its left). Therefore, every element of D has a limit position after completing infinite such operations.

Let $k_i = 2^i$, as in Lemma 4.4 these will denote the ratio by which we decrease the area of our triangles in each step.

Apply the proof of Lemma 4.4 to ABC with $k = k_1$, denote the resulting set by T_1. If we would just apply the lemma to every triangle we would have no upper bound to the size of the resulting set, since segments will be moved too far from their current position. Let n_1 be the number of triangles in T_1. For every triangle in T_1 extend it to the left by a parallelogram with a base of length $\frac{|AB|}{2k_1n_1}$, as seen on Figure 4. Call this enlarged set P_1. Easy calculation shows that the area of P_1 is less than $\frac{2a}{k_1}$. Cut every triangle in T_1 into triangles with equal base length less than $\frac{|AB|}{4k_1n_1}$ with C as their third vertex.

Now apply the lemma to these new triangles with $k = k_2$. Every segment moves at most $\frac{|AB|}{4k_1n_1}$ and therefore remains in P_1. Call the resulting set T_2 with n_2 triangles. Now again extend each triangle of T_2 with a parallelogram of base of $\frac{|AB|}{2k_2n_2}$. Call the resulting set P_2. The area of P_2 is less than $\frac{2a}{k_2}$. Now we need to cut every triangle in T_2 into smaller triangles with bases shorter than $\frac{|AB|}{4k_2n_2}$. If we apply the lemma again to these triangles then not only will every segment remain in P_2 they will also remain inside P_1.

Repeat this process of cutting the triangles of the previous step, then slightly enlarging the set and then cutting the triangles into even smaller ones.

In the i-th operation every segment will move at most $\frac{|AB|}{4k_{i-1}n_{i-1}}$. Therefore, after the i-th step it will move at most

$$\frac{|AB|}{4k_i n_i} + \frac{|AB|}{4k_{i+1}n_{i+1}} + \frac{|AB|}{4k_{i+2}n_{i+2}} + \dots < \frac{|AB|}{4n_i}(\frac{1}{k_i} + \frac{1}{2k_i} + \frac{1}{4k_i} + \dots) = \frac{|AB|}{2k_i n_i}.$$

Therefore, the limit position of every segment lies in P_i for all i. Since the Jordan measure of P_i is less than $\frac{2a}{2k_i}$, the Jordan measure of the union of the limit positions must be zero. $\square$

Originally Besicovitch used Theorem 4.6 to answer a seemingly unrelated question. Consider a Riemann integrable function f on $[0,1]^2$. If for example f is continuous then

$$\int_{[0,1]^2} f dA = \int_{[0,1]} \left(\int_{[0,1]} f(x,y)dx \right) dy.$$

This of course does not hold for general Riemann integrable functions as $\int_{[0,1]} f(x,y)dx$ may not exist for all y. A natural question is whether this is the fault of our coordinate system, can we always choose a coordinate system in which this Fubini-like statement holds?

Corollary 4.7. *There exists a Riemann integrable function f such that its repeated integral is undefined in every coordinate system.*

Proof. Since the set given in Theorem 4.6 is Jordan measurable it is also bounded. As such there exits an $n \in \mathbb{N}$ such that the set can be shrunk into $[0,1]^2$ and it contains a line segment of length $\frac{1}{n}$ in every direction. Using translations we can also assume that there are vertical and horizontal segments in our set which are at an irrational distance from the corresponding axes. Let this modified set be E. Let $E_0 = E \cap ((\mathbb{Q} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Q}))$ and let f be the characteristic function of E_0. Since E has Jordan measure zero f is Riemann integrable. Suppose that in our new coordinate system the x-axis is in direction v. There exists a segment e in E parallel to v. Since f is equal to 1 on a dense subset of e and 0 on a dense subset, the repeated integral is undefined (for a given y we are asked to integrate f on e). $\square$

The proof of Theorem 4.6 constructed a compact Besicovitch set of measure zero. As an intermediate step we constructed a continuous rotation of the unit line segment during which it sweeps an area less than ε. Now we will present the other direction, given a compact Besicovitch set of measure zero, we will rotate the unit line segment in an arbitrarily small area. This shows a strong connection between the two problems, a Besicovitch set can be thought of as a set in which the unit line segment can be fully rotated (reaching all directions) in a discontinuous manner.

Proof of Theorem 4.1 supposing Theorem 4.6. The closure of the set given in Theorem 4.6 is a compact Besicovitch set of measure zero, call it A. Therefore, for every ε there exists a δ such that $\lambda(A + B(\delta)) < \varepsilon$. For every unit line segment e contained in A the set $e + B(\delta) \subset A + B(\delta)$ contains a triangle containing e. By Observation 4.3 we are done.

□

Now we will give a different proof to Lemma 4.4, which is a yet unpublished result of A. Gáspár.

Lemma 4.8. *Let ABC be a triangle with base length w. As before, ABC can be cut, with lines through C, into 2^n smaller triangles so that their bases have length $\frac{w}{2^n}$. These triangles can be horizontally translated so that their union U has the following properties:*

1. *The intersection of U with the segment AB is contained in a line segment of length $\frac{w}{n+1}$.*
2. *The intersection of U with any horizontal line segment has total length less than $\frac{w}{n+1}$.*

Proof. We will proceed with induction on n, going from $n-1$ to n. The case of $n = 0$ is trivial. Using affine transformations, we can assume $A = (-n-1, -1)$, $B = (n+1, -1)$, and $C = (0, n)$, therefore $w = 2(n+1)$. Also denote the points $A' = (-n, 0)$, $B' = (n, 0)$, and $O = (0, 0)$. We can now apply the inductive hypothesis to $A'OC$ and $B'OC$, giving us sets U_1' and U_2' respectively. The intersection of U_1' and any horizontal line segment has length less than $\frac{n}{n-1+1} = 1 = \frac{w}{2(n+1)}$, similarly for U_2'. This means that if we translate U_1' and U_2' horizontally then their union will always intersect every horizontal line in a set of length less than $\frac{w}{n+1}$.

The cutting and translating of $A'OC$ and $B'OC$ naturally induces cuts and a set of translations on ABC, which cuts the triangle into 2^n triangles. Call the resulting sets U_1 and U_2. From the above argument we only need to pay attention to the strip $S = \{y \in [-1, 0]\}$. Let the intersection of U_1 and the x-axis be contained in a segment e_1, from the hypothesis we know that the length of e_1 is less than 1, we get e_2 for U_2 similarly. Translate U_1 by vector v_1 so that e_1 becomes a part of $[0, 1]$ on the x-axis, and translate U_2 by v_2 so that e_2 becomes a part of $[-1, 0]$ (note that they "switched sides"). Since U_1 is the union of segments with slope at least 1, $(v_1 + U_1) \cap S$ is contained in the trapezoid defined by the points $(-1, -1), (1, -1), (1, 0), (0, 0)$. Similarly $(v_1 + U_1) \cap S$ is contained in the trapezoid defined by the points $(-1, 0), (0, 0), (-1, -1), (1, -1)$. Therefore, it is easy to verify that $(U_1 + v_1) \cup (U_2 + v_2)$ satisfies both conditions. □

We now present results about whether there are sets in which a unit line segment can be rotated while possessing some additional property. These results are due to Cunningham [3].

The use of Pál joins in the previous section results in the diameter of the sets we construct rapidly increasing as ε tends to zero, and their fundamental group also becomes large. It is natural to ask if the area scraped can tend to zero if we have an absolute bound on the diameter of the sets or a bound on their fundamental group.

It is easy to verify that a unit line segment cannot be rotated inside a circle of radius less than 1 in a way such that its swept area is arbitrarily small. Thus, the following theorem is the best we can hope for.

Theorem 4.9 (Cunningham)**.** *For every ε there exists a simply connected subset of the unit circle of measure less than ε in which the unit line segment can be rotated.*

Call a set A star-shaped if it has a point O such that for every $P \in A$ the segment OP is contained in A.

Theorem 4.10 (Cunningham)**.** *Let A be a star-shaped set in which the unit line segment can be rotated. Then A has measure at least $\frac{\pi}{108}$.*

Remark 4.11. It is not known whether this lower bound is optimal.

5 Kakeya property

Recall Definition 3.2 and Definition 3.3 for the definition of the Kakeya property and the strong Kakeya property respectively. We will often refer to the Kakeya property as property (K) and the strong Kakeya property as property (K^S)

This section will contain fewer proofs, its main purpose is to build the readers intuition for the Kakeya property. First, I would like to present an argument by Davies [11], showing that the union of two non-parallel line segments does not possess the strong Kakeya property. This argument gives a strong intuition about which sets can have the Kakeya property. We will later see that this intuition does in fact hold for nice enough sets but unfortunately fails in the general case.

Theorem 5.1 (Davies)**.** *The union of two non-parallel line segments cannot be fully rotated in an arbitrarily small area.*

Argument. Call the segments A_1A_2 and B_1B_2, and their union E. Suppose that for every ε there exists a motion M_ε, that rotates E in area less than ε. Let the intersection of lines of A_1A_2 and B_1B_2 be C. Without loss of generality we can suppose that C is disjoint from the segments (we can decrease E if necessary). Now take a very small disk D centred at C. If C does not leave D during a rotation then E must sweep a complete ring almost the the width of A_1A_2. Therefore, for small enough ε, during the motion M_ε, the point C must leave D. Take the first moment this happens and let the new position of E be $E' = A_1'A_2' \cup B_1'B_2'$. Depending on the radius of D, there exists an ε_0 such that either the area of the quadrilateral $A_1A_2A_1'A_2'$ or of $B_1B_2B_1'B_2'$ is greater than ε_0. If D is small enough then E must sweep almost all of these quadrilaterals, and we get a lower bound on the area swept up to this moment.

□

To more deeply understand this argument, we must define free movements.

Definition 5.2. We say that an $\alpha \in \mathrm{Isom}^+(\mathbb{R}^2)$ is a *free movement* of a set E if during the motion M^α the set E sweeps zero area.

Sets with property (K) are almost characterized in [8] by Marianna Csörnyei, Kornélia Héra, Miklós Laczkovich. We now present some of their results.

If a set has a free motion, then it trivially possesses the Kakeya property. In particular, it is easy to see that if a set can be covered by a nullset consisting of concentric circles or parallel lines then it has a free motion and thus has property (K), we will call these trivial (K)-sets.

The main idea of the above argument can be generalized the following way: Let E be a set with no free movement. Take the first moment t_0 when the motion $M(t)$ leaves the ε neighbourhood of the identity. Let $\alpha = M(t_0)$. The movement up to this moment can be approximated by a rotation or a translation, namely M^α. If E has no free movements then it must sweep a positive area during M^α. If ε is small and E is nice enough then E must also sweep most of this area during M. This suggests that if a nice enough set has no free motions then it cannot have the Kakeya property.

Unfortunately this intuition does not hold for all sets.

Remark 5.3 (Csörnyei, Héra, Laczkovich [8]). There exists a compact set which possesses property (K), but is not a trivial (K)-set.

Sketch of proof. If the set A has $\lambda(p_v(A)) > 0$ for all directions v, where p_v denotes the projection in direction v, and its distance set

$$D(A;p) = \{|x-p| \in \mathbb{R} : x \in A\}$$

has positive measure for every point p then A is not a trivial (K)-set. Let e_n be a sequence of distinct directions with limit e. By Talagrand's theorem detailed in [10] we can construct A such that $\lambda(p_{e_n}(A)) = \frac{1}{n}$ and $\lambda(p_v(A)) > 0$ for all directions v. This means that A cannot be covered by a nullset of parallel lines. Guaranteeing that the distance set has positive measure requires more exact details of the construction but is not difficult.

The set A has the (K) property, since it can be moved to its translated copy in direction e in an arbitrarily small area. We can achieve this by translating it by the same amount in direction e_n, sweeping an arbitrarily small area if n is large enough, and finishing the movement with a small translation.

$\square$

Nonetheless the previous intuition yields true results in the case of highly connected sets.

Theorem 5.4 (Csörnyei, Héra, Laczkovich [8]). *Let $A \subset \mathbb{R}^2$ be a closed set having property (K) such that all of its connected components contain at least 2 points. Then A is a trivial (K)-set. In particular if A is connected then it is a line segment, a half line, a line, a circular arc, a circle or a singleton.*

If a set has the property (K), then an isometry naturally belongs to this set from the definition. The definition does not guarantee anything about the motions that sweep small area while arriving at this position. The following lemma shows that we can "stay close" to a fixed isometry, while the area swept tends to zero. Recall that for an $\alpha \in \mathrm{Isom}^+(\mathbb{R}^2)$ the natural motion induced by α is denoted by $M^\alpha(t)$, see Section 2.

Lemma 5.5. *If $E \subset \mathbb{R}^2$ has property (K), then there exists an $\alpha \in \mathrm{Isom}^+(\mathbb{R}^2)$ such that α^2 is not the identity, and the following condition is satisfied. For every $\varepsilon > 0$ there is a continuous motion M such that $M(1) = \alpha$, the area swept by E during M is less than ε, and $|M(t) - M^\alpha(t)| < \varepsilon$ for every $t \in [0,1]$.*

The proof is technical, and as such it is omitted.

We will prove a special case of theorem 5.4, where each connected component of A is a curve. The more general case requires statements of topological nature, and do not add much to our intuition.

Theorem 5.6. *Let $E \subset \mathbb{R}^2$ be a closed set having property (K) such that all of its connected components are curves. Then E is a trivial (K)-set. In particular if E is a curve then it is a line segment, a half line, a line, a circular arc, a circle or a singleton.*

(Proof based on the original proof of Theorem 5.4). Let E_1 be a connected component of E. Lemma 5.5 yields an isometry α satisfying the conditions of the lemma. We will prove that if α is a translation in direction v then every connected component of E can covered by a line parallel to v. We can suppose v is horizontal.

Suppose $\{e \cdot v^{\perp} | e \in E_1\}$ has two distinct elements $a_1 < a_2$. We can reduce the curve E_1 so that $\{e \cdot v^{\perp} | e \in E_1\}$ achieves its unique extrema at the endpoints, and it becomes a simple curve. Let B_1 be the vertical ray upward from the upper endpoint of E_1 and similarly B_2 be the vertical downward ray from the lower endpoint of E_1. Let B be the union of B_1 and B_2. Take $b_1, b_2 \in (a_1, a_2)$. If the motion M is close enough to M^{α} then the strip $S = \{(x, y) | (x, y) \cdot v^{\perp} \in [y_1, y_2]\}$ stays disjoint from B. It is easy to see that $E_1 \cup B$ cuts the plane into two connected components. The subset of the strip S which changes components between $M_0(E_1 \cup B)$ and $M_1(E_1 \cup B)$ has area $(y_2 - y_1)||\alpha||$. All of these points must be swept by E_1 during M.

A similar argument proves that if α is a rotation around c then every connected of A can be covered by a circle centred at c.

This completes the proof as every $\alpha \in \operatorname{Isom}^+ \mathbb{R}^2$ is either a translation or a rotation. $\square$

6 Strong Kakeya property

6.1 Classical results

We have seen in the previous section that there are quite few natural sets with the Kakeya property, and as such we only need to check a few cases to characterize sets with the strong Kakeya property (K^S). In section 4 we already saw that a single line segment has property (K^S). In [11] R. Davies proved a more general result.

Theorem 6.1 (Davies)**.** *A finite union of parallel line segments has property* (K^S)*.*

We will present the full original proof of this theorem as it uses a wide range of classical techniques.

Lemma 6.2. *For every rectangle* R *and* $\delta > 0$ *there exists a closed set of arbitrarily small measure which contains a translate of every segment in* R *at most* δ *distance from the original segment.*

Proof. Such a set can be easily obtained from Lemma 4.4 and Remark 4.5. □

Definition 6.3. Call a set an *n-set* if it is the union of n line segments. Note that they do not have to be parallel.

Lemma 6.4. *Let* F *be a compact set and* G *be an open neighbourhood of* F*. Given an integer* n *there exists a compact set* $F' \subset G$ *of measure less than* ε *such that* F' *contains a translate of every* n*-set in* F*.*

Proof. Let

$$\mathbb{R}^4 \supset S(F) = \{(a,b) | a, b \in F \subset \mathbb{R}^2, \text{the segment between } a \text{ and } b \text{ is contained in } F\}.$$

It is easy to verify that $S(F)$ is a compact set. For a closed rectangle $R \subset G$ the set $S(int(R))$ is an open set and sets of this form form an open covering of $S(F)$. Therefore, by the compactness of $S(F)$ we can select a finite number of rectangles

$$R_1, \ldots, R_N \subset G$$

such that every segment in F is also contained in one of the rectangles. Choose closed rectangles R_i' such that $R_i \subset \text{int}(R_i') \subset G$, where $\text{int}(R_i')$ denotes the interior of R_i'.

Denote by $S_1, \ldots, S_M$ the sequence we get by listing every rectangle R_i' n times ($M = nN$). For every S_k we will recursively define a closed set F_k and an open set G_k such that $F_k \subset G_k \subset \overline{G_k} \subset S_k$ and $F' = \cup_k \overline{G_k}$ will be suitable.

Let δ_0 be such that $\cup R_i + B(\delta_0) \subset \cup R_i'$ and let $\delta_1 = \frac{\delta_0}{n}$. Recursively define

$$\delta_k = \min\left\{\delta_1, \frac{d_1}{n}, \frac{d_2}{n}, \ldots, \frac{d_{k-1}}{n}\right\},$$

where d_i is the distance of F_i and the complement of G_i. For each S_k let F_k be a set given by Lemma 6.2 for rectangle S_k, $\delta = \delta_k$, and measure less than $\frac{\varepsilon}{2M}$ and G_k be an open neighbourhood of F_k such that its closure has measure less than $\frac{\varepsilon}{M}$.

We will show that $F' = \cup_k \overline{G_k}$ is suitable. It is clear that it has measure less then ε. Let E be an n-set in F. By the definition of $R_1, \ldots, R_N$ there exists a list, of length n, consisting of distinct indices j_i such that T_i, the i-th segment, is contained in the rectangle corresponding to S_{j_i}.

Since $T_1 \subset S_{j_1}$ there exists a translation ϕ_1 which is shorter then δ_{j_1} and translates T_1 in to F_{j_1}. Similarly if we have defined translations $\phi_1, \phi_2, \ldots, \phi_{k-1}$, where ϕ_i is shorter than δ_{j_i}, then $\phi_1\phi_2\ldots\phi_{k-1}T_k \subset S_{j_i}$, since $\phi_1\phi_2\ldots\phi_{k-1}$ is shorter than

$$\sum_{i=1}^{k-1} \delta_{j_i} < n\delta_1 \leq \delta_0.$$

Therefore, there exists a translation ϕ_k shorter than δ_{k_j} that moves $\phi_1\phi_2\ldots\phi_{k-1}T_k$ into F_k.

Our final translation will be $\phi_1\phi_2\ldots\phi_n$. We only need to prove that for every k the segment $\phi_1\phi_2\ldots\phi_nT_k$ is within $\overline{G_k}$. This follows from the fact that the translation $\phi_{k+1}\phi_{k+2}\ldots\phi_n$ is shorter than

$$\sum_{i=k+1}^{n} \delta_{j_i} \leq (n-k)\frac{\delta_{k_j}}{n} < \delta_{k_j}$$

. □

Theorem 6.5 (Davies). *Given an integer n there exists a compact set F of measure 0 such that F contains a translate of every n-set with diameter less than n.*

Proof. Let F_0 be a closed set containing all n-sets of diameter less than n, for example a large enough disk. Let G_0 be a bounded neighbourhood of F_0. Now by repeatedly applying Lemma 6.4 we get $F_i \subset G_{i-1}$ such that F_i still contains every n-set of diameter less than n and $\lambda(F_i) < \frac{1}{i}$ and we can choose G_i such that $F_i \subset \overline{G_i} \subset G_{i-1}$ and $\lambda(\overline{G_i}) < \frac{1}{i}$. It is easy to check that $F = \bigcap_i^\infty \overline{G_i}$ is suitable. □

Corollary 6.6. *There exists an F_σ subset of the plane which contains a translate of every polygonal arc.*

Proof. Apply Theorem 6.5 for every n and take the union of the sets given by the theorem. Every polygonal arc is an n-set for some n. □

Proof of Theorem 6.1. Let E be the finite union of parallel line segments. Similarly to the proof of Theorem 4.1, we can translate E in an arbitrarily small area using the same idea as in Observation 4.2. Therefore, it is enough to show that E can be rotated between any of its two orientations in an arbitrarily small area. We will show that it can be fully rotated in an arbitrarily small area.

By Theorem 6.5 there exists a compact null set which contains E in every direction. Therefore, there exists and open set G of arbitrarily small measure which contains E in every direction. For any position of E in G we can rotate it a small amount in both directions. This defines open intervals on S^1, which form an open cover, select a finite cover. This corresponds to a finite number of positions of E, choose one of them as the staring position. We can always rotate E until its direction reaches the interval corresponding to the next position and we can translate it there. □

6.2 Recent developments

A full circle does not possess property (K^S) as to reach a disjoint position it needs to sweep every inner point.

The question whether all incomplete circular arcs possess property (K^S) was first asked by Cunningham in [4] but remains an open question to date. Progress has been made on smaller circular arcs. In [7] Kornélia Héra and Miklós Laczkovich proved that circular arcs of radius 1 and length shorter than 1.32 do in fact have property (K^S).

The most recent results, including the strongest result on Cunningham's question, in the topic were achieved by Alan Chang and Marianna Csörnyei in [1]. We will mostly just present the results of this article.

Theorem 6.7 (Chang-Csörnyei [1]). *For every $\varepsilon > 0$ the set $\{x^2 + y^2 = 1\colon |x| > \varepsilon\}$ has property (K^S).*

Let $\mathcal{H}^1$ denote the one-dimensional Hausdorff measure.

Every rectifiable set in $\mathbb{R}^2$ has a tangent field that is defined $\mathcal{H}^1$ almost everywhere. To state some theorems, we will define it at every point, but it turns out that the choice does not affect the results. Denote the tangent direction at x by θ_x and denote the normal line by ν_x.

Theorem 6.7 easily follows from the next theorem.

Theorem 6.8 (Chang-Csörnyei [1]). *Let $E \subset \mathbb{R}^2$ be a rectifiable set of finite $\mathcal{H}^1$ measure. Then for every ε and point g there exists a polygonal path $\mathbb{R}^2 \supset P = \cup_i^n L_i$ from the origin to g and directions θ_i, such that*

$$\lambda(\bigcup_i \bigcup_{p \in L_i} (p + \{x \in E | \theta_x \notin B(\theta_i, \varepsilon)\})) \leq \varepsilon.$$

That is, for a fixed end position we can choose a finite number of translations and during every translation we can chose a direction and ignore points of E whose tangents are close to that direction in such a way that the remaining points sweep small area.

Theorem 6.8 applies to parallel line segments as well, but the statement is empty since we can delete the whole set during each translation. It is also easy to see that translations are not enough to achieve any surprising result about line segments, we need to include rotations. To unify translations and rotations we consider translations as rotations around a point at "infinity", as such every translation and rotation has a projective centre in the projective plane, $\mathbb{P}^2$. The polygonal path in Theorem 6.8 is a series of direction vectors and corresponding lengths, the natural analogue for rotations is a series of centres and angles of rotation.

An additional ambiguity arises when dealing with rotations: when we perform a rotation the centre of all other rotations move. To avoid this problem we will always specify our rotations as compared to our set. This means that given rotations r_1 and r_2 around z_1 and z_2 we will perform r_1 and the next rotation will be performed around $r_1(z_2)$. Therefore, a polygonal path now means a path in $\mathrm{Isom}^+(\mathbb{R}^2)$ such that it is the concatenation of a finite number of "line segments", where each line segment is of the form M^α.

Theorem 6.9 (Chang-Csörnyei [1]). *Let E be a bounded rectifiable set of finite $\mathcal{H}^1$ measure. Let $\varepsilon > 0$ and $p \in \mathrm{Isom}^+(\mathbb{R}^2)$ be arbitrary and let $l \subset \mathbb{P}^2$ be a line through the projective centre of p. Then there is a polygonal path $\cup_{i=1}^n L_i$ such that it connects the identity to p and the projective centres of L_i lie within the ε neighbourhood of l and we can choose $u_i \in l$ in a way such that*

$$\lambda(\bigcup_i \bigcup_{r \in L_i} r(\{x | x \in E, \nu_x \cap l \cap B(u_i, \varepsilon) = \emptyset\})) < \varepsilon.$$

This is almost a generalization of Theorem 6.8, since if p is a translation, then we can choose l to be the ideal line. We get a very similar statement to that of Theorem 6.8, the difference being in whether the projective centres lie on the ideal line (and are therefore translations) or in the ε neighbourhood of the ideal line.

If E is a line segment, then at every moment we delete only a small segment form E, since ν_x has the same direction for all $x \in E$.

Corollary 6.10. *If E is a bounded union of parallel line segments which have finite total length, then E has property (K^S).*

Proof. Let $E' = E_0 \cup E_1$ be the union of two copies of E that are far away from each other and can be translated into each other while sweeping zero area. Now apply Theorem 6.9 to E'. This gives a polygonal path during which E' sweeps small area, but during every segment of the path we delete some part of E'. It is easy to see that the deleted points must either lie in E_0 or E_1 entirely. Therefore, between each subsequent segments of the path we can trivially translate E into a copy such that its points do not get deleted during the next segment. □

This result generalizes Theorem 6.1. The other notable aspect of this result is that it applies to sets which have closures of positive measure.

We get new results if we consider what happens when $\varepsilon \to 0$ in Theorem 6.9. The case of a convex function is interesting since we can choose l to be "under" the graph and then $\nu_x \cap l$ defines x.

Theorem 6.11 (Chang-Csörnyei [1]). *Let E be a rectifiable set. We can rotate E by $360°$ while sweeping a set of zero area, if at every moment we are allowed to delete a $\mathcal{H}^1$-null subset of E. In the special case when E is the graph of a convex function, we only need to delete a single point at every moment.*

Unfortunately the choice of deleted point cannot be always continuous.

If we choose E to be a line then the set $\cup_t(E_t \setminus x_t)$ is Lebesgue null and contains a line segment in every direction, therefore it is a Besicovitch set.

7 Rotating a square

Theorem 6.1 states that finitely many parallel line segments can be rotated in an arbitrarily small area. This is of course impossible if we consider a union of parallel line segments which has positive measure, but we can ask whether there exists a rotating motion such that every segment sweeps small area. The most natural case is when we consider the vertical lines in a square. The following results are due to the author [9].

Theorem 7.1. *For every $\varepsilon > 0$ there exists a continuous motion of the unit square during which every initially vertical line segment sweeps at most ε area, while the square does a full rotation.*

This result easily implies Theorem 6.1. This section will deal with the proof of Theorem 7.1. We will see applications of this theorem in Section 8.

Let V be the set of planes, which have a $45°$ angle with the $\{y = 0\}$ plane. In Section 7.1 we will prove the following lemma.

Lemma 7.2. *There exists a closed set A in $\mathbb{R}^3$, which is the union of planes in V at a bounded distance from the origin, contains a translate of every plane in V, and intersects every vertical line in a set of measure zero.*

The proof relies on a duality argument and utilizes the construction of Talagrand [10], for an English description see Appendix A. in [13].

Lemma 7.2 is closely related to the question of rotating a square. The connection can be seen if we think of the plane as the $\{y = 0\}$ plane in $\mathbb{R}^3$ and raise every vertical segment of the unit square by its distance from the origin. During the motion of the unit square, the motion of each initially vertical segment corresponds to the motion of the raised rectangle at a given height.

Thus Lemma 7.2 can be thought of as giving a discontinuous motion of the unit square, during which it achieves every direction. In Section 7.2 we turn this motion into a continuous one and prove Theorem 7.1.

7.1 Planes of certain angle, proof of Lemma 7.2

This section will exclusively deal with the proof of Lemma 7.2.

Let $a \cdot b$ denote the standard dot product of a and b. For a vector v let

$$p_v(X) = \{x \cdot v \mid x \in X\}.$$

It is clear that it is sufficient to construct a set fulfilling the conditions, with the exception that it only contains planes in V, whose directions form an interval.

We will use a duality argument and encode planes with points of $\mathbb{R}^3$. Similar encodings are often used in planar cases, see for examples [12],[6]. To the point (a, b, c) assign the plane, which contains $(0, 0, a)$, has slope b in the $y = 0$ plane, slope c in the $x = 0$ plane. It is easy to see that a point (x, y, z) is on the plane corresponding to (a, b, c) if and only if $a + bx + cy = z$. Through straightforward calculation a plane is in V when its triple has $c^2 - b^2 = 1$.

7.1.1 Requirements for our codeset and some basic observations

Let K be a set of triples, and A the union of the corresponding planes. If K is compact, then A is closed and every plane is at a bounded distance from the origin. The point (x, y, z) is in A if K contains a triple (a, b, c), which satisfies $a + bx + cy = z$, meaning K has a point in this plane. The set $(x, y, -) \cap A$ is $p_{(1,x,y)}(K)$, hence their measure is equal. Hence if K is such that for all vectors v, of the form $(1, x, y)$ we have $\lambda(p_v(K)) = 0$, then A intersects every vertical line in a set of measure 0. For the directions of the planes in A to form an interval, we need $p_{(0,1,0)}(K)$ to be an interval.

Let the 3 coordinates of the space containing K be a, b, c. By the above argument in order to prove Lemma 7.2, it is enough to prove the following lemma.

Let $H = \{(a, b, c)|c^2 - b^2 = 1, c > 0\}$.

Lemma 7.3. *There exists a compact set $K \subset H$ such that $p_{(0,1,0)}(K)$ is an interval and $\lambda(p_{(1,x,y)})(K) = 0$ for all x, y.*

The remainder of this section deals with the proof of Lemma 7.3.

7.1.2 Preliminaries for constructing K

Let $f : \mathbb{R}^2 \to H$, $f(a, b) = (a, b, \sqrt{1 + b^2})$.

We will construct $K' \subset \mathbb{R}^2$ such that $K = f(K')$ has the desired properties.

For any x, y, s we have $s \in p_{(1,x,y)}(K)$ if and only if K has a point r, for which

$$r \cdot (1, x, y) = s.$$

Such points r on H form a curve:

$$c_{x,y,s} = \{(s - x \sinh(t) - y \cosh(t), \sinh(t), \cosh(t)) : t \in \mathbb{R}\}.$$

Define

$$C_{x,y,s} = \{(s - x \sinh(t) - y \cosh(t), \sinh(t)) : t \in \mathbb{R}\}.$$

Then

$$s \in p_{(1,x,y)}(K) \iff (K \cap c_{x,y,s} \neq \varnothing) \iff (K' \cap C_{x,y,s} \neq \varnothing).$$

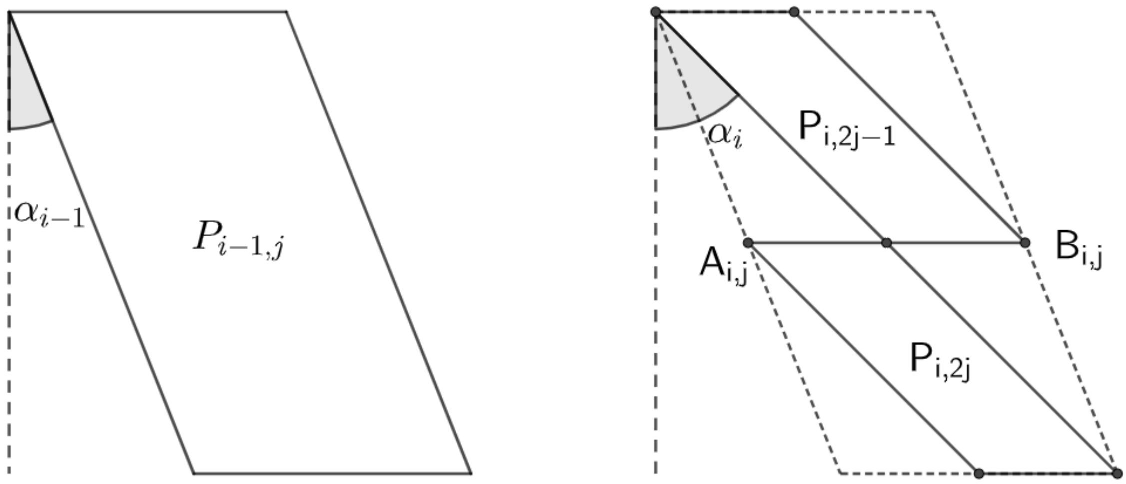

Figure 5: From $P_{i-1,j}$ to $P_{i,2j-1}$ and $P_{i,2j}$.

For each x, y we define a function $\alpha_{x,y}$ on the plane: To calculate $\alpha_{x,y}(a, b)$, take the curve $C_{x,y,s}$ through (a, b) and take the signed angle between the tangent of this curve at (a, b) and the b axis. Observe that $\alpha_{x,y}$ is never $\pm 90°$. The function $\tan(\alpha_{(x,y)}(a, b))$ is continuous as a function of x, y, a, b. Thus, it is uniformly continuous on a compact set, and so

$$\begin{aligned} &\forall G \forall \varepsilon \exists \delta \, \forall r_1, r_2 \in [0,1]^2, \ |x|, |y| \leq G, \ |r_1 - r_2| < \delta \\ &\implies |\tan(\alpha_{x,y}(r_1)) - \tan(\alpha_{x,y}(r_2))| < \epsilon. \end{aligned} \tag{7.1}$$

7.1.3 Construction of K'

We will prove that the set constructed by Talagrand [10] (see also in [13]) is a suitable set. For the sake of completeness, we repeat the construction.

By induction, we shall construct a decreasing sequence (K'_m) of finite unions of closed rectangles such that $K' = \cap_{i=1}^{\infty} K'_m$ has the desired properties. To begin with let $\varepsilon_1 = 1$ and $K'_1 = [0, 1]^2$. We now suppose that $m \in \mathbb{N}, 0 < \varepsilon_m \leq 1/m$, and that K'_m is a union of rectangles

$$R_n = [a_n, a_n + \frac{\varepsilon_m}{N}] \times \left[\frac{n-1}{N}, \frac{n}{N}\right], \quad 1 \leq n \leq N,$$

where $N \in \mathbb{N}^+$, $a_1, \ldots, a_N \in \mathbb{R}$. We will now construct K'_{m+1}.

We fix $1 \leq n \leq N$, define $P_{0,1} = R_n$, and choose $k_m \in \mathbb{N}$ such that $k_m \geq \frac{2m}{\varepsilon_m}$. Starting with $P_{0,1}$, we construct parallelograms $P_{i,j}, 1 \leq i \leq k, 1 \leq j \leq 2^i$. We take the midpoint of every side of $P_{i-1,j}$, let the midpoint of the left side be $A_{i,j}$, and the midpoint of the right side $B_{i,j}$. Take the midpoint of $A_{i,j}B_{i,j}$ and let the parallelograms $P_{i,2j-1}$, $P_{i,2j}$, and the angle α_i be as indicated by Fig. 5.

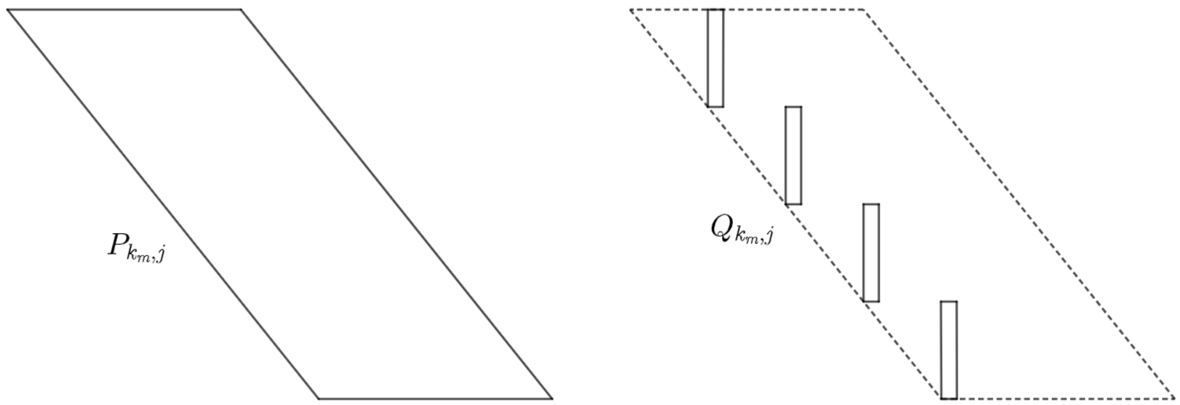

Figure 6: From $P_{k_m,j}$ to $Q_{k_m,j}$.

It follows by induction from the construction that the two intervals $p_{(0,1)}(P_{i,2j-1})$ and $p_{(0,1)}(P_{i,2j})$ have length $2^{-i}/N$ and that $|A_{i,j} - B_{i,j}| = 2^{-i}\varepsilon_m/N$. Therefore, in the last step also using that $k_m \geq \frac{2m}{\varepsilon_m}$, we obtain

$$\begin{aligned} \tan\alpha_i &= \frac{\varepsilon_m}{2} + \tan\alpha_{i-1}, \\ \implies \tan\alpha_i &= \frac{i\varepsilon_m}{2}, \\ \implies \tan\alpha_{k_m} &\geq m. \end{aligned} \tag{7.2}$$

Next we shall replace each $P_{k_m,j}$ by a union of rectangles $Q_{k_m,j}$ (see Fig. 6) such that $p_{(0,1)}(Q_{k_m,j}) = p_{(0,1)}(P_{k_m,j})$ and their union $T(K'_m)$ satisfies $\lambda(p_{(1,0)}(T(K'_m))) \leq \frac{1}{m+1}$. To that end we choose a suitable multiple N' of $2^{k_m}N$, an $\varepsilon_{m+1} \in (0, \frac{1}{m+1}))$, which is sufficiently small, and replace each of the parallelograms $P_{k_m,j}$, $1 \leq j \leq 2^{k_m}$, by a subset $Q_{k_m,j}$, which is a union of rectangles of the form $[t,\ t + \varepsilon_{m+1}/N'] \times [u,\ u + 1/N']$ as indicated by Fig. 2. Using (7.1) we can choose N' to be large enough, such that the oscillation of $\tan(\alpha_{x,y})$ is less than $\frac{\varepsilon_{m+1}}{2}$ in each rectangle, whenever $x, y < m$.

If m is odd, let $K'_{m+1} = T(K'_m)$. If m is even, we take $K'_{m+1} = S(T(S(K'_m)))$, where S denotes the reflection about the line $\{b = 1/2\}$.

7.1.4 Showing that K is suitable

Recall that $K = f(K')$. It is clear that $K \subset H$ and $p_{(0,1,0)}(K)$ is an interval, so it remains be proved that $\lambda(p_{(1,x,y)}(K)) = 0$ for any x, y. We now fix x, y. Let $F \subset \mathbb{R}^2$ be the set on which $\alpha_{x,y}$ is non-negative. It is easy to verify that F is a half-plane with a defining line parallel to the a-axis, or the whole plane or possibly the empty set. We prove that

$$\lambda(p_{(1,x,y)}(K \cap f(F)) = 0. \tag{7.3}$$

The reasoning is similar if $\alpha_{x,y}$ is negative, in that case we need to use that for even m we have $K'_{m+1} = S(T(S(K'_m)))$. Thus (7.3) implies that K is a suitable set.

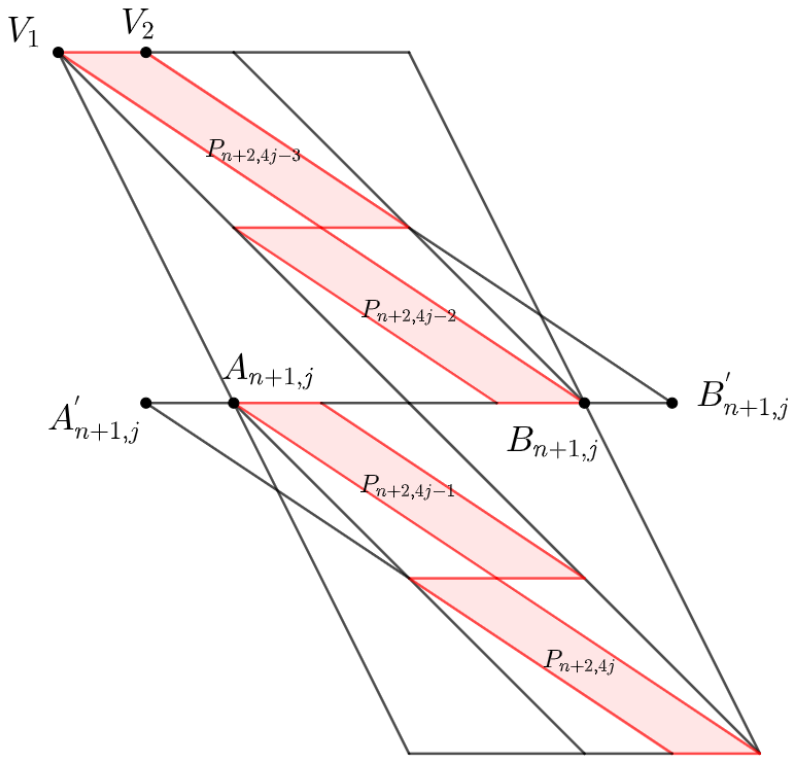

Figure 7: Two generations of parallelograms

The following claim clearly implies (7.3), therefore it will complete the proof of Lemma 7.3 and thus the proof of Lemma 7.2.

Claim 7.4. *If m is large enough and even, then*

$$\lambda(p_{(1,x,y)}(f(K'_m \cap F)) \leq \frac{3\sqrt{1+x^2+y^2}}{2(m-1)}.$$

Proof. Let M be the maximum of $\tan(\alpha_{x,y})$ on $[0,1]^2$. We can choose m to be large enough, such that $x, y < m$ and $n+2 < k_m$ for every $n < \frac{2M}{\varepsilon_{m-1}}$, since $k_m > \frac{2m}{\varepsilon_{m-1}}$. The set $(K'_{m-1} \cap F)$ is a union of rectangles, this follows from the observation made on the possible shapes of F. Let R be one of these rectangles. The oscillation of $\tan(\alpha_{x,y})$ on R is less than $\frac{\varepsilon_{m-1}}{2}$, since $x, y < m$. Let n be such that

$$\tan(\alpha_{x,y}) \subset \left[\frac{n\varepsilon_{m-1}}{2}, \frac{(n+2)\varepsilon_{m-1}}{2}\right]$$

on R.

Let $A'_{n+1,j}B'_{n+1,j}$ be the segment we obtain by scaling $A_{n+1,j}B_{n+1,j}$ by $\frac{3}{2}$ from its midpoint (Fig. 7). We claim that every curve $C_{x,y,s}$ intersecting $P_{n+2,4j-i}$, where $i = 0, 1, 2, 3$ must intersect the segment $A'_{n+1,j}B'_{n+1,j}$. This can be verified for each parallelogram separately. We will only check this for $P_{n+2,4j-3}$, the others can be done similarly. Suppose $C_{x,y,s}$ intersects $P_{n+2,4j-3}$. We claim that if it intersects the line of $A'_{n+1,j}B'_{n+1,j}$

right of $B'_{n+1,j}$, then it must be right of the segment $B'_{n+1,j}V_2$ in the horizontal strip between $B'_{n+1,j}$ and V_2. Indeed, this follows from the fact that by (7.2) the segment $B'_{n+1,j}V_2$ has an angle of $\arctan\left(\frac{(n+2)\varepsilon_{m-1}}{2}\right)$ with the b-axis, since the tangent of the curve has a smaller angle with the b-axis on R. On the left side, a slightly stronger statement is true. If the curve intersects the line of $A'_{n+1,j}B'_{n+1,j}$ left of $A_{n+1,j}$, then it must be left of the segment $A_{n+1,j}V_1$ in the horizontal strip between $A_{n+1,j}$ and V_1 (this segment has an angle of $\arctan(\frac{n\varepsilon_{m-1}}{2})$ with the b-axis).

The segment $A'_{n+1,j}B'_{n+1,j}$ is of length $\frac{2^{-(n+1)}3\varepsilon_{m-1}}{N}$. Note that $f(A'_{n+1,j}B'_{n+1,j})$ is a segment of the same length. This means that

$$\lambda(p_{(1,x,y)}(f(\cup_{i=0}^{3}P_{n+2,4j-i}))) \leq \lambda(p_{(1,x,y)}(f(A'_{n+1,j}B'_{n+1,j}))) \leq$$
$$\leq \frac{2^{-(n+1)}3\varepsilon_{m-1}}{N}\sqrt{1+x^2+y^2},$$

$$\lambda(p_{(1,x,y)}(f(R\cap K'_m))) \leq \sum_{j=1}^{2^n}\lambda(p_{(1,x,y)}(f(\cup_{i=0}^{3}P_{n+2,4j-i}))) \leq \frac{3\varepsilon_{m-1}}{2N}\sqrt{1+x^2+y^2},$$

$$\lambda(p_{(1,x,y)}(f(K'_m\cap F))) \leq \sum_{i=1}^{N}\lambda(p_{(1,x,y)}(f(R_i))) \leq$$
$$\leq \frac{3\varepsilon_{m-1}}{2}\sqrt{1+x^2+y^2} \leq \frac{3\sqrt{1+x^2+y^2}}{2(m-1)}.$$

□

7.2 Rotating a square, the proof of Theorem 7.1

7.2.1 Neighbourhoods of A

Call a rectangle in $\mathbb{R}^3$ *interesting* if its sides are of length 1 and $\sqrt{2}$, one of the shorter sides lies in the plane $\{y=0\}$, it has a $45°$ angle with the $\{y=0\}$ plane and is in the $y\geq 0$ half-space. By the *neighbourhood of an interesting rectangle* we will mean a set that contains the rectangle and is relatively open in $\mathbb{R}\times[0,1]\times\mathbb{R}$.

Claim 7.5. *For every $\varepsilon>0$ there exists a set U such that for every possible direction U contains a neighbourhood of an interesting rectangle in that direction, while it intersects the $\{y=h\}$ plane in a set of measure less than ε for all h.*

Proof. Let A be the set provided by Lemma 7.2. Fix an N large enough that

$$A\cap((-N,N)\times[0,1]\times(-N,N))$$

contains an interesting rectangle in every direction. For any $\delta>0$ let A_δ be $A+\{(-\delta,\delta)\times\{0\}\times(-\delta,\delta)\}$, where + denotes the Minkowski sum and define A_0 to be A. The set A_δ

is open for all $\delta > 0$, since A is a union of planes and we replace each plane with an open set. Define

$$B_\delta = A_\delta \cap ([-N, N] \times [0, 1] \times [-N, N]).$$

The set B_δ is a relatively open in $[-N, N] \times [0, 1] \times [-N, N]$ for all $\delta > 0$, therefore B_δ contains a neighbourhood of an interesting rectangle in every possible direction.

Let $f_\delta(h) = \lambda(B_\delta \cap \{y = h\})$. Using the fact that B_0 is compact, it is easy to verify that the function $f_\delta(h)$ is continuous in δ. Since $f_0(h) = 0, f_\delta(h)$ converges to 0 as δ tends to 0.

We claim that

$$f_{\delta+t}(h_1) \geq f_\delta(h_2) \tag{7.4}$$

whenever $|h_1 - h_2| < t$. Indeed for every point p in $B_\delta \cap \{y = h_2\}$ the set A contains a plane that intersects $\{y = h_2\}$ in a line, which is less distance away from p than δ. This plane has a $45°$ angle with the $\{y = h_2\}$ plane and intersects $\{y = h_1\}$ as well.

By (7.4), $f_\delta(h)$ is upper semicontinuous in h. Hence $f_{\frac{1}{n}}$ is a sequence of upper semicontinuous functions on $[0, 1]$, which is pointwise monotonically decreasing, and pointwise converges to 0. It is easy to prove that such a sequence must uniformly converge to 0. Therefore, there exists a $\delta_0 > 0$ such that $U = B_{\delta_0}$ has all the required properties. $\square$

7.2.2 Rotating an interesting rectangle

Let R be an interesting rectangle. Call a continuous motion M of R *interesting* if at every moment it keeps R interesting.

Lemma 7.6. *For all $\varepsilon > 0$ there exists an interesting motion of R, during which R does a full a rotation, but $R \cap \{y = t\}$ sweeps an area less than ε for all t.*

Proof. We will again use the idea of Pál joins to move between translated copies. We claim that for any interesting rectangle R' parallel to R there exists an interesting motion of R during which $R \cap \{y = t\}$ sweeps less than ε area for all t and R is translated to R'.

There is a direction in which we can translate R in such a way, that it sweeps 0 area at every height, call this its free direction. We will move R in an N-like shape. We translate it in its free direction, there we rotate it by a small angle, then we translate it in its free direction, and rotate it in the opposite direction, by the same angle. If we translate it far enough, then the angle of rotation can be arbitrarily small, and so the area swept at each height will be small.

Let U be the set given by Claim 7.5. For any possible direction d, U contains the neighbourhood of an interesting rectangle R_d, which is in the direction of d. For each R_d, there is an interesting motion, that rotates the rectangle a small amount within U. Since S^1 is compact, we can choose finitely many directions, whose neighbourhoods cover all directions. We have seen that we can move between parallel copies of R

while sweeping an arbitrarily small area at all heights, therefore these will only add an arbitrarily small area to the arbitrarily small area of U at each height. □

7.2.3 Proof of the main result

Proof of Theorem 7.1. Let M be the motion of an interesting rectangle R given by Lemma 7.6. Let $T = R \cap \{y = 0\}$. If at every moment we project R onto the $\{y = 0\}$ plane, then we get M', a motion of the unit square. Observe that during M' the segment at t distance from T sweeps the same area, as $R \cap \{y = t\}$ does during M.

□

8 Sets with the strong Kakeya property in $\mathbb{R}^3$

The results in this section are due to the author [9].

We say that a compact set $K \subset \mathbb{R}^3$ is *cylinderlike* if there exists $n \in \mathbb{N}$ such that for almost all t the set $\{x = t\} \cap K$ can be covered by n vertical lines.

Theorem 8.1. *If K is cylinderlike from two non-parallel directions d_1, d_2, then K has the strong Kakeya property.*

Lemma 8.2. *If ϕ is a rotation around the x-axis, K is cylinderlike, then K can be moved to $\phi(K)$ in an arbitrarily small volume.*

Proof. We will look at K from the direction of the x-axis, and give a motion that keeps the x-coordinate of every point constant. The set K is bounded, so its projection onto the $\{x = 0\}$ plane can be covered by a square. Apply Theorem 7.1 to this square. This naturally induces a motion on K. For almost all planes perpendicular to the x-axis the swept area will be less than $n\varepsilon$, since if a plane contains n vertical segments, then the motion of each segment corresponds to the motion of a vertical segment of the square and these all sweep an area less than ε. Applying Fubini's theorem we are done.

$\square$

Remark 8.3. It is easy to check that the proof of the above lemma also works for *general cylinderlike* sets defined as follows: For a compact set $K \subset \mathbb{R}^3$ let $n(t)$ be the minimum (possibly infinite) number of vertical lines required to cover $K \cap \{x = t\}$. Say that K is *general cylinderlike* if $\int_{\mathbb{R}} n(t)dt < \infty$.

We say that a compact set K is *cylinderlike from direction d* if there is an orthogonal coordinate system, in which the x-axis is parallel to d, and K is cylinderlike.

By Lemma 8.2: If K is cylinderlike from direction d, then K can be rotated around d in an arbitrarily small volume.

Proof of Theorem 8.1. Translations can be done again with Pál joins: if K is cylinderlike from some direction, then there exists a direction in which K can be translated in 0 volume, we can bring it very far from its original position sweeping 0 volume, there we rotate it by a very small angle, translate it in the free direction and rotate it back by the same angle.

Let $\phi(K)$ be the desired position, by using translations we can suppose that the origin is a fixed point of ϕ. Since d_1 and d_2 are non-parallel, the vectors $\phi(d_1), \phi(d_2)$ uniquely determine ϕ. Let their distance on S^2 be t. From this point forward every distance is the arc distance on S^2. By Lemma 8.2 we can rotate K around the vectors d_1, d_2 in an arbitrarily small volume. Therefore, it is enough to solve the following problem: Take S^2 and stick two needles (n_1 and n_2) into it, at directions d_1 and d_2 respectively. The following step is allowed: take a needle and rotate the sphere around it. The other needle moves accordingly. We need to prove that we can get the needles to any pair of fixed points p_1 and p_2, which are t distance apart, in a finite number of steps.

First case: Suppose n_1 and p_1 are further than t apart. Then we rotate around n_1 in a way such that n_2 moves onto a short arc between n_1 and p_1. Then we rotate around n_1 by π. It is clear that after a finite number of steps n_1 and p_1 will be at most t distance apart.

Second case: Suppose n_1, p_1 are at most t apart. We rotate around n_1 in a way such that n_2 moves onto the circle centred around p_1 of radius t. We can rotate around n_2 in such a way that n_1 moves into p_1. Now we can rotate n_2 into p_2 by rotating around n_1. □

Corollary 8.4. *If for a compact set $A \subset \mathbb{R}^2$ there exists an $n \in \mathbb{N}$ and non-parallel directions d_1 and d_2, such that every line perpendicular to d_i intersects A in at most n points for $i \in \{1, 2\}$, then $A \times [0, 1]$ has the strong Kakeya property.*

Proof. The set $A \times [0, 1]$ is cylinderlike from directions d_1 and d_2, therefore by Theorem 8.1 it has the strong Kakeya property. □

Corollary 8.5. *If A can be covered by a finite union of graphs of Lipschitz functions, then $A \times [0, 1]$ possesses the strong Kakeya property.*

Proof. The graph of a Lipschitz function intersects every steep enough line in one point, therefore we can use Corollary 8.4. □

Corollary 8.6. *If $K = A \times [0, 1]$ is compact, and A is such that it can be covered by a finite number of monotonic functions, then K possesses the strong Kakeya property.*

Proof. Under such conditions A intersects every horizontal and vertical line in at most n points, where n is the number of monotonic functions required to cover A. Again Corollary 8.4 can be applied. □

Corollary 8.7. *The curved surface of a cylinder has the strong Kakeya property. Moreover, the finite union of parallel curved cylinder surfaces possesses the strong Kakeya property.*

Corollary 8.8. *If K can be covered by a finite set of planes that have normal vectors in a common plane, then K has the strong Kakeya property.*

References

[1] A. Chang, M. Csörnyei, The Kakeya needle problem and the existence of Besicovitch and Nikodym sets for rectifiable sets, *Proc. Lond. Math. Soc.*, vol. 118, no. 5, pp. 1084–1114, 2019.

[2] A. S. Besicovitch, On Kakeya's problem and a similar one, *Math. Z.*, vol. 27, no. 1, pp. 312–320, 1928.

[3] F. Cunningham Jr., The Kakeya Problem for Simply Connected and for Star-Shaped Sets, *The American Mathematical Monthly* vol. 78, no. 2 (1971): 114–29.

[4] F. Cunningham Jr., Three Kakeya Problems, *The American Mathematical Monthly*, vol. 81, no. 6, pp. 582–592, 1974.

[5] H. Wang, J. Zahl, Volume estimates for unions of convex sets, and the Kakeya set conjecture in three dimensions, *arXiv:* 2502.17655

[6] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley & Sons, 2014.

[7] K. Héra and M. Laczkovich, The Kakeya Problem for Circular Arcs, *Acta Math. Hungar.*, vol. 150, pp. 479–511, 2016.

[8] M. Csörnyei, K. Héra, and M. Laczkovich, Closed sets with the Kakeya property, *Mathematika*, vol. 63, no. 1, pp. 184–195, 2017.

[9] M. Kökényesi, An improvement on a result of Davies, *arXiv:* 2411.11083

[10] M. Talagrand, Sur la mesure de la projection d'un compact et certaines familles de cercles, *Bull. Math. Sci.*, vol. 2, no. 104, p. 3, 1980.

[11] R. O. Davies, Some remarks on the Kakeya problem, *Math. Proc. Cambridge Philos. Soc.* 69 (1971), 417–421.

[12] T. Kátay, The intersection of typical Besicovitch sets with lines, *Real Anal. Exchange*, vol. 45, no. 2, pp. 451-461, 2020.

[13] W. Hansen, Littlewood's one-circle problem, revisited, *Expo. Math.*, vol. 26, no. 4, pp. 365-374, 2008.

MI nyilatkozat

Alulírott Kökényesi Márk Péter nyilatkozom, hogy szakdolgozatom elkészítése során az alább felsorolt feladat elvégzésére a megadott MI alapú eszközt alkalmaztam:

Feladat	Felhasznált eszköz	Felhasználás helye	Megjegyzés
Táblázat készítés az MI nyilatkozathoz	GPT-4o	MI nyilatkozat	LaTeX kód generálása

A felsoroltakon kívül más MI alapú eszközt nem használtam.

Dátum: 2025. 05. 30.

Aláírás: