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Classical and projective Fraïssé theory

Thesis

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1 Introduction

The primary objective of this thesis is to present an introduction to classical Fraïssé theory and its recently developed dual, projective Fraïssé theory. Fraïssé's classical theorem describes a way of building a countable, universal and homogeneous limit structure called the *Fraïssé limit* from a well-behaved class of small (finitely generated) structures called a *Fraïssé class.* Fraïssé introduced this framework in the 1950s, and it has since become a cornerstone of model theory as well as a key tool in other areas of mathematics, such as descriptive set theory and algebra.

Notable examples of Fraïssé limits include:

- the *random (or Rado) graph*, arising as the limit of the class of finite graphs;
- the *Hall universal group*, obtained as the limit of the class of finite groups;
- the *rational Urysohn space*, whose metric completion is the Urysohn universal metric space.

In the first part of this thesis, we focus on classical Fraïssé theory; we explore its various (primarily model-theoretic) applications, such as *ω-categoricity.* We will see that in certain cases ω-categoricity can be checked fairly easily for a given Fraïssé limit. However, we will also see that not every Fraïssé limit is ω-categorical.

In the second part of this thesis, we develop the dual theory commonly known as *projective Fraïssé theory.* Solecki and Irwin published the foundational paper on this topic in 2006 [4], in which they describes a way of building a compact homogeneous inverse limit structure called the *projective Fraïssé* limit from a well-behaved class of compact structures called a *projective Fraïssé class.* As an application we show that a well-known continuum the *pseudo-arc* can be realized as a suitable quotient of a projective Fraïssé limit, then we prove several nontrivial topological properties of the pseudo-arc using the framework we have developed. Projective Fraïssé theory is already very successful, by owing to its great potential for dualizing the results of classical Fraïssé theory.

At this point we remark that throughout this thesis, whenever we refer to topological spaces, we implicitly assume they are <u>Hausdorff</u>.

Notation

We assume that the reader is familiar with the basic definitions and results of set theory and first-order logic such as first-order languages, structures, the definition of satisfaction, etc.

Throughout this thesis, we work exclusively with countable languages. Most of our notation is standard: let ω denote the least infinite ordinal (which we identify with the set of natural numbers), and for each $n \in \omega$ we let

$$n = \{0, 1, 2, \dots, n-1\}.$$

In general, first-order (and, in Section 3, topological) L-structures denoted by calligraphic letters (such as $\mathcal{A}$), and their domains by the corresponding uppercase Latin letters (such as A). This convention may be overridden by tradition or by the customary names of particular structures. For example, the set of rational numbers with its usual ordering is denoted by $\mathbb{Q}$ and fields (viewed as first-order structures) by the standard blackboard-bold symbols $\mathbb{F}$ or $\mathbb{K}$ ect.

If f is a mapping, we denote its range by $\mathrm{Ran}(f)$ and its domain by $\mathrm{Dom}(f)$. In Section 2, the notions of embedding, isomorphism, epimorphism, etc. are understood in the usual sense. In Section 3, where we will work with topological structures we will need to modify them. We write

$$\mathcal{A} \cong \mathcal{B}$$

to indicate that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic first-order (or topological) L-structures, and

$$\mathcal{A} \leq \mathcal{B}$$

to indicate that $\mathcal{A}$ is a substructure of $\mathcal{B}$. In this case, we often use $\leq$ to denote the inclusion map of $\mathcal{A}$ into $\mathcal{B}$. We also note that if f is a function with domain A and B is a subset of A, then its restriction $f|_B$ has domain B and satisfies

$$f(b) = f|_B(b) \quad \text{for all } b \in B.$$

Similarly, if $R \subseteq A^n$ is an n-ary relation on A, then $R|_B$ denotes the restriction of R to B^n, i.e.,

$$R|_B \;=\; R \cap B^n.$$

We denote by $\mathrm{Form}(L)$ the set of all formulas in a given first-order language L. Furthermore, we introduce the operations Mod and Th as follows: if $\Sigma \subseteq \mathrm{Form}(L)$ is a set of L-formulas (i.e., an L-theory), then

$$\mathrm{Mod}(\Sigma)$$

is the class of all L-structures in which every formula of Σ is satisfied and if $\mathcal{A}$ is an L-structure, then

$$\mathrm{Th}(\mathcal{A}) \;=\; \{\varphi \in \mathrm{Form}(L) \mid \mathcal{A} \models \varphi\}$$

denotes the *theory* of $\mathcal{A}$.

If X is a topological space, then let $C[X]$ denote the space of all continuous function from X into the unit interval endowed with the uniform metric d_U. Where for any $f, g \in C[X]$

$$d_U(f, g) = \sup\{|f(x_0) - f(x_1)| : x_0, x_1 \in X\}$$

and $|.|$ is the the absolute value on the reals. If (Y, d) is a metric space, a number $\delta > 0$ and H is a subset of Y then for a δ neighborhood of H denoted by H_δ, where

$$H_\delta = \bigcup_{h \in H} \{y \in Y : d(h, y) < \delta\}.$$

Finally, S_n denotes the symmetric group of degree n. Moreover, if K is an arbitrary set, then Sym_K denotes the full permutation group on K.

2 Classical Fraïssé Theory

In this section, as already mentioned, we review the fundamental concepts of classical Fraïssé theory. First, we describe a method for constructing countable homogeneous universal structures by form certain well-behaved classes of finitely generated structures. In the second part, we present examples of Fraïssé classes and apply the developed framework to these classes to prove properties of their limit structures.

2.1 Construction

First, we recall the basic definitions of the Hereditary Property (HP), the Joint Embedding Property (JEP), and the Amalgamation Property (AP). As we shall see in the second part of this section, these class properties correspond to natural concepts.

Definition 2.1 *(Age) Let $\mathcal{A}$ be a countable L-structure. The age of $\mathcal{A}$ is the class Δ of all finitely generated L-structures that can be embedded into $\mathcal{A}$. Somewhat, we also refer to a class Δ as the age of $\mathcal{A}$ if it consists, up to isomorphism, exactly the finitely generated substructures of $\mathcal{A}$. For a countable L-structure we detonate by*

$$\mathrm{Age}(\mathcal{A})$$

the age of $\mathcal{A}$.

Definition 2.2 *(HP) A class Δ of L-structures has the Hereditary Property if for any $\mathcal{A} \in \Delta$ and any finitely generated substructure $\mathcal{B}$ of $\mathcal{A}$ there exists a model $\mathcal{C} \in \Delta$ such that $\mathcal{B}$ is isomorphic to $\mathcal{C}$.*

Definition 2.3 *(JEP) A class Δ of L-structures has the Joint Embedding Property if for any $\mathcal{A}, \mathcal{B} \in \Delta$ there exists a structure $\mathcal{C} \in \Delta$ such that both $\mathcal{A}$ and $\mathcal{B}$ can be embedded into $\mathcal{C}$.*

$$\begin{tikzcd}
\mathcal{A} \arrow[dr, hook] & \\
& \mathcal{C} \\
\mathcal{B} \arrow[ur, hook] &
\end{tikzcd}$$

Remark 2.4 *It is easy to see that if Δ is the age of $\mathcal{A}$, then Δ has the hereditary property (HP) and the joint embedding property (JEP).*

Remark 2.5 *For any structures $\mathcal{A}$ and $\mathcal{C}$, if $f : \mathcal{A} \to \mathcal{C}$ is an embedding, then there is $\mathcal{B} \geq \mathcal{A}$ and an isomorphism $g : \mathcal{B} \to \mathcal{C}$, such that f can be written as $g \circ i$, where $i : \mathcal{A} \to \mathcal{B}$ is the inclusion map. Thus, whenever we have an embedding $f : \mathcal{A} \to \mathcal{C}$, we may assume that $\mathcal{A}$ is a substructure of $\mathcal{C}$.*

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, hook, "f"] \arrow[dr, hook, "i"'] & & \mathcal{C} \\
& \mathcal{B} \arrow[ur, "g"'] &
\end{tikzcd}$$

We will now show that the converse of Remark 2.4 is also true.

Theorem 2.6 *Let Δ be a countable set of finitely generated L-structures that has both the JEP and the HP. Then Δ is the age of a countable structure $\mathcal{A}$.*

Proof. Without loss of generality, assume that Δ contains at most one element from each isomorphism class. Let $\{\mathcal{A}_i : i \in \omega\}$ be an enumeration of the elements of Δ. We define the set of L-structures $\{\mathcal{B}_i : i \in \omega\}$ recursively as follows:

- Let $\mathcal{B}_0 = \mathcal{A}_0$.
- Assume that we have already defined $\mathcal{B}_i$. By the JEP, there exists an L-structure $\mathcal{B}'_{i+1} \in \Delta$ such that $\mathcal{B}_i$ and $\mathcal{A}_{i+1}$ can be embedded into $\mathcal{B}'_{i+1}$. By the previous remark, there exists an L-structure $\mathcal{B}_{i+1}$ that extends $\mathcal{B}_i$ and is isomorphic to $\mathcal{B}'_{i+1}$.

Now let

$$\mathcal{A} = \bigcup_{i \in \omega} \mathcal{B}_i$$

be a countable structure (countable because its domain is a countable union of countable structures). By construction, all models in Δ can be embedded into $\mathcal{A}$. If $\mathcal{C}$ is a finitely generated substructure of $\mathcal{A}$, then there exists $i \in \omega$ such that $\mathcal{C}$ can be embedded into $\mathcal{B}'_i$. By the HP, there exists a model $\mathcal{C}' \in \Delta$ such that $\mathcal{C}$ is isomorphic to $\mathcal{C}'$. Hence, Δ is the age of $\mathcal{A}$. ∎

Definition 2.7 *(AP) A class Δ of L-structures has the Amalgamation Property if for any structures $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \Delta$ and embeddings $f : \mathcal{A} \to \mathcal{B}$, $g : \mathcal{A} \to \mathcal{C}$ there exists a model $\mathcal{D} \in \Delta$ and embeddings $h : \mathcal{B} \to \mathcal{D}$, $e : \mathcal{C} \to \mathcal{D}$ such that $e \circ g = h \circ f$.*

$$\begin{tikzcd}
& \mathcal{B} \arrow[dr, dashed, "h"] & \\
\mathcal{A} \arrow[ur, "f"] \arrow[dr, "g"'] & & \mathcal{D} \\
& \mathcal{C} \arrow[ur, dashed, "e"'] &
\end{tikzcd}$$

Definition 2.8 *(Fraïssé class) A countable nonempty class Δ of finitely generated L-structures is a Fraïssé class if it has the HP, the JEP and the AP.*

The following notion is called ultra-homogenecity in many sources, such as our primary reference [2]. In this theses, however, we will omit the prefix "ultra".

Definition 2.9 *(Homogeneous L-structure) Let $\mathcal{A}$ be an L-structure. We say that $\mathcal{A}$ is homogeneous if every isomorphism between finitely generated substructures of $\mathcal{A}$ can be extended to an automorphism of $\mathcal{A}$.*

Definition 2.10 *(Chain of structures) A sequence of L-structures $(\mathcal{B}_i)_{i \in \omega}$ is a chain (of structures) if $\mathcal{B}_i \leq \mathcal{B}_{i+1}$ for all $i \in \omega$.*

Definition 2.11 *(Weakly homogeneous L-structure) An L-structure $\mathcal{D}$ is weakly homogeneous if for any finitely generated substructures $\mathcal{A}, \mathcal{B} \leq \mathcal{D}$ with $\mathcal{A} \leq \mathcal{B}$ and any embedding $f : \mathcal{A} \to \mathcal{D}$, there exists an embedding $g : \mathcal{B} \to \mathcal{D}$ such that $f \subseteq g$.*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "f"'] \arrow[d, "\leq"'] & \mathcal{D} \\
\mathcal{B} \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

Remark 2.12 *Clearly, if $\mathcal{D}$ is homogeneous, then it is also weakly homogeneous.*

Lemma 2.13 *Let $\mathcal{C}$ and $\mathcal{D}$ be countable structures. Suppose that $\mathrm{Age}(\mathcal{C}) \subseteq \mathrm{Age}(\mathcal{D})$ and $\mathcal{D}$ is weakly homogeneous. Then $\mathcal{C}$ can be embedded into $\mathcal{D}$. In fact, any embedding of a finitely generated substructure of $\mathcal{C}$ into $\mathcal{D}$ can be extended to an embedding of $\mathcal{C}$ into $\mathcal{D}$.*

Proof. Since $\mathcal{C}$ is countable, we have

$$\mathcal{C} = \bigcup_{i\in\omega} \mathcal{A}_i$$

where $(\mathcal{A}_i)_{i\in\omega}$ is a chain of finitely generated structures. By recursion, we define embeddings $f_n : \mathcal{A}_n \to \mathcal{D}$ such that $f_n \subseteq f_{n+1}$ for all $n \in \omega$. Since $\mathrm{Age}(\mathcal{C}) \subseteq \mathrm{Age}(\mathcal{D})$, there is an embedding $f_0 : \mathcal{A}_0 \to \mathcal{D}$, let $\mathcal{B}_0 = f_0(\mathcal{A}_0)$.

Assume that f_{n-1} is already defined. There exists an embedding $g_n : \mathcal{A}_n \to \mathcal{B}_n$ because $\mathrm{Age}(\mathcal{C}) \subseteq \mathrm{Age}(\mathcal{D})$. Let $\mathcal{B}_n = g_n(\mathcal{A}_n)$.

Hence $\mathcal{D}$ is weakly homogeneous, there exists an embedding $h_n : \mathcal{B}_n \to \mathcal{D}$ such that

$$f_{n-1} \circ (g_n^{-1}|_{g_n(\mathcal{A}_{n-1})}) \subseteq h_n.$$

Let $f_n = h_n \circ g_n$. Then $f_{n-1} \subseteq f_n$, hence $f = \bigcup_{n\in\omega} f_n$ is an embedding on : $\mathcal{C}$ into $\mathcal{D}$.

For the "in fact" part of the statement, note that we can choose the structure $\mathcal{A}_0 \in \mathrm{Age}(\mathcal{C})$ and an embedding $f_0 : \mathcal{A}_0 \to \mathcal{D}$ arbitrary. ∎

Lemma 2.14

(a) Let $\mathcal{C}$ and $\mathcal{D}$ be countable weakly homogeneous L-structures with the same age. Then $\mathcal{C}$ and $\mathcal{D}$ are isomorphic. Moreover, any embedding from a finitely generated substructure of $\mathcal{C}$ can be extended to an isomorphism from $\mathcal{C}$ to $\mathcal{D}$.

(b) A countable L-structure is homogeneous if and only if it is weakly homogeneous.

Proof. (a) Let us write

$$\mathcal{C} = \bigcup_{i\in\omega} \mathcal{A}_i \quad \text{and} \quad \mathcal{D} = \bigcup_{i\in\omega} \mathcal{B}_i$$

where $(\mathcal{A}_i)_{i\in\omega}$ and $(\mathcal{B}_i)_{i\in\omega}$ are chains of finitely generated L-structures.

It is suffices to define a sequence (f_n) of maps such that for every $n \in \omega$

(a) $f_{n-1} \subseteq f_n$,

(b) $\mathrm{Dom}(f_{2n+1}) \supseteq \mathcal{A}_n$,

(c) $\mathrm{Ran}(f_{2n+1} \supseteq \mathcal{B}_n$,

(d) f_n is an isomorphism between finitely generated substructures of $\mathcal{C}$ and $\mathcal{D}$.

First fix any embedding $f_0 : \mathcal{A}_0 \to \mathcal{D}$. Recursively, assume $f_0, ..., f_{2n-1}$ are defined. Choose $k \geq n$ so that $\mathrm{Dom}(f_{2n+1}) \supseteq \mathcal{A}_k$. By lemma 2.13, there exists an embedding $f_{2n} : \mathcal{A}_k \to \mathcal{D}$ that extends f_{2n-1}. Similarly, choose $l \geq n$ that $\mathrm{Ran}(f_{2n}) \supseteq \mathcal{B}_l$. Again, by lemma 2.13 there is an embedding $g : \mathcal{B}_l \to \mathcal{C}$ that extends f_{2n}^{-1}. We let $f_{2n+1} = g^{-1}$, witch concludes the proof of the first statement. Since $\mathcal{A}_0$ and f_0 were chosen arbitrarily, the moreover part also follows.

(b) We have already noted that homogeneous structures are weakly homogeneous. The converse follows from the moreover part of (a). ∎

Remark 2.15 *If $\mathcal{C}$ and $\mathcal{D}$ are not countable L-structures, then (a) fails in the previous theorem.*

Proof. Let $\mathcal{A} = \langle \omega_1 \times \mathbb{Q}, < \rangle$ where $<$ is the lexicographic order and let $\mathcal{B}$ be the mirror image of $\mathcal{A}$. Then $\mathcal{A}$ and $\mathcal{B}$ are homogeneous and have the same age, but they are not isomorphic, because every initial segment of $\mathcal{A}$ is countable, while every initial segment of $\mathcal{B}$ is uncountable. ∎

The following theorem is the fundamental theorem of classical Fraïssé theory.

Theorem 2.16 *(Fraïssé's Theorem)*
Let Δ be Fraïssé class of L-structures. Then there exists a unique (up to isomorphism) countable homogeneous L-structure $\mathcal{D}$ such the $\mathrm{Age}(\mathcal{D}) = \Delta$.

We also say that $\mathcal{D}$ is the Fraïssé limit of Δ.
The statement consists of two parts: existence and uniqueness. Note that uniqueness already follows from part (a) of Lemma 2.14, which asserts that countable weakly homogeneous structures with the same age are isomorphic. To prove existence, we will construct a direct limit from our Fraïssé class using recursion. Furthermore, we must ensure that the limit structure is both universal and homogeneous. At each step, a countable many tasks will arise, which we will list using bookkeeping, and we will resolve these tasks by applying the AP and teh JEP of the Fraïssé class.

Proof. We may assume that Δ ruct a chain $(\mathcal{B}_i)_{i \in \omega} \subseteq \Delta$ such that the following holds.

($\star$) If $\mathcal{A}, \mathcal{C} \in \Delta$ are structures such that $\mathcal{A} \leq \mathcal{C}$ and there is an embedding $f : \mathcal{A} \to \mathcal{B}_i$ for some $i \in \omega$, then there is $j > i$ and an embedding $g : \mathcal{C} \to \mathcal{B}_j$ such that $f \subseteq g$.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "f"] \arrow[d, "\leq"'] & \mathcal{B}_i \arrow[d, "\leq"] \\ \mathcal{C} \arrow[r, dotted, "g"'] & \mathcal{B}_j \end{tikzcd}$$

Assume that we have already defined chain $\{\mathcal{B}_i : i \in \omega\}$ with the property ($\star$). Let $\mathcal{D} = \bigcup_{i \in \omega} \mathcal{B}_i$. Every finitely generated substructure of $\mathcal{D}$ lies in $\mathcal{B}_i$ for some $i \in \omega$. Since Δ has the HP it follows that $\mathrm{Age}(\mathcal{D}) \subseteq \Delta$.

To show $\Delta \subseteq$ Age let $\mathcal{C} \in \Delta$ be a structure. Then by the JEP, there exists a $\mathcal{C}' \in \Delta$ such that $\mathcal{B}_0 \leq \mathcal{C}'$ and $\mathcal{C}$ is embeddable in $\mathcal{C}'$. By property ($\star$), the identity map of $\mathcal{B}_0$ extends to an embedding of $\mathcal{C}'$ into $\mathcal{B}_j$ for some $j > 0$, so $\mathcal{C}$ and $\mathcal{C}'$ are in the age of $\mathcal{D}$.

$$\begin{tikzcd} \mathcal{B}_0 \arrow[r, "\mathrm{Id}"] \arrow[d, "\leq"'] & \mathcal{B}_0 \arrow[d, "\leq"] \\ \mathcal{C}' \arrow[r, dotted, "g"'] & \mathcal{B}_j \end{tikzcd}$$

To construct the chain $\{\mathcal{B}_i : i \in \omega\}$ with the property ($\star$), let Γ be the class of all $\langle \mathcal{E}, \mathcal{F} \rangle$ such $\mathcal{E}, \mathcal{F} \in \Delta$ and $\mathcal{E} \leq \mathcal{F}$, and let $\Gamma \subseteq \Gamma'$ be a countable set that contain exactly one representative of each isomorphism type of pairs. Let $\iota : \omega^2 \to \omega$ be a bijection such that $\iota(i,j) \geq i$ for every $i, j \in \omega$, and let $\mathcal{B}_0 \in \Delta$ be an arbitrary element. Assume that we have already defined $\mathcal{B}_0, ..., \mathcal{B}_k$. For each $i \leq k$ let $\{\langle f_{ij}, \mathcal{E}_{ij}, \mathcal{F}_{ij} \rangle : j \in \omega\}$ be an enumeration of all triples $\langle f, \mathcal{E}, \mathcal{F} \rangle$, such that $\langle \mathcal{E}, \mathcal{F} \rangle \in \Gamma$ and $f : \mathcal{E} \to \mathcal{B}_i$ is an embedding.

By the AP, for every $k \in \omega$ there exists $\mathcal{B}_{k+1} \in \Delta$ such that $\mathcal{B}_k \leq \mathcal{B}_{k+1}$ and if $k = \iota(i,j)$, then f_{ij} extends to an embedding $g_{ij} : \mathcal{F}_{ij} \to \mathcal{B}_{k+1}$.

$$\begin{tikzcd} & \mathcal{F}_{ij} \arrow[dr, dashed, "g_{ij,k+1}"] & \\ \mathcal{E}_{ij} \arrow[ur, "\leq"] \arrow[dr, "f_{ij}"'] & & \mathcal{B}_{k+1} \\ & \mathcal{B}_k \arrow[ur, "\leq"'] & \end{tikzcd}$$

The chain $(\mathcal{B}_n)_{n\in\omega}$ has property $(\star)$, and $\mathcal{D} = \bigcup_{n\in\omega} \mathcal{B}_n$ is weakly homogeneous. Because if for any finitely generated substructures $\mathcal{A}, \mathcal{C} \leq \mathcal{D}$ with $\mathcal{A} \leq \mathcal{C}$ and any embedding $f : \mathcal{A} \to \mathcal{D}$ then there exist $i \in \omega$ such $\mathrm{Ran}(f) \subseteq B_i$ using property $(\star)$ there is $j > i$ and an embedding $g : \mathcal{C} \to \mathcal{B}_j$ such that $f \subseteq g$, while $\mathcal{D} = \bigcup_{n\in\omega} \mathcal{B}_n$ then g is also an embedding into $\mathcal{D}$. Then by Lemma 2.14 (b), $\mathcal{D}$ is homogeneous.

By Lemma 2.14 (a), $\mathcal{D}$ is unique up to isomorphism. ∎

Theorem 2.17 *Let $\mathcal{D}$ be a countable homogeneous L-structure with the age Δ. Then Δ is a Fraïssé class.*

Proof. It is obvious that the $\mathrm{Age}(\mathcal{D})$ has the HP and the JEP, so we need to show that Δ has the AP. Let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2 \in \Delta$ and suppose that

$$f_1 \colon \mathcal{A}_0 \to \mathcal{A}_1, \quad f_2 \colon \mathcal{A}_0 \to \mathcal{A}_2$$

are embeddings. Since $\mathcal{D}$ is homogeneous, the isomorphism

$$f_2 \circ f_1^{-1} \colon f_1(\mathcal{A}_0) \longrightarrow f_2(\mathcal{A}_0)$$

extends to an automorphism α of $\mathcal{D}$. Let $\mathcal{A}_3$ be the substructure of $\mathcal{D}$ generated by $A_2 \cup \alpha(A_1)$. Then the inclusion

$$i \colon \mathcal{A}_2 \to \mathcal{A}_3$$

and the restriction

$$\alpha' = \alpha\big|_{A_1} \colon \mathcal{A}_1 \to \mathcal{A}_3$$

are embeddings satisfying

$$\alpha' \circ f_1 = i \circ f_2.$$

$$\begin{tikzcd}
& \mathcal{A}_1 \arrow[dr, "\alpha'"] & \\
\mathcal{A}_0 \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & \mathcal{A}_3 \\
& \mathcal{A}_2 \arrow[ur, "i"'] &
\end{tikzcd}$$

Therefore, Δ has the AP, so the proof is complete. ∎

2.2 Applications

Fraïssé theory has numerous applications. In this subsection, we will examine a few examples of Fraïssé classes and investigate their model-theoretic properties. It is worth noting that there are many other applications in descriptive set theory and abstract analysis. First, let us recall some model-theoretic definitions. One of the central concepts in model theory is categoricity.

Definition 2.18 *(κ-categoricity) Let κ be a cardinal, and Σ an L-theory. We say that Σ is κ-categorical if it has, up to isomorphism exactly one model of carinality κ.*

The motivation for the following definition is to control the size of the elements in the Fraïssé class Δ. This will allow us to ensure that the limit of an appropriate Fraïssé class is categorical.

Definition 2.19 *(Uniform local finiteness) We say that a structure $\mathcal{A}$ is uniformly locally finite if there exists a function $f : \omega \to \omega$ such that for every $n \in \omega$ and substructure $\mathcal{B} \leq \mathcal{A}$, such $\mathcal{B}$ can be generated by at most n elements, then $|B| \leq f(n)$.*

We say that a class Δ of structures is uniformly locally finite if there exists a function $f : \omega \to \omega$ such that for every $\mathcal{A} \in \Delta$ for every $n \in \omega$ and substructure $\mathcal{B} \leq \mathcal{A}$, such $\mathcal{B}$ can be generated by at most n elements, then $|B| \leq f(n)$.

Remark 2.20 *Note that any class of relational structures is uniformly locally finite.*

Another interesting property is quantifier elimination.

Definition 2.21 *(Quantifier elimination) An L-theory Σ is said to have quantifier elimination if for every L-formula ϕ , there exists a quantifier-free formula ψ such that*

$$\Sigma \models (\phi \iff \psi).$$

The following theorem proves ω-categoricity and quantifier elimination for the theory of the limit of any uniformly locally finite Fraïssé class.

Theorem 2.22 *Suppose that L is a finite language and Δ is a uniformly locally finite Fraïssé class. Let $\mathcal{D}$ be the Fraïssé limit of Δ and $\Sigma = \mathrm{Th}(\mathcal{D})$. Then:*

1. *Σ is ω-categorical,*
2. *Σ has quantifier elimination.*

Proof. First, we will show that there is a $\forall_2$ L-theory Γ whose models are homogeneous structures with age Δ. There are two key facts here. One is that if $\mathcal{A}$ is a finite L-structure with n generators $\overline{a}$, then there exists a quantifier-free formula $\psi = \psi_{\mathcal{A},\overline{a}}$ such that for any L-structure $\mathcal{B}$ and $\overline{b} \in B^n$:

(*) $\mathcal{B} \models \psi(\overline{b})$ if and only if there is an isomorphism from $\mathcal{A}$ to $\langle \overline{b} \rangle_{\mathcal{B}}$ which takes $\overline{a}$ to $\overline{b}$.

In fact $\psi = \psi_{\mathcal{A},\overline{a}}$ is the conjunction of all atomic formulas and negation of formulas that are satisfied by $\overline{a}$ in $\mathcal{A}$. The second is that by uniform local finiteness, for each $n \in \omega$, there are finitely many isomorphism types in Δ generated by n elements.

Now, let

$$\Gamma_0 = \{\forall \overline{x}(\psi_{\mathcal{A},\overline{a}}(\overline{x}) \implies \exists y \psi_{\mathcal{B},\overline{a}b}(\overline{x}, y)) : \mathcal{A} \in \Delta \text{ generated by } \overline{a},\ \mathcal{B} \in \Delta \text{ generated by } \overline{a}b\}$$

and

$$\Gamma_1 = \bigcup_{n\in\omega} \{\forall x_0 \dots \forall x_n (\bigvee_{\mathcal{A},\overline{a}} \psi_{\mathcal{A},\overline{a}}(\overline{x})) : \mathcal{A} \in \Delta \text{ generated by } \overline{a} \text{ and } |\overline{a}| = n + 1\},$$

where $|.|$ denotes the length of a tuple. Then uniform local finiteness implies that for every element of Γ_1, the disjunction is finite (up to logical equivalence). Let us note that in the formulas of Γ_0, $\overline{a} = \emptyset$ is also allowed. Let $\Gamma = \Gamma_0 \cup \Gamma_1$. It is clear that $\mathcal{D} \in \mathrm{Mod}(\Gamma)$. Suppose that $\mathcal{D}'$ is another model of Γ. Note that, since Δ is a Fraïssé class, then in every Δ-structure the constants generate the same substructure $\mathcal{C}_0$. Hence, from $\mathcal{D}' \models \Gamma_1$ it follows that $\mathcal{C}_0 \leq \mathcal{D}'$. Now if $\overline{a} = \emptyset$, then the sentences in Γ_0 give the condition that every one-generator structure in Δ can be embedded into $\mathcal{D}'$. The general sentences in Γ_0 state that:

(**) If $\mathcal{A}, \mathcal{B}$ are finitely generated substructures of $\mathcal{D}'$, and $\mathcal{B}$ is obtained from $\mathcal{A}$ by adding one more generator, and $f : \mathcal{A} \to \mathcal{D}'$ is an embedding, then there exists an embedding $g : \mathcal{B} \to \mathcal{D}'$ that extends f.

By induction on the number of generators and using that Δ has the HP, we get that every element of Δ can be embedded into $\mathcal{D}'$. The sentences for Γ_1 tell us that the age of $\mathcal{D}'$ is exactly Δ. If $\mathcal{A}, \mathcal{B}$ are finitely generated substructures of $\mathcal{D}'$ and $h : \mathcal{A} \to \mathcal{B}$ is an isomorphism, then, by recursion (using the formulas of Γ_0), it is easy to construct an automorphism of $\mathcal{D}'$ that extends h, so $\mathcal{D}'$ is homogeneous. Therefore, by Theorem 2.16, $\mathcal{D}' \cong \mathcal{D}$. Thus, we have shown that 1. holds.

Now suppose that $\phi(\overline{x})$ is an L-formula and $\overline{x} \neq \emptyset$ and $\mathcal{D} \in \mathrm{Mod}(\Gamma)$. Let

$$X = \{\overline{a} \in D^{|\overline{x}|} : \mathcal{D} \models \phi(\overline{a})\}.$$

If $\overline{a} \in X$ and $\overline{b}$ is tuple in $\mathcal{D}$ such that there exists an isomorphism $g : \langle\overline{a}\rangle_{\mathcal{D}} \to \langle\overline{b}\rangle_{\mathcal{D}}$, then g extends to an automorphism of $\mathcal{D}$, so $\overline{b} \in X$ as well. Since the set

$$\{\langle\overline{a}\rangle_{\mathcal{D}} : \overline{0} \in X\}$$

is close under isomorphism, among the substructures of $\mathcal{D}$, it follows that $\phi(x)$ is equivalent modulo Σ to the disjunction of all formulas $\psi_{\mathcal{A},\overline{a}}$ with $(\mathcal{A}, \overline{a})$ rancing over all isomorphism types of pairs such that $\overline{a} \in X$, $\mathcal{A} \geq \mathcal{D}$ and $\langle\overline{a}\rangle_{\mathcal{D}} = \mathcal{A}$. Which is a finite quantifier-free formula. Finally, if ϕ is a sentence of L, then since Σ is complete, ϕ is equivalent modulo Σ to either $\perp$ or $\neg \perp$ (where $\perp$ is the identically false formula). Therefore, we have shown that 2. holds, thus completing the proof. ∎

In the following, we will go through classes of structures and show that they are Fraïssé classes. Two of them are uniformly locally finite, allowing us to apply Theorem 2.22. Therefore, the theories of their limits are ω-categorical and have quantifier elimination. Furthermore, we will also see an example where the theory of a Fraïssé limit is not ω-categorical.

2.2.1 Finite graphs

In this subsection, consider L as the language of graphs which contains only one binary relation symbol R, except for equality.

Definition 2.23 *Let $\mathcal{G} = Mod(\Sigma)$ be the class of simple graphs, where Σ consists of the axioms of simple graphs:*

- $\forall v(\neg R(v, v))$ *(irreflexivity),*
- $\forall v, u(R(v, u) \implies R(u, v))$ *(symmetry).*

Claim 2.23.1 *The class $\mathcal{K} \subseteq \mathcal{G}$ of all finite graphs is a Fraïssé class.*

Proof. Straightforward. ∎

Definition 2.24 *The random graph $\mathcal{R} = \langle V, E\rangle \in \mathcal{G}$ is the unique, up to isomorphism, countable L-structure such that*

$$\mathrm{Mod}(\Sigma \cup \{\phi_{n,m} \ : \ n, m \in \omega\}) = \{\mathcal{R}\}$$

where

$$\phi_{n,m} \equiv \forall x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1} \exists z \left(\bigwedge_{i \neq j} (x_i \neq y_j) \implies \bigwedge_{i \in n} R(x_i, z) \wedge \bigwedge_{j \in m} \neg R(y_j, z) \right).$$

Remark 2.25 *It is well known that $\Sigma \cup \{\phi_{n,m} \ : \ n, m \in \omega\}$ is a consistent ω-categorical theory.*

Theorem 2.26 *$\mathcal{R}$ is the Fraïssé limit of $\mathcal{K}$.*

Proof. First, $\mathrm{Age}(\mathcal{R}) \subseteq \mathcal{K}$ is clear. The fact that $\mathcal{R}$ contains every finite graph $\mathcal{G}_0$, follows by induction of the cardinality of $\mathcal{G}_0$ using that $\phi_{n,m}$ holds in $\mathcal{R}$. To prove that $\mathcal{R}$ is homogeneous, we will using back and forth argument. Let $f : \mathcal{A} \to \mathcal{B}$ be a partial isomorphism between two finite subgraphs of $\mathcal{R}$. Enumerate the vertices of the random graph $R = \{r_i\}_{i \in \omega}$. We will build an increasing chain of partial isomorphism

$$f_0 = f \subseteq f_1 \subseteq f_2 \subseteq ...$$

whose union will be an automorphism of $\mathcal{R}$, and for every $n \in \omega$ we get that $r_n \in \mathrm{Dom}(f_{2n})$ and $r_n \in \mathrm{Ran}(f_{2n+1})$. Assume that we already defined f_{2n-1}. Let

$$U = \{x \in \mathrm{Dom}(f_{2n-1} : R(x, r_n)\}$$

$$V = \{x \in \mathrm{Dom}(f_{2n-1} : \neg R(x, r_n)\}.$$

Using that $\phi_{n,m}$ holds in $\mathcal{R}$, we can find $z \in R$, such z is connects with every element of $f_{2-n}(U)$ and avoids the elements of $f_{2-n}(V)$. Let

$$f_{2n}|_{\mathrm{Dom}(f_{2n-1})} = f_{2n} \quad \text{and} \quad f_{2n}(r_n) = z.$$

At each odd sage (2n+1) we symmetrically extend the partial map to cover the n-th vertex in the range. Therefore

$$g = \bigcup_{n \in \omega} f_n$$

is an automorphism of $\mathcal{R}$.

∎

2.2.2 Finite linear orders

Next we take a look at the class of finite linear orders and we prove that it is a Fraïssé class. We also show that its Fraïssé limit is $\langle \mathbb{Q}, < \rangle$.

Definition 2.27 *Let L_{ord} be a language that consists of only one relation symbol $<$, except for equality. Let Γ be the theory of linear orders, which consists of the following formulas:*

- $\forall x \neg (x < x)$ *(irreflexivity),*
- $\forall x \forall y \forall z ((x < y) \wedge (y < z) \implies (x < z))$ *(transitivity),*
- $\forall x \forall y (\neg (x = y) \implies (x < y) \vee (y < x))$ *(linearity).*

If A is an arbitrary set, then $\mathcal{A} = \langle A, < \rangle$ is a linear order if $\mathcal{A} \models \Gamma$.

Theorem 2.28 *The class Δ of all finite linear orders is a Fraïssé class.*

Proof. Clearly, Δ is countable, has the HP and the JEP.

AP: Let $\mathcal{A},\mathcal{B},\mathcal{C} \in \Delta$, where $\mathcal{A} = \langle A, <_{\mathcal{A}} \rangle$, $\mathcal{B} = \langle B, <_{\mathcal{B}} \rangle$, $\mathcal{C} = \langle C, <_{\mathcal{C}} \rangle$, and $f_0 : \mathcal{A} \to \mathcal{B}$, $f_1 : \mathcal{A} \to \mathcal{C}$ are embeddings. Using isomorphic copies, we can assume that $A = B \cap C$ and

$$<_{\mathcal{A}} = <_{\mathcal{B}} |_A = <_{\mathcal{C}} |_A.$$

We will define a structure $\mathcal{D} \in \Delta$ with domain $B \cup C$, and the relation $<_{\mathcal{D}}$ will extend $<_{\mathcal{B}}$ and $<_{\mathcal{C}}$. So let $<_{\mathcal{D}} \subseteq {}^2(B \cup C)$ such that $<_{\mathcal{B}}, <_{\mathcal{C}} \subseteq <_{\mathcal{D}}$ and satisfies the following:

$$\text{if } a, a' \in A, \text{ then } a <_{\mathcal{D}} a' \text{ if and only if } a <_{\mathcal{A}} a';$$

$$\text{if } b, b' \in B, \text{ then } b <_{\mathcal{D}} b' \text{ if and only if } b <_{\mathcal{B}} b';$$

$$\text{if } c, c' \in C, \text{ then } c <_{\mathcal{D}} c' \text{ if and only if } c <_{\mathcal{C}} c';$$

Suppose that $|A| = n$. Then the elements of A partition both B and C into $n+1$ sections

$$B_0, ..., B_n \text{ and } C_0, ..., C_n.$$

So that a section with a larger index contains larger elements. If $b \in B \setminus A$ and $c \in C \setminus A$ and there is $i \in n+1$ such that $b \in B_i$ and $c \in C_i$ then let

$$b <_{\mathcal{D}} c;$$

otherwise there are $i, j \in n+1$, with $j \neq j'$ (WLOG $j < j'$) such that $b \in B_j$ and $c \in C_{j'}$ then let

$$b <_{\mathcal{D}} c.$$

Then it is easy to check that $\mathcal{D} \models \Gamma$, so $\mathcal{D} \in \Delta$. Thus, we have shown that Δ has AP.

∎

Definition 2.29 *We call an L-structure $\mathcal{A} \in \mathrm{Mod}(\Gamma)$ a dense linear order without endpoints if it satisfies the L-theory Γ' that consists of the formulas of Γ and:*

- $\forall x \forall y \exists z (x < y \implies x < z < y)$ *(dense in itself),*
- $\forall x \exists y \exists z (y < x < z)$ *(without endpoints).*

it is a nice exercise to prove that $\langle \mathbb{Q}, < \rangle$ is unique model of Γ'. Now, Theorem 2.22 confirms this fact.

Theorem 2.30 $\langle \mathbb{Q}, < \rangle$ *is the Fraïssé limit of the class Δ (finite linear orders).*

Proof. We need to show that $\mathbb{Q}$ is homogeneous and the age of $\mathbb{Q}$ is Δ.

$Age(\mathbb{Q}) = \Delta$: any element of $Age(\mathbb{Q})$ is a finite substructure of $\mathbb{Q}$, so it is a finite linear order as well. On the other hand, it is clear that any element of Δ can be embedded into $\mathbb{Q}$, since the order of $\mathbb{Q}$ is dense.

To show that $\mathbb{Q}$ is homogeneous, we will using back and forth argument (as we did in the case of the random graph). Let $f : \mathcal{A} \to \mathcal{B}$ a partial isomorphism between two finite substructure of $\mathbb{Q}$. Enumerate the elements rationals, $\mathbb{Q} = \{q_i\}_{i \in \omega}$. We will build an increasing chain of partial isomorphism

$$f_0 = f \subseteq f_1 \subseteq f_2 \subseteq ...$$

whose union will be an automorphism of $\mathbb{Q}$, and for every $n \in \omega$ we get that $q_n \in \mathrm{Dom}(f_{2n})$ and $q_n \in \mathrm{Ran}(f_{2n+1})$. Assume that we already defined f_{2n-1}. Let

$$U = \{x \in \mathrm{Dom}(f_{2n-1} : x < q_n\}$$

$$V = \{x \in \mathrm{Dom}(f_{2n-1} : x > q_n\}$$

(Wlog $q_n \notin \mathrm{Dom}(f_{2n-1})$). Using that $\mathbb{Q}$ is dense in itself, we can find $y \in \mathbb{Q}$, such y is strictly less than the elements $f_{2-n}(U)$ and strictly greater then the elements of $f_{2-n}(V)$. Let

$$f_{2n}|_{\mathrm{Dom}(f_{2n-1})} = f_{2n} \quad \text{and} \quad f_{2n}(q_n) = y.$$

At each odd sage (2n+1) we symmetrically extend the partial map to cover the n-th vertex in the range. Therefore

$$g = \bigcup_{n \in \omega} f_n$$

is an automorphism of $\mathbb{Q}$. ∎

2.2.3 Finite groups

Let us now consider other languages the simple relational languages. Let L be the language of groups, so L contains exactly an binary function symbol $\cdot$, an unary function symbol $^{-1}$, a constant symbol 1, and consists the equality relation symbol.

Definition 2.31 *The theory* Π *of groups consists of the following formulas:*

- $\forall x \forall y \forall z (x \cdot (y \cdot z) = (x \cdot y) \cdot z)$ *(associativity),*
- $\forall x ((x \cdot 1) = (1 \cdot x) = x)$ *(neutral element),*
- $\forall x ((x \cdot x^{-1}) = x^{-1} \cdot x = 1)$ *(inverse).*

A structure $\mathcal{G} = \langle G, \cdot, ^{-1}, 1 \rangle$ *is a group if* $\mathcal{G} \models \Pi$.

Let Δ be the class of finite groups. It is obvious, that Δ has the HP. It is easy to see that Δ has the JEP, because the direct product of two finite group is also finite group.

Definition 2.32 *Let* $\mathcal{G}$ *be a finite group and* $\mathcal{H} \leq \mathcal{G}$ *a subgroup. For any element* $g \in G$, *the st*

$$gH = \{g \cdot h : h \in H\}$$

is called a left coset of $\mathcal{H}$ *in* $\mathcal{G}$.

Definition 2.33 *Let us* $\mathcal{G}$ *be a finite group and* $\mathcal{H} \leq \mathcal{G}$ *a subgroup, we say that* $S \subseteq H$ *is the left transversal of* $\mathcal{H}$ *is* $|S \cap gH| = 1$ *for every* $g \in G$. *For any element* $g \in \mathcal{G}$ *and its unique product decomposition* $g = s \cdot h$ *with* $s \in S$ *and* $h \in H$, *we define* $s = g^{\sigma}$ *and* $h = g^{-\sigma+1}$.

Theorem 2.34 *The class* Δ *of finite groups has the AP.*

Proof. Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \Delta$ be such that $\mathcal{A}$ can be embedded into $\mathcal{B}$ and $\mathcal{C}$. Without loss of generality, we can assume that $A = B \cap C$. We have to find a finite group $\mathcal{K}$, such that there are embeddings $\rho : \mathcal{B} \to \mathcal{K}$ and $\rho' : \mathcal{C} \to \mathcal{K}$ such that $\rho(a) = \rho'(a)$ for all $a \in A$.

$$\begin{tikzcd}
 & \mathcal{B} \arrow[dr, "\rho"] & \\
\mathcal{A} \arrow[ur, "\leq"] \arrow[dr, "\leq"'] & & \mathcal{K} \\
 & \mathcal{C} \arrow[ur, "\rho'"'] &
\end{tikzcd}$$

Choose a left transversal S, T of $\mathcal{A}$ for $\mathcal{B}$ and $\mathcal{C}$. Reintroduce the decompositions:

$$\text{for } b \in B,\ b = s \cdot h,\ s = b^{\sigma} \in S,\ h = b^{-\sigma+1} \in A$$

and

$$\text{for } c \in C,\ c = t \cdot h,\ t = c^{\tau} \in T,\ h = c^{-\tau+1} \in A.$$

Let $K = S \times T \times A$. For every $b \in B$ we define the permutation ρ_b as follows:

$$\rho_b : (s, t, a) \mapsto (s', t', a'),$$

where $t = t'$ and $s' \cdot a' = s \cdot a \cdot b$.

Since the decomposition of elements of $\mathcal{B}$ is unique, this is well defined. We can also write it as

$$(s, t, a)^{\rho(b)} = ((s \cdot a \cdot b)^{\sigma}, t, (s \cdot a \cdot b)^{-\sigma+1}).$$

Analogously, for any $c \in C$ we can define $\rho'(c) : (s, t, a) \mapsto (s, t', a')$, so that

$$(s, t, a)^{\rho'(c)} = (s, (t \cdot a \cdot c)^{\tau}, (t \cdot a \cdot c)^{-\tau+1}).$$

If $a_0 \in A$, then $\rho(a_0) = \rho'(a_0)$ because $a \cdot a_0 \in A$ and the decompositions σ and τ fix the representatives:

$$(s \cdot a \cdot a_0)^{\sigma} = s \text{ and } (t \cdot a \cdot a_0)^{\tau} = t,$$

so

$$(s, t, a)^{\rho(a_0)} = (s, t, a \cdot a_0) = (s, t, a)^{\rho'(a_0)}.$$

Now we also can view ρ as a map from $\rho : \mathcal{B} \to \mathrm{Sym}_K$. We claim that $\rho \circ {}^{-1}$ is an embedding where

$$\rho \circ {}^{-1} : b \mapsto \rho(b^{-1}).$$

ρ is injective because it has a trivial kernel: If $\rho(b) \in Id_K$, then for all $s \in S$ and $a \in A$:

$$(s \cdot a \cdot b)^{\sigma} = s \text{ and } (s \cdot a \cdot b)^{-\sigma+1} = a,$$

so $s \cdot a \cdot b = s \cdot a$, which implies that $b = 1$.

ρ is an anti-homomorphism ($\rho(x \cdot y) = \rho(y) \circ \rho(x)$) because if $b_0, b_1 \in B$ and $(s, t, a) \in K$, then

$$(s, t, a)^{\rho(b_1) \circ \rho(b_1)} = ((s \cdot a \cdot b_0)^{\sigma}, t, (s \cdot a \cdot b_0)^{-\sigma+1})^{\rho(b_1)} =$$
$$= (((s \cdot a \cdot b_0)^{\sigma} \cdot (s \cdot a \cdot b_0)^{-\sigma+1} \cdot b_1)^{\sigma}, t, ((s \cdot a \cdot b_0)^{\sigma} \cdot (s \cdot a \cdot b_0)^{-\sigma+1} \cdot b_1)^{-\sigma+1}) =$$
$$= ((s \cdot a \cdot b_0 \cdot b_1)^{\sigma}, t, (s \cdot a \cdot b_0 \cdot b_1)^{-\sigma+1}) = (s, t, a)^{\rho(b_0 \cdot b_1)}.$$

Therefore $\rho \circ {}^{-1}$ is an injective homomorphism. The same proof works for $\rho' \circ {}^{-1} : \mathcal{C} \to \mathrm{Sym}_K$. Therefore, we have shown that $\mathcal{B}$ and $\mathcal{C}$ can be embedded into $\mathrm{Sym}_K \in \Delta$, such that $\rho(a^{-1}) = \rho'(a^{-1})$ for all $a \in A$, so the proof is complete. ∎

Definition 2.35 *A group $\mathcal{G}$ is locally finite if every finitely generated subgroup of $\mathcal{G}$ is finite.*

Definition 2.36 *Hall's universal group is the unique, up to isomorphism, countable locally finite group $\mathcal{U}$ that satisfies the following conditons:*

- *every finite group can be embedded into $\mathcal{U}$;*
- *If $\mathcal{G}_0, \mathcal{G}_1 \in \Delta$ and $f_i : \mathcal{G}_i \to \mathcal{U}$ for $i \in 2$ are embeddings, then they are conjugate by some inner automorphism of $\mathcal{U}$.*

Hall's universal group was defined by Hall in 1959, who prove its existence in paper [13]. Note, by Theorem 2.16, that $\mathcal{U}$ is the Fraïssé limit of Δ. Since $\mathcal{U}$ is homogeneous, by definition. The class of finite groups is not uniformly locally finite, hence Hall's universal group does not satisfy the conditions of Theorem 2.22 . Next, we will see that the theory of this group is not ω-categorical, thus providing an example of a Fraïssé limit that is not ω-categorical.

Definition 2.37 *Let $\mathcal{G}$ be a group. The exponent of $\mathcal{G}$ is the least common multiple of the orders of all the elements of $\mathcal{G}$.*

The proof of the following theorem requires further model-theoretic tools that are far-reaching and not closely related to our topic, therefore we will state this theorem without proof. We refer the interested reader to [7].

Theorem 2.38 *If the theory $Th(\mathcal{G})$ of a group $\mathcal{G}$ is ω-categorical, then $\mathcal{G}$ has finite exponent.*

It is clear that Hall's universal group does not have finite exponent, since every finite group can be embedded in it, and thus there exist elements of arbitrarily large finite order. Consequently, from the previous theorem, it follows that the theory of Hall's universal group is not ω-categorical.

2.2.4 Further examples

There are numerous other examples of Fraïssé classes. Here we will list a few of them and their Fraïssé limits, without claiming completeness.

1. The Fraïssé limit of the class of finite K_n-free graphs $\Delta_{\neg(K_n)}$ is the $\mathcal{H}_n$ Henson graph. Since class $\Delta_{\neg(K_n)}$ is a uniformly locally finite, we have that $\mathrm{Th}(\mathcal{H}_n)$ is ω-categorical and has quantifier elimination. As we can also see in [7].

Algebraic structures:

Note that we need to be careful about the choice of signature. For example in the case of groups, if $L = \{1, \cdot\}$, then the substructures of a group will be submonoids, closed under $\cdot$ but not necessary containing inverses. To ensure that substructures we also need to put symbol for $^{-1}$. A similar situation holds for other algebraic structures as well, such as fields.

2. Let $L = \{0, 1, +, -, \cdot, ^{-1}\}$. Then the Fraïssé limit of the Fraïssé class of finite fields with characteristic-p Δ_p is $\overline{\mathbb{F}_p}$, where $\overline{\mathbb{F}_p}$ is the algebraic closure of $\mathbb{F}_p$. Since the class Δ_p is uniformly locally finite, we have that $\mathrm{Th}(\overline{\mathbb{F}_p})$ is ω-categorical and has quantifier elimination. See [7] and [11].

3. The Fraïssé limit of the class of finite abelian p-groups is $\mathbb{Z}[p^\infty]^{(\omega)}$ (the direct sum of countably many copies of the Prüfer p-group), where

$$\mathbb{Z}[p^\infty] = \{z \in \mathbb{C} : z^{p^n} = 1 \text{ for some } n \in \mathbb{Z}^+\}.$$

4. The Fraïssé limit of the class of finite abelian groups is $\bigoplus_{p \text{ is prime}} \mathbb{Z}[p^\infty]^{(\omega)}$.

Metric structures:
It is surprising that the classical Fraïssé theory has applications to metric spaces, as metric spaces are not typically viewed as first-order structures. However, the field of connecting classical Fraïssé theory with metric structures is indeed very active. We will now mention a few metric Fraïssé limits, many of whose properties we have understood through the lens of Fraïssé theory. However, we will not delve into these applications in this thesis; for those interested, we recommend the paper [12].

5. A metric space in which all distances are rational is called a rational-metric space. The Fraïssé limit of the class of finite rational metric spaces is called the rational Urysohn space. As we we can also see in [11]. In fact, the completion of the rational Urysohn space is the Urysohn universal space. Where the Urysohn universal space $(\mathcal{U}, d)$ is a well know complete separable metric space, such it contains an isometric copy of any finite metric space and any finite partial isometry can be extended to an isometry on the whole space.

6. The Fraïssé limit of the class of finite dimensional Hilbert spaces is the space l^2. See [12].

7. The Gurarij space. We recall that a Gurarij space is a separable Banach space $\mathcal{G}$ having the property that for any $\epsilon > 0$, finite dimensional Banach spaces $\mathcal{E} \subseteq \mathcal{F}$, and isometric embedding $\psi : \mathcal{E} \to \mathcal{G}$, there is a linear embedding $\phi : \mathcal{F} \to \mathcal{G}$ extending ψ such that in addition, for all $x \in F$,

$$(1 - \epsilon)\|x\| < \|\phi(x)\| < (1 + \epsilon)\|x\|.$$

   Note that the Gurarij space is unique up to isometric isomorphism. The Fraïssé limit of the class of finite dimensional Banach spaces is the Gurarij space $\mathcal{G}$. As we can see in paper [12].

3 Projective Fraïssé theory

Projective Fraïssé theory was initiated by Solecki [4] and Irwin in 2006 as the dual counterpart to classical Fraïssé theory. This framework provides a method for constructing a compact, homogeneous structure , known as the *projective Fraïssé limit*, as an inverse limit, from a suitably chosen class of compact structures called a *projective Fraïssé class*. Moreover, the authors of [4] demonstrated that the so-called pseudo-arc, a well-studied object from continuum theory, arises as a quotient of the projective Fraïssé limit. Using this construction, they proved that every chainable continuum can be realized as the continuous image of the pseudo-arc (see Theorem 3.28). They also obtained a new characterization of the pseudo-arc, which, for instance, implies that the group of homeomorphisms of the pseudo-arc is dense in the space of all onto continuous self-maps.

3.1 Preliminaries

In section 3, we will develop the dual theory following [4]. It is a natural idea that the duals of discrete first-order structures should be certain compact spaces, as seen, for instance, in Stone duality or Pontryagin duality.

Definition 3.1 *(Topological L-structure) Let L be a first order language. $\mathcal{D}$ is a topological L-structure if*

- *it is an L-structure;*
- *$\mathcal{D}$ is a 0-dimensional, second countable compact Hausdorff space;*
- *the functions on $\mathcal{D}$ are continuous;*
- *for every $k \in \omega$, the k-ary relations on $\mathcal{D}$ are closed subsets of D^k.*

Warning: The following definition of epimorphism requires more then what the name suggests. However it turns out to be a natural concept on our context . For instance, in Section 3.3, where we examine the class of finite linear graphs as topological L-structures, a morphism will be an epimorphism in the usual sense if and only if it is an epimorphism in the new sense as well.

Definition 3.2 *Let $\mathcal{E}$, $\mathcal{F}$ be topological L-structures. Then $\phi : \mathcal{E} \to \mathcal{F}$ is an epimorphism if it is a surjective continuous homomorphism, and for every relation R in L, if $\overline{y} \in R^{\mathcal{F}}$, then there exists $\overline{x} \in R^{\mathcal{E}}$ such that $\phi(\overline{x}) = \overline{y}$.*

Definition 3.3 *For topological L-structures $\mathcal{E}$ and $\mathcal{F}$, we say that $f : \mathcal{E} \to \mathcal{F}$ is an isomorphism if it is an injective epimorphism. The terminology is justified by the fact that such a map is a model-theoretic isomorphism and a homeomorphism at the a same time.*

As mentioned earlier, in the dual theory, instead of constructing the limit structure as a direct limit, we will build it using an appropriate inverse limit construction. Let us now turn to the technical preparations for this.

Definition 3.4 *(Inverse limit) Let $(\mathcal{D}_n)_{n\in\omega}$ be a sequence of topological L-structures and let*

$$\pi_n : \mathcal{D}_{n+1} \to \mathcal{D}_n$$

be epimorphisms for each n. The inverse limit of $(\mathcal{D}_n, \pi_n)_{n\in\omega}$, denoted by

$$\mathcal{D} = \varprojlim \mathcal{D}_n,$$

is the topological L-structure defined as follows:

- *The domain is*
$$D = \{\, x \in \prod_{n\in\omega} D_n : \pi_n(x_{n+1}) = x_n \textit{ for all } n \in \omega\},$$
equipped with the subspace topology inherited from $\prod_{n\in\omega} D_n$*.*

- *For each* m*-ary relation symbol* $R \in L$*, a tuple* $\bar{x} = (x^0, \ldots, x^{m-1}) \in D^m$ *lies in* $R^{\mathcal{D}}$ *if and only if*
$$(x^0_n, \ldots, x^{m-1}_n) \in R^{\mathcal{D}_n} \quad \textit{for every } n \in \omega,$$
where $x^i = (x^i_n)_{n\in\omega}$*.*

- *For each* m*-ary function symbol* $f \in L$*, the interpretation*
$$f^{\mathcal{D}} \colon D^m \to D$$
is given by
$$f^{\mathcal{D}}(x^0, \ldots, x^{m-1}) = (\, f^{\mathcal{D}_n}(x^0_n, \ldots, x^{m-1}_n)\,)_{n\in\omega}.$$

For each n*, let*
$$\pi^\infty_n \colon \mathcal{D} \to \mathcal{D}_n, \quad \pi^\infty_n(x) = x_n,$$
and for $n < m$ *define*
$$\pi^m_n = \pi_n \circ \pi_{n+1} \circ \cdots \circ \pi_{m-1} \colon \mathcal{D}_m \to \mathcal{D}_n.$$
Then for every relation R *and tuple* $(x^0, \ldots, x^{m-1}) \in D^m$*,*
$$(x^0, \ldots, x^{m-1}) \in R^{\mathcal{D}} \iff (\pi^\infty_n(x^0), \ldots, \pi^\infty_n(x^{m-1})) \in R^{\mathcal{D}_n} \quad \textit{for all } n,$$
and for every function f*,*
$$f^{\mathcal{D}}(x^0, \ldots, x^{m-1}) = (f^{\mathcal{D}_n}(\pi^\infty_n(x^0), \ldots, \pi^\infty_n(x^{m-1})))_{n\in\omega}.$$

Remark 3.5 *It is easy to sheck the following:*

(i) *the sets of the form* $B_{n,U} = \{x \in D : \pi^\infty_n(x) \in U\}$ *with* $n \in \omega$ *and* U *a basic clopen in* D_n*, is a basis for* D*;*

(ii) *the inverse limit* $\mathcal{D}$ *is a topological* L*-structure;*

(iii) *the maps* π^∞_n *and* π^m_n *is an epimorphism;*

(iv) $\mathcal{D} \cong \varprojlim \mathcal{D}_{n_k}$ *for every subsequence* $(n_k)_{k\in\omega}$*, where* $\pi_{n_k} = \pi^{n_{k+1}}_{n_k}$*.*

Naturally, Definition 3.4 can be generalized to the limit of an inverse system. We now attempt to dualize the definition of a Fraïssé class: from the JEP and the AP, we will obtain the projective JEP and projective AP (see (F1), (F2)), by replacing embeddings with epimorphisms and thereby reversing the direction of the morphisms (i.e., reversing the arrows).

Definition 3.6 *(Projective Fraïssé class) A class* Δ *of topological* L*-structures is a projective Fraïssé class if the following hold.*

(F1) *For any* $\mathcal{D}, \mathcal{E} \in \Delta$*, there exists* $\mathcal{F} \in \Delta$ *and epimorphisms* $\phi_0 : \mathcal{F} \to \mathcal{D}$ *and* $\phi_1 : \mathcal{F} \to \mathcal{E}$*.*

$$\begin{tikzcd}
\mathcal{D} & \\
 & \mathcal{F} \arrow[ul, "\phi_0"'] \arrow[dl, "\phi_1"] \\
\mathcal{E} &
\end{tikzcd}$$

(F2) *For any $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \Delta$ and epimorphisms $\phi_0 : \mathcal{D} \to \mathcal{C}$ and $\phi_1 : \mathcal{E} \to \mathcal{C}$, there exists $\mathcal{F} \in \Delta$ and epimorphisms $\psi_0 : \mathcal{F} \to \mathcal{D}$ and $\psi_1 : \mathcal{F} \to \mathcal{E}$ such that $\phi_0 \circ \psi_0 = \phi_1 \circ \psi_1$.*

$$\begin{tikzcd}
 & \mathcal{D} \arrow[ld, "\phi_0"'] & \\
\mathcal{C} & & \mathcal{F} \arrow[lu, dashed, "\psi_0"'] \arrow[ld, dashed, "\psi_1"] \\
 & \mathcal{E} \arrow[lu, "\phi_1"] &
\end{tikzcd}$$

As we have seen, the definition of a projective Fraïssé class does not explicitly require the HP or its dual. However, in the definition of a projective Fraïssé limit, in addition to universality (L1) and projective homogeneity (L3), we also require an additional property(L2).

Definition 3.7 *(Projective Fraïssé limit) Let Δ be a class of topological L-structures. Then a topological L-structure $\mathcal{D}$ is a projective Fraïssé limit of Δ if:*

(L1) *For any $\mathcal{B} \in \Delta$, then there exists an epimorphism $\phi : \mathcal{D} \to \mathcal{B}$.*

(L2) *For any a finite discrete topological space A and continuous map $f : \mathcal{D} \to A$, there exists a structure $\mathcal{B} \in \Delta$, an epimorphism $\phi' : \mathcal{D} \to \mathcal{B}$, and a function $f' : \mathcal{B} \to A$ such that $f = f' \circ \phi'$.*

$$\begin{tikzcd}
 & \mathcal{D} \arrow[ld, "f"'] \arrow[rd, "\phi'"] & \\
A & & \mathcal{B} \arrow[ll, dashed, "f'"]
\end{tikzcd}$$

(L3) *For any epimorphisms $\mathcal{B} \in \Delta$ and any $\phi_0 : \mathcal{D} \to \mathcal{B}$, $\phi_1 : \mathcal{D} \to \mathcal{B}$, there exists an isomorphism $\psi : \mathcal{D} \to \mathcal{D}$ such that $\phi_0 = \phi_1 \circ \psi$.*

$$\begin{tikzcd}
 & \mathcal{D} \arrow[ld, "\phi_0"'] \arrow[rd, dashed, "\psi"] & \\
\mathcal{B} & & \mathcal{D} \arrow[ll, "\phi_1"]
\end{tikzcd}$$

Let us prove three lemmas.

Lemma 3.8 *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be topological L-structures, let $\phi : \mathcal{C} \to \mathcal{B}$ be an epimorphism, and let $f : \mathcal{B} \to \mathcal{A}$ and $g : \mathcal{C} \to \mathcal{A}$ be maps such that $g = f \circ \phi$. Then g is an epimorphism if and only if f is an epimorphism.*

$$\begin{tikzcd}
 & \mathcal{B} \arrow[ld, "f"'] \\
\mathcal{A} & \\
 & \mathcal{C} \arrow[lu, "g"] \arrow[uu, "\phi"']
\end{tikzcd}$$

Proof. Most of the proof involves checking the definitions.

The only nontrivial implication is that if g is continuous, then f is also continuous. Assume the contrary, that g is continuous but there exists a sequence $z_n \to z$ in $\mathcal{B}$ such that $\lim f(z_n) \neq f(z)$. Pick a sequence $w_n \in \phi^{-1}(z_n)$. By the compactness, we may assume that $w_n \to w$ for some $w \in \mathcal{C}$; since ϕ is continuous, we have $\phi(w) = z$, and we obtain:

$$f(z_n) = g(w_n) \to g(w) = f(z),$$

which is a contradiction. ∎

Definition 3.9 *(Epimorphism refines an open cover) Let A be a topological space, and $\mathcal{V}$ is an open cover on it, another open cover $\mathcal{V}'$ is refines $\mathcal{V}$, if for every $V' \in \mathcal{V}'$ there exist $V \in \mathcal{V}$ such $V' \subseteq V$. Let $\mathcal{A}, \mathcal{B}$ be topological L-structures, an epimorphism $\phi : \mathcal{A} \to \mathcal{B}$ refines an open cover $\mathcal{U}$ of A if the open cover $\{\phi^{-1}(b) : b \in B\}$ refines $\mathcal{U}$.*

Lemma 3.10 *For a toplological L-structure $\mathcal{D}$, property (L2) is equivalent to the following.*

(L2') For all open covers $\mathcal{U}$ of $\mathcal{D}$, there exists $\mathcal{B} \in \Delta$ and an epimorphism $\phi : \mathcal{D} \to \mathcal{B}$ such that ϕ refines $\mathcal{U}$.

Proof. $(L2) \implies (L2')$ Let $\mathcal{U}$ be an open cover of $\mathcal{D}$, we may assume that $\mathcal{U}$ is a finite clopen partition of $\mathcal{D}$, sine in a compact zero dimensional space any open cover is refined by a finite clopen partition.

We can view $\mathcal{U}$ as a finite discrete space. For any $d \in D$, let $U_d \in \mathcal{U}$ be the unique element such that $d \in U_d$. Define $f : \mathcal{D} \to \mathcal{U}$ by $f(d) = U_d$, which is continuous. By (L2), there exists a structure $\mathcal{B} \in \Delta$, an epimorphism $\phi' : \mathcal{D} \to \mathcal{B}$, and a function $f' : \mathcal{B} \to \mathcal{U}$ such that $f = f' \circ \phi'$. Then ϕ' refines $\mathcal{U}$, since for $d \in B$, we have

$$\phi'^{-1}(d) \subseteq (f' \circ \phi')^{-1}(f'(d)) = f^{-1}(f'(d)) \in \mathcal{U}.$$

$$\begin{tikzcd} \mathcal{D} \arrow[rr, "f"] \arrow[dr, dashed, "\phi'"'] & & \mathcal{U} \\ & \mathcal{B} \arrow[ur, dashed, "f'"'] & \end{tikzcd}$$

$(L2') \implies (L2)$ Let A be a finite discrete space, and let $f : \mathcal{D} \to \mathcal{A}$ be continuous. Then $\mathcal{U} = \{f^{-1}(a) \mid a \in A\}$ is a clopen partition of $\mathcal{D}$. By (L2'), there exists $\mathcal{B} \in \Delta$ and an epimorphism $\phi : \mathcal{D} \to \mathcal{B}$ that refines $\mathcal{U}$. So for every $b \in B$, there exists a unique $a_b \in A$ such that $\phi^{-1}(b) \subseteq f^{-1}(a_b)$. Define $f' : \mathcal{B} \to \mathcal{A}$ by $f'(b) = a_b$. Then, for any $x \in \mathcal{D}$, there exists $b \in B$ such that $x \in \phi^{-1}(b) \subseteq f^{-1}(a_b)$. Therefore, we have $f(x) = a_b = f'(b) = f'(\phi(x))$. ∎

Lemma 3.11 *Let Δ be a projective Fraïssé class of finite topological L-structures, and let $\mathcal{D}$ be the projective Fraïssé limit of Δ. Then, for any $\mathcal{A}, \mathcal{B} \in \Delta$ and epimorphisms $\phi : \mathcal{A} \to \mathcal{B}$, $\psi : \mathcal{D} \to \mathcal{B}$, there exists an epimorphism $\chi : \mathcal{D} \to \mathcal{A}$ such $\phi \circ \chi = \psi$.*

$$\begin{tikzcd} & \mathcal{A} \arrow[d, "\phi"] \\ \mathcal{D} \arrow[ur, dashed, "\chi"] \arrow[r, "\psi"'] & \mathcal{B} \end{tikzcd}$$

Proof. By $(L1)$, there exists an epimorphism $\chi' : \mathcal{D} \to \mathcal{A}$, and by $(L3)$, there exists an isomorphism $\alpha : \mathcal{D} \to \mathcal{D}$ such that $\phi \circ \chi' \circ \alpha = \psi$. Let $\chi = \chi' \circ \alpha$.

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\chi'"] & \mathcal{A} \arrow[d, "\phi"] \\ \mathcal{D} \arrow[u, "\alpha"] \arrow[r, "\psi"'] & \mathcal{B} \end{tikzcd}$$

∎

3.2 Construction

In this subsection, we will prove the following theorem witch is the main theorem of projective Fraïssé theory.

Theorem 3.12 *Let Δ be a countable projective Fraïssé class of finite topological L-structures. Then there exists a projective Fraïssé limit of Δ, which is unique up to isomorphism of topological L-structures.*

The proof will consist of two parts: existence and uniqueness. In the first part, we will recursively construct an inverse limit from our projective Fraïssé class. We will need to ensure that the limit structure satisfies conditions (L1), (L2), and (L3). At each step, countably many tasks will arise, which, as in the classical Fraïssé construction (Theorem 2.16), we will list and solve with bookkeeping. In the second part of the proof, we will show that the resulting structure is unique up to isomorphism. To do this, we will take two structures that satisfy conditions (L1), (L2), and (L3), and using these conditions, we will recursively construct a third structure that is isomorphic to both. Now, let us proceed with the proof.

Proof. Existence:
We build an inverse sequence $(\mathcal{D}_n, \pi_n)$:

$$\mathcal{D}_0 \xleftarrow{\pi_0} \mathcal{D}_1 \xleftarrow{\pi_1} \mathcal{D}_2 \xleftarrow{\pi_2} \ldots,$$

where $\mathcal{D}_n \in \Delta$ and π_n is an epimorphism for all $n \in \omega$ such that the following hold.

(A) For any $\mathcal{B} \in \Delta$, there exists $n \in \omega$ and an epimorphism $\phi : \mathcal{D}_n \to \mathcal{B}$.

(B) For any $\mathcal{E}, \mathcal{F} \in \Delta$ and epimorphisms $\phi_0 : \mathcal{F} \to \mathcal{E}$, $\phi_1 : \mathcal{D}_n \to \mathcal{E}$, there exists $m > n$ and an epimorphism $\psi : \mathcal{D}_m \to \mathcal{F}$ such that $\phi_0 \circ \psi = \phi_1 \circ \pi^m_n$.

$$\begin{tikzcd}
\mathcal{D}_m \arrow[r, "\pi_n^m"] \arrow[d, "\psi"'] & \mathcal{D}_n \arrow[d, "\phi_1"] \\
\mathcal{F} \arrow[r, "\phi_0"'] & \mathcal{E}
\end{tikzcd}$$

We will construct $(\mathcal{D}_n, \pi_n)$ recursively. First, enumerate the elements of Δ as $(\mathcal{B}_n)_n$. We will use the odd steps to, ensure that condition (A) is satisfied. Assume that we already defined $\mathcal{D}_{2k}$. By (F1) we can find a structure $\mathcal{D}_{2k+1} \in \Delta$ a $\phi : D_{2k+1} \to \mathcal{B}_k$ and epimorphism $\pi_{2k} : \mathcal{D}_{2k+1} \to \mathcal{D}_{2k}$. To achieve (B) we will use bookkeeping to list and solve the tasks: for every $k \in \omega$ at stage k, countable many task form

$$\begin{tikzcd}
? \arrow[r, "?"] \arrow[d, "?"'] & \mathcal{D}_k \arrow[d, "\phi_1"] \\
\mathcal{F} \arrow[r, "\phi_0"'] & \mathcal{E}
\end{tikzcd}$$

arise, witch we enumerate as $(T_{k,j})_{j \in \omega}$.

Let $\iota : \omega^2 \to 2\mathbb{N}$ be a bijection (where $2\mathbb{N}$ is the set of nonnegative even numbers) such that for every $i, j \in \omega$ we have $\iota(i, j) \geq i$. In step $2n = \iota(i, j)$, we need to solve task $T_{i,j}$: for $\mathcal{E}_{i,j}, \mathcal{F}_{i,j} \in \Delta$, and epimorphisms $\phi^0_{i,j} : \mathcal{F}_{i,j} \to \mathcal{E}_{i,j}$, $\phi^1_{ij} : \mathcal{D}_i \to \mathcal{E}_{i,j}$ we need to find a structure $\mathcal{D}_{2n}$ and epimorphisms $\psi : \mathcal{D}_{2n} \to \mathcal{F}_{i,j}$ and $\pi^{2n}_i : \mathcal{D}_{2n} \to \mathcal{D}_i$ such that $\phi^0_{i,j} \circ \psi = \phi^1_{i,j} \circ \pi^{2n}_i$. We must also be careful because the epimorphism $\pi^{2n-1}_i : \mathcal{D}_{2n-1} \to \mathcal{D}_i$

is already given, hence we looking for an epimorphism $\pi_{2n-1} : \mathcal{D}_{2n} \to \mathcal{D}_{2n-1}$ for witch $\phi^1_{i,j} \circ \pi^{2n-1}_i \circ \pi_{2n-1} = \phi^0_{i,j} \circ \psi$ holds. Applying (F2),

$$\begin{tikzcd}
\mathcal{D}_{2n} \arrow[r, dashed, "\pi_{2n-1}"] \arrow[d, dashed, "\psi"'] & \mathcal{D}_{2n-1} \arrow[r, "\pi^{2n-1}_i"] & \mathcal{D}_i \arrow[d, "\phi^1_{i,j}"] \\
\mathcal{F}_{i,j} \arrow[rr, "\phi^0_{i,j}"'] & & \mathcal{E}_{i,j}
\end{tikzcd}$$

we obtain a topological L-structure $\mathcal{D}_{2n}$ and epimorphisms $\pi_{2n-1} : \mathcal{D}_{2n} \to \mathcal{D}_{2n-1}$ and $\psi : \mathcal{D}_{2n} \to \mathcal{F}_{i,j}$ we solve task $T_{i,j}$.

We claim that the inverse limit $\mathcal{D} = \lim_{\leftarrow}(\mathcal{D}_n, \pi_n)$ is a projective Fraïssé limit of Δ. By Remark 3.5 (ii), it is a topological L-structure. We need to verify that it satisfies (L1), (L2), and (L3).

(L1) By (A), for every $\mathcal{B} \in \Delta$, there exists $n \in \omega$ and an epimorphism $\phi : \mathcal{D}_n \to \mathcal{B}$. since π^∞_n is an epimorphism, $\phi \circ \pi^\infty_n$ witnesses property (L1).

(L2) By Lemma 3.10, fix a finite discrete topological space A. Let $f : \mathcal{D} \to A$ be a continuous map. Then

$$\mathcal{U} = \{f^{-1}(a) : a \in A\}$$

is a finite clopen partition of $\mathcal{D}$. It is easy to see that for large enough $n \in \omega$ the partition $\mathcal{V} = \{(\pi^\infty_n)^{-1}(d) : d \in D_n\}$ is a refinement of $\mathcal{U}$. Fix such $n \in \omega$. Now for any $d \in D_n$, there exists a unique $a_d \in A$ such that $(\pi^\infty_n)^{-1}(d) \subseteq f^{-1}(a_d)$. Then $f' : \mathcal{D}_n \to A$, $f'(d) = a_d$ is a good choice, that is, $f = f' \circ \pi^\infty_n$ hold.

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "f"] \arrow[dr, "\pi^\infty_n"'] & & A \\
& \mathcal{D}_n \arrow[ur, "f'"'] &
\end{tikzcd}$$

(L3) To show that $\mathcal{D}$ satisfies the condition, let us first consider the following claim.

Claim. If $k < n$ and $\chi_0 : \mathcal{D}_n \to \mathcal{D}_k$ is an epimorphism, then there exists an isomorphism $\chi : \mathcal{D} \to \mathcal{D}$ such that $\chi_0 \circ \pi^\infty_n = \pi^\infty_k \circ \chi$.

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\chi"] \arrow[d, "\pi^\infty_n"'] & \mathcal{D} \arrow[d, "\pi^\infty_k"] \\
\mathcal{D}_n \arrow[r, "\chi_0"'] & \mathcal{D}_k
\end{tikzcd}$$

Proof. We will construct two sequences $(n_i)_{i\in\omega}, (k_i)_{i\in\omega} \in \mathbb{N}^\omega$ by recursion so that $k_i < n_i < k_{i+1}$ for all $i \in \omega$. Let $k_0 = k$ and $n_0 = n$. We will define epimorhisms $\chi_i : \mathcal{D}_{n_i} \to \mathcal{D}_{k_i}$ and $\alpha_i : \mathcal{D}_{k_{i+1}} \to \mathcal{D}_{n_i}$ so that $\chi_i \circ \alpha_i = \pi^{k_{i+1}}_{k_i}$ and $\alpha_i \circ \chi_i = \pi^{n_{i+1}}_{n_i}$.

$$(*) \quad \begin{tikzcd}
\mathcal{D}_{k_{i+2}} \arrow[r, "\alpha_{i+1}"] \arrow[rr, bend right, "\pi^{k_{i+2}}_{k_{i+1}}"'] & \mathcal{D}_{n_{i+1}} \arrow[r, "\chi_{i+1}"] \arrow[rr, bend left, "\pi^{n_{i+1}}_{n_i}"] & \mathcal{D}_{k_{i+1}} \arrow[r, "\alpha_i"] \arrow[rr, bend right, "\pi^{k_{i+1}}_{k_i}"'] & \mathcal{D}_{n_i} \arrow[r, "\chi_i"] & \mathcal{D}_{k_i}
\end{tikzcd}$$

Let $\chi_0 = \chi$, and assume $k_0, n_0, .., k_i, n_i$ and $\chi_0, \alpha_0, \ldots, \alpha_{i-1}, \chi_i$ are defined. By property (B), we can find k_{i+1} and an epimorphism $\alpha_i : \mathcal{D}_{k_{i+1}} \to \mathcal{D}_{n_i}$ such that $\chi_i \circ \alpha_i = \pi^{k_{i+1}}_{k_i}$ and similarly for $\mathcal{D}_{n_{i+1}}$ and χ_{i+1}.

$$\begin{tikzcd} \mathcal{D}_{k_{i+1}} \arrow[r, "\pi^{k_{i+1}}_{k_i}"] \arrow[d, "\alpha_i"'] & \mathcal{D}_{k_i} \arrow[d, "\mathrm{Id}"] \\ \mathcal{D}_{n_i} \arrow[r, "\chi_i"'] & \mathcal{D}_{k_i} \end{tikzcd}$$

Subclaim. The sequences $(\chi_n)_{n\in\omega}$ and $(\alpha_n)_{n\in\omega}$ induce epimorphisms $\chi : \mathcal{D} \to \mathcal{D}$ and $\alpha : \mathcal{D} \to \mathcal{D}$ such that $\alpha \circ \chi = Id$.

Proof. Let

$$\pi_{(n_i)} : \varprojlim(\mathcal{D}_n, \pi_n) \to \varprojlim(\mathcal{D}_{n_i}, \pi^{n_{i+1}}_{n_i}) = \mathcal{D}_1$$

be the projection, which is an isomorphism by Remark 3.5 (iv), and similarly for $\pi_{(k_i)}$ and $\pi_{(k_{i+1})}$. Also,

$$\sigma : \mathcal{D}_0 = \varprojlim(\mathcal{D}_{k_i}, \pi^{k_{i+1}}_{k_i}) \to \varprojlim(\mathcal{D}_{k_{i+1}}, \pi^{k_{i+2}}_{k_{i+1}}) = \mathcal{D}_2, \quad \sigma(x)(i) = x(i+1)$$

is clearly an isomorphism. Define $\alpha' : \mathcal{D}_2 \to \mathcal{D}_1$ by $\alpha'(x)(i) = \alpha_i(x(i))$ and $\chi' : \mathcal{D}_1 \to \mathcal{D}_0$ by $\chi'(x)(i) = \chi_i(x(i))$.

We need to check that α' and χ' indeed map to $\mathcal{D}_1$ and $\mathcal{D}_0$ respectively and they are epimorphisms. (We only show these for α', the proof for χ' is similar.)

Relations. Let R be an m-ary relation symbol in L. Then $(x_0, \dots, x_{m-1}) \in R^{\mathcal{D}_2}$ if and only if

$$\text{for all } i \in \omega\ (x_0(i), \dots, x_{m-1}(i)) \in R^{\mathcal{D}_{k_{i+1}}},$$

which implies that

$$\text{for all } i \in \omega\ (\alpha_i(x_0(i)), \dots, \alpha_i(x_{m-1}(i))) \in R^{\mathcal{D}_{n_i}}.$$

By definition, this is equivalent to

$$\forall i \in \omega \quad (\alpha(x_0(i))_{i\in\omega}, \dots, \alpha(x_{m-1}(i))_{i\in\omega}) \in R^{\mathcal{D}_1}$$

By the definition of α', this is the same as

$$(\alpha'(x_0), ..., \alpha'(x_{n-1})) \in R^{\mathcal{D}_1}.$$

Let $(y_0, \dots, y_{m-1}) \in R^{\mathcal{D}_1}$. Then for all $i \in \omega$, we have

$$(y_0(i), \dots, y_{m-1}(i)) \in R^{\mathcal{D}_{n_i}}.$$

Now observe that for every $i \in \omega$ and $j \in m$ we have

$$\chi_{i+1}(y_j(i+1)) \in \alpha_i^{-1}(y_j(i))$$

by the commutativity of $(*)$. Since χ_{i+1} is a homomorphism, we also have

$$((\chi_{i+1}(y_0(i+1)))_{i\in\omega}, ..., (\chi_{i+1}(y_{m-1}(i+1)))_{i\in\omega}) \in R^{\mathcal{D}_2}$$

is a good preimage for $(y_0, ..., y_{m-1})$.

Functions. Similar to the case of relations (using that all α_i are epimorphisms).

α' is continuous. It is continuous coordinatewise since the α_i are continuous.

α' **is surjective.** Let $y \in \mathcal{D}_1$. Now $\chi_{i+1}(y_j(i+1)) \in \alpha_i^{-1}(y_j(i))$ shows that

$$(\chi_{i+1}(y(i+1)))_{i\in\omega} =$$

$$= (\alpha_i(\chi_{i+1}(y(i+1))))_{i\in\omega} = (y_{n_i})_{i\in\omega} = y,$$

hence α' is surjective. We conclude that α' is indeed an epimorphism.

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\alpha"] \arrow[d, "\pi_{(k_{i+1})}"'] & \mathcal{D} \arrow[r, "\chi"] \arrow[d, "\pi_{(n_i)}"] & \mathcal{D} \arrow[d, "\pi_{(k_i)}"] \\
\mathcal{D}_2 \arrow[r, "\alpha'"'] & \mathcal{D}_1 \arrow[r, "\chi'"'] & \mathcal{D}_0
\end{tikzcd}$$

Let $\alpha = \pi_{(n_i)}^{-1} \circ \alpha' \circ \pi_{(k_{i+1})}$ and $\chi = \pi_{(k_i)}^{-1} \circ \chi' \circ \pi_{(n_i)}$. Then $\chi \circ \alpha = Id$ is clear from the definitions of α', χ' and the commutativity of $(*)$, so χ and α are isomorphisms, which proves the subclaim. Now $\chi_0 \circ \pi_n^\infty = \pi_k^\infty \circ \chi$ follows from the definitions, which proves the claim.

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\chi"] \arrow[d, "\pi^\infty_{n_0}"'] & \mathcal{D} \arrow[d, "\pi^\infty_{k_0}"] \\
\mathcal{D}_{n_0} \arrow[r, "\chi_0"'] & \mathcal{D}_{k_0}
\end{tikzcd}$$

Now we return to the proof of property (L3). Fix any structure $\mathcal{B} \in \Delta$. Let $\phi_1 : \mathcal{D} \to \mathcal{B}$ and $\phi_2 : \mathcal{D} \to \mathcal{B}$ be epimorphisms. By (L2), there exist $n_1, n_2 \in \omega$, and there exist functions $\phi_1' : \mathcal{D}_{n_1} \to \mathcal{B}$ and $\phi_2' : \mathcal{D}_{n_2} \to \mathcal{B}$ such that $\phi_1 = \phi_1' \circ \pi^\infty_{n_1}$ and $\phi_2 = \phi_2' \circ \pi^\infty_{n_2}$. By Lemma 3.8, ϕ_1' and ϕ_2' are epimorphisms, and (B) implies that there exists $m > n_1, n_2$ and, $\psi : \mathcal{D}_m \to \mathcal{D}_{n_1}$ such that $\phi_1' \circ \psi = \phi_2' \circ \pi^m_{n_2}$. By the previous claim there is an isomorphism $\chi : \mathcal{D} \to \mathcal{D}$, such that $\pi^\infty_{n_1} \circ \chi = \psi \circ \pi^\infty_m$. We conclude that the following diagram commutates, hence $\phi_2 = \phi_1 \circ \chi$ holds.

$$\begin{tikzcd}
\mathcal{D} \arrow[dr, "\pi^\infty_{n_1}"] \arrow[dddr, bend right, "\phi_1"'] & & & \mathcal{D} \arrow[lll, "\chi"'] \arrow[dl, "\pi^\infty_m"'] \arrow[dddll, bend left, "\phi_2"] \\
& \mathcal{D}_{n_1} \arrow[dd, "\phi_1'"] & \mathcal{D}_m \arrow[l, "\psi"'] \arrow[d, "\pi^m_{n_1}"'] & \\
& & \mathcal{D}_{n_2} \arrow[dl, "\phi_2'"'] & \\
& \mathcal{B} & &
\end{tikzcd}$$

Thus, we have proven the existence part of the theorem.

UNIQUENESS:

Let $\mathcal{D}'$ and $\mathcal{D}''$ be projective Fraïssé limits of Δ. The plan is as follows: we will construct an inverse limit $\mathcal{D}$ from elements of Δ, so that $\mathcal{D}' \cong \mathcal{D} \cong \mathcal{D}''$.

Fix sequences $(U_n')_{n\in\omega}$ and $(U_n'')_{n\in\omega}$ of clopen sets in $\mathcal{D}'$ and $\mathcal{D}''$ that separate the points of $\mathcal{D}'$ and $\mathcal{D}''$ respectively. We construct an inverse sequence $(\mathcal{D}_n, \psi_n)_{n\in\omega}$, where $\mathcal{D}_n \in \Delta$ and $\psi_n : \mathcal{D}_{n+1} \to \mathcal{D}_n$ is an epimorphism for every $n \in \omega$. Simultaneously, we also define epimorphisms $\phi_n' : \mathcal{D}' \to \mathcal{D}_n$, $\phi_n'' : \mathcal{D}'' \to \mathcal{D}_n$ so that the following holds:

(1) $\phi_n' = \psi_n \circ \phi_{n+1}'$,

(2) $\phi''_i = \psi_i \circ \phi''_{i+1}$,

(3) $\phi''_i = \psi_i \circ \phi''_{i+1}$, ϕ'_{2n} refines $\{U'_n, D' \setminus U'_n\}$,

(4) ϕ''_{2n+1} refines $\{U''_n, D'' \setminus U''_n\}$.

.

$$\begin{tikzcd}
 & \vdots \arrow[d, "\psi_3"] & \\
\mathcal{D}' \arrow[ur] \arrow[r, "\phi'_3"] \arrow[dr, "\phi'_2"] \arrow[ddr, "\phi'_1"] \arrow[dddr, "\phi'_0"'] & \mathcal{D}_3 \arrow[d, "\psi_2"] & \mathcal{D}'' \arrow[ul] \arrow[l, "\phi''_3"'] \arrow[dl, "\phi''_2"'] \arrow[ddl, "\phi''_1"'] \arrow[dddl, "\phi''_0"] \\
 & \mathcal{D}_2 \arrow[d, "\psi_1"] & \\
 & \mathcal{D}_1 \arrow[d, "\psi_0"] & \\
 & \mathcal{D}_0 &
\end{tikzcd}$$

Let $\mathcal{D}_0 \in \Delta$ be arbitrary. By (L1), there is an epimorphism $\phi'_0 : \mathcal{D}' \to \mathcal{D}_0$, which refines $\{\emptyset, D'\}$. Assume we have already defined ϕ'_i for $i \leq 2n$ and ϕ''_j, ψ_j for all $j < 2n$. Then we have the following commutative diagram.

$$\begin{tikzcd}
 & \mathcal{D}_{2n} \arrow[d, "\psi_{2n-1}"] & \\
\mathcal{D}' \arrow[ur, "\phi'_2"] \arrow[r, "\phi'_{2n-1}"] \arrow[dr, "\phi'_{2n-2}"] \arrow[ddr] & \mathcal{D}_{2n-1} \arrow[d, "\psi_{2n-2}"] & \mathcal{D}'' \arrow[l, "\phi''_{2n-1}"'] \arrow[dl, "\phi_{2n-2}"'] \arrow[ddl] \\
 & \mathcal{D}_{2n-2} \arrow[d] & \\
 & \vdots &
\end{tikzcd}$$

By Lemma3.11, there exists $\phi''_{2n} : \mathcal{D}'' \to \mathcal{D}_{2n}$ such that $\phi''_{2n-1} = \psi_{2n-1} \circ \phi''_{2n}$, and by Lemma 3.10, there exists a $\mathcal{D}_{2n+1} \in \Delta$ and an epimorphism $\phi''_{2n+1} : \mathcal{D}'' \to \mathcal{D}_{2n+1}$, which refines

$$\mathcal{U}''_n = \{{\phi''_{2n}}^{-1}(d) \cap U''_n : d \in D_{2n}\} \cup \{{\phi''_{2n}}^{-1}(d) \cap (U''_n)^c : d \in D_{2n}\}.$$

Then we can defined ψ_{2n} as a follows, for any $d \in \mathcal{D}_{2n+1}$, let $\psi_{2n}(d)$ be the unique element of $\phi''_{2n}(\phi''^{-1}_{2n+1}(d))$. Then ψ_{2n} is well defined since ϕ''_{2n} refines $\{\phi''^{-1}_{2n}(d) : d \in D_{2n}\}$, and it is an epimorphism by Lemma 3.8. The same works for constructing $\phi'_{2n+1}, \psi_{2n+1}$ and ϕ'_{2n+2}. Let $\mathcal{D} = \varprojlim \mathcal{D}_n$. By symmetry, it remains to prove the following.

Claim: $\phi' : \mathcal{D}' \to \mathcal{D}$, $x \mapsto (\phi'_n(x))_{n \in \omega}$, is an isomorphism.

Proof:

ϕ' maps to $\mathcal{D}$. Since we have $\phi'_n = \psi_n \circ \phi'_{n+1}$ for all $n \in \omega$.

ϕ' is injective. If $x, y \in D'$ with $x \neq y$, then there exists $n \in \omega$ such that U'_n separates them, so $\phi'_{2n}(x) \neq \phi'_{2n}(y)$.

ϕ' **is surjective.** Pick any $z \in D$. Then $\bigcap_{n\in\omega} {\phi'_{2n}}^{-1}(z(n)) \neq \emptyset$, because it is a decreasing intersection of non-empty compact sets. So let $x \in \bigcap_{n\in\omega} {\phi'_{2n}}^{-1}(z(n))$, then $\phi'(x) = z$.

ϕ' **is continuous.** First note that sets of the form

$$U_{d,n} = \{z \in D : z(n) = d\}$$

form a basis in $\mathcal{D}$ by remark3.5 (v). on the other hand, for every $n \in \omega$ and $d \in D_n$, we have ${\phi'}^{-1}(U_{d,n}) = {\phi'_n}^{-1}(U_{d,n})$ is clopen because ϕ'_n is continuous.

ϕ' **is a homomorphism.** Since each ϕ'_n is a homomorphism.

Relations. Let R be a relation symbol in L and $(z_0, \ldots, z_{k-1}) \in R^{\mathcal{D}}$. For each $n \in \omega$, since ϕ'_n is an epimorphism, the set

$$K_n = \left(\phi_n'^{-1}(z_0(n)) \times ... \times \phi_n'^{-1}(z_{k-1}(n))\right) \cap R^{\mathcal{D}'}$$

is nonempty compact set in ${\mathcal{D}'}^k$. Also $K_{n+1} \subseteq K_n$ by $\phi'_n = \psi_n \circ \phi'_{n+1}$. Thus, there exists

$$(x_0, .., x_{k-1}) \in \bigcap_{n\in\omega} K_n.$$

Now $(x_0, .., x_{k-1}) \in R^{\mathcal{D}'}$ and $\phi'(x_j) = z_j$ for each $j \in k$.

Hence, we have established that $\mathcal{D}'$ and $\mathcal{D}''$ are isomorphic, which implies the uniqueness part of the theorem, this completes the proof. ∎

3.3 Example and applications

In this subsection, we will apply the projective Fraïssé theory we have developed. First, we show that finite linear graphs equipped with the discrete topology form a projective Fraïssé class. Then we demonstrate that a suitable (and very natural) quotient of its projective Fraïssé limit is the pseudo-arc. Furthermore, using the tools of projective Fraïssé theory, we present some interesting properties and a characterization of the pseudo-arc. This illustrates the potential of dual theory for proving non-trivial topological results. Since the publication of [4], a number of authors have explored projective Fraïssé theory and obtained interesting topological results. See, for example, [14].

3.3.1 Finite Linear Graphs

Now we fix the language L until the end of the thesis: let R be the only (binary) relation symbol apart from equality (=) in L.

Let Δ_0 be the class of all L-structures $\mathcal{A}$ such that:

1. $R^{\mathcal{A}}$ is reflexive: $\mathcal{A} \models \forall x(R(x, x))$,
2. $R^{\mathcal{A}}$ is symmetric $\mathcal{A} \models \forall x \forall y(R(x, y) \implies R(y, x))$,
3. for all $a \in A$, a has ≤ 3 neighbors (including itself)

$$\mathcal{A} \models \forall a, x_0, x_1, x_2 \big((\bigwedge_{i\in 3} R(a, x_i)) \implies (\bigvee_{i,j\in 3;\ i\neq j} x_i = x_j)\big),$$

4. there are exactly 2 elements $a \in A$ that have ≤ 2 neighbors (including themselves),

5. $\langle \mathcal{A}, R^{\mathcal{A}} \rangle$ is connected (as a graph).

We defined Δ_0 so that i elements of Δ_0 each vertex has a loop for technical reason. We my assume without loss of generality that if $\mathcal{A} \in \Delta_0$, then $A = \{0, 1, \ldots, n-1\}$ and $(\mathcal{A} \models R(x, y)) \iff |x - y| < 2$.

Remark 3.13 *Let $\mathcal{A}, \mathcal{B} \in \Delta_0$. it is easy to prove that a surjective map $f : \mathcal{A} \to \mathcal{B}$ is an epimorphism if and only if for all $i, j \in A$ with $|i - j| < 2$ we have $|f(i) - f(j)| < 2$. In particular, if $\mathcal{A}, \mathcal{B} \in \Delta_0$, then there exists an epimorphism $\phi : \mathcal{A} \to \mathcal{B}$ if and only if $|A| \geq |B|$.*

Theorem 3.14 *Δ_0 is a projective Fraïssé class.*

Proof.

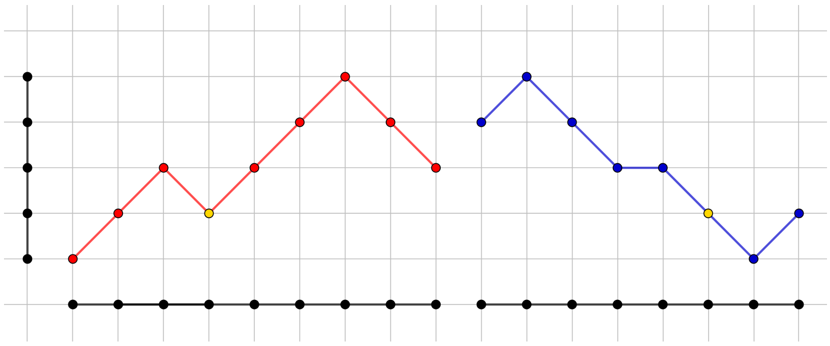
(F1): Let $\mathcal{D}, \mathcal{E} \in \Delta_0$ by arbitrary. Then there exists a structure $\mathcal{F} \in \Delta_0$, such that $|\mathcal{F}| > |\mathcal{D}|$ and $|\mathcal{F}| > |\mathcal{E}|$. By the previous remark, there exist epimorphisms $\phi_0 : \mathcal{F} \to \mathcal{D}$ and $\phi_1 : \mathcal{F} \to \mathcal{E}$.

(F2): Let $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \Delta_0$ be structures with epimorphisms $\phi_0 : \mathcal{D} \to \mathcal{C}$ and $\phi_1 : \mathcal{E} \to \mathcal{C}$. We need to find a structure $\mathcal{F} \in \Delta_0$ and epimorphisms $\psi_0 : \mathcal{F} \to \mathcal{D}$ and $\psi_1 : \mathcal{F} \to \mathcal{E}$ such that

$$(*) \quad \phi_0 \circ \psi_0 = \phi_1 \circ \psi_1.$$

In what follows follows, we reduce this task to another combinatorial problem.

The previous task is equivalent to the discrete mountain climbinging problem: we can view the graphs of functions ϕ_0 and ϕ_1 as mountain chains (see the figure). The two climbers begin climbing from points on the mountain chain that lie at the same elevation. At each step, each climber may move one unit left or right along the graph, or stay in place. We must provide a finite sequence of moves such that both climbers traverse their entire mountain chain while staying at the same elevation throughout.



Observe that if we denote the functions describing the horizontal movement of the two climbers by ψ_0 and ψ_1, then the pair (ψ_0, ψ_1) is a solution for the mountain climber problem then (ψ_0, ψ_1) is a witnesses to $(*)$ holds. Thus, to prove that the (F2) holds in Δ_0, it suffices to show that the discrete mountain climbing problem is solvable. In the follows, we will prove this.

Claim. We may assume there are no plateaus on the graphs ϕ_0 and ϕ_1, that is , for any $k \in 2$ and $i < \max(\mathrm{Dom}(\phi_k))$ we have $\phi_k(i) \neq \phi_k(i+1)$.

Proof. Cut out the plateaus. Any solution to the new problem gives a solution to the original problem: since any of the climber can stay still while the other climber can move through the plateau at any time. This prove the claim.

From this point on, we assume that the graphs of ϕ_0 and ϕ_1 contain no plateaus. The lemma below establishes the solvability of a special case of the discrete mountain climbing problem, we will use this result to tackle the more complex, general case.

Now we assume that $C = \{0, ...n_C\}$, $D = \{0, ...n_D\}$ and $E = \{0, ...n_E\}$.

Lemma 3.15 *Assume the following:*

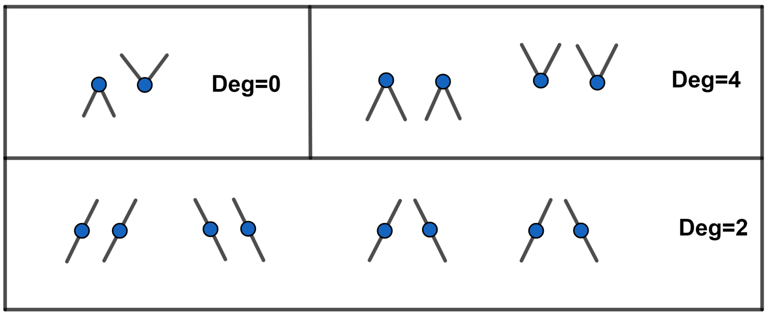
1. $\phi_0(i) = 0$ *and* $\phi_1(j) = 0$ *if and only if* $i, j = 0$,
2. $\phi_0(i) = n_C - 1$ *and* $\phi_1(j) = n_C - 1$ *if and only if* $i = n_D - 1$ *and* $j = n_E - 1$.

Then there is a solution to the discrete mountain climbing problem.

Proof. Consider the following graph $G = (V, H)$:

$$V = \{(i, j) \in \mathbb{N}^2 \colon i < n_D, j < n_E, \phi_0(i) = \phi_1(j)\},$$

and $\langle (i, j); (i', j') \rangle \in H$ if and only if $|i - i'| = |j - j'| = 1$. Observe that any path in G between $(0, 0)$ and (n_D, n_E) describes a solution to the discrete mountain climbing problem, thus it suffices to prove that $(0, 0)$ and (n_D, n_E) are in a same connected component. Also note that if $(i, j) \in V \setminus \{(0, 0), (n_D, n_E)\}$, then $\mathrm{Deg}(i, j)$ is even.



If $(i, j) \in \{(0, 0), (n_D, n_E)\}$, then $\mathrm{Deg}(i, j) = 1$.

Since the sum of degrees is even in every connected component of G, the vertices $(0, 0)$ and (n_D, n_E) must be in the same component, so there is a solution, therefore we proved the lemma.

We can assume that $n_D \leq n_E$. Let $a < b < n_D$ such that $\phi_1(a) = 0$ and $\phi_1(b) = n_C$. (The case when $\phi_1(a) = n_C$ and $\phi_1(b) = 0$ is similar).

Claim. To prove the theorem, it suffices to show that climber 0 can legally climb the graph of ϕ_0 while climber 1 only uses $\phi_1|_{[a,b]}$ (we refer to this climb as a restricted climb). By a legal climb, we mean that the climber ascends their mountain while remaining at the same elevation as the other climber throughout the process.

Proof. Without loss of generality, climber 0 can finish a restricted climb at the endpoint of an interval $[c, d]$ which has the same property for ϕ_0 as $[a, b]$ for ϕ_1. They can switch roles, so climber 1 can also climb the full graph of ϕ_1.

Claim. Climber 0 can legally climb the graph of ϕ_0 while climber 1 only uses $\phi_1|_{[a,b]}$.

Proof: Induction on n_C. For $n_C = 1, 2$, it is obvious. Assume it holds for $n_C \geq k$. The extrenum points of ϕ_0 divide $[0, n_D - 1]$ into subintervals. First, note that it suffices to solve the problem for these subintervals exactly. Second also note that each of these subintervals I falls into one of two categories: either one endpoint is a minimum point and the other a maximum point, in which case we can apply Lemma 3.15 to solve this subproblem, or either either both end pints are minimum points or both end points are maximum points, in witch case $\mathrm{Ran}(\phi_0|_I) \subsetneq \{0, ..., n_C - 1\}$, allowing us to apply the inductive hypothesis. This completes the proof of Theorem 3.14. ∎

3.3.2 Pseudo-arc

In this subsection, we construct the pseudo-arc as a quotient of the projective Fraïssé limit of Δ_0. As previously discussed, projective Fraïssé limits are universal and homogeneous structures. It is therefore not surprising that a factor of such a limit also yields a universal topological space.

The pseudo-arc is a well-studied chainable continuum whose universality and various topological characterizations have long been known. Projective Fraïssé theory provides a framework in which this space can be obtained as a quotient of an inverse limit, offering a new construction of the pseudo-arc. We begin this subsection by recalling several fundamental definitions from continuum theory, after which we proceed with the construction of the pseudo-arc.

Definition 3.16 *(Continuum) A continuum K is a nonempty compact, connected metric space. If $L \subset K$ is also a continuum, we say L is a subcontinuum of K, and if $L \neq K$, it is a proper subcontinuum of K. If K has more than one point, we say it is nondegenerate.*

Here are some examples of continua.

Examples 3.17

- *The closed interval* $[0, 1]$.
- *The circle.*
- *An arc (homeomorphic image of* $[0, 1]$*).*
- *The n-cell:* $[0, 1]^n$ *for some* $n \in \omega$.
- *The Hilbert cube:* $[0, 1]^\omega$.
- *The topologist's sine curve:* $(\{0\} \times [0, 1]) \cup \{(x, \sin(\frac{1}{x})) \colon x \in (0, 1]\}$.

Definition 3.18 *(Indecomposable continuum) A continuum K is decomposable if there exist proper subcontinua $L, L' \subset K$ such that $K = L \cup L'$. If K is not decomposable, we say it is indecomposable. If every subcontinuum of K is also indecomposable, we say K is hereditarily indecomposable.*

Remark 3.19 *All the previous examples (3.17) are decomposable.*

Definition 3.20 *(Chain of open subsets) Let X be a metric space. A chain is a finite sequence of open subsets of X, $\mathbf{C} = \{C_0, C_1, \ldots, C_{n-1}\}$ such that*

$$C_i \cap C_j \neq \emptyset \text{ if and only if } |i - j| \leq 1.$$

The open sets $C_0, C_1, \ldots, C_{n-1}$ are called the links of $\mathbf{C}$, C_0 and C_{n-1} are called the end links of $\mathbf{C}$, and $C_1, \ldots, C_{n-2}$ are called the interior links of $\mathbf{C}$. A chain $\mathbf{C}$ is an ϵ-chain if the diameter of all $C \in \mathbf{C}$ is less than ϵ.

Definition 3.21 *(Chainable continuum) A continuum K is chainable if for every $\epsilon > 0$, there is an ϵ-chain covering K.*

Let Δ_0 be the class of finite linear graphs as before.

Lemma 3.22 *Let $\mathcal{P}$ be the projective Fraïssé limit of Δ_0. Then $R^{\mathcal{P}}$ is an equivalence relation with classes of size at most 2.*

Proof. Recall that $\mathcal{P}$ can be written as an inverse limit of structures from Δ_0. Since $R^{\mathcal{A}}$ is reflexive and symmetric for every element $\mathcal{A} \in \Delta_0$, it follows that $R^{\mathcal{P}}$ is also reflexive and symmetric. thus it suffices to show that there are no distinct elements $x, y, z \in \mathcal{P}$ such that

$$R^{\mathcal{P}}(x, y) \wedge R^{\mathcal{P}}(y, z) \text{ holds.}$$

Suppose there exist such elements $x, y, z \in \mathcal{P}$, and let $X \cup Y \cup Z = \mathcal{P}$ be a clopen a partition such that $x \in X$, $y \in Y$, and $z \in Z$. By (L2'), there is $\mathcal{D} \in \Delta_0$ and an epimorphism $\phi : \mathcal{P} \to \mathcal{D}$ that refines the partition. So, $a = \phi(x)$, $b = \phi(y)$, and $c = \phi(z)$ are pairwise distinct elements, and

$$R^{\mathcal{D}}(a, b) \wedge R^{\mathcal{D}}(b, c) \text{ holds.}$$

Fix $\mathcal{D}' \in \Delta_0$ and distinct elements $a', b', b'', c' \in D'$, such $R^{\mathcal{D}'}(a', b')$, $R^{\mathcal{D}'}(b', b'')$, and $R^{\mathcal{D}'}(b'', c')$ hold and an epimorphism $\psi : \mathcal{D}' \to \mathcal{D}$ such that $\psi^{-1}(a) = \{a'\}$, $\psi^{-1}(b) = \{b', b''\}$, and $\psi^{-1}(c) = \{c'\}$ holds. (Such an epimorphism clearly exists.)

By Lemma 3.11, there exists an epimorphism $\chi : \mathcal{P} \to \mathcal{D}'$ such $\phi \circ \chi = \psi$.

$$\begin{tikzcd} & \mathcal{D}' \arrow[d, "\phi"] \\ \mathcal{P} \arrow[ur, dashed, "\chi"] \arrow[r, "\psi"'] & \mathcal{D} \end{tikzcd}$$

Then $\chi(x) = a'$, $\chi(z) = c'$, and either $\chi(y) = b'$ or $\chi(y) = b''$, hence $(\chi(x), \chi(y)) \notin R^{\mathcal{D}'}$ or $(\chi(y), \chi(z)) \notin R^{\mathcal{D}'}$, contradicting the assumption that χ is an epimorphism. ∎

Lemma 3.23 *Let $\mathcal{D}$ be a topological L-structure, and let $R^{\mathcal{D}}$ be an equivalence relation. If every open cover of $\mathcal{D}$ is refined by an epimorphism $\phi : \mathcal{D} \to \mathcal{D}'$ for some $\mathcal{D}' \in \Delta_0$ then $\mathcal{D}/R^{\mathcal{D}}$ is a chainable continuum.*

Proof. It follows from well-known facts from basic topology, that $\mathcal{D}/R^{\mathcal{D}}$ is a second countable compact Hausdorff space [15]. We will prove the following.

(1.) $\mathcal{D}/R^{\mathcal{D}}$ is connected.

(2.) $\mathcal{D}/R^{\mathcal{D}}$ is chainable.

(1.) To prove connectivity it suffices to show that for any nontrivial clopen partition $U \cup V = D$, there are $x \in U$ and $y \in V$ such that $R^{\mathcal{D}}(x, y)$. Fix $\mathcal{D}' \in \Delta_0$ and an epimorphism $\phi : \mathcal{D} \to \mathcal{D}'$ that refines the cover $\{U, V\}$. Pick $d_0 \in \phi(U)$ and $d_1 \in \phi(V)$ such that $(d_0, d_1) \in R^{\mathcal{D}'}$. Since ϕ is an epimorphism, there exist $x \in \phi^{-1}(d_0)$ and $y \in \phi^{-1}(d_1)$ such that $(x, y) \in R^{\mathcal{D}}$, and since ϕ refines $\{U, V\}$, we have $\phi^{-1}(d_0) \subseteq U$ and $\phi^{-1}(d_1) \subseteq V$.

(2.) Now, fix any $\epsilon > 0$. Since $\mathcal{D}$ is compact and $\mathcal{D}/R^{\mathcal{D}}$ is metrizable, the map $\sigma : \mathcal{D} \to \mathcal{D}/R^{\mathcal{D}}$ is uniformly continuous. Let $B_0 \cup B_1 \cup \cdots \cup B_{n-1} = D$ be a clopen partition such that $\mathrm{diam}(\sigma(B_i)) < \epsilon$ for all $i \in \{0, 1, \ldots, n-1\}$. Fix a $\mathcal{D}' \in \Delta_0$ and an epimorphism $\phi : \mathcal{D} \to \mathcal{D}'$ that refines the cover $\{B_0, B_1, \ldots, B_{n-1}\}$.

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\phi} & \mathcal{D}' \\ {\scriptstyle\sigma}\downarrow & & \\ \mathcal{D}/R^{\mathcal{D}} & & \end{array}$$

For every $d \in \mathcal{D}'$, let $U_d = \phi^{-1}(d)$. Then $\{U_d : d \in D'\}$ is a clopen cover of D. Note that:

$$\sigma(U_{d_0}) \cap \sigma(U_{d_1}) \neq \emptyset \iff \text{ there exist } x \in U_{d_0} \text{ and } y \in U_{d_1} \text{ such that } (x, y) \in R^{\mathcal{D}} \iff$$

$$\iff (d_0, d_1) \in R^{\mathcal{D}'}.$$

The first equivalence holds by the definiton of σ and the second eqvivalence follows from the fact that ϕ is an epimorphism.

Since ϕ refines the cover $\{B_0, B_1, \ldots, B_{n-1}\}$, we have that $\{\sigma(U_{d_0}), \ldots, \sigma(U_{d_{m-1}})\}$ is an "ϵ-chain of compact sets" that covers $\mathcal{D}/R^{\mathcal{D}}$. Therefore, there exists a $\delta > 0$ such that $\{\sigma(U_{d_0})_\delta, \ldots, \sigma(U_{d_{m-1}})_\delta\}$ is an ϵ-chain. ∎

Theorem 3.24 *(Bing [16]) The pseudo-arc is the unique (up to homeomorphism) non-degenerate chainable, hereditarily indecomposable continuum.*

We do not prove this theorem.

Theorem 3.25 *$\mathcal{P}/R^{\mathcal{P}}$ is homeomorphic to the pseudo-arc.*

Proof. By Lemma 3.23, $\mathcal{P}/R^{\mathcal{P}}$ is a chainable continuum, thus by Bing's theorem, it remains to prove that $\mathcal{P}/R^{\mathcal{P}}$ is hereditarily indecomposable.

Definition 3.26 *For a topological L-structure $\mathcal{D}$, a set $A \subseteq \mathcal{D}$ is:*

- *R-invariant, if for any $x \in A$ and $y \in D$ we have $R^{\mathcal{D}}(x, y) \implies y \in A$,*
- *R-connected if there is no decomposition $A = A_0 \cup A_1$ into relatively closed sets such that if $x \in A_i$ and $y \in A_j$ and $R^{\mathcal{D}}(x, y)$ then $i = j$.*

Let $\sigma : \mathcal{P} \to \mathcal{P}/R^{\mathcal{P}}$ be the quotient map. Suppose for contradiction that there exists a subcontinuum $X \subsetneq \mathcal{P}/R^{\mathcal{P}}$ with a nontrivial decomposition $X = X_0 \cup X_1$ is into subcontinua. Let $F = \sigma^{-1}(X)$, $F_0 = \sigma^{-1}(X_0)$, $F_1 = \sigma^{-1}(X_1)$.

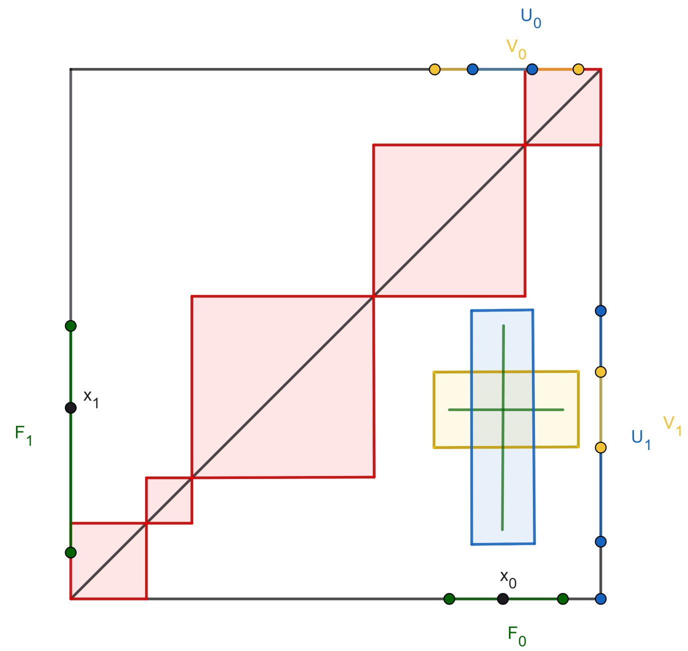
Claim 1. F, F_0, and F_1 are R-invariant and R-connected in $\mathcal{P}$.

Proof: It is clear that F, F_0, and F_1 are unions of R-classes, so they are R-invariant. Suppose that F is not R-connected (the same argument works for F_0 and F_1). Let $F = F' \cup F''$ be a decomposition, that witnesses this. Since σ is onto and continuous, we have $X = \sigma(F') \cap \sigma(F'')$, where $\sigma(F')$ and $\sigma(F'')$ are compact. Since there is no $x \in F'$ and $y \in F''$ with $R^{\mathcal{P}}(x, y)$, the sets $\sigma(F')$ and $\sigma(F'')$ are disjoint. This contradict the fact that X is connected, witch proves the claim.

Since X is connected, we have that $X_0 \cap X_1 \neq \emptyset$, which also implies that $F_0 \cap F_1 \neq \emptyset$. Since σ is surjective, it suffices to show that $F_0 \subseteq F_1$ or $F_1 \subseteq F_0$, as this implies $X_0 \subseteq X_1$ or $X_1 \subseteq X_0$. Suppose for contradiction that $F_0 \setminus F_1 \neq \emptyset$ and $F_1 \setminus F_0 \neq \emptyset$. Fix $x_0 \in F_0 \setminus F_1$ and $x_1 \in F_1 \setminus F_0$. Since $R^{\mathcal{P}}$ is closed, $\mathcal{P}$ is zero dimensional, and $\{x_0\} \times F_1$ and $F_0 \times \{x_1\}$ are disjoint from $R^{\mathcal{P}}$, there are clopen sets $U_0, U_1, V_0, V_1 \subseteq P$ such that:

$$\{x_0\} \times F_1 \subseteq U_0 \times U_1 \subseteq P^2 \setminus R^P \quad \text{and} \quad F_0 \times \{x_1\} \subseteq V_0 \times V_1 \subseteq P^2 \setminus R^{\mathcal{P}}.$$

Note that $U_0 \cap U_1 = \emptyset$ and $V_0 \cap V_1 = \emptyset$ since $R^{\mathcal{P}}$ contains the diagonal of P^2.

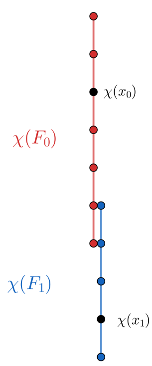


Fix $\mathcal{D} \in \Delta_0$ and an epimorphism $\chi : \mathcal{P} \to \mathcal{D}$ that refines the partition by the atoms of the set algebra $\mathcal{B} = \langle U_0, U_1, V_0, V_1 \rangle$. Since F_0, F_1, and F are R-connected and χ is an epimorphism, it is easy to check that $\chi(F_0)$, $\chi(F_1)$, and $\chi(F)$ are also R-connected, since the image of an R-connected set under an epimorphism is also R-connected. Moreover, it is easy to see that in finite lienar fraphs, the R-connected sets are exactly the intervals. Alos note that $X = X_0 \cup X_1 \implies F = F_0 \cup F_1 \implies \chi(F) = \chi(F_0) \cup \chi(F_1)$.

Since χ refines the atoms of $\mathcal{B} = \langle U_0, U_1, V_0, V_1 \rangle$, we have $\chi(x_0) \notin \chi(F_1)$ and $\chi(x_1) \notin \chi(F_0)$, since $U_0 \times U_1 \cap R^{\mathcal{P}} = \emptyset$ and $R^{\mathcal{P}}$ contain the diagonal, thus the open sets U_0, U_1 separate x_0 and F_1 from each other, similar the open sets V_0, V_1 separate x_1 and F_0, which implies that $\chi(F_0) \setminus \chi(F_1) \neq \emptyset$ and $\chi(F_1) \setminus \chi(F_0) \neq \emptyset$. Also, we have observed that $F_0 \cap F_1 \neq \emptyset$, so $\chi(F_0 \cap F_1) \subseteq \chi(F_0) \cap \chi(F_1) \neq \emptyset$.

Claim 2. There is no $y \in \chi(F_1)$ such that y and $\chi(x_0)$ are neighbors in $\mathcal{D}$. Similarly, there is no $y' \in \chi(F_0)$ such that y' and $\chi(x_1)$ are neighbors in $\mathcal{D}$.

Proof: Assume for contradiction that there is $y \in \chi(F_1)$ such that y and $\chi(x_0)$ are neighbors in $\mathcal{D}$. Since χ refines the partition by the atoms of $\mathcal{B} = \langle U_0, U_1, V_0, V_1 \rangle$, we have $\chi^{-1}(\chi(x_0)) \subseteq U_0$ and $\chi^{-1}(y) \subseteq U_1$, and thus $(\chi^{-1}(\chi(x_0)) \times \chi^{-1}(y)) \cap R^{\mathcal{P}} = \emptyset$, which is a contradiction because χ is an epimorphism.

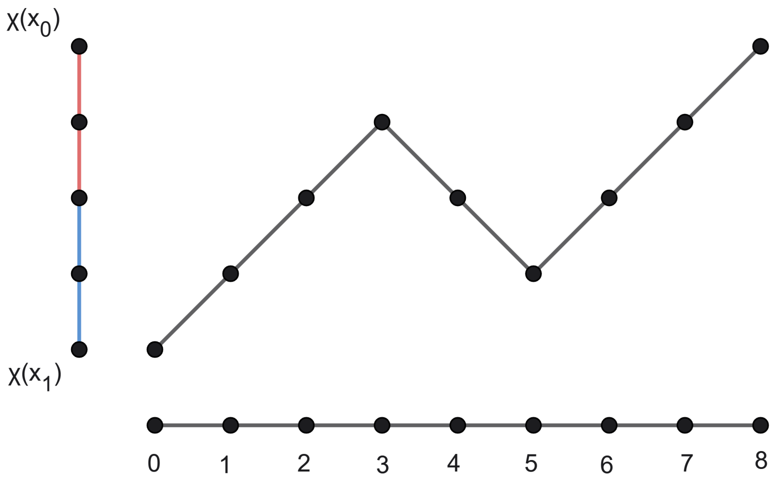


By Claim 2 and $\chi(F_0) \cap \chi(F_1) \neq \emptyset$, we have $|\chi(F)| \geq 5$. By composing an epimorphism after χ if necessary, we may assume that $D = \{0, 1, 2, 3, 4\}$ and $(i, j) \in R^{\mathcal{D}}$ if and only if $|i - j| \leq 1$ and

$$\chi(F_0) = \{2, 3, 4\}, \quad \chi(F_1) = \{0, 1, 2\}, \quad \chi(x_0) = 4, \quad \chi(x_1) = 0.$$

Potentially we lose the property that χ is refines the partition induced by the atoms of $\mathcal{B}$ but at this point that no matters. Now consider the following finite lieal graph $\mathcal{C} \in \Delta$, where $C = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and $(i, j) \in R^{\mathcal{C}}$ if and only if $|i - j| \leq 1$. Define the epimorphism $\phi : \mathcal{C} \to \mathcal{D}$ as follows:

$$\phi(0) = 0,\ \phi(1) = 1,\ \phi(2) = 2,\ \phi(3) = 3,\ \phi(4) = 2,\ \phi(5) = 1,\ \phi(6) = 2,\ \phi(7) = 3,\ \phi(8) = 4.$$



By Lemma 3.11, there is an epimorphism $\psi : \mathcal{P} \to \mathcal{C}$such that $\chi = \phi \circ \psi$.

$$\begin{tikzcd} \mathcal{P} \arrow[r, dashed, "\psi"] \arrow[dr, "\chi"'] & \mathcal{C} \arrow[d, "\phi"] \\ & \mathcal{D} \end{tikzcd}$$

Then we must have $\psi(x_0) = 8$ and $\psi(x_1) = 0$, so $\psi(F) = C$ since F is R-connected. Also note that

$$\{0, 1, 5\} \subseteq \psi(F_1) \subseteq \{0, 1, 2, 4, 5, 6\},$$

since $\{0, 1\}\chi(F_0) = \emptyset$ and $\chi(F_1) \subseteq \{0, 1, 2\}$. Thus $\psi(F_1)$ cis not an interval, contradiction. Therefore the proof of Theorem 3.25 is complete. ∎

3.3.3 Applications

In this section, we will apply the projective theory to prove topological properties of the pseudo-arc. In the main theorem (3.28), we provide a new proof of the fact that every chainable continuum is the continuous image of the pseudo-arc, and, in addition, we prove an interesting characterization of the pseudo-arc, which was not known before the paper [4] by Solecki and Irwin.

Let $\mathcal{P}$ be the projective Fraïssé limit of the class Δ_0 of finite linear graphs and $\mathcal{P}/R^{\mathcal{P}}$ be the pseudo-arc, as before.

Consider the following "approximate projective homogeneity" property:

Definition 3.27 *(Approximate projective homogeneity)*
Let C be a compact metric space. Then C has the property $(*)$ if

($*$) *for every (X,d) chainable continuum (X,d), surjective continuous maps $f_0 : C \to X$, $f_1 : C \to X$ and f $\epsilon > 0$, there exists a homeomorphism $h : C \to C$ such that $d_U(f_0, f_1 \circ h) < \epsilon$, where d_U detonates the uniform metric.*

$$\begin{tikzcd} C \arrow[rr, dashed, "h"] \arrow[dr, "f_0"'] & & C \arrow[dl, "f_1"] \\ & X & \end{tikzcd}$$

We will also use the notation $(*)_C$ whenever C has the property $(*)$.

Let us note that if a chainable continuum C has property $(*)$, then homeomorphisms form a dense set in the space of continuous surjections from C to itself for any continuous surjective $f_0 : C \to C$ and $\epsilon > 0$, apply $(*)$ with $f_1 = \mathrm{Id}$.

Theorem 3.28

1. *Every chainable continuum is the continuous image of the pseudo-arc.*
2. *$\mathcal{P}/R^{\mathcal{P}}$ has the property of $(*)$.*
3. *Up to homeomorphism there is at most one non degenerate chainable continuum $\mathcal{C}$, that has the property $(*)$.*

Definition 3.29 *(Special L-structure) An L-structure $\mathcal{D}$ is special if the following holds:*

(α) *Each open cover of $\mathcal{D}$ is refined by an epimorphism $\phi : \mathcal{D} \to \mathcal{C}$ onto some $\mathcal{C} \in \Delta_0$.*

(β) *$R^{\mathcal{D}}$ is an equivalence relation with classes of size at most 2.*

Definition 3.30 *Let (X,d) be a chainable continuum, then a chain $\boldsymbol{C} = \{U_0, U_1, ..., U_{n-1}\}$ on X is δ-fine if the following holds:*

(C1) $dist(U_i, U_j) > \delta$ if $|i-j| > 1$,

(C2) for all $i \in n$, there is $x \in U_i$ such that $dist(\{x\}, \bigcup_{i \neq j} U_j) > \delta$,

(C3) for any $A \subseteq X$, with $\mathrm{diam}(A) < \delta$ then there is $i \in n$ such that $A \subseteq U_i$.

A chain $\boldsymbol{C}$ is fine if it is δ-fine for some $\delta > 0$.
The chain $\boldsymbol{C}$ closure-refines a cover $\mathcal{V}$ of X if for all $i \in n$ the set $\overline{U_i}$ lies in some element of $\mathcal{V}$.

The previous definition is very natural. Using the Lebesgue number lemma, it is easy to see that a continuum is chainable if and only if it is δ-fine chainable with enough small δ.

Lemma 3.31 *Let $\mathcal{D}_0$ and $\mathcal{D}_1$ be special L-structures, $\phi : \mathcal{D}_0 \to \mathcal{D}_1$ be an epimorphism, $\rho_0 : \mathcal{D}_0 \to \mathcal{D}_0/R^{\mathcal{D}_0}$ and $\rho_1 : \mathcal{D}_1 \to \mathcal{D}_1/R^{\mathcal{D}_1}$ be quotient maps. Then:*

1. *there is a continuous surjection map $\phi^* : \mathcal{D}_0/R^{\mathcal{D}_0} \to \mathcal{D}_1/R^{\mathcal{D}_1}$ such that $\phi^* \circ \rho_0(x) = \rho_1 \circ \phi(x)$ for all $x \in D_0$,*

2. if ϕ is an isomorphism, then ϕ^* is a homeomorphism.

$$\begin{tikzcd} \mathcal{D}_0 \arrow[r, "\phi"] \arrow[d, "\rho_0"'] & \mathcal{D}_1 \arrow[d, "\rho_1"] \\ \mathcal{D}_0/R^{\mathcal{D}_0} \arrow[r, "\phi^*"'] & \mathcal{D}_1/R^{\mathcal{D}_1} \end{tikzcd}$$

Proof. Just checking the definitions. ∎

Remark 3.32 *If X is a chainable continuum, and $\mathcal{U}$ is an open cover of X, then $\mathcal{U}$ can be closure refined by a fine chain.*

Proof. Let $\mathcal{U}$ is an open cover of X, then by the Lebesgue number lemma, there exists $\delta > 0$ such every subset X with diameter less then δ is covered by an element of $\mathcal{U}$. Since X is chainable then there is a $\frac{\delta}{2}$-fine chain $\mathbf{C}$. Therefore $\mathbf{C}$ is closure refines $\mathcal{U}$. ∎

Lemma 3.33 *Let X be a chainable continuum. Then there exists a special L-structure $\mathcal{C}$, such that $X = \mathcal{C}/R^{\mathcal{C}}$.*

Proof. We will construct a sequence $(\mathbf{C}_n)_{n\in\omega}$ of chains on X such that $\mathbf{C}_{n+1}$ refines $\mathbf{C}_n$ for all $n \in \omega$. Then we will associate an inverse limit sequence $(\mathcal{C}_n)_{n\in\omega}$ of finite linear graphs to $(\mathbf{C}_n)_{n\in\omega}$ and define $\mathcal{C}$ as the inverse limit of $(\mathcal{C}_n)_{n\in\omega}$.

Let $\mathbf{C}(k)$ detonate the k-th link of $\mathbf{C}$, and let $\mathrm{Mesh}(\mathbf{C}) = \max\{\mathrm{diam}(C) : C \in \mathbf{C}\}$. We will recursively construct a sequence $(\mathbf{C}_n)_{n\in\omega}$ of chains on X such that:

(1) $\mathbf{C}_{n+1}$ closure refines $\mathbf{C}_n$,

(2) $\mathrm{Mesh}(\mathbf{C}_n) < \frac{1}{n+1}$,

(3) $\mathbf{C}_n$ is fine,

(4) if $\mathbf{C}_{n+1}(i) \subseteq \mathbf{C}_n(k)$ and $\mathbf{C}_{n+1}(j) \subseteq \mathbf{C}_n(l)$ such that $|k-l| > 1$, then $|i-j| > 2$,

(5) for any link $\mathbf{C}_n(k)$, there exists i such that $\mathbf{C}_{n+1}(i) \subseteq \mathbf{C}_n(k) \setminus \left(\bigcup_{l\neq k} \mathbf{C}_n(l)\right)$.

Let $\mathbf{C}_0$ be any chain covering X. Suppose $\mathbf{C}_n$ is already defined for some $n \in \omega$ and it is δ-fine. Note that by $(C3)$, any chain with sufficiently small mesh closure refines $\mathbf{C}_n$. We will construct a fine chain $\mathbf{C}_{n+1}$ a fine chain, that closure refines $\mathbf{C}_n$ and satisfies $\mathrm{Mesh}(\mathbf{C}_{n+1}) < \min\left\{\frac{\delta}{3}, \frac{1}{n+2}\right\}$. Cover X by balls of radius less than $\min(\frac{1}{2(n+1)}, \frac{\delta}{6})$. Let $\mathbf{C}_{n+1}$ be a fine chain closure refining this cover. Observe that for each link $\mathbf{C}_{n+1}(i)$ of $\mathbf{C}_{n+1}$

$$\mathrm{diam}(\overline{\mathbf{C}_{n+1}(i)}) < \frac{\delta}{3},$$

by (C3) $\mathbf{C}_{n+1}$ is closure refines $\mathbf{C}_n$.

Then conditions (2) and (3) holds by definition, (C1) implies condition (4) and (C2) implies condition (5).

Now we build the inverse limit. Let $\mathcal{C}_n$ be the Finite linear graph:

- $C_n = \{0, 1, ..., |\mathbf{C}_n| - 1\}$ with the discrete topology;
- for any $i, j < |\mathbf{C}_n|$, let $(i, j) \in R^{\mathcal{C}_n}$ if and only if $|i-j| \leq 1$.

Let $\phi_n : \mathcal{C}_{n+1} \to \mathcal{C}_n$ be defined by

$$\phi_n(i) = \min \left\{ k \in |\mathbf{C}_n| : \overline{\mathbf{C}_{n+1}(i)} \subseteq \mathbf{C}_n(k) \right\}.$$

By condition (1), ϕ_n is well-defined. By condition (5), ϕ_n is surjective, and by condition (4), ϕ_n preserves the relations. Recall that in the class Δ_0 these are sufficient for ϕ_n to be an epimorphism. Let $\mathcal{C} = \lim_{\leftarrow}(\mathcal{C}_n, \phi_n)$ be the inverse limit. It remains to prove the following:

(a) $\mathcal{C}$ is special,

(b) $X \cong \mathcal{C}/R^{\mathcal{C}}$.

(a) Condition (α) follows easily from the fact that $\mathcal{C}$ is an inverse limit of structures in Δ_0.

(a)Condition (β) follows by the inverse limit construction, it is clearly that $R^{\mathcal{C}}$ is reflexive and symmetric, so it suffices to show that if $x, y, z \in C$, $x \neq z$, $(x, y) \in R^{\mathcal{C}}$, and $(y, z) \in R^{\mathcal{C}}$, then $x = y$ or $y = z$. There are two cases.

CASE 1. $|x(n), y(n)| = 1$ for all but finitely many n. Then either $x(n) = y(n)$ for all but finitely many n or $y(n) = z(n)$ for all but finitely many n because $\mathcal{C}_n \in \Delta_0$. So we have $x = y$ or $y = z$.

CASE 2. $|x(n) - z(n)| = 2$ for all but finitely many n. This cannot happen because by condition (4),

$$|x(n) - z(n)| = 2 \implies |x(n+1) - z(n+1)| > 2,$$

so either $(x(n+1), y(n+1)) \notin R^{\mathcal{C}_{n+1}}$ for all but finitely many n or $(y(n+1), z(n+1)) \notin R^{\mathcal{C}_{n+1}}$ for all but finitely many n, any of which contradicts our assumptions.

(b) Define $f : \mathcal{C} \to X$ as follows: for any $c \in C$ let $f(c)$ be the unique point in $\bigcap_{n\in\omega} \overline{\mathbf{C}_n(c(n))}$, witch is a singleton by conditions (1) and (2). Also, condition (2) implies that f is continuous. To show that f is surjective: fix $x \in X$, and let

$$T_x = \bigcup_{n\in\omega} \{(m_0, m_1, ..., m_n) \in \prod_{i\in n+1} C_i : x(n) \in \mathbf{C}_n(m_n) \text{ and for all } i \in n, \phi_i(m_{i+1}) = m_i\} \cup \{\emptyset\}.$$

Then clearly T_x is a tree, and for all $n \in \omega$ we have $T_x \cap \prod_{i\in n+1} C_i \neq \emptyset$. By König's Lemma, T_x has an infinite branch c, and clearly $f(c) = x$.

Claim. $f(c_0) = f(c_1)$ if and only if $(c_0, c_1) \in R^{\mathcal{C}}$.

Proof. Observe that $(c_0, c_1) \in R^{\mathcal{C}}$ if and only if for all $n \in \omega$ we have $(c_0(n), c_1(n)) \in R^{\mathcal{C}_n}$ if and only if for all $n \in \omega$ we have $\mathbf{C}_n(c_0(n)) \cap \mathbf{C}_n(c_1(n)) \neq \emptyset$ if and only if $f(c_0) = f(c_1)$. Therefore we proved the claim.

Now let us define $\overline{f} : \mathcal{C}/R^{\mathcal{C}} \to X$ by $\overline{f}(\rho(x)) = f(x)$, where $\rho : \mathcal{C} \to \mathcal{C}/R^{\mathcal{C}}$ is the quotient map. First, $\overline{f}$ is well-defined bijection by the claim. It is continuous by the definition of the quotient topology. Since $\mathcal{C}/R^{\mathcal{C}}$ is compact, it follows that $\overline{f}$ is a homeomorphism, which proves the lemma. ∎

Proposition 3.1 *Let Δ be a projective Fraïssé class of finite L-structures, and let $\mathcal{D}$ be the projective Fraïssé limit of Δ. Let $\mathcal{B}$ be a topological L-structure such that every open cover of $\mathcal{B}$ is refined by an epimorphism $\phi : \mathcal{B} \to \mathcal{C}$ onto some $\mathcal{C} \in \Delta$. Then there exists an epimorphism $\chi : \mathcal{D} \to \mathcal{B}$.*

Proof. We will write $\mathcal{B}$ as the inverse limit of an inverse sequence from Δ. Enumerate all clopen partitions of $\mathcal{B}$ as $(\mathcal{U}_n)_{n\in\omega}$. Let $\phi_0 : \mathcal{B} \to \mathcal{B}_0$ be an epimorphism that refines the open cover $\mathcal{U}_0$ of $\mathcal{B}$. Assume that we have already defined $\phi_n : \mathcal{B} \to \mathcal{B}_n$. Pick $\mathcal{B}_{n+1} \in \Delta$ and epimorphism $\phi_{n+1} : \mathcal{B} \to \mathcal{B}_{n+1}$ that refines the open cover

$$\{u \cap \phi_n^{-1}(b) : b \in \mathcal{B}_n \text{ and } u \in U_{n+1}\}.$$

Let $\psi_n : \mathcal{B}_{n+1} \to \mathcal{B}_n$ be a map such that $\phi_n = \psi_n \circ \phi_{n+1}$.

$$\begin{tikzcd}
 & \vdots \arrow[d, "\psi_3"] \\
 & \mathcal{B}_3 \arrow[d, "\psi_2"] \\
\mathcal{B} \arrow[ur, "\phi_3"] \arrow[r, "\phi_2"] \arrow[dr, "\phi_1"] \arrow[ddr, "\phi_0"'] & \mathcal{B}_2 \arrow[d, "\psi_1"] \\
 & \mathcal{B}_1 \arrow[d, "\phi_0"] \\
 & \mathcal{B}_0
\end{tikzcd}$$

By Lemma 3.8, the maps ψ_n are epimorphisms. By property (L1), we can find an epimorphism $\phi'_0 : \mathcal{D} \to \mathcal{B}_0$, and, by using Lemma 3.11, we can recursively find $\phi'_n : \mathcal{D} \to \mathcal{B}_n$ such that $\phi'_{n-1} = \psi_{n-1} \circ \phi'_n$.

$$\begin{tikzcd}
\mathcal{B}_n \arrow[d, "\psi_{n-1}"'] & \mathcal{D} \arrow[l, "\phi'_n"'] \arrow[ddl, "\phi'_0"] \\
\vdots \arrow[d, "\psi_0"'] & \\
\mathcal{B}_0 &
\end{tikzcd}$$

Now let

$$\phi : \mathcal{B} \to \varprojlim(\mathcal{B}_n, \psi_n), \quad \phi(x) = (\phi_n(x))_{n\in\omega}$$

and

$$\phi' : \mathcal{D} \to \varprojlim(\mathcal{B}_n, \psi_n), \quad \phi'(x) = (\phi'_n(x))_{n\in\omega}.$$

The same argument as in the uniqueness part of the proof of Theorem 3.8 shows that ϕ and ϕ' are epimorphisms. Since, in addition, ϕ separates the points of $\mathcal{B}$, ϕ is injective as well. Hence it is an isomorphism. Thus $\phi^{-1} \circ \phi' : \mathcal{D} \to \mathcal{B}$ is an epimorphism, witch concludes the proof.

∎

We note that the proposition holds in general for any projective Fraïssé class.

Proof. (Of Theorem 3.28)

1. Let (X, d) be a chainable continuum. By Lemma 3.33, there exists a special topological L-structure $\mathcal{C}$ such that $X \cong \mathcal{C}/R^{\mathcal{C}}$. By Proposition3.1, there is an epimorphism $\phi : \mathcal{P} \to \mathcal{C}$, so by Lemma 3.31, we can find an epimorphism $\phi^* : \mathcal{P}/R^{\mathcal{P}} \to \mathcal{C}/R^{\mathcal{C}}$. It remains to recall that $\mathcal{P}/R^{\mathcal{P}}$ is homeomorphic to the pseudo-arc by Theorem 3.25.

2. Let (X, d) be a chainable continuum, and let $f_0 : \mathcal{P}/R^{\mathcal{P}} \to X$ and $f_1 : \mathcal{P}/R^{\mathcal{P}} \to X$ be surjective continuous maps. Fix $\epsilon > 0$. It suffices to find an isomorphism $\phi : \mathcal{P} \to \mathcal{P}$ such that $d_U(f_0 \circ \rho, f_1 \circ \rho \circ \phi) < \epsilon$ since for the homeomorphism $\phi^* : \mathcal{P}/R^{\mathcal{P}} \to \mathcal{P}/R^{\mathcal{P}}$ given by Lemma 3.31. We get that $d_U(f_0, f_1 \circ \phi^*) < \epsilon$ (where d_U is the uniform metric).

$$\begin{tikzcd}
\mathcal{P} \arrow[rr, "\phi"] \arrow[d, "\rho"'] & & \mathcal{P} \arrow[d, "\rho"] \\
\mathcal{P}/R^{\mathcal{P}} \arrow[rr, dashed, "\phi^*"] \arrow[dr, "f_0"'] & & \mathcal{P}/R^{\mathcal{P}} \arrow[dl, "f_1"] \\
& X &
\end{tikzcd}$$

Let $\mathbf{U} = \{U_0, \ldots, U_{n-1}\}$ be a δ-fine ϵ-chain on X. Let $\mathcal{D} \in \Delta_0$ be such that $|D| = n$. Since P is compact, both $f_0 \circ \rho$ and $f_1 \circ \rho$ are uniformly continuous. So there are structures $\mathcal{E}_0, \mathcal{E}_1 \in \Delta_0$ and epimorphisms $\phi_0 : \mathcal{P} \to \mathcal{E}_0$, $\phi_1 : \mathcal{P} \to \mathcal{E}_1$ such for all $\in 2$ and $e \in E_i$ we have $\operatorname{diam}(f_i(\rho(\phi_i^{-1}(e)))) < \delta$. Let $\psi_i : \mathcal{E}_i \to \mathcal{D}$ be maps defined by

$$\psi_i(e) \mapsto \min\{j \in n : f_i(\rho(\phi_i^{-1}(e))) \subseteq U_j\}$$

for $i \in 2$.

$$\begin{tikzcd}
\mathcal{P} \arrow[rr, "\phi"] \arrow[d, "\phi_0"'] & & \mathcal{P} \arrow[d, "\phi_1"] \\
\mathcal{E}_0 \arrow[dr, "\psi_0"'] & & \mathcal{E}_1 \arrow[dl, "\psi_1"] \\
& \mathcal{D} &
\end{tikzcd}$$

Claim. Both ψ_i is an epimorphism.

Proof. By (C3), since the sets $f_i(\rho(\phi_i^{-1}(e)))$ of diameter less then δ cover X and $\mathcal{U}$ satisfies $(C2)$, ψ_i is surjective. Let $e, e' \in E_i$ such that $(e, e') \in R^{\mathcal{E}_i}$. Since ϕ_i is an epimorphism, there exist $x \in \phi_i^{-1}(e)$ and $x' \in \phi_i^{-1}(e')$ such that $(x, x') \in R^{\mathcal{P}}$, which implies that $\rho(x) = \rho(x')$, and thus $f_i(\rho(\phi_i^{-1}(e))) \cap f_i(\rho(\phi_i^{-1}(e'))) \neq \emptyset$ since $\mathcal{U}$ satisfies $(C1)$ we conclude that $(\psi_i(e), \psi_i(e')) \in R^{\mathcal{D}}$. We have verified that ψ_i is an epimorphism, which proves the claim.

Since $\mathcal{P}$ is a projective Fraïssé limit, it has property (L3), which gives an isomorphism $\phi : \mathcal{P} \to \mathcal{P}$ such that $\psi_0 \circ \phi_0 = \psi_1 \circ \phi_1 \circ \phi$. Fix any $x \in P$. Then $\psi_0(\phi_0(x)) = \psi_1(\phi_1(x))$ implies that the sets $f_0(\rho(\phi_0^{-1}(\phi_0(x))))$ and $f_1(\rho(\phi_1^{-1}(\phi_1(\phi(x)))))$ lie in the same link U_j. Since $f_0(\rho(x)) \in f_0(\rho(\phi_0^{-1}(\phi_0(x))))$ and $f_1(\rho(\phi(x)))) \in f_1(\rho(\phi_1^{-1}(\phi_1(\phi(x)))))$ and $\operatorname{dima}(U_j) < \epsilon$, we conclude that $d(f_0(\rho(x)), f_1(\rho(\phi(x)))) < \epsilon$ as desired.

Recall some facts form continuum theory. If (X, d_X) and (X, d_X) are compact metric spaces with then a continuous map $f : X \to Y$ is called a δ-map if $\operatorname{diam}(f^{-1}(f(x))) < \delta$ for each $x \in X$. It is easy to see that, if $f : X \to Y$ is a δ-map, then there is a $\xi > 0$ such that $\operatorname{diam}(f^{-1}(A)) < \delta$ if $\operatorname{diam}(A) < \xi$ for any $A \subseteq Y$ so in particular if $d_X(x_0, x_1) \geq \delta$, then $d_Y(f(x_0), f(x_1)) \geq \xi$. It is well known that a non-degenerate continuum (X, d_X)

is chainable if and only if for every $\delta > 0$ there is a δ-map from X onto the closed unit interval. See [5].

3. Let (X, d_X) and (Y, d_Y) be nondegenerate chainable continua such that both X and Y has property $(*)$. Let $I_n = [0,1]$ for all $n \in \omega$. We will construct continuous surjections $\phi_n : I_{n+1} \to I_n$, $f_n : X \to I_n$, and $g_n : Y \to I_n$, as well as number $\epsilon_n > 0$ for all $n \in \omega$ such that:

(a_n) $\epsilon_n < \frac{1}{n+1}$;

(b_n) $d_{U_X}(\phi_{k,n-1} \circ f_n, \phi_{k,m-1} \circ f_m) < \epsilon_m$ for all $k \leq m \leq n$, where d_{U_X} is the uniform metric on $C[X]$ and $\phi_{m_1,m_2} = \phi_{m_1} \circ \phi_{m_1+1} \circ \ldots \circ \phi_{m_2}$ for $m_1 \leq m_2$, and $\phi_{i,i-1}$ is the identity map on I_i;

(c_n) $d_{U_Y}(\phi_{k,n-1} \circ g_n, \phi_{k,m-1} \circ g_m) < \epsilon_m$ for all $k \leq m \leq n$, where d_{U_Y} is the uniform metric on $C[Y]$;

(d_n) for all $x, y \in X$, if n is even and $d_X(x,y) \geq \frac{1}{n+1}$, then $|f_n(x) - f_n(y)| > 2\epsilon_n$;

(e_n) for all $x, y \in X$, if n is odd and $d_Y(x,y) \geq \frac{1}{n+1}$, then $|g_n(x) - g_n(y)| > 2\epsilon_n$.

Let $f_0 : X \to I_0$ be a continuous surjection (a δ-map with $\delta = 1$) and ϵ_0 be any positive real number less than 1, such that for all $x, y \in X$ with $d_X(x,y) \geq 1$ implies that $|f_0(x) - f_0(y)| > 2\epsilon_0$. Let $g_0 : Y \to I_0$ be a continuous surjection. Clearly, we can find (f_0, ϵ_0) such that $(a_0) - (e_0)$ holds (the conditions (c_i) and (e_i) hold vacuously no matter how g_0 is chosen).

$$\begin{tikzcd}
& \vdots \arrow[d] & \\
& I_{2n+1} \arrow[d, "\phi_{2n}"] & \\
X \arrow[ur, "f_{2n+1}"] \arrow[r, "f_{2n}"] \arrow[dr, "f_{2n-1}"'] & I_{2n} \arrow[d, "\phi_{2n-1}"] & Y \arrow[ul, "g_{2n+1}"'] \arrow[l, "g_{2n}"'] \arrow[dl, "g_{2n-1}"] \\
& I_{2n-1} \arrow[d] & \\
& \vdots &
\end{tikzcd}$$

Assume we have already found (f_i, ϵ_i) such that $(a_i), (b_i), (d_i)$ hold for every $i \leq 2n$ and (g_i, ϕ_i) such that (c_i) and (e_i) hold for $i < 2n$. Now we find $g_{2n}, \phi_{2n}, g_{2n+1}$, and ϵ_{2n+1} (for $f_{2n+1}, \phi_{2n+1}, f_{2n+2}$, and ϵ_{2n+2} the same arguments work).

Claim. For any $\epsilon > 0$ and any continuous surjections $\phi : [0,1] \to [0,1]$ and $g : Y \to [0,1]$, there is a continuous surjection $g' : Y \to [0,1]$ such that $d_U(\phi \circ g', g) < \epsilon$.

$$\begin{tikzcd}
& Y \arrow[dl, dashed, "g'"'] \arrow[dr, "g"] & \\
{[0,1]} \arrow[rr, "\phi"'] & & {[0,1]}
\end{tikzcd}$$

Proof. Applying $(*)_Y$: for $\epsilon > 0$ and two continuous surjections $g : Y \to [0,1]$ it follows that $\phi \circ g : Y \to [0,1]$, there exists a homeomorphism $\psi : Y \to Y$ such that the following diagram commutes with ϵ error, that is, $|g(y) - \phi(g(\psi(y)))| < \epsilon$ for every $y \in Y$.

$$
\begin{tikzcd}
Y \arrow[rr, dashed, "\psi"] \arrow[dr, "g"'] & & Y \arrow[d, "g"] \\
& {[0,1]} & {[0,1]} \arrow[l, "\phi"]
\end{tikzcd}
$$

Let $g' = g \circ \psi$, which confirms the claim.

By our inductive assumption $(c)_{2n-1}$, we can find $\epsilon > 0$ such that for all $k \leq m \leq 2n-1$ we have

$$d_{U_Y}(\phi_{k,2n-2} \circ g_{2n-1}, \phi_{k,m-1} \circ g_m) + \epsilon < \epsilon_m \tag{1}$$

By the previous claim, we can find a continuous surjection $g_{2n} : Y \to [0,1]$ such that $d_{U_Y}(\psi_{2n-1} \circ g_{2n}, g_{2n-1}) < \gamma$, where γ can be made arbitrarily small, and we choose it to be sufficiently small to satisfy

$$d_{U_Y}(\phi_{k,2n-1} \circ g_{2n}, \phi_{k,2n-2} \circ g_{2n-1}) = d_{U_Y}(\phi_{k,2n-2} \circ (\phi_{2n-1} \circ g_{2n}), \phi_{k,2n-2} \circ g_{2n-1}) < \epsilon \tag{2}$$

for all $k \leq 2n-1$, which is possible by the uniform continuity of $\phi_{k,2n-2}$. Consequently, inequalities (1), (2), and the triangle inequality yield

$$d_{U_Y}(\phi_{k,2n-1} \circ g_{2n}, \phi_{k,m-1} \circ g_m) < \epsilon_m \tag{3}$$

for all $k \leq m \leq n-1$. since the inequality (3) also holds for $m = 2n$, the map g_{2n} satisfies condition (c_{2n}). Next, we construct g_{2n+1}, ϕ_{2n}, and ϵ_{2n+1} using the inductive hypothesis (c_{2n}), and we can select an $\epsilon' > 0$ such that for all $k \leq m \leq 2n$

$$d_{U_Y}(\phi_{k,2n-1} \circ g_{2n}, \phi_{k,m-1} \circ g_m) + \epsilon' < \epsilon_m \tag{4}$$

We will define ϕ_{2n} and g_{2n+1} so that $g_{2n+1} : Y \to [0,1]$ ontinuous surjection such that $d_{U_Y}(\psi_{2n} \circ g_{2n+1}, g_{2n})$ can be made arbitrarily small, and we choose it to be sufficiently small to satisfy

$$d_{U_Y}(\phi_{k,2n} \circ g_{2n+1}, \phi_{k,2n-1} \circ g_{2n}) = d_{U_Y}(\phi_{k,2n-1} \circ (\phi_{2n} \circ g_{2n+1}), \phi_{k,2n-1} \circ g_{2n}) < \epsilon' \tag{5}$$

for all $k \leq 2n$.

We will need the following lemma, witch we do not prove. The reader can find the proof in book [5] (Lemma 12.17).

Lemma 3.34 [1] *If (X, d_X) is a compact metric space, $f : X \to [0,1]$ is a continuous surjection, and $\epsilon > 0$. Then there exists $\delta = \delta(f, \epsilon)$ so that if $g : X \to [0,1]$ is a δ-map, then there is a continuous surjection $\phi : [0,1] \to [0,1]$ so that $|\phi(g(x)) - f(x)| < \epsilon$ for all $x \in X$.*

By uniform continuity, we can fix $\epsilon'' > 0$ so that if $|x - y| < \epsilon$, then $|\phi_{k,2n-1}(x) - \phi_{k,2n-1}(x)(y)| < \epsilon'$ for each $k \leq 2n$. Then let $\delta' < \min\{\frac{1}{2n+2}, \delta(g_{2n}, \epsilon'')\}$, where $\delta(g_{2n}, \epsilon'')$ is given by the lemma. Fix any continuous surjective δ'-map $g_{2n+1} : Y \to I_{2n+1}$. Now by the lemma, we can define $\phi_{2n} : [0,1] \to [0,1]$ so that $d_U(g_{2n}, \phi_{2n} \circ g_{2n+1}) < \epsilon''$. Then (5) holds by the choice of ϵ''. By applying the triangle inequality to (4) and (5), we obtain that

[1] This lemma in [5] explicitly states that ϕ need not be onto, but an analysis of the proof shows that if f is onto, then ϕ is onto.

$$d_{U_Y}(\phi_{k,2n} \circ g_{2n+1}, \phi_{k,m-1} \circ g_m) < \epsilon_m \tag{6}$$

for all $k \leq m \leq 2n$. Let $\frac{1}{2n+2} > \epsilon_{2n+1} > 0$ be such that the condition (e_{2n+1}) holds. Note that this is possible since g_{2n+1} is a δ-map for some $\delta < \frac{1}{2n+2}$. Thus, conditions $(a_n) - (e_n)$ are stisfied and the reculsive construction is complete.

It remains to show that X and Y are homeomorphic. We will prove that X and Y are homeomorphic to the inverse limit $Z = \varprojlim(I_n, \phi_n)$. By symmetry, it suffices to show this for X. By (a_n) and (b_n), for every $k \in \omega$ the sequence of functions

$$(\phi_{k,n-1} \circ f_n)_{n \geq k}$$

converges uniformly to a continuous function $\overline{f_k} : X \to I_k$. In particular, we have

$$d_{U_X}(\overline{f_k}, \phi_{k,m-1} \circ f_m) < \epsilon_m \tag{7}$$

for all $k \in \omega$ and $m \geq k$. We claim that $F : X \to \prod_{n \in \omega} I_n$

$$x \mapsto (\overline{f_0}(x), \overline{f_1}(x), \overline{f_2}(x), \dots)$$

is a homeomorphism between X and Z.

- $(\overline{f_0}(x), \overline{f_1}(x), \overline{f_2}(x), \dots) \in Z$ holds because $\phi_k \circ \phi_{k+1,n-1} = \phi_{k,n-1}$ for $n > k+1$, and ϕ_k is continuous.
- F is continuous since each $\overline{f_k}$ is continuous.
- To show that F is surjective, it is suffices to show that for each $k \in \omega$, the range of $\overline{f_k}$ is dense in I_k since F is continuous and X is compact. This follows from the fact that for every $m \geq k$ the map $\phi_{k,m-1} \circ f_m$ is surjective, equation (7) holds and condition (a_m) is satisfied.
- We show that F is injective. Let $x, y \in X$ with $x \neq y$. Then we can find $k_0 \in \omega$ such that $d_X(x,y) > \frac{1}{2k_0+1}$. By the condition $(d)_{2k_0}$, we have that $|f_{2k_0}(x) - f_{2k_0}(y)| > 2\epsilon_{2k_0}$, which implies that $\overline{f_{2k_0}}(x) \neq \overline{f_{2k_0}}(y)$ by (7). Hence $F(x) \neq F(y)$.

Since X is compact and F is a continuous bijection, it follows that F is a homeomorphism. Therefore we proved the the last part of Theorem 3.28.

∎

References

[1] C.C. Chang, H.J. Keisler, *Model Theory.* North–Holland, Amsterdam (1990).

[2] Hodges, Wilfrid. *Model theory.* Cambridge university press, (1993).

[3] Hodges, Wilfrid. *A shorter model theory.* Cambridge university press, (1997).

[4] Irwin, Trevor, and Sławomir Solecki. *"Projective Fraïssé limits and the pseudo-arc."* Transactions of the American Mathematical Society 358.7 (2006): 3077-3096.

[5] Nadler Jr, Sam B. *"Continuum theory and graph theory: disconnection numbers."* Journal of the London Mathematical Society 2.1 (1993): 167-181.

[6] Rogers Jr, James Ted. *"Pseudo-circles and universal circularly chainable continua."* Illinois Journal of Mathematics 14.2 (1970): 222-237.

[7] Asensi Arranz, Roger. *"Fraïssé Limits."* (2022).

[8] Kechris, Alexander. *Classical descriptive set theory.* Vol. 156. Springer Science & Business Media, (2012).

[9] Gábor, Sági. *"Válogatott Fejezetek a Modellelméletből és Határterületeiből."*

[10] Neumann, Bernhard H. *"Permutational products of groups."* Journal of the Australian Mathematical Society 1.3 (1960): 299-310.

[11] Kabluchko, Zakhar, and Katrin Tent. *"Universal-homogeneous structures are generic."* arXiv preprint arXiv:1710.06137 (2017).

[12] Yaacov, Itaï Ben. *"Fraïssé limits of metric structures."* The Journal of Symbolic Logic 80.1 (2015): 100-115.

[13] Hall, Philip. *"Some constructions for locally finite groups."* Journal of the London Mathematical Society 1.3 (1959): 305-319.

[14] Bartosova, Dana, and Aleksandra Kwiatkowska. *"Lelek fan from a projective Fraïssé limit."* arXiv preprint arXiv:1312.7514 (2013).

[15] Armstrong, Mark Anthony. *"Basic topology."* Springer Science & Business Media, (2013).

[16] Bing, R. H. *"Concerning hereditarily indecomposable continua."* (1951): 43-51.

Alulírott Ivanyos János Balázs nyilatkozom, hogy szakdolgozatom elkészítése során az alább felsorolt feladatok elvégzésére a megadott MI eszközöket alkalmaztam:

Table 1:

Feladat	**Felhasznált eszköz**	**Felhasználás helye**	**Megjegyzés**
Nyelvhelyesség ellenorzése, fogalmazás gördülékenyebbé tétele, Angol-Magyar fordítás	GPT-4o	Teljes dolgozatban	
Diagram készítés	quiver	Teljes dolgozatban	Latex kód generálása

A felsoroltakon túl más MI alapú eszközöket nem használtam.