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Alternating sign matrices

Thesis

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Contents

1 Introduction

Alternating sign matrices (ASMs for short) are $(0, \pm 1)$-matrices such that the sum of the entries in each row and column equals one and the non-zero entries in each row and column alternate in sign. Every permutation matrix is an ASM, so we can consider ASMs a natural and exciting generalization of permutation matrices. The aim of this thesis is to investigate various questions concerning alternating sign matrices and other related classes of matrices using the tool set of polyhedral combinatorics and combinatorial optimization that are well known among mathematicians interested in operations research, but seem to be unknown among those involved in the study of ASMs.

Most of the results we discuss here also appear in [4]. The present thesis discusses the questions from the literature in more detail, while the paper focuses more on the connection between ASMs (PBMs) and g-polymatroids, as well as feasible circulations. In some cases, we also provide alternative proofs to those in the paper.

In Section 2, we introduce the concept of *prefix bounded matrices* (PBMs for short), which generalize alternating sign matrices and several other related matrix classes, and investigate the existence of such matrices under various constraints, relying on the connection we establish between PBMs and feasible circulations. We also show some interesting properties of these matrices which are motivated either by related questions in the literature or simply by our own curiosity.

In Sections 3 and 4, we discuss several problems proposed in the literature using the general framework of PBMs. We provide simple answers for open questions that were previously believed to be difficult to solve.

In Section 5, we discuss generalizations that turned out to be NP-hard. We conclude this thesis in Section 6, by proposing some interesting questions that we plan to answer in the near future.

1.1 Historical overview

The interest in ASMs started in the early 1980s with the alternating sign matrix conjecture [26, 27] by Mills, Robbins, and Rumsey, which provided the following formula for the number of $n \times n$ ASMs:

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

The conjecture has since been proved using several methods [22, 29], showing that there are connections between ASMs and various interesting combinatorial objects. The first such connection was to *descending plane partitions*, which inspired the proof the Macdonald conjecture [26]. Another important connection is between ASMs and the so-called *square ice model* used in statistical phyisics. The fact that there is a one-to-one correspondence between ASMs and the states of square ice played a key role in Kuperberg's alternative proof of the ASM conjecture [22]. A state of the square ice model is an $n \times n$ grid with directed arcs, such that the in- and out-degree of every vertex is 2, where we imagine that there are 4 arcs incident to each vertex, including those on the border. The extra horizontal arcs of the vertices on the border must point inward, and the extra vertical arcs must point outward. We ignore the other endpoints of these extra arcs, thus when we refer to a vertex, we always mean one of those in the $n \times n$ grid. The model is called square-ice because it is used to model 2-dimensional ice. The oxygen atoms correspond to the vertices of the digraph, and the hydrogen atoms correspond to the arcs, in a way that every hydrogen atom belongs to the oxygen whose vertex the arc representing the hydrogen points to. Therefore, the in- and out-degree constraints ensure that 2 hydrogen atoms belong to every oxygen atom. A state of the square ice model can be seen in Figure 2 below. The in- and out-degree constraints imply that the arcs around every vertex must be one the following six configurations, called the state of the vertex.

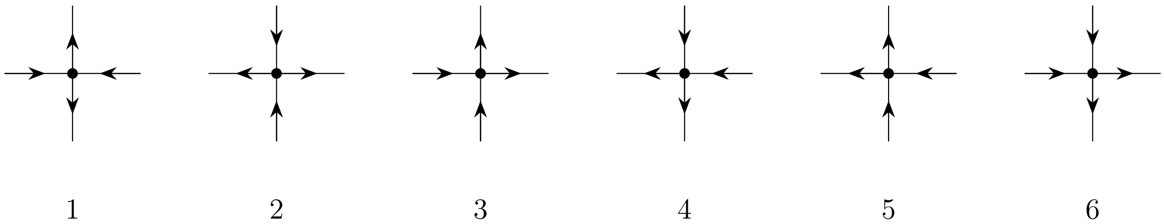
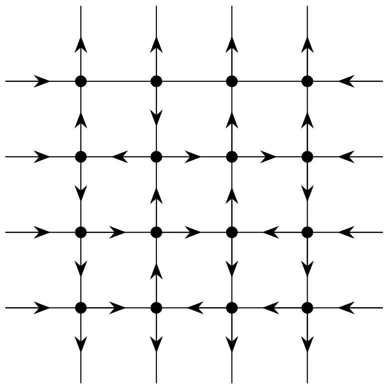


Figure 1: The six configurations.

Given a state of the square ice model, the corresponding ASM $A = (a_{i,j})$ is the matrix, where the entries represent the states of the vertices, such that $a_{i,j} = +1$ if and only if the vertex at the position (i, j) in the grid is of state 1, $a_{i,j} = -1$ if and only if the corresponding vertex is of state 2, and $a_{i,j} = 0$ otherwise. It can be seen that this is a bijection, as there is a unique state of square ice, from which we get a given ASM.



$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Figure 2: A state of 4×4 square ice and the corresponding ASM.

In the past decade, there has been a growing interest in questions concerning the existence of ASMs with various restrictions, most of which can be formulated as the existence of an ASM with additional lower and upper bounds on the matrix entries. Solutions were only found for a few special cases [8, 11]. We answer such questions in general by providing a necessary and sufficient condition for the existence of an ASM with bounds on its entries using Hoffman's theorem on feasible circulations. A key observation about ASMs is that the alternating sign property can be expressed as lower and upper bounds for the sum of entries on the *prefixes* (sets consisting of the first few entries of a row/column), which leads us to introduce the concept of PBMs. A different direction of the study of ASMs concerned the alternating sign matrix polytope, as well as various similar polytopes, with special focus on their faces and vertices. In this thesis, we discuss one such problem, namely we characterize the vertices of a class of integer polytopes, which generalizes a similar result concerning the alternating sign matrix polytope and at the same time provides a more elegant proof.

1.2 Notations

For any $n \in \mathbb{Z}_{>0}$, let $[n]$ denote the set $\{1, \dots, n\} \subseteq \mathbb{Z}_{>0}$. For an $m \times n$ matrix, let S be the set of positions $S = [m] \times [n]$. We call the set of the first j entries of the i-th row the horizontal prefix ending at the position (i, j). Similarly, we call the first i entries of the j-th column the vertical prefix ending at the position (i, j). We consider the empty set to be a prefix. We call the difference of two prefixes of the same row (column) a horizontal (vertical) *interval*. For a matrix M, we denote by $M_{i,\cdot}$ the i-th row of the matrix and by $M_{\cdot,j}$ its j-th column.

Let $D = (V, A)$ be a digraph and $f : A \to \mathbb{R}$ be an arbitrary function defined on its arcs. For a subset $U \subseteq V$, we denote by $\varrho_f(U)$ the sum of f on the arcs entering U, and by $\delta_f(U)$ the sum of f on the arcs leaving U, that is:

$$\varrho_f(U) = \sum_{uv \in A: u \notin U, v \in U} f(uv), \quad \delta_f(U) = \sum_{uv \in A: u \in U, v \notin U} f(uv).$$

Let $f : S \to \mathbb{R}$ be an arbitrary function defined on the elements of S. From f, we define the set function $\widetilde{f} : 2^S \to \mathbb{R}$, where

$$\widetilde{f}(Z) = \sum_{z \in Z} f(z)$$

for every subset $Z \subseteq S$.

2 Prefix bounded matrices

In this section, we introduce the concept of prefix bounded matrices, which is a natural generalization of alternating sign matrices. We discuss their relationship in Section 3.

Definition 1. *Let $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, \alpha, \beta)$ be an 8-tuple, where $\Gamma^1, \Gamma^2, \Phi^1, \Phi^2, f, g$ are $m \times n$ integer matrices where Φ^1, Φ^2, f might have $-\infty$ entries and Γ^1, Γ^2, g might have ∞ entries. Let α, β be integers where α might be $-\infty$ and β might be ∞. We assume that $\Phi^1 \leq \Gamma^1$, $\Phi^2 \leq \Gamma^2$, $f \leq g$ and $\alpha \leq \beta$.*

We call an $m \times n$ integer matrix $X = (x_{i,j})$ a prefix bounded matrix *(PBM for short) with bounds $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, \alpha, \beta)$ if it satisfies the following conditions: $f \leq X \leq g$ and for every $i \in [m], j \in [n]$ $\Phi^1_{i,j}$ is lower and $\Gamma^1_{i,j}$ is upper bound for the sum of the first j entries in the i-th row. Similarly, $\Phi^2_{i,j}$ is lower and $\Gamma^2_{i,j}$ is upper bound for the first i entries in the j-th column. Furthermore, α and β are lower and upper bounds for the sum of the entries in X. We only use the term PBM and omit the 8-tuple of the bounds when it is clear from the context.*

The integer solutions to the following inequalities are, by definition, the PBMs. Surprisingly, the polyhedron described by these inequalities is the convex hull of PBMs in $\mathbb{R}^{m\times n}$.

$$x \in \mathbb{R}^S$$

$$f_{i,j} \leq x_{i,j} \leq g_{i,j} \qquad \forall (i,j) \in S \tag{1a}$$

$$\Phi^1_{i,j} \leq \sum_{j'=1}^{j} x_{i,j'} \leq \Gamma^1_{i,j} \qquad \forall (i,j) \in S \tag{1b}$$

$$\Phi^2_{i,j} \leq \sum_{i'=1}^{i} x_{i',j} \leq \Gamma^2_{i,j} \qquad \forall (i,j) \in S \tag{1c}$$

$$\alpha \leq \sum_{(i,j)\in S} x_{i,j} \leq \beta \tag{1d}$$

Let Q be the coefficient matrix of inequalities (1b), (1c), and (1d). Note that the prefixes of the rows and columns form two laminar set families on the ground set S, moreover, if we add S to any of these families, then we still get two laminar set families. Therefore, our inequality system can be formulated as $f \leq x \leq g, a \leq Qx \leq b$, where Q is a network matrix [15, p. 151] and the bounds a and b are determined by (1b), (1c), and (1d). The fact that Q is a network matrix implies that it is also totally unimodular (TU for short) [15, p. 150]. This implies that the polyhedron is an integer polyhedron, therefore, the feasibility of the system is equivalent to the existence of a PBM.

Using the fact that the matrix Q is TU, we derive a decomposition theorem for PBMs, which can be considered a generalization of Kőnig's edge coloring theorem [21].

Theorem 1. *For every $k \in \mathbb{Z}_{>0}$ and every PBM A satisfying the bounds $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, \alpha, \beta)$, there exist PBMs $A^1, \ldots, A^k$ satisfying the bounds $\left(\left\lfloor \frac{\Phi^1}{k} \right\rfloor, \left\lceil \frac{\Gamma^1}{k} \right\rceil, \left\lfloor \frac{\Phi^2}{k} \right\rfloor, \left\lceil \frac{\Gamma^2}{k} \right\rceil, \left\lfloor \frac{f}{k} \right\rfloor, \left\lceil \frac{g}{k} \right\rceil, \left\lfloor \frac{\alpha}{k} \right\rfloor, \left\lceil \frac{\beta}{k} \right\rceil\right)$ such that $A = A^1 + \cdots + A^k$, furthermore, for every position $(i,j) \in S$ and every index $l \in [k]$, if $a_{i,j} \geq 0$, then $a^l_{i,j} \geq 0$ and if $a_{i,j} \leq 0$, then $a^l_{i,j} \leq 0$, where $A^l = \left(a^l_{i,j}\right)$ for $l \in [k]$.*

Proof. We prove the statement by induction on k. The statement obviously holds for $k = 1$. We show that there is always a matrix A^1 such that the matrix $A - A^1$ can be decomposed into $k-1$ matrices, which together with A^1 form the desired decomposition of A. To show this, consider the following system of inequalities.

$$
\begin{aligned}
& x \in \mathbb{R}^S && && \text{(LP)}\\
& \left\lfloor \frac{f_{i,j}}{k} \right\rfloor \leq x_{i,j} \leq \left\lceil \frac{g_{i,j}}{k} \right\rceil && \forall (i,j) \in S && \text{(2a)}\\
& \left\lfloor \frac{\Phi^1_{i,j}}{k} \right\rfloor \leq \sum_{j'=1}^{j} x_{i,j'} \leq \left\lceil \frac{\Gamma^1_{i,j}}{k} \right\rceil && \forall (i,j) \in S && \text{(2b)}\\
& \left\lfloor \frac{\Phi^2_{i,j}}{k} \right\rfloor \leq \sum_{i'=1}^{i} x_{i',j} \leq \left\lceil \frac{\Gamma^2_{i,j}}{k} \right\rceil && \forall (i,j) \in S && \text{(2c)}\\
& \left\lfloor \frac{\alpha}{k} \right\rfloor \leq \sum_{(i,j)\in S} x_{i,j} \leq \left\lceil \frac{\beta}{k} \right\rceil && && \text{(2d)}\\
& 0 \leq x_{i,j} \leq a_{i,j} && \forall (i,j) \in S : a_{i,j} \geq 0 && \text{(2e)}\\
& a_{i,j} \leq x_{i,j} \leq 0 && \forall (i,j) \in S : a_{i,j} \leq 0 && \text{(2f)}\\
& (k-1)\left\lfloor \frac{f_{i,j}}{k} \right\rfloor \leq a_{i,j} - x_{i,j} \leq (k-1)\left\lceil \frac{g_{i,j}}{k} \right\rceil && \forall (i,j) \in S && \text{(2g)}\\
& (k-1)\left\lfloor \frac{\Phi^1_{i,j}}{k} \right\rfloor \leq \sum_{j'=1}^{j} (a_{i,j'} - x_{i,j'}) \leq (k-1)\left\lceil \frac{\Gamma^1_{i,j}}{k} \right\rceil && \forall (i,j) \in S && \text{(2h)}\\
& (k-1)\left\lfloor \frac{\Phi^2_{i,j}}{k} \right\rfloor \leq \sum_{i'=1}^{i} (a_{i',j} - x_{i',j}) \leq (k-1)\left\lceil \frac{\Gamma^2_{i,j}}{k} \right\rceil && \forall (i,j) \in S && \text{(2i)}\\
& (k-1)\left\lfloor \frac{\alpha}{k} \right\rfloor \leq \sum_{(i,j)\in S} (a_{i,j} - x_{i,j}) \leq (k-1)\left\lceil \frac{\beta}{k} \right\rceil && && \text{(2j)}
\end{aligned}
$$

It is easy to check that $x_{i,j} = \frac{a_{i,j}}{k}$ is a solution to the system and therefore, there also exists an integer solution to the system, as the coefficient matrix of the system is TU and the bounding vectors are integer-valued.

Let x be an integer solution to the system and let $a^1_{i,j} = x_{i,j}$ for each position $(i,j) \in S$. Because of the inequalities (2a) - (2d), the matrix A^1 we just defined, is a PBM with bounds $\left(\left\lfloor \frac{\Phi^1}{k} \right\rfloor, \left\lceil \frac{\Gamma^1}{k} \right\rceil, \left\lfloor \frac{\Phi^2}{k} \right\rfloor, \left\lceil \frac{\Gamma^2}{k} \right\rceil, \left\lfloor \frac{f}{k} \right\rfloor, \left\lceil \frac{g}{k} \right\rceil, \left\lfloor \frac{\alpha}{k} \right\rfloor, \left\lceil \frac{\beta}{k} \right\rceil\right)$. Moreover, the inequalities (2e) and (2f) imply that if $a_{i,j} \geq 0$, then $a^1_{i,j} \geq 0$, and if $a_{i,j} \leq 0$, then $a^1_{i,j} \leq 0$ for every position $(i,j) \in S$, thus A^1 has the properties listed in the theorem.

Furthermore, because of the inequalities (2g) - (2j), the matrix $A - A^1$ is a PBM with bounds $\left((k-1)\left\lfloor \frac{\Phi^1}{k} \right\rfloor, (k-1)\left\lceil \frac{\Gamma^1}{k} \right\rceil, (k-1)\left\lfloor \frac{\Phi^2}{k} \right\rfloor, (k-1)\left\lceil \frac{\Gamma^2}{k} \right\rceil, (k-1)\left\lfloor \frac{f}{k} \right\rfloor, (k-1)\left\lceil \frac{g}{k} \right\rceil, (k-1)\left\lfloor \frac{\alpha}{k} \right\rfloor, (k-1)\left\lceil \frac{\beta}{k} \right\rceil\right)$. The inequalities (2e) and (2f) imply that if $a_{i,j} \geq 0$, then $a_{i,j} - a^1_{i,j} \geq 0$ and if $a_{i,j} \leq 0$, then $a_{i,j} - a^1_{i,j} \leq 0$ for every position $(i,j) \in S$. Therefore, by induction, $A - A^1$ can be decomposed into $k-1$ matrices $A^2, \ldots, A^k$ such that $A^1, A^2, \ldots, A^k$ form the desired decomposition of the matrix A. □

Remark 1. *One can also prove this theorem using the* integer decomposition property *of polyhedra described by a TU matrix and an integer bounding vector [2]. The reason why we chose to show this slightly more complicated proof is that it directly provides an algorithm to find the* *desired decomposition in polynomial time, as we can find* A^1 *and then inductively* $A^2, \dots, A^k$ *using the fact that an integer solution of* (LP) *is a PBM with bounds determined by the inequalities of* (LP) *and as we show later in this section, there is a one-to-one correspondence between PBMs and feasible circulations in a certain digraph, which we can find using standard network flow algorithms.*

Another consequence of Q being a network matrix is that there is a bijection between the solutions to the system and feasible circulations in a digraph, moreover, there is a bijection between PBMs and integer-valued feasible circulations in the same digraph [15, p. 158]. A detailed proof of this bijection can be found in [4]. Now we explicitly define the circulation problem equivalent to the existence of a PBM. Let us define a digraph $D = (V, A)$, where $V = V^1 \cup V^2$ and $A = A^1 \cup A^2 \cup N \cup \{v_0^2 v_0^1\}$. We define V^1, V^2 as the following:

$$V^1 = \{v_{i,j}^1 : (i,j) \in S\} \cup \{v_0^1\},$$

$$V^2 = \{v_{i,j}^2 : (i,j) \in S\} \cup \{v_0^2\}.$$

For $i \in [m]$ and $j \in [n-1]$, we define the arc

$$a_{i,j}^1 = v_{i,j+1}^1 v_{i,j}^1$$

representing the horizontal prefix ending at the position (i,j). For $i \in [m]$, let the arc

$$a_{i,n}^1 = v_0^1 v_{i,n}^1$$

represent the prefix ending at the position (i,n). Let A^1 be the set of arcs representing the horizontal prefixes, that is,

$$A^1 = \{a_{i,j}^1 : (i,j) \in S\}.$$

These correspond to the inequalities in (1b).

Similarly, for $i \in [m-1]$ and $j \in [n]$, we define the arc

$$a_{i,j}^2 = v_{i,j}^2 v_{i+1,j}^2$$

representing the vertical prefix ending at the position (i,j). For $j \in [n]$, let the arc

$$a_{m,j}^2 = v_{m,j}^2 v_0^2$$

represent the prefix ending at the position (m,j). Let A^2 be the set of arcs representing the vertical prefixes, that is,

$$A^2 = \{a_{i,j}^2 : (i,j) \in S\}.$$

These correspond to the inequalities in (1c).

We also add an arc

$$a_0 = v_0^2 v_0^1,$$

which represents the sum of all entries of the matrix, which corresponds to (1d).

The arcs in $A^1 \cup A^2 \cup \{a_0\}$ are called the *tree arcs* that belong to the spanning tree determined by the network matrix Q.

Furthermore, let

$$a_{i,j} = v_{i,j}^1 v_{i,j}^2,$$

$$N = \{a_{i,j} : (i,j) \in S\}$$

be the *non-tree arcs* that represent the entries in the matrix. Figure 3 shows the construction for $m = n = 4$. Tree arcs are represented as continuous green arrows and non-tree arcs are represented as dotted black arrows in the digraph.

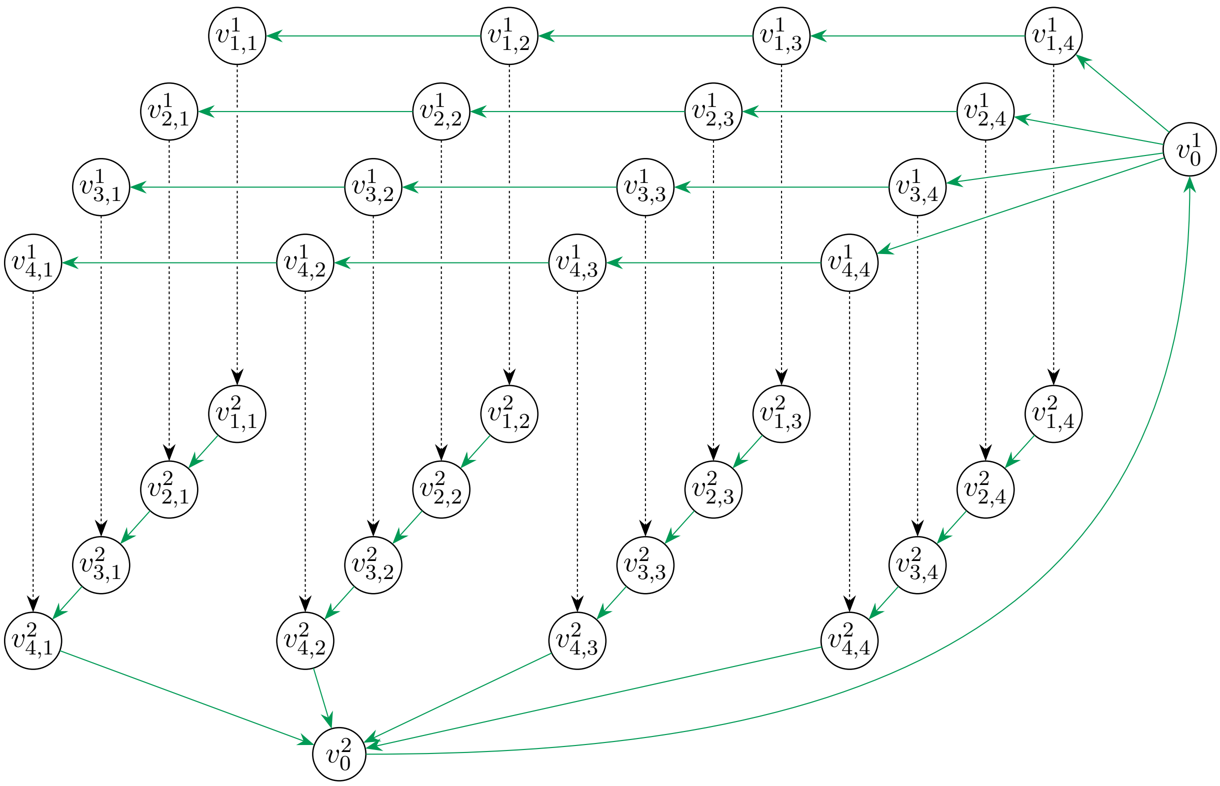


Figure 3: The circulation network for a 4×4 PBM.

For every $(i,j) \in S$, we define the lower and upper bounds $l, u : A \to \mathbb{Z}$ as

$$l(a^1_{i,j}) = \Phi^1_{i,j},\ u(a^1_{i,j}) = \Gamma^1_{i,j},$$

$$l(a^2_{i,j}) = \Phi^2_{i,j},\ u(a^2_{i,j}) = \Gamma^2_{i,j},$$

$$l(a_{i,j}) = f_{i,j},\ u(a_{i,j}) = g_{i,j},$$

$$l(a_0) = \alpha,\ u(a_0) = \beta.$$

Theorem 2 (Hoffman's circulation theorem [19, 25])**.** *Let $D = (V, A)$ be a digraph and $l, u : A \to \mathbb{R}$ lower and upper bounds defined on its arcs, such that $l \le u$. There exists an (l,u)-feasible circulation $z : A \to \mathbb{R}$, that is, $l \le z \le u$ and $\varrho_z(v) = \delta_z(v)$ for every vertex $v \in V$, if and only if $\varrho_u(W) \ge \delta_l(W)$ holds for every subset $W \subseteq V$ of the vertices. If l and u are integer-valued, then z can also be chosen integer-valued.*

Now we know that there is a PBM if and only if the condition of Hoffman's circulation theorem holds for the corresponding circulation problem. In the next sections, we rely on this relationship to derive necessary and sufficient conditions for the existence of a PBM with various bounds.

Remark 2. *For any linear cost function $c : S \to \mathbb{R}$, we can model the problem $\min\{cx : f \le x \le g, a \le Qx \le b\}$, where Q is the matrix defined above, as a minimum cost circulation problem, with $c'(a_{i,j}) = c_{i,j}$ $\forall (i,j) \in S$ and $c'(a) = 0$ $\forall a \in A \setminus N$, thus we can find a minimum cost PBM in polynomial time.*

2.1 A general theorem about the existence

Let $\Phi^1, \Phi^2, \Gamma^1, \Gamma^2, f, g, \alpha, \beta$ be the bounds introduced in Definition 1, and $D = (V, A)$ be the digraph with bounds l, u on the arcs defined above. Let $\mathcal{P}_1$ be the set of all horizontal prefixes of S, and $p_1, b_1 : \mathcal{P}_1 \to \mathbb{Z}$ be the lower and upper bounds defined by Φ^1 and Γ^1. Similarly, let $\mathcal{P}_2$ be the set of all vertical prefixes of S and $p_2, b_2 : \mathcal{P}_2 \to \mathbb{Z}$ the lower and upper bounds defined by Φ^2 and Γ^2. We extend the definitions of p_1, b_1, p_2 and b_2 to intervals, and we refer to the extended functions as p_1^*, b_1^*, p_2^*, and b_2^*. Let $Z \subseteq S$ be a horizontal interval in the i-th row, let (i,h) and (i,k) be its first and last positions, respectively. Then let

$$\begin{aligned} b_1^*(Z) &= \Gamma^1_{i,k} - \Phi^1_{i,h-1}, \\ p_1^*(Z) &= \Phi^1_{i,k} - \Gamma^1_{i,h-1}, \end{aligned} \tag{3}$$

where $\Phi^1_{i,0} = \Gamma^1_{i,0} = 0$ for all $i \in [m]$. For the empty set, $p_1^*(\emptyset) = b_1^*(\emptyset) = 0$. Similarly, if $Z \subseteq S$ is a vertical interval in the j-th column and (h,j), (k,j) are its first and last positions, then let

$$\begin{aligned} b_2^*(Z) &= \Gamma^2_{k,j} - \Phi^2_{h-1,j}, \\ p_2^*(Z) &= \Phi^2_{k,j} - \Gamma^2_{h-1,j}, \end{aligned} \tag{4}$$

where $\Phi^2_{0,j} = \Gamma^2_{0,j} = 0$ for all $j \in [n]$. For the empty set, $p_2^*(\emptyset) = b_2^*(\emptyset) = 0$. We extend the definition of p_1^*, b_1^*, p_2^*, and b_2^* to every subset $X \subseteq S$. For an arbitrary set $X \subseteq S$, let $p_1^*(X)$ and $b_1^*(X)$ be the sum of the bounds on the maximal horizontal intervals in X. The bounds p_2^* and b_2^* can be defined similarly as the sum of the bounds on the maximal vertical intervals in X.

Theorem 3. *Let $S = [m] \times [n]$. There exists an $m \times n$ integer matrix with its entries between the bounds f, g, the sum of its entries between α and β, and satisfying the prefix bounds defined by $\Phi^1, \Phi^2, \Gamma^1, \Gamma^2$ if and only if the following conditions hold for every $X_1, X_2 \subseteq S$*

$$p_1^*(X_1) + \widetilde{f}(X_2 - X_1) \le b_2^*(X_2) + \widetilde{g}(X_1 - X_2), \tag{5a}$$

$$p_2^*(X_2) + \widetilde{f}(X_1 - X_2) \le b_1^*(X_1) + \widetilde{g}(X_2 - X_1), \tag{5b}$$

$$b_1^*(X_1) + b_2^*(X_2) + \widetilde{g}(\overline{X}_1 \cap \overline{X}_2) - \widetilde{f}(X_1 \cap X_2) \ge \alpha, \tag{5c}$$

$$p_1^*(X_1) + p_2^*(X_2) + \widetilde{f}(\overline{X}_1 \cap \overline{X}_2) - \widetilde{g}(X_1 \cap X_2) \le \beta. \tag{5d}$$

We can naturally replace the sets X_1, X_2 with a family of intervals $\mathcal{I} = \mathcal{I}^H \uplus \mathcal{I}^V$, where $\mathcal{I}^H$ consists of the maximal horizontal intervals in X_1 and $\mathcal{I}^V$ consists of the maximal vertical intervals in X_2. Note that $\mathcal{I}^H$ and $\mathcal{I}^V$ may contain the same entry as a one-element horizontal and vertical interval.

Example 1. *Consider the bounds defined below:*

$$\Phi^1 = \begin{bmatrix} -\infty & -\infty & 1 \\ -\infty & -\infty & 1 \\ -\infty & -\infty & -\infty \end{bmatrix}, \Phi^2 = \begin{bmatrix} -\infty & 1 & -\infty \\ -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty \end{bmatrix}, \Gamma^1 = \begin{bmatrix} \infty & \infty & \infty \\ -1 & \infty & \infty \\ \infty & \infty & \infty \end{bmatrix}, \Gamma^2 = \begin{bmatrix} \infty & \infty & \infty \\ \infty & \infty & \infty \\ \infty & -1 & \infty \end{bmatrix},$$

$$f = \begin{bmatrix} 0 & 0 & 0 \\ -\infty & -\infty & 0 \\ -\infty & 0 & -\infty \end{bmatrix}, \ g = \begin{bmatrix} 1 & 1 & 1 \\ \infty & \infty & 1 \\ \infty & \infty & \infty \end{bmatrix}.$$

Figure 4 shows an example of a family of intervals that proves that there is no PBM with the bounds $(\Phi^1, \Gamma^1 \Phi^2, \Gamma^2, f, g, -\infty, \infty)$.

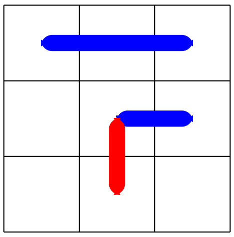


Figure 4: A family of intervals $\mathcal{I} = \mathcal{I}^H \uplus \mathcal{I}^V$ violating (5a).

Proof. We call two subsets $X_1, X_2 \subseteq S$ a *violating pair* if they do not satisfy at least one of the inequalities (5a), (5b), (5c), and (5d).

First, we prove that if there exists a violating pair X_1, X_2, then there exists no PBM with bounds $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, \alpha, \beta)$. It is easy to see that the left-hand sides of the inequalities (5a) and (5b), are both lower bounds for the sum of the entries in $X_1 \cup X_2$, while the right-hand sides of the inequalities are both upper bounds for the sum of the entries in $X_1 \cup X_2$, thus, if one of them does not hold, it means that we have a lower and upper bound, where the upper bound is smaller than the lower bound, therefore, there exists no PBM. In (5c), the left-hand side is an upper bound for the sum of the entries in the matrix, thus, if it is smaller than α, then there cannot exist a PBM. The left-hand side of (5d) is a lower bound for the sum of the entries, thus if it is greater than β, then there exists no PBM.

Second, we show that if there exists no PBM with bounds $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, \alpha, \beta)$, then there exists a violating pair X_1, X_2. If there is no PBM, then there cannot be an (l, u)-feasible circulation either, since we showed that there exists a circulation if and only if there exists a PBM. It follows from Hoffman's circulation theorem that there exists a subset $W \subseteq V$ such that

$$\varrho_u(W) - \delta_l(W) < 0. \tag{6}$$

For a digraph $D = (V, A)$ and for a subset $W \subseteq V$, let $\Delta^{\text{in}}(W) := \{uv \in A : u \notin W, v \in W\}$ denote the set of arcs entering W, and let $\Delta^{\text{out}}(W) := \{uv \in A : u \in W, v \notin W\}$ denote the set of arcs leaving W.

Let Z be a horizontal interval in the i-th row, and denote its first and last positions by (i, h)

and (i,k), respectively. Then

$$\begin{aligned} b_1^*(Z) &= \Gamma^1_{i,k} - \Phi^1_{i,h-1} = u(a^1_{i,k}) - l(a^1_{i,h-1}), \\ p_1^*(Z) &= \Phi^1_{i,k} - \Gamma^1_{i,h-1} = l(a^1_{i,k}) - u(a^1_{i,h-1}), \end{aligned} \tag{7}$$

where we define $u(a^1_{i,0}) = l(a^1_{i,0}) = 0$ for all $i \in [m]$, so that we get the bounds for the prefixes for $h = 1$.

Similarly, if Z is a vertical interval in the j-th column and (h,j) and (k,j) are its first and last elements, respectively. Then

$$\begin{aligned} b_2^*(Z) &= \Gamma^2_{k,j} - \Phi^2_{h-1,j} = u(a^2_{k,j}) - l(a^2_{h-1,j}), \\ p_2^*(Z) &= \Phi^2_{k,j} - \Gamma^2_{h-1,j} = l(a^2_{k,j}) - u(a^2_{h-1,j}), \end{aligned} \tag{8}$$

where we define $u(a^2_{0,j}) = l(a^2_{0,j}) = 0$ for all $j \in [n]$, so that we get the bounds for the prefixes for $h = 1$.

To prove our theorem, we investigate four different cases depending on the set W that violates the condition of Hoffman's circulation theorem. Each of these cases corresponds to an inequality in Theorem 3.

Case 1. $v_0^1, v_0^2 \notin W$
Let X_1, X_2 be defined as

$$X_1 = \{(i,j) \in S : v^1_{i,j} \in W\} \text{ and } X_2 = \{(i,j) \in S : v^2_{i,j} \in W\}.$$

We know that a_0 does not contribute to $\varrho_u(W) - \delta_l(W)$, as neither of its endpoints are in W.

The contribution of the arcs in A^1 to $\varrho_u(W) - \delta_l(W)$ is

$$\sum_{a^1_{i,j} \in \Delta^{\text{in}}(W) \cap A^1} u(a^1_{i,j}) - \sum_{a^1_{i,j-1} \in \Delta^{\text{out}}(W) \cap A^1} l(a^1_{i,j-1}) = b_1^*(X_1).$$

The contribution of the arcs in A^2 is

$$\sum_{a^2_{i-1,j} \in \Delta^{\text{in}}(W) \cap A^2} u(a^2_{i-1,j}) - \sum_{a^2_{i,j} \in \Delta^{\text{out}}(W) \cap A^2} l(a^2_{i,j}) = -p_2^*(X_2).$$

An arc $a_{i,j} \in N$ contributes $(-f_{i,j})$ if $v^1_{i,j} \in W$ and $v^2_{i,j} \notin W$, that is, $(i,j) \in X_1 - X_2$. It contributes $g_{i,j}$ if $v^1_{i,j} \notin W$ and $v^2_{i,j} \in W$, that is, $(i,j) \in X_2 - X_1$. Otherwise, it contributes 0. From these, we get

$$\varrho_u(W) - \delta_l(W) = b_1^*(X_1) - p_2^*(X_2) - \widetilde{f}(X_1 - X_2) + \widetilde{g}(X_2 - X_1) < 0,$$

which, by rearrangements, means that X_1 and X_2 violate (5b).

Case 2. $v_0^1, v_0^2 \in W$
Let X_1, X_2 be defined as

$$X_1 = \{(i,j) : v^1_{i,j} \notin W\} \text{ and } X_2 = \{(i,j) : v^2_{i,j} \notin W\}.$$

We know that a_0 does not contribute to $\varrho_u(W) - \delta_l(W)$, since both of its endpoints are in W.

The contribution of the arcs in A^1 is

$$\sum_{a^1_{i,j-1}\in\Delta^{\mathrm{in}}(W)\cap A^1} u(a^1_{i,j-1}) - \sum_{a^1_{i,j}\in\Delta^{\mathrm{out}}(W)\cap A^1} l(a^1_{i,j}) = -p^*_1(X_1).$$

The contribution of the arcs in A^2 is

$$\sum_{a^2_{i,j}\in\Delta^{\mathrm{in}}(W)\cap A^2} u(a^2_{i,j}) - \sum_{a^2_{i-1,j}\in\Delta^{\mathrm{out}}(W)\cap A^2} l(a^2_{i-1,j}) = b^*_2(X_2).$$

An arc $a_{i,j} \in N$ contributes $(-f_{i,j})$ if $v^1_{i,j} \in W$ and $v^2_{i,j} \notin W$, that is, $(i,j) \in X_2 - X_1$. It contributes $g_{i,j}$ if $v^1_{i,j} \notin W$ and $v^2_{i,j} \in W$, that is, $(i,j) \in X_1 - X_2$. Otherwise, it contributes 0. From these, we get

$$\varrho_u(W) - \delta_l(W) = b^*_2(X_2) - p^*_1(X_1) - \widetilde{f}(X_2 - X_1) + \widetilde{g}(X_1 - X_2) < 0,$$

which, by rearrangements, means that X_1 and X_2 violate (5a).

Case 3. $v^1_0 \in W, v^2_0 \notin W$
Let X_1, X_2 be defined as

$$X_1 = \{(i,j) : v^1_{i,j} \notin W\} \text{ and } X_2 = \{(i,j) : v^2_{i,j} \in W\}.$$

In this case, the arc a_0 enters W, thus its contribution to $\varrho_u(W) - \delta_l(W)$ is β.

The contribution of the arcs in A^1 is

$$\sum_{a^1_{i,j-1}\in\Delta^{\mathrm{in}}(W)\cap A^1} u(a^1_{i,j-1}) - \sum_{a^1_{i,j}\in\Delta^{\mathrm{out}}(W)\cap A^1} l(a^1_{i,j}) = -p^*_1(X_1).$$

The contribution of the arcs in A^2 is

$$\sum_{a^2_{i-1,j}\in\Delta^{\mathrm{in}}(W)\cap A^2} u(a^2_{i-1,j}) - \sum_{a^2_{i,j}\in\Delta^{\mathrm{out}}(W)\cap A^2} l(a^2_{i,j}) = -p^*_2(X_2).$$

An arc $a_{i,j} \in N$ contributes $(-f_{i,j})$ if $v^1_{i,j} \in W$ and $v^2_{i,j} \notin W$, that is, $(i,j) \in \overline{X}_1 \cap \overline{X}_2$. It contributes $g_{i,j}$ if $v^1_{i,j} \notin W$ and $v^2_{i,j} \in W$, that is, $(i,j) \in X_1 \cap X_2$. Otherwise, it contributes 0. From these, we get

$$\varrho_u(W) - \delta_l(W) = -p^*_1(X_2) - p^*_2(X_1) - \widetilde{f}(\overline{X}_1 \cap \overline{X}_2) + \widetilde{g}(X_1 \cap X_2) + \beta < 0,$$

wich, by rearrangements, means that X_1 and X_2 violate (5d).

Case 4. $v^1_0 \notin W, v^2_0 \in W$
Let X_1, X_2 be defined as

$$X_1 = \{(i,j) : v^1_{i,j} \in W\} \text{ and } X_2 = \{(i,j) : v^2_{i,j} \notin W\}.$$

In this case, the arc a_0 leaves W, thus its contribution to $\varrho_u(W) - \delta_l(W)$ is $(-\alpha)$.

The contribution of the arcs in A^1 is

$$\sum_{a^1_{i,j}\in\Delta^{\mathrm{in}}(W)\cap A^1} u(a^1_{i,j}) - \sum_{a^1_{i,j-1}\in\Delta^{\mathrm{out}}(W)\cap A^1} l(a^1_{i,j-1}) = b^*_1(X_1).$$

The contribution of the arcs in A^2 is

$$\sum_{a^2_{i,j}\in\Delta^{\text{in}}(W)\cap A^2} u(a^2_{i,j}) - \sum_{a^2_{i-1,j}\in\Delta^{\text{out}}(W)\cap A^2} l(a^2_{i-1,j}) = b^*_2(X_2).$$

An arc $a_{i,j} \in N$ contributes $(-f_{i,j})$ if $v^1_{i,j} \in W$ and $v^2_{i,j} \notin W$, that is, $(i,j) \in X_1 \cap X_2$. It contributes $g_{i,j}$ if $v^1_{i,j} \notin W$ and $v^2_{i,j} \in W$, that is, $(i,j) \in \overline{X}_1 \cap \overline{X}_2$. Otherwise, it contributes 0. From these, we get

$$\varrho_u(W) - \delta_l(W) = b^*_1(X_1) + b^*_2(X_2) - \widetilde{f}(X_1 \cap X_2) + \widetilde{g}(\overline{X}_1 \cap \overline{X}_2) - \alpha < 0,$$

which, by rearrangements, means that X_1 and X_2 violate (5c). □

2.2 Simpler bounds

A straightforward special case is when we do not have any bounds on the sum of all entries. We often use this corollary instead of Theorem 3, since in this case the conditions are simpler.

Corollary 1. *Let $S = [m] \times [n]$. There exists an $m \times n$ integer matrix with entries between the bounds f, g satisfying the prefix bounds defined by $\Phi^1, \Phi^2, \Gamma^1, \Gamma^2$ if and only if the following conditions hold for every $X_1, X_2 \subseteq S$*

$$p^*_1(X_1) + \widetilde{f}(X_2 - X_1) \le b^*_2(X_2) + \widetilde{g}(X_1 - X_2), \tag{5a}$$

$$p^*_2(X_2) + \widetilde{f}(X_1 - X_2) \le b^*_1(X_1) + \widetilde{g}(X_2 - X_1). \tag{5b}$$

Proof. We can see that the statement is a special case of Theorem 3 with $\alpha = -\infty$ and $\beta = \infty$. Because of this, (5c) and (5d) automatically hold for every $X_1, X_2 \subseteq S$, thus (5a) and (5b) hold for every $X_1, X_2 \subseteq S$ if and only if there exists a PBM. □

We also state the theorem in the special case when we only have bounds on the prefixes, with no bounds on the matrix entries or their sum.

Corollary 2. *Let $S = [m] \times [n]$. There exists an $m \times n$ integer matrix satisfying the prefix bounds defined by $\Phi^1, \Phi^2, \Gamma^1, \Gamma^2$ if and only if the following conditions hold for every* connected *subset $X \subseteq S$:*

$$p^*_1(X) \le b^*_2(X), \tag{9a}$$

$$p^*_2(X) \le b^*_1(X), \tag{9b}$$

where we call a subset $X \subseteq S$ connected if, for every $s, s' \in X$, there exists a sequence of positions $x_0, x_1, \ldots, x_k \in X$, where $x_0 = s, x_k = s'$ and for all $0 \le i < k$ x_i and x_{i+1} are adjacent positions.

Proof. In this special case of Corollary 1 the entry bounds are $f \equiv -\infty$ and $g \equiv \infty$.

It is obvious that if there exists a subset X, for which (9a) or (9b) does not hold, then there exists no PBM.

If there exists no PBM, we know that there exists a pair of subsets X_1, X_2 for which (5a) or (5b) does not hold. Suppose it is (5a). Observe that if $X_1 \ne X_2$, then either $\widetilde{f}(X_2 - X_1) = -\infty$ or $\widetilde{g}(X_1 - X_2) = \infty$, thus (5a) holds. This implies $X_1 = X_2 =: X$, and (5a) is equivalent to $p^*_1(X) \le b^*_2(X)$. What remains to show is that if there is such an X, then there is a connected X' for which (5a) still does not hold. Let $X^1, \ldots, X^k$ be the connected components of X. It is easy to see that $p^*_1(X) = p^*_1(X^1) + \cdots + p^*_1(X^k)$ and $b^*_2(X) = b^*_2(X^1) + \cdots + b^*_2(X^k)$, thus at least one connected component X^i must violate (5a). The same argument works if (5b) does not hold. □

2.3 Exact prescriptions for the rows and columns

We investigate an interesting case of PBMs, when there are exact prescriptions on the sum of the entries in each full row and column of the matrix. This means that

$$\begin{aligned} -\infty < \Phi^1_{i,n} = \Gamma^1_{i,n} < \infty \ \forall i \in [m], \\ -\infty < \Phi^2_{m,j} = \Gamma^2_{m,j} < \infty \ \forall j \in [n]. \end{aligned} \tag{10}$$

We suppose that

$$\sum_{i\in[m]} \Gamma^1_{i,n} = \sum_{j\in[n]} \Gamma^2_{m,j} =: H, \tag{11}$$

otherwise, there could not exist a PBM.

Lemma 1. *Let $S = [m] \times [n]$ be the ground set and let $p_1^*, p_2^*, b_1^*, b_2^*$ be the bounds defined by $\Phi^1, \Phi^2, \Gamma^1, \Gamma^2$ at the beginning of Section 2.1, where $\Phi^1, \Phi^2, \Gamma^1, \Gamma^2$ satisfy* (10) *and* (11)*. Then for every subset $X \subseteq S$*

$$p_1^*(X) + b_1^*(S - X) = H, \tag{12a}$$
$$p_2^*(X) + b_2^*(S - X) = H. \tag{12b}$$

Proof. We only show the proof of (12a) as (12b) can be proven similarly. Recall that $S_{i,\cdot}$ denotes the i-th row of S, and let X^i denote the positions in X belonging to the i-th row, that is, $X^i = X \cap S_{i,\cdot}$. As we know that

$$p_1^*(X) = \sum_{i\in[m]} p_1^*(X^i),$$

it suffices to show that (12a) holds for every row $S_{i,\cdot}$. The statement for $S_{i,\cdot}$ is the following:

Claim 1. *For every subset $Z \subseteq S_{i,\cdot}$,*

$$p_1^*(Z) + b_1^*(S_{i,\cdot} - Z) = b_1^*(S_{i,\cdot}), \tag{13}$$

where $b_1^*(S_{i,\cdot}) = \Gamma^1_{i,n} = \Phi^1_{i,n} = p_1^*(S_{i,\cdot})$.

Proof. We prove this statement by induction on the number of maximal intervals $Z_1, \dots, Z_k$ in Z. If there are 0 intervals in Z, then $Z = \emptyset$ and the statement clearly holds. Let $Z = \bigcup_{i\in[k]} Z_i$, where $Z_1, \dots, Z_k$ are the disjoint maximal intervals in Z. By induction, we assume that the statement holds for all subsets that contain fewer than k maximal intervals. Let $Z' = \bigcup_{i\in[k-1]} Z_i$ be the union of the first $k-1$ maximal intervals. We know that (13) holds for Z', thus

$$p_1^*(Z') + b_1^*(S_{i,\cdot} - Z') = b_1^*(S_{i,\cdot}).$$

We also know that

$$p_1^*(Z) = p_1^*(Z') + p_1^*(Z_k).$$

Let us notice that when we add Z_k to Z', we remove Z_k from $S_{i,\cdot} - Z$ and from the maximal interval that contained Z_k in $S_{i,\cdot} - Z$ we get two intervals. Let (i,h) and (i,k) be the first and last entries of Z_k. This way, we add $\Phi^1_{i,k} - \Gamma^1_{i,h-1}$ to p_1^* and we add $\Gamma^1_{i,h-1} - \Phi^1_{i,k}$ to b_1^*, thus their sum remains $b_1^*(S_{i,\cdot})$. □

Lemma 1 immediately follows from Claim 1. □

Theorem 4. *Let $S = [m] \times [n]$. There exists an $m \times n$ integer matrix with entries between the bounds f, g satisfying the prefix bounds defined by $\Phi^1, \Phi^2, \Gamma^1, \Gamma^2$, where $\Phi^1, \Phi^2, \Gamma^1, \Gamma^2$ satisfy* (10) *and* (11) *if and only if the following condition holds for every $X_1, X_2 \subseteq S$:*

$$H \le b_1^*(X_1) + b_2^*(X_2) + \widetilde{g}(\overline{X}_1 \cap \overline{X}_2) - \widetilde{f}(X_1 \cap X_2). \tag{14}$$

Proof. Let us notice that the right-hand side of (14) is an upper bound for the sum of all entries, thus, if (14) does not hold, then there exists no PBM.

Now we need to prove that, if there exists no PBM, then there exists a violating pair of subsets, for which (14) does not hold. We know from Corollary 1 that there exists a pair $X_1, X_2 \subseteq S$, for which (5a) or (5b) does not hold. If (5a) does not hold, that is,

$$p_1^*(X_1) + \widetilde{f}(X_2 - X_1) > b_2^*(X_2) + \widetilde{g}(X_1 - X_2).$$

Let $X_1' = S - X_1$ be the complement set of X_1. Using this and Lemma 1, we get

$$H - b_1^*(X_1') + \widetilde{f}(X_2 \cap X_1') > b_2^*(X_2) + \widetilde{g}(\overline{X}_1' \cap \overline{X}_2),$$

which obviously implies that (14) does not hold for X_1', X_2.

If (5b) does not hold for X_1, X_2, then with a similar argument we get that (14) does not hold for X_1, X_2', where $X_2' = S - X_2$ is the complement set of X_2. □

2.4 Another approach based on g-polymatroids

The problems discussed so far in this section can also be handled with *g-polymatroids* [14]. In this thesis, we only provide a brief summary on how we get the above theorems using g-polymatroids and only define the most essential concepts. The details of this approach can be found in [4] and the background of g-polymatroids can be found in [15]. Let S be a finite set, which we will refer to as the ground set. Let $p, b : 2^S \to \mathbb{R}$ be a *strongly paramodular pair*, that is, p is supermodular, b is *submodular*, and they satisfy the *cross-inequality* for every pair of subsets $X_1, X_2 \subseteq S$. We also suppose that $p(\emptyset) = b(\emptyset) = 0$.

Definition 2. *We call the polyhedron $Q(p, b) = \{x \in \mathbb{R}^S : p(Z) \le \widetilde{x}(Z) \le b(Z)\ \forall Z \subseteq S\}$ a generalized polymatroid (g-polymatroid).*

For a g-polymatroid $Q(p, b)$, the bordering paramodular pair is unique, namely

$$b(Z) = \max\{\widetilde{x}(Z) : x \in Q\}, \quad p(Z) = \min\{\widetilde{x}(Z) : x \in Q\}.$$

Observe that the bounding functions defined in (7) and (8) are paramodular pairs, thus the matrices satisfying the bounds defined by Φ^1 and Γ^1 form a g-polymatroid $Q(p_1^*, b_1^*)$. Similarly, the matrices satisfying the bounds defined by Φ^2 and Γ^2 form a g-polymatroid $Q(p_2^*, b_2^*)$. This means that the polyhedron of PBMs is the intersection $Q(p_1^*, b_1^*) \cap Q(p_2^*, b_2^*) \cap T(f, g) \cap K(\alpha, \beta)$, where $T(f, g) = \{x \in \mathbb{R}^S : f(s) \le x(s) \le g(s)\ \forall s \in S\}$ and $K(\alpha, \beta) = \{x \in \mathbb{R}^S : \alpha \le \widetilde{x}(S) \le \beta\}$. We know that if $Q(p, b)$ is a g-polymatroid, then both $Q(p, b) \cap T(f, g)$ and $Q(p, b) \cap K(\alpha, \beta)$ are g-polymatroids. Our Theorem 3, Corollary 1, Corollary 2, and Theorem 4 can all be proved using these and the following theorem.

Theorem 5 (Intersection theorem for g-polymatroids)**.** *Let (p_1, b_1) and (p_2, b_2) be two paramodular pairs. The intersection $M = Q(p_1, b_1) \cap Q(p_2, b_2)$ is non-empty if and only if*

$$p_1 \le b_2 \text{ and } p_2 \le b_1.$$

Using tools related to g-polymatroids, we can also prove Lemma 1 a lot simpler, as we know from the definition of strongly paramodular pairs that the cross-inequality holds for (p_1^*, b_1^*) and (p_2^*, b_2^*), and the statement of the lemma can be easily derived from it.

2.5 Linking property

We say that a decision problem has the *linking property* if we there exist some relaxed subproblems in a way that the original problem is feasible if and only if all the relaxed subproblems are feasible. An example for the linking property is the following theorem of Mendelsohn and Dulmage.

Theorem 6 (Mendelsohn, Dulmage [24]). *Let $G = (S, T, E)$ be a bipartite graph and $X \subseteq S, Y \subseteq T$ two subsets of the vertex sets. If there is a matching covering X and there is a matching covering Y, then there exists a matching covering $X \cup Y$.*

In this subsection, we discuss some linking properties of the problem of deciding whether there exists a PBM with given bounds. One possible way to split the problem into two relaxed problems is that we relax the bounds on the full rows of the matrix. Of course, we could say the same with relaxing the bounds on the full columns instead of the rows. We only show the case when we relax the bounds on full rows as the same argument could be used for the columns.

Theorem 7. *There exists a PBM with bounds $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, \alpha, \beta)$ if and only if*

7.1 there exists a PBM with bounds $(\overline{\Phi}^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, \alpha, \infty)$, and

7.2 there exists a PBM with bounds $(\Phi^1, \overline{\Gamma}^1, \Phi^2, \Gamma^2, f, g, -\infty, \beta)$,

where $\overline{\Phi}^1_{i,j} = \Phi^1_{i,j}$ for every $i \in [m], j \in [n-1]$ and $\overline{\Phi}^1_{i,n} = -\infty$ for every $i \in [m]$. Similarly $\overline{\Gamma}^1_{i,j} = \Gamma^1_{i,j}$ for every $i \in [m], j \in [n-1]$ and $\overline{\Gamma}^1_{i,n} = \infty$ for every $i \in [m]$.

Proof. We show that if there is no PBM with bounds $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, \alpha, \beta)$, then one of the two relaxed subproblems must be infeasible. In Theorem 3, we provided a necessary and sufficient condition on the existence of a PBM. It suffices to show that if there exists a pair of subsets $X_1, X_2 \subseteq S$ violating one of the inequalities in Theorem 3, then one of the relaxed problems must be infeasible.
Note that the upper bounds b_1^*, b_2^* in the first subproblem 7.1 are exactly the same as in the original problem, thus (5c) remains the same if we use Theorem 3 for the bounds $(\overline{\Phi}^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, \alpha, \infty)$. The values of p_2^* remain the same for every $X_2 \subseteq S$, therefore (5b) remains the same in 7.1. It follows that if the pair X_1, X_2 violates (5b) or (5c) in Theorem 3, then it violates these inequalities in 7.1 as well.
In the second subproblem 7.2, the bounds p_1^*, p_2^* are the same as in the original problem, thus (5d) is the same for the bounds $(\Phi^1, \overline{\Gamma}^1, \Phi^2, \Gamma^2, f, g, -\infty, \beta)$. The values of b_2^* also remain the same for every $X_2 \subseteq S$, therefore (5a) is the same for the subproblem 7.2 and the original problem.

We showed that if the pair X_1, X_2 violates (5b) or (5c), then 7.1 is infeasible and if the pair X_1, X_2 violates (5a) or (5d) then 7.2 is infeasible. Obviously, the opposite direction holds as well, that is, if the original problem is feasible, then the relaxed subproblems are also feasible. □

A natural next step is to try to relax the bounds on both the full rows and columns. In the next theorem we show that the linking property holds if we do not have bounds on the sum of all entries. Then we provide a counterexample in Remark 3 for the case when we do have bounds for the total sum.

Theorem 8. *There exists a PBM with bounds* $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, -\infty, \infty)$ *if and only if*

8.1 there exists a PBM with bounds $(\overline{\Phi}^1, \Gamma^1, \Phi^2, \overline{\Gamma}^2, f, g, -\infty, \infty)$*, and*

8.2 there exists a PBM with bounds $(\Phi^1, \overline{\Gamma}^1, \overline{\Phi}^2, \Gamma^2, f, g, -\infty, \infty)$*, where*

$$\overline{\Phi}^1_{i,j} = \Phi^1_{i,j} \ \forall i \in [m], j \in [n-1] \text{ and } \overline{\Phi}^1_{i,n} = -\infty \ \forall i \in [m],$$

$$\overline{\Gamma}^2_{i,j} = \Gamma^2_{i,j} \ \forall i \in [m-1], j \in [n] \text{ and } \overline{\Gamma}^2_{m,j} = \infty \ \forall j \in [n],$$

$$\overline{\Phi}^2_{i,j} = \Phi^2_{i,j} \ \forall i \in [m-1], j \in [n] \text{ and } \overline{\Phi}^2_{m,j} = -\infty \ \forall j \in [n],$$

$$\overline{\Gamma}^1_{i,j} = \Gamma^1_{i,j} \ \forall i \in [m], j \in [n-1] \text{ and } \overline{\Gamma}^1_{i,n} = \infty \ \forall i \in [m].$$

Proof. We only show that if there is no PBM satisfying the original bounds, then one of the two subproblems must be infeasible, as the reverse direction is trivial. If there is no such PBM, then Corollary 1 states that there must be a pair of subsets X_1, X_2 violating (5a) or (5b).

If the pair X_1, X_2 violates (5a), then the subproblem 8.2 must be infeasible since b_2^* is the same as in the original problem for every $X_2 \subseteq S$ and p_1^* is the same for every $X_1 \subseteq S$.

Similarly b_1^* is the same in the subproblem 8.1 as in the original problem for every $X_1 \subseteq S$ and p_2^* is the same for every $X_2 \subseteq S$, thus if the pair X_1, X_2 violates (5b) in Corollary 1, then 8.1 is infeasible. □

Remark 3. *If we have bounds* α, β *on the sum of all entries, then the linking property shown in Theorem 8 no longer holds, even if we include them in both subproblems. Let* $n = m = 2, \alpha = 5$*,* $\beta = \infty$*, and* $\Phi^1 = \Phi^2 \equiv -\infty$*. Let the upper bounds for the prefixes be*

$$\Gamma^1 = \Gamma^2 = \begin{bmatrix} \infty & \infty \\ \infty & 1 \end{bmatrix}.$$

For every position (i, j) *let* $f_{i,j} = 0$ *and* $g_{i,j} = 2$*. Then it is easy to see that there is no PBM with bounds* $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, \alpha, \beta)$*, since the pair*

$$X_1 = \{(2, 1), (2, 2)\}, X_2 = \{(1, 2), (2, 2)\}$$

violates (5c)*, however the matrices*

$$M_1 = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}, \ M_2 = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$$

are PBMs with bounds $(\overline{\Phi}^1, \Gamma^1, \Phi^2, \overline{\Gamma}^2, f, g, \alpha, \beta)$ *and* $(\Phi^1, \overline{\Gamma}^1, \overline{\Phi}^2, \Gamma^2, f, g, \alpha, \beta)$*, respectively.*

One might wonder whether we can extend Theorem 8 to every prefix if there are no bounds on the sum of all entries, that is, in one subproblem we only have lower bounds on the horizontal prefixes, and upper bounds on the vertical ones, and conversely in the other subproblem. The answer is negative, as we show a counterexample in Remark 4.

Remark 4. *It is possible that there exist PBMs with bounds* $(\Phi^1, \infty, -\infty, \Gamma^2, -\infty, \infty, -\infty, \infty)$ *and* $(-\infty, \Gamma^1, \Phi^2, \infty, -\infty, \infty, -\infty, \infty)$, *but there exists no PBM with bounds* $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, -\infty, \infty, -\infty, \infty)$.

Let the bounds be

$$\Phi^1 = \Gamma^1 = \begin{bmatrix} 1 & 3 \end{bmatrix}, \ \Phi^2 = \begin{bmatrix} -\infty & 1 \end{bmatrix}, \text{ and } \Gamma^2 = \begin{bmatrix} \infty & 1 \end{bmatrix}.$$

Let the matrices A *and* B *be defined as*

$$A = \begin{bmatrix} 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

It is easy to see that A *is a PBM with bounds* $(\Phi^1, \infty, -\infty, \Gamma^2, -\infty, \infty, -\infty, \infty)$ *and* B *is a PBM with bounds* $(-\infty, \Gamma^1, \Phi^2, \infty, -\infty, \infty, -\infty, \infty)$, *but there is no PBM with bounds* $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, -\infty, \infty, -\infty, \infty)$.

Theorem 9. *Let* Φ^1 *and* Γ^1 *be such that in every row there is at most one position where* Φ^1 *and* Γ^1 *might be finite. All other entries in* Φ^1 *and* Γ^1 *are* $-\infty$ *and* ∞, *respectively. Similarly, let* Φ^2 *and* Γ^2 *be such that in every column there is exactly one position where* Φ^2 *and* Γ^2 *might be finite. All the other entries in* Φ^2 *and* Γ^2 *are* $-\infty$ *and* ∞, *respectively. Then there exists a PBM with bounds* $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, -\infty, \infty)$ *if and only if*

9.1 there exists a PBM with bounds $(\Phi^1, \infty, -\infty, \Gamma^2, f, g, -\infty, \infty)$, *and*

9.2 there exists a PBM with bounds $(-\infty, \Gamma^1, \Phi^2, \infty, f, g, -\infty, \infty)$.

Proof. Once again we use Corollary 1 to prove that if the original problem is infeasible, then one of the relaxed subproblems must be infeasible as well. Let $X_1, X_2 \subseteq S$ be a pair of subsets violating (5a) or (5b). In this special problem for a horizontal interval Z, $b_1^*(Z) < \infty$ if and only if Z is the only prefix in its row with finite bound, and $p_1^*(Z) > -\infty$ if and only if it is the only prefix in its row with finite bounds. Similarly, in every column there is at most one interval, with finite bounds, and it must be the prefix with finite bounds. It is easy to see now that (5a) is the same for the subproblem 9.1 as for the original problem, and (5b) is the same for the subproblem 9.2 as for the original problem. This shows that if the original problem is infeasible, then one of the relaxed subproblems must be infeasible.

Obviously, the reverse direction also holds. □

We derive the following well-known theorem of Ford and Fulkerson as a special case of Theorem 9.

Corollary 3 (Ford, Fulkerson [15, p. 71])**.** *Let* $G = (S, T, E)$ *be a bipartite graph,* $V = S \cup T$, *and let* $p, b : V \to \mathbb{Z}_{\geq 0}$ *be two functions with* $p \leq b$. *There exists a subgraph* $H = (S, T, F)$ *of* G *for which* $p(v) \leq d_F(v) \leq b(v)\ \forall v \in V$ *if and only if*

3.1 there is a subgraph of G *such that the degrees meet the lower bounds on* S *and the upper bounds on* T, *and*

3.2 there is a subgraph of G *such that the degrees satisfy the upper bounds on* S *and the lower bounds on* T.

Proof. We construct a PBM problem that is equivalent to the subgraph problem in the statement. Let the vertices of G be $S = \{s_1, \dots, s_m\}, T = \{t_1, \dots, t_n\}$. Let $A \in \mathbb{Z}^{m\times n}$ be the biadjacency matrix of G, where the i-th row represents s_i and the j-th column represents t_j. Let $f_{i,j} = 0$ for every $(i, j) \in [m] \times [n]$ and $g_{i,j} = a_{i,j}$ for all $(i, j) \in [m] \times [n]$. Let the bounds on the prefixes be

- $\Phi^1_{i,j} = -\infty$ for all $i \in [m], j \in [n-1]$, $\Phi^1_{i,n} = p(s_i)$ for all $i \in [m]$,
- $\Gamma^1_{i,j} = \infty$ for all $i \in [m], j \in [n-1]$, $\Gamma^1_{i,n} = b(s_i)$ for all $i \in [m]$,
- $\Phi^2_{i,j} = -\infty$ for all $i \in [m-1], j \in [n]$, $\Phi^2_{m,j} = p(t_j)$ for all $j \in [n]$,
- $\Gamma^2_{i,j} = \infty$ for all $i \in [m-1], j \in [n]$, $\Gamma^2_{m,j} = b(t_j)$ for all $j \in [n]$.

By applying Theorem 9, we get the statement of the corollary, since the relaxed PBM problems are equivalent to the relaxed subgraph problems in our statement. □

In the case when $\alpha = \beta$, we can relax the bounds on the sum of the entries of the matrix and get the following linking property:

Theorem 10. *There exists a PBM with bounds* $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, \gamma, \gamma)$ *if and only if*

10.1 there exists a PBM with bounds $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, -\infty, \gamma)$*, and*

10.2 there exists a PBM with bounds $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, \gamma, \infty)$*.*

Proof. We only show that if there exists a PBM $A = (a_{i,j})$ with bounds $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, -\infty, \gamma)$ and a PBM $B = (b_{i,j})$ with bounds $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, \gamma, \infty)$, then there exists a PBM with bounds $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, \gamma, \gamma)$. Let us define

$$\alpha = \sum_{(i,j)\in S} a_{i,j}, \text{ and } \beta = \sum_{(i,j)\in S} b_{i,j},$$

where $\alpha \le \gamma$ and $\beta \ge \gamma$. Let P be the polyhedron of the PBMs. In Section 2, we showed the inequality system describing P. Let Q be the coefficient matrix of these inequalities, and let the inequalities describing P be $f \le x \le g, a \le Qx \le b$. The convexity of the polyhedron P implies that there is an $x \in \mathbb{R}^{m\times n}$ such that $x \in P$ and $\sum x_{i,j} = \gamma$. Remember that Q is a network matrix, thus if we add $\sum x_{i,j} = \gamma$ to the set of inequalities describing P, the coefficient matrix Q' is also a network matrix, therefore, if there is a solution, then there is an integer solution, which is a PBM in which the sum of the entries equals γ. □

The proof we showed actually proves the following slightly stronger statement.

Corollary 4. *Let α be the smallest integer such that there exists a PBM in which the sum of the entries equals α. Similarly, let β be the greatest integer such that there exists a PBM in which the sum of the entries equals β. Then, for every integer $\gamma \in \mathbb{Z} : \alpha \le \gamma \le \beta$, there exists a PBM in which the sum of the entries equals γ.*

It is natural to ask whether we can relax the bounds f and g on the entries, similarly to the linking property of g-polymatroids, where $Q(p, b) \cap T(f, g)$ is non-empty if and only if $Q(p, b) \cap T(-\infty, g)$ and $Q(p, b) \cap T(f, \infty)$ are both non-empty. Our problem, however, is the intersection of two g-polymatroids, thus it does not necessarily hold.

Remark 5. *Consider the following problem:*

$$m = n = 3, \Phi^1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \Phi^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \Gamma^1 = \Gamma^2 \equiv 1,$$

and

$$f = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}, g = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

It is easy to see that

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a PBM with bounds $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, f, \infty, -\infty, \infty)$*, and*

$$M_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

is a PBM with bounds $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, -\infty, g, -\infty, \infty)$*. There is, however, no PBM with bounds* $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, -\infty, \infty)$*.*

The PBMs satisfying these prefix bounds are actually the well-known alternating sign matrices, which we discuss in the next section.

3 Alternating sign matrices

We shift our focus towards a well-known special case of the PBMs, the alternating sign matrices.

Definition 3. *We call a* $(0, \pm 1)$*-valued matrix* $A \in \mathbb{Z}^{n \times n}$ *an alternating sign matrix (ASM for short) if in each row and column the non-zero entries alternate in sign and their sum equals* 1*.*

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Figure 5: The 3×3 "diamond" ASM, the only 3×3 ASM that is not a permutation matrix.

In 2007, Striker [28] as well as Behrend and Knight [3] provided the following polyhedral description of the convex hull of $n \times n$ ASMs.

$$x \in \mathbb{R}^{n\times n}$$

$$0 \le \sum_{j'=1}^{j} x_{i,j'} \le 1 \qquad \forall i \in [n], j \in [n-1] \tag{15a}$$

$$0 \le \sum_{i'=1}^{i} x_{i',j} \le 1 \qquad \forall i \in [n-1], j \in [n] \tag{15b}$$

$$\sum_{j'=1}^{n} x_{i,j} = 1 \qquad \forall i \in [n] \tag{15c}$$

$$\sum_{i'=1}^{n} x_{i',j} = 1 \qquad \forall j \in [n] \tag{15d}$$

In fact, they also proved that the vertices of this polytope are exactly the ASMs. In Section 4.1 we show a characterization of the vertices of certain polytopes, from which we get this theorem as a special case.

It is easy to see that the integer solutions of the system are exactly the $n \times n$ ASMs. Observe that the integer solutions of the system, thus the ASMs, are PBMs with bounds $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, -\infty, \infty, -\infty, \infty)$, where

$$\Phi^1 = \begin{bmatrix} 0 & \dots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 1 \end{bmatrix}, \Phi^2 = \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 1 & \dots & 1 & 1 \end{bmatrix}, \text{ and } \Gamma^1 = \Gamma^2 = \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix}. \tag{16}$$

This means that the existence of an $n \times n$ ASM with bounds f and g on its entries is equivalent to the existence of an $n \times n$ PBM with bounds $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, f, g, -\infty, \infty)$.

3.1 ASMs with restricted entries

The question of whether there exists an ASM between the bounds f and g is a common generalization of various problems investigated in the literature. After showing a theorem about the existence of ASMs between lower and upper bounds on their entries, we discuss some of these questions in detail.

The bounds Φ^1, Γ^1, Φ^2, and Γ^2 defined in (16) satisfy the inequalities (10) and (11) with $H = n$. This means we obtain a necessary and sufficient condition for the existence of an ASM between bounds f and g using Theorem 4.

Definition 4. *We call a family $\mathcal{I}^H$ of horizontal intervals* separated *if, for any two distinct intervals $I^1, I^2 \in \mathcal{I}^H$ that lie in the same row $S_{i,\cdot}$, there exists an element $(i,j) \in S_{i,\cdot}$ such that for every $(i,j_1) \in I^1$ $j_1 < j$ and for every $(i,j_2) \in I^2$ $j < j_2$; or the same holds if we swap I^1 and I^2. We call a family of vertical intervals $\mathcal{I}^V$ separated, if its transpose ${\mathcal{I}^V}^T = \{I^T : I \in \mathcal{I}^V\}$ is a family of separated horizontal intervals. We call a family of intervals $\mathcal{I} = \mathcal{I}^H \uplus \mathcal{I}^V$ separated if both $\mathcal{I}^H$ and $\mathcal{I}^V$ are separated.*

Let I_0 denote the set of positions that are not covered by $\mathcal{I}$ and let I_2 denote the set of positions that are covered twice by $\mathcal{I}$. The following theorem gives a characterization for the existence of an ASM between given lower and upper bounds.

Theorem 11. *Let $f \leq g$ be lower and upper bounds on $S = [n] \times [n]$. There exists an ASM between the bounds f and g if and only if the following holds for every family of separated horizontal and vertical intervals $\mathcal{I} = \mathcal{I}^H \uplus \mathcal{I}^V$*

$$|\mathcal{I}| \geq n + \widetilde{f}(I_2) - \widetilde{g}(I_0). \tag{17}$$

Proof. Observe that the bounds in Theorem 4 are $b_1^*(I) = 1$ for every horizontal interval, and $b_2^*(I) = 1$ for every vertical interval, thus, by rearrangements, inequality (14) is equivalent to (17). □

For ease of application, we reformulate Theorem 11 in an equivalent but slightly more tangible form.
Let $\mathcal{S} = \{S_0, S_{-1}, S_{+1}, S_-, S_+, S_L\}$ be a partition of the ground set S.

Definition 5. *We call an ASM A $\mathcal{S}$-*compatible *if the entries of A at the positions in S_i are i for $i = 0, +1, -1$, the entries at the positions of S_- are non-positive, and the entries at the positions of S_+ are non-negative.*

Definition 6. *We call a family $\mathcal{I}$ of intervals $\mathcal{S}$-*feasible *if*

$$\begin{aligned} d_{\mathcal{I}}(i,j) = 0 &\Rightarrow (i,j) \in S_0 \cup S_{-1} \cup S_-, \\ d_{\mathcal{I}}(i,j) = 2 &\Rightarrow (i,j) \in S_0 \cup S_{+1} \cup S_+, \end{aligned} \tag{18}$$

where $d_{\mathcal{I}}(i,j)$ denotes the number of intervals in $\mathcal{I}$ that cover the position (i,j). Note that $\mathcal{S}$-feasibility is equivalent to requiring that $\mathcal{I}$ covers the positions of $S_- \cup S_{-1}$ at most once, $S_+ \cup S_{+1}$ at least once, and S_L exactly once.

Theorem 12. *There exists an $\mathcal{S}$-compatible ASM if and only if*

$$|\mathcal{I}| \geq n + |S_{-1} \cap I_0| + |S_{+1} \cap I_2| \tag{19}$$

holds for every $\mathcal{S}$-feasible family $\mathcal{I} = \mathcal{I}^H \uplus \mathcal{I}^V$ of intervals.

Proof. The $\mathcal{S}$-compatibility can be described by the following bounds

$$\begin{aligned} f_{i,j} &= \begin{cases} 0 & \text{if } (i,j) \in S_0 \cup S_+, \\ 1 & \text{if } (i,j) \in S_{+1}, \\ -\infty & \text{if } (i,j) \in S_- \cup S_{-1} \cup S_L, \end{cases} \\ g_{i,j} &= \begin{cases} 0 & \text{if } (i,j) \in S_- \cup S_0, \\ -1 & \text{if } (i,j) \in S_{-1}, \\ \infty & \text{if } (i,j) \in S_+ \cup S_{+1} \cup S_L. \end{cases} \end{aligned} \tag{20}$$

Observe that $f_{i,j}$ is finite only if $(i,j) \in S_0 \cup S_{+1} \cup S_+$ and $g_{i,j}$ is finite only if $(i,j) \in S_0 \cup S_{-1} \cup S_-$. This means that $\mathcal{S}$-feasibility is equivalent to $\widetilde{f}(I_2)$ and $\widetilde{g}(I_0)$ being finite, moreover $\widetilde{f}(I_2) = |S_{+1} \cap I_2|$ and $\widetilde{g}(I_0) = -|S_{-1} \cap I_0|$. If $\widetilde{f}(I_2) = -\infty$ or $\widetilde{g}(I_0) = \infty$ (17) automatically holds, so it suffices to require it in the case when they are both finite. In that case, (17) is equivalent to (19). □

In [11], Brualdi and Kim investigated the problem of a matrix having an *ASM completion.* For an $n \times n$ $(0, \pm 1)$-matrix A, a matrix B obtained from A by replacing some 0s by +1s is a *completion* of A. If B is an ASM, then it is called an ASM completion of A. Their main result is that every *bordered permutation matrix* has an ASM completion.

Definition 7. *For $n \geq 2$, a matrix $A \in \{0, -1\}^{n \times n}$ is a bordered permutation matrix if the first and last rows and columns only contain 0 entries and the middle $(n-2) \times (n-2)$ submatrix is the negative of a permutation matrix.*

Theorem 13 (Brualdi, Kim [11])**.** *Every bordered permutation matrix has an ASM completion.*

In their paper, they provide an inductive proof, which also gives an algorithm to construct such an ASM completion. Here we show an alternative proof using Theorem 12.

Proof. Let $A = (a_{i,j})$ be an $n \times n$ bordered permutation matrix and P be the $(n-2) \times (n-2)$ permutation matrix such that $-P$ is the middle $(n-2) \times (n-2)$ submatrix of A. It is easy to see that an ASM-completion of A is an $\mathcal{S}$-compatible ASM, where $\mathcal{S} = \{S_{-1}, S_{+}\}$ and $S_{-1} = \{(i,j) \in S : a_{i,j} = -1\}$, $S_{+} = \{(i,j) \in S : a_{i,j} = 0\}$. By Theorem 12, there exists an $\mathcal{S}$-compatible ASM if and only if (19) holds for every $\mathcal{S}$-feasible family of intervals, thus it suffices to show that (19) holds for every $\mathcal{S}$-feasible family of intervals. Suppose indirectly that there is an $\mathcal{S}$-feasible family $\mathcal{I}$ for which (19) does not hold, that is,

$$|\mathcal{I}| < n + |S_{-1} \cap I_0|. \tag{21}$$

We show that there is an $\mathcal{S}$-feasible family $\mathcal{I}'$ of intervals such that it does not cover any position in S_{-1} and it satisfies (21). We construct such an $\mathcal{I}'$ from $\mathcal{I}$ the following way. If there is a position $(i,j) \in S$ and an interval $I \in \mathcal{I}$ such that $(i,j) \in I$, then we replace I with the two intervals of $I - (i,j)$. (Note that these intervals might be empty.) By doing this, we still get an $\mathcal{S}$-feasible family of intervals and (21) still holds, because the $\mathcal{S}$-feasibility of $\mathcal{I}$ implies that the only interval covering (i,j) was I and by the replacement we added (i,j) to the set I_0, and thus the right-hand side of the inequality increased by 1, while the left-hand side increased by at most 1. We can repeat this process until we get an $\mathcal{I}'$ that does not cover any positions of S_{-1}. We get

$$|\mathcal{I}'| < n + |S_{-1}| = 2n - 2$$

and we can replace the conditions of the $\mathcal{S}$-feasibility by requiring $\mathcal{I}'$ to cover the positions of S_{+} at least once, and to not cover the positions in S_{-1}. This shows that we only need to show that it is not possible to cover the 0-s of A with less than $2n-2$ horizontal and vertical intervals that do not cover the -1 entries of A. Before we show this, we need to prove a lemma.

Lemma 2. *Let $P \in (0,1)^{m \times m}$ be a permutation matrix and $\mathcal{I}$ be a set of horizontal and vertical intervals such that $\mathcal{I}$ covers every 0 entry and does not cover any 1 entry. Then $|\mathcal{I}| \geq 2m - 2$.*

Proof. Note that if there is a family of intervals $\mathcal{I}$ satisfying the conditions above, then there is a family of intervals $\mathcal{I}'$ such that it still satisfies the conditions, $|\mathcal{I}'| \leq |\mathcal{I}|$ and every interval starts and ends at the border of the matrix or next to a 1 entry. From this point, when we refer to a family of intervals, we always suppose it has these properties.
We prove our statement by contradiction. Let $P \in \{0,1\}^{m \times m}$ be such that there is a cover $\mathcal{I}$ that satisfies the conditions and $|\mathcal{I}| < 2m - 2$, and the statement of the lemma holds for $m' < m$. If there is a 1 entry in one of the corners (we can assume it is the top left) and $p_{i,2} = p_{2,j} = 1$,

then the intervals covering $p_{i,1}$ and $p_{1,j}$ do not cover any elements of the $(m-1)\times(m-1)$ minor, thus it can be covered with $2(m-1)-3$ intervals, which contradicts the choice of P.
If there are no 1 entries in the corners, then let $p_{i,1}=1$ and $p_{j,2}=1$. We suppose $i<j$. The case $i>j$ can be solved with a similar argument. Similarly to the previous case, the interval covering $p_{1,j}$ does not cover any entry of the $(m-1)\times(m-1)$ minor. If there is another vertical interval in the first column, then we can leave that out and cover the $(m-1)\times(m-1)$ minor with $2(m-1)-3$ intervals. If there is a horizontal interval in the row $P_{i,\cdot}$, then we can leave that out too, and cover the $(m-1)\times(m-1)$ minor with $2(m-1)-3$ intervals.
If there are no horizontal intervals in $P_{i,\cdot}$, then there must be vertical intervals covering every entry of the row. The first few intervals start from the top, let $P_{\cdot,k}$ be the first column, where the interval covering $p_{i,k}$ does not start at the top of the column, meaning there is a 1 entry above it, let it be $p_{l,k}$. If there is no horizontal interval between $p_{l,1}$ and $p_{l,k}$ then there must be a vertical interval covering $p_{1,l}$ which we can leave out and cover the $(m-1)\times(m-1)$ minor with $2(m-1)-3$ intervals. If there is a horizontal interval between $p_{l,1}$ and $p_{l,k}$ then we can leave it out since $p_{l,2},\dots,p_{l,k-1}$ are all covered by vertical intervals. That means we can cover the $(m-1)\times(m-1)$ minor with $2(m-1)-3$ intervals which contradicts the choice of P. □

From Lemma 2, we know that we need at least $2(n-2)-2=2n-6$ intervals to cover the 0 entries of the negated permutation $-P$. Now we prove by contradiction that we need $2n-2$ intervals to cover the zeros of the bordered permutation matrix without covering the -1s. Suppose we can cover all the 0s with $2n-3$ intervals. Note that there is exactly one -1 in the first and last rows and columns of $-P$. To cover the 0s that are next to these in the border, we need 4 distinct intervals and these intervals cannot cover any 0s of $-P$, thus $-P$ can be covered with $2n-7=2(n-2)-3$ intervals, which contradicts Lemma 2. □

The primal-dual relationship between families of intervals and ASM completions in special cases, including Lemma 2, is mentioned in a later paper of Brualdi and Dahl [6] in relation to the notion of *A-interval matrices*. They show a theorem about the existence of A-interval matrices, which implies a necessary and sufficient condition for a $(0,-1)$-matrix having an ASM completion. Before we show this theorem, we recall a few definitions they introduced.

Definition 8. *Let A be an $n\times n$ $(0,*)$-matrix. An $n\times n$-matrix M is an A-interval matrix if it can be obtained from A by replacing one $*$ by a 1 in every maximal horizontal and vertical interval of $*$ entries of A and setting all other $*$ entries to 0.*

It is easy to see that the existence of an ASM-completion of a $(0,-1)$-matrix $M=(m_{i,j})$ is equivalent to the existence of an A-interval matrix, where

$$A=(a_{i,j}) \text{ and } a_{i,j}=\begin{cases} 0 & \text{if } \ m_{i,j}=-1, \\ * & \text{if } \ m_{i,j}=0. \end{cases}$$

The key observation they made (which already appeared in a slightly different form in [11]) is that there is a bijection between A-interval matrices and perfect matchings in a certain bipartite graph $G_A=(\mathcal{S},\mathcal{T},E)$. They define the graph the following way. Let $\mathcal{S}$ be the set of maximal horizontal intervals of $*$ entries and let $\mathcal{T}$ be the set of maximal vertical intervals of $*$ entries. There is an edge between $X\in\mathcal{S}$ and $Y\in\mathcal{T}$ whenever $X\cap Y\neq\emptyset$. By applying Kőnig's theorem [20], we get their theorem, which we rephrase using our own terminology.

Theorem 14. *Let A be an $n \times n$ $(0, *)$-matrix. There exists an A-interval matrix if and only if the $*$ entries of A cannot be covered with fewer than $n + k$ horizontal and vertical intervals without covering any 0s, where k is the number of 0s in A.*

The following corollary about ASM completion is an immediate consequence of this theorem.

Corollary 5. *An $n \times n$ $(0, -1)$-matrix A has an ASM-completion if and only if the 0 entries cannot be covered with less than $n + k$ intervals without covering any -1 entry of A, where k denotes the number of -1s in A.*

In a recent paper [8], Brualdi and Dahl investigated further problems concerning ASMs, which can also be solved with the help of Theorem 12. In the paper they seek an analogus theorem to the Frobenius-Kőnig theorem [17] concerning ASMs instead of permutation matrices.

Theorem 15 (Frobenius-Kőnig). *Let X be an $n \times n$ $(0, 1)$-matrix. Then there exists an $n \times n$ permutation matrix $P \leq X$ if and only if X does not have an $r \times s$ zero submatrix O_{rs} with $r + s = n + 1$.*

They first ask the question: When does an $n \times n$ $(0, \pm 1)$-matrix X contain an ASM A? By their terms a matrix X "contains" an ASM A, in other words, A is subordinate to X $(A \trianglelefteq X)$, if A can be obtained from X by replacing some of the ± 1s of X by 0s, similarly to the Frobenius-Kőnig theorem, where we can replace 1s with 0s. They provide such a theorem for the case when X belongs to a special class of matrices and for the general case they remark:

"In this generality, there is probably no simple answer..."

However, using Theorem 12, we can easily provide such an answer for the general case. Observe that for a given $n \times n$ matrix $X = (x_{i,j})$, $A \trianglelefteq X$ is equivalent to A being $\mathcal{S}$-compatible for $\mathcal{S} = \{S_0, S_-, S_+\}$, where

$$\begin{aligned} S_0 &= \{(i, j) : x_{i,j} = 0\}, \\ S_- &= \{(i, j) : x_{i,j} = -1\}, \\ S_+ &= \{(i, j) : x_{i,j} = +1\}. \end{aligned}$$

Now we see that the existence of an ASM A such that $A \trianglelefteq X$ is equivalent to the existence of an $\mathcal{S}$-compatible ASM, thus we can apply Theorem 12 and get the following corollary answering the question of Brualdi and Dahl:

Corollary 6. *For a given $n \times n$ $(0, \pm 1)$-matrix X there exists an ASM A such that $A \trianglelefteq X$ if and only if there is no family $\mathcal{I}$ of intervals that covers the $+1$s of X at least once, the -1s of X at most once, and $|\mathcal{I}| < n$.*

In the same paper they also propose the following questions for future research:

Question 3.1.

(I) *Given an $n \times n$ $(0, 1)$-matrix $X = (x_{i,j})$, when does there exist an ASM A obtained by replacing some of the 0's of X by -1's?*

(II) *Given an $n \times n$ $(0, -1)$-matrix $X = (x_{i,j})$, when does there exist an ASM A obtained by replacing some of the 0's of X by $+1$'s?*

We can easily answer these question, as the following corollary can be derived directly from Theorem 12.

Corollary 7.

1. *Given an $n \times n$ $(0,1)$-matrix $X = (x_{i,j})$, let $S_{+1} = \{(i,j) : x_{i,j} = 1\}$ be the set of positions with 1 entries and $S_{-} = \{(i,j) : x_{i,j} = 0\}$ the set of positions with 0 entries in X. There exists an ASM A obtained by replacing some 0s of X with -1s if and only if for every family $\mathcal{I}$ covering every position of S_{-} at most once and every position in S_{+1} at least once*

$$|\mathcal{I}| \geq n + |S_{+1} \cap I_2|.$$

2. *Given an $n \times n$ $(0,-1)$-matrix $X = (x_{i,j})$, let $S_{-1} = \{(i,j) : x_{i,j} = -1\}$ be the set of positions with -1 entries and $S_{+} = \{(i,j) : x_{i,j} = 0\}$ the set of positions with 0 entries in X. There exists an ASM A obtained by replacing some 0s of X with $+1$s if and only if for every family $\mathcal{I}$ covering every position of S_{+} at least once and every position of S_{-1} at most once*

$$|\mathcal{I}| \geq n + |S_{-1} \cap I_0|.$$

They introduce the class $\mathcal{C}_n$ of $n \times n$ *near permutation matrices* (NPMs for short) and provide a necessary and sufficient condition for the existence of matrix $C \in \mathcal{C}_n$ such that $C \trianglelefteq X$ for a given $(0,\pm 1)$-matrix X. A $(0,\pm 1)$-matrix is an NPM if the sum of its entries in each row and column equals 1. It is easy to see that NPMs are by definition PBMs, moreover, the bounds defining them satisfy (10) and (11), thus the condition they provide can be derived from Theorem 4. To provide this condition, they model this special case of the problem as a feasible circulation problem in a much simpler digraph than ours, and use Hoffman's circulation theorem to derive the necessary and sufficient condition.

4 Generalizations of ASMs

4.1 Weak alternating sign matrices

We call an $m \times n$ $(0,\pm 1)$-matrix a *weak alternating sign matrix* (WASM) if its non-zero entries in each row and column alternate in sign. Note that, as opposed to ASMs, a WASM does not need to be a square matrix, and its first and last non-zero entries are not necessarily $+1$s. It is easy to see that a matrix is a WASM if and only if it can be made an ASM by adding rows and columns before the first and after the last row and column.

The polytope of $m \times n$ WASMs is a projection of the polyhedron of $(m+1) \times (n+1)$ prefix bounded matrices with bounds $(\Phi^1, \Gamma^1, \Phi^2, \Gamma^2, -\infty, \infty, -\infty, \infty)$ defined below, where the indices of rows and columns start from 0.

$$\Phi^1_{0,j} = -\infty \qquad \forall j \in \{0\} \cup [n] \tag{22a}$$
$$\Phi^1_{i,j} = 0 \qquad \forall i \in [m]\ j \in \{0\} \cup [n] \tag{22b}$$
$$\Phi^2_{i,0} = -\infty \qquad \forall i \in \{0\} \cup [m] \tag{22c}$$
$$\Phi^2_{i,j} = 0 \qquad \forall i \in \{0\} \cup [m]\ j \in [n] \tag{22d}$$
$$\Gamma^1_{0,j} = \infty \qquad \forall j \in \{0\} \cup [n] \tag{22e}$$
$$\Gamma^1_{i,j} = 1 \qquad \forall i \in [m]\ j \in \{0\} \cup [n] \tag{22f}$$
$$\Gamma^2_{i,0} = \infty \qquad \forall i \in \{0\} \cup [m] \tag{22g}$$
$$\Gamma^2_{i,j} = 1 \qquad \forall i \in \{0\} \cup [m]\ j \in [n] \tag{22h}$$

It is easy to see that if an $(m+1) \times (n+1)$ matrix is a PBM with these bounds, then its bottom right $m \times n$ submatrix is a WASM. If a matrix is a WASM, then we get a PBM by adding an extra row and column full of 0s before the first row and column of the matrix, and in the rows and columns, in which the first non-zero entry is a -1, we change the leading 0 to 1. In [12], Brualdi and Kim investigated the question of the existence of a so-called $(u, u'|v, v')$-*ASM*, which are basically WASMs whose first and last non-zeros are prescribed in every row and column. There is, however, a slight difference, which we point out after the definition of $(u, u'|v, v')$-ASMs. Let $u, u' \in \{\pm 1\}^n$ and $v, v' \in \{\pm 1\}^m$ be ± 1-valued vectors. For an $m \times n$ matrix A, we define A' to be the following $(m+2) \times (n+2)$ matrix.

$$A' = \begin{array}{|c|c|c|} \hline 0 & u_1 \dots u_n & 0 \\ \hline v_1 & & v'_1 \\ \vdots & A & \vdots \\ v_m & & v'_m \\ \hline 0 & u'_1 \dots u'_n & 0 \\ \hline \end{array}$$

We say that A is an $(u, u'|v, v')$-ASM if the non-zero entries of A' alternate in sign in each row or column except the first and last ones. It is easy to see that $(u, u'|v, v')$-ASMs are WASMs, in which the first and last non-zero entries are prescribed in each row and column, except the case when $u_j = u'_j$ and the column $A_{.,j}$ contains only zeros or $v_i = v'_i$ and the row $A_{i,.}$ contains only zeros. In these two cases it is true that the first and last non-zero entries of the column (row) are $-u_j$ and $-u'_j$ ($-v_i$ and $-v'_i$) as there are no non-zero entries in the column (row), thus such matrices are technically WASMs with prescribed first and last non-zero entries. However, these are not $(u, u'|v, v')$-ASMs, because the non-zeros in $A'_{.,j}$ ($A'_{i,.}$) do not alternate. In the case $u = u' \equiv -1$ and $v = v' \equiv -1$ the $(u, u'|v, v')$-ASM are exactly the alternating sign matrices. We show a more detailed example below.

Example 2. *Let* $u = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^T$, $u' = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T$, $v = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T$, *and* $v' = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}^T$. *Then the following matrix A is an $(u, u'|v, v')$-ASM, with A' denoting the extended matrix as above. The matrix A'' is a PBM whose projection is A.*

$$A = \begin{array}{|ccc|}\hline 0 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ \hline\end{array}\,, \quad A' = \begin{array}{|c|ccc|c|}\hline 0 & -1 & 1 & 1 & 0 \\ \hline 1 & 0 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 & 1 \\ 1 & -1 & 1 & 0 & -1 \\ \hline 0 & 1 & -1 & 1 & 0 \\ \hline\end{array}\,, \quad A'' = \begin{array}{|c|ccc|}\hline 0 & 0 & 1 & 1 \\ \hline 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ \hline\end{array}$$

The authors show a necessary and sufficient condition for the existence of an $(u, u'|v, v')$-ASM, and remark that if the vectors u, u', v, v' may contain 0s, then a similar theory could be derived, but it seems technically difficult to formulate. Here we provide an alternative proof of their theorem about the existence of $(u, u'|v, v')$-ASMs.

Similarly to the WASMs, the convex hull of $m \times n$ $(u, u'|v, v')$-ASMs is a projection of the polytope of $(m+1) \times (n+1)$ PBMs with bounds $(\overline{\Phi}^1, \overline{\Gamma}^1, \overline{\Phi}^2, \overline{\Gamma}^2, -\infty, \infty, -\infty, \infty)$, where define $\overline{\Phi}^1, \overline{\Gamma}^1, \overline{\Phi}^2$, and $\overline{\Gamma}^2$ similarly to the bounds defined in (22a) - (22h) the following way.

$$\overline{\Phi}^1_{i,0} = \begin{cases} 1 & \text{if } v_i = +1 \\ 0 & \text{otherwise} \end{cases} \quad \overline{\Phi}^1_{i,n} = \begin{cases} 1 & \text{if } v'_i = -1 \\ 0 & \text{otherwise} \end{cases} \qquad \forall i \in [m] \tag{23a}$$

$$\overline{\Phi}^1_{i,j} = 0 \qquad \forall i \in [m]\ j \in [n-1] \tag{23b}$$

$$\overline{\Phi}^1_{0,j} = -\infty \qquad \forall j \in \{0\} \cup [n] \tag{23c}$$

$$\overline{\Phi}^2_{0,j} = \begin{cases} 1 & \text{if } u_j = +1 \\ 0 & \text{otherwise} \end{cases} \quad \overline{\Phi}^2_{m,j} = \begin{cases} 1 & \text{if } u'_j = -1 \\ 0 & \text{otherwise} \end{cases} \qquad \forall j \in [n] \tag{23d}$$

$$\overline{\Phi}^2_{i,j} = 0 \qquad \forall i \in [m-1]\ j \in [n] \tag{23e}$$

$$\overline{\Phi}^2_{i,0} = -\infty \qquad \forall i \in \{0\} \cup [m] \tag{23f}$$

$$\overline{\Gamma}^1_{i,0} = \begin{cases} 0 & \text{if } v_i = -1 \\ 1 & \text{otherwise} \end{cases} \quad \overline{\Gamma}^1_{i,n} = \begin{cases} 0 & \text{if } v'_i = +1 \\ 1 & \text{otherwise} \end{cases} \qquad \forall i \in [m] \tag{23g}$$

$$\overline{\Gamma}^1_{i,j} = 1 \qquad \forall i \in [m]\ j \in [n-1] \tag{23h}$$

$$\overline{\Gamma}^1_{0,j} = \infty \qquad \forall j \in \{0\} \cup [n] \tag{23i}$$

$$\overline{\Gamma}^2_{0,j} = \begin{cases} 0 & \text{if } u_j = -1 \\ 1 & \text{otherwise} \end{cases} \quad \overline{\Gamma}^2_{m,j} = \begin{cases} 0 & \text{if } u'_j = +1 \\ 1 & \text{otherwise} \end{cases} \qquad \forall j \in [n] \tag{23j}$$

$$\overline{\Gamma}^2_{i,j} = 1 \qquad \forall i \in [m-1]\ j \in [n] \tag{23k}$$

$$\overline{\Gamma}^2_{i,0} = \infty \qquad \forall i \in \{0\} \cup [n] \tag{23l}$$

One can see that the bottom right $m \times n$ submatrices of PBMs with these bounds are exactly the $m \times n$ $(u, u'|v, v')$-ASMs, thus the existence of an $(u, u'|v, v')$-ASM is equivalent to the existence of such a PBM. Note that this is still true if the vectors u, u', v, and v' contain 0 components. In fact, we only need the projection if we allow 0 components, otherwise we could define bounds such that the $(u, u'|v, v')$-ASMs are exactly the PBMs with those bounds.

We state the theorem of Brualdi and Kim in a slightly altered form, so that we do not need to define too many new concepts. For two vectors $w, w' \in \{\pm 1\}^k$ and an integer $1 \le i \le k$, let $r_i^+(w, w') = |\{j \in [i] : w_j = w'_j = +1\}|$ be the numbers of components that are $+1$ in both w and w'. Similarly, let $r_i^- = |\{j \in [i] : w_j = w'_j = -1\}|$. For a vector $w \in \{\pm 1\}^k$, let $w^+ = |\{j \in [k] : w_j = +1\}|$ be the number of its $+1$ components, and $w^- = |\{j \in [k] : w_j = -1\}|$ be the number of its -1 components.

Theorem 16. *Let $u, u' \in \{\pm 1\}^n$ and $v, v' \in \{\pm 1\}^m$ be ± 1-vectors. There exists an $(u, u'|v, v')$-ASM if and only if the following inequalities hold.*

$$r_m^-(v,v') - r_m^+(v,v') = r_n^-(u,u') - r_n^+(u,u') \tag{24a}$$
$$-u^+ \leq r_i^-(v,v') - r_i^+(v,v') \leq u^- \qquad \forall i \in [m] \tag{24b}$$
$$-v^+ \leq r_j^-(u,u') - r_j^+(u,u') \leq v^- \qquad \forall j \in [n] \tag{24c}$$

Proof. First, we prove that these are necessary conditions for the existence of an $(u, u'|v, v')$-ASM, that is, if they do not hold, then there cannot exist an $(u, u'|v, v')$-ASM. Observe that $r_i^-(v,v') - r_i^+(v,v')$ is exactly the sum of the entries in the first i rows of the matrix, and similarly $r_j^-(u,u') - r_j^+(u,u')$ is the sum of the entries in the first j columns of the matrix. From this, it is easy to see that both sides of (24a) equal the sum of the entries of the matrix, thus if it does not hold, then there exists no $(u, u'|v, v')$-ASM. To see that (24b) is necessary, it suffices to show that $-u^+$ is lower bound and u^- is upper bound for the sum of the entries in the first i rows. Consider the vertical prefixes that contain the first i entries of their columns. The entries in these prefixes are exactly those in the first i rows of the matrix. The sum of the first i entries of the j-th column can only be positive if $u_j = -1$, in which case, this sum is at most one, thus u^- is an upper bound for the sum of the first i rows. Similarly, the sum of the first i entries of the j-th columns can only be negative if $u_j = +1$, in which case it is at least -1, thus $-u^+$ is lower bound for the sum of the first j rows. It can be shown analogously that (24c) is necessary.

Next, we prove the sufficiency of these inequalities, that is, if there exists no $(u, u'|v, v')$-ASM, then one of the inequalities (24a), (24b), and (24c) does not hold. If (24a) does not hold, we are done, so we assume that (24a) holds. If there exists no $(u, u'|v, v')$-ASM, then there exists no PBM with the bounds defined in (23a)-(23l). Corollary 2 implies that there is a connected subset $X \subseteq S$, for which (9a) or (9b) does not hold. Looking at the bounds, it is easy to see that $X \subseteq [m] \times [n]$, that is, X is a subset of the positions of the $m \times n$ matrix and does not contain any of the additional positions. Next, we observe that a subset X that does not contain any full row or column of the $m \times n$ matrix, satisfies both (9a) and (9b), as $p_i^*(X) \leq 0$ and $b_i^*(X) \geq 0$ for $i = 1, 2$. This implies that any $X \subseteq S$ violating (9a) or (9b) must contain every position of at least one row or column. If X violates one of the inequalities and it contains every entry of a row, then for every row $S_{i,\cdot}$ such that $X \cap S_{i,\cdot} \neq \emptyset$, $X \cup S_{i,\cdot}$ also violates the same inequality. Similarly, if X contains every entry of a column, then adding every column, in which there is a position in X, to X we get a subset violating one of inequalities (9a) or (9b). This implies that if there is no $(u, u'|v, v')$-ASM, then the subset X violating (9a) or (9b) can be chosen to be the union of some rows or columns, and because of the connectivity, these rows or columns are adjacent. It is easy to see that (24a) implies that X violates (9a) if and only if $\overline{X}$ violates (9b), thus we only need to check those cases when X is the union of the first or last few rows or columns and violates (9a). By checking these cases, using (24a) we get that one of the four inequalities in (24b) and (24c) does not hold. □

In the case when we allow the vectors u, u', v, and v' to have 0 components, that is, there may be rows in which the first and/or last non-zero is not prescribed, the main difference is that (24a) is no longer a necessary condition, as the sum of the entries of the matrix is no longer determined, thus it is no longer true that X violates (9a) if and only if $\overline{X}$ violates (9b), therefore we have to investigate a lot more cases.

As the WASMs are an extension of ASMs, it is natural to ask, whether we can generalize theorems about ASMs to WASMs. In Section 3, we mentioned that Striker [28] and Behrend and Knight [3] independently showed a linear inequality system describing the polytope of $n \times n$ ASMs and also proved that the vertices of the polytope are exactly the $n \times n$ ASMs. Here, we provide a generalization of this theorem to $(u, u'|v, v')$-ASMs, where $u, u' \in \{0, \pm 1\}^n$ and $v, v' \in \{0, \pm 1\}^m$. In the case of the polytope of $(u, u'|v, v')$-ASMs the theorem is not that simple, as there are $(u, u'|v, v')$-ASMs that are not vertices of the polytope. For example consider the polytope of $m \times n$ $(u, u'|v, v')$-ASMs, and a matrix M that is an element of the polytope, such that the i-th row and j-th column of M contains only 0s, moreover, $u_j = u'_j = 0$ and $v_i = v'_i = 0$. It is easy to see that M is not a vertex of the polytope, as it is the convex combination of the following $(u, u'|v, v')$-ASMs.

$$M = \left[\begin{array}{c|c|c} & 0 & \\ & \vdots & \\ \hline 0\dots0 & 0 & 0\dots0 \\ \hline & \vdots & \\ & 0 & \end{array}\right] = \frac{1}{2}\left[\begin{array}{c|c|c} & 0 & \\ & \vdots & \\ \hline 0\dots0 & -1 & 0\dots0 \\ \hline & \vdots & \\ & 0 & \end{array}\right] + \frac{1}{2}\left[\begin{array}{c|c|c} & 0 & \\ & \vdots & \\ \hline 0\dots0 & 1 & 0\dots0 \\ \hline & \vdots & \\ & 0 & \end{array}\right]$$

We show that actually these are the only $(u, u'|v, v')$-ASMs that are not vertices of the polytope.

Theorem 17. *Let the vectors $u, u' \in \{0, \pm 1\}^n$ and $v, v' \in \{0, \pm 1\}^m$ be as before. An $m \times n$ $(u, u'|v, v')$-ASM M is a vertex of the polytope of $m \times n$ $(u, u'|v, v')$-ASMs if and only if it does not contain both a row $M_{i,\cdot}$ and a column $M_{\cdot,j}$ full of 0s, such that $u_j = u'_j = 0$ and $v_i = v'_i = 0$.*

Proof. We showed that if there is a row $M_{i,\cdot}$ and a column $M_{\cdot,j}$ full of 0s in the matrix, such that $u_j = u'_j = 0$ and $v_i = v'_i = 0$, then the matrix cannot be a vertex of the polytope. It is left to prove the reverse direction, that is, if a matrix does not have such a row and column, then it is a vertex of the polytope.

Let $M = (m_{i,j})$ be an $m \times n$ $(u, u'|v, v')$-ASM, and assume that M does not have a row $M_{i,\cdot}$ full of 0s, such that $v_i = v'_i = 0$. The case where the same holds for the columns instead of the rows can be proven analogously. By the definition of vertices of a polyhedron [15, p. 142], it suffices to show that there is a linear cost function $c : S \to \mathbb{R}$, such that M is the only element of that politope at which $\widetilde{c}$ takes its maximum, that is, for every element M' of the polytope, if $M' \neq M$, then $\widetilde{c}(M') < \widetilde{c}(M)$. Since the polytope is an integer polyhedron, we only need to show this for every integral M', which are exactly the $m \times n$ $(u, u'|v, v')$-ASMs. We define the cost function c the following way. For a position $(i, j) \in S$ let $c_{i,j} := K$ if $m_{i,j} = +1$ and $c_{i,j} := -K$ if $m_{i,j} = -1$, where $K := n^3 + 1$ For the positions with 0 entries, we define the cost c based on their distance from the non-zero entries in the same row. If a position (i, j) is between a -1 and a $+1$ entry, where either $M_{i,j-k}$ or $M_{i,j+k}$ is the -1 entry next to it, then $c_{i,j} := k$. In case (i, j) is before the first non-zero, if the first non-zero is -1 at position $(i, j + k)$, then $c_{i,j} := k - n - 1$ and if the first non-zero is $+1$ at position $(i, j + k)$, then $c_{i,j} := n + 1 - k$. Similarly, in case (i, j) is after the last non-zero position, if the last non-zero is -1 at position $(i, j - k)$, then $c_{i,j} := k - n - 1$ and if the last non-zero is $+1$, then $c_{i,j} := n + 1 - k$. If the i-th row only contains 0s, then at either $u_i \neq 0$ or $u'_i \neq 0$ and we can handle it like it was one of the previous cases with a non-zero at the position $(i, 0)$ or at the position $(i, n+1)$. To show that $\widetilde{c}(M') < \widetilde{c}(M)$ for every $M' \neq M$, we first note that, for any matrix $M^* = (m^*_{i,j})$, where c takes its maximum, for every position (i, j) where M has a non-zero entry $m^*_{i,j} = m_{i,j}$ since we chose the cost K of these positions large enough, so that if we change any of these positions, the total cost decreases regardless the entries at the other positions. The only thing left to show is that if we replace some 0 entries of M, then the cost also decreases. Assume that we change a some 0 entries to ± 1 in the i-th row. First, consider the new non-zero entries that lie between a $+1$ and a -1 entry of M. Obviously, the new non-zero entries we get between these, must alternate and must start with -1 if the non-zero entry of M before them is a $+1$, and with a $+1$ if the non-zero before them is a -1, otherwise the matrix we get would no longer be an $(u, u'|v, v')$-ASM. It can easily be checked that because of the monotonity of c in this interval, and the order of the new non-zero entries, the total contribution of these entries to the cost is negative. Next, we consider the new non-zeros we get before the first non-zero entry of M. If the number of new non-zero entries we get is even, then the same argument proves that their contribution to the cost is also negative. If we get an odd number of new non-zero entries, then we only show that the contribution of the last one is negative, as the rest together has a negative contribution, as we have already seen. Note that the last of these must be -1 if the first non-zero entry of M is $+1$, and -1 if the first non-zero of M is $+1$, and the sign of the cost implies that the contribution of this entry is also negative. It can be shown similarly that the contribution of the new non-zero entries we get after the last non-zero entry of M is also negative implying that if we replace some zeros with non-zeros, then the total cost decreases, thus M is the only matrix with maximum cost, therefore it is a vertex. $\square$

4.2 k-regular ASMs

Brualdi and Dahl [7] introduced the notion of *k-regular alternating sign matrices*. We call an $n \times n$ $(0, \pm 1)$-matrix a k-regular ASM if the sum of the entries in each prefix falls between 0 and k, moreover, the sum of the entries in every row and column equals k. One can consider the k-regular ASMs an extension of ASMs, as for $k = 1$ the 1-regular ASMs are exactly the ASMs. They investigate the relationship of k-regular ASMs and ASMs, especially questions about decomposition of k-regular ASMs into ASMs. They prove that k-regular ASMs can be decomposed into k ASMs and conjecture that these k ASMs can always be chosen to be pairwise *pattern disjoint*, that is, their supports are pairwise disjoint. We prove their conjecture, showing that it is a corollary of Theorem 1.

Corollary 8. *For every positive integer k and every $n \times n$ k-regular ASM A, there exist $A^1, \dots, A^k$ pairwise pattern disjoint $n \times n$ ASMs such that $A = A^1 + \cdots + A^k$.*

Proof. It is easy to see that k-regular ASMs are PBMs in the special case when the lower bound Φ^1 is identically k in the last column and 0 elsewhere; Φ^2 is identically k in the last row and 0 elsewhere; the upper bounds Γ^1 and Γ^2 are identically k; and the entry bounds f and g are identically -1 and $+1$, respectively. Applying Theorem 1 to PBMs with the bounds defining the k-regular ASMs, we get the corollary. $\square$

In the same paper, the authors also conjecture that every 2-regular ASM contains an ASM, that is, we can obtain an ASM by changing some ± 1 entries to 0. Of course, this is an immediate consequence of Corollary 8, as the 2 pattern disjoint ASMs, into which we decompose it, are both such matrices.

They also pose the following question as another possible relationship between ASMs and 2-regular ASMs.

Question 1. *Given an $n \times n$ ASM A, is it possible to replace some of its -1 entries with 0s and some of its 0 entries with $+1$s, so that the resulting matrix A' is a 2-regular ASM?*

We provide an affirmative answer to this question for $n \geq 2$. We show a more general lemma, from which this follows as a corollary.

Lemma 3. *Let $k \geq 1$ be an integer and let A be an $n \times n$ k-regular ASM. There exists an $n \times n$ $(k+1)$-regular ASM A' that can be obtained from A by changing some of the 0 entries of A to $+1$ and some of its -1 entries to 0 if and only if for every $r \times s$ submatrix of A that contains only $+1$ entries, $r + s \leq n$.*

Proof. Let $k \geq 1$ be an integer and let $A = (a_{i,j})$ be an $n \times n$ k-regular ASM. First, observe that changing some 0 entries to $+1$ and some -1 entries to 0 is equivalent to adding a $(0,1)$-matrix to A. Moreover, the matrix we add, must be a permutation matrix, otherwise we cannot get a $(k+1)$-regular ASM. Therefore, we need to decide whether there exists a permutation matrix $P = (p_{i,j})$ such that $A' = A + P$ is a $(k+1)$-regular ASM. Obviously, the sum of the entries in every prefix of A' is between 0 and $k+1$, since this sum is between 0 and k in A, and between 0 and 1 in P, for any permutation matrix P. Similarly, the sum of the entries in each row and column of A' equals $k+1$. The only restriction we have for P is that for every position $(i,j) \in [n] \times [n]$, if $a_{i,j} = +1$, then $p_{i,j}$ must be 0, that is $P \leq M$, where $M = (m_{i,j})$ and $m_{i,j} = 0$ if $a_{i,j} = +1$, and $m_{i,j} = 1$ otherwise. By applying Theorem 15 to M, the lemma immediately follows. ☐

We can also find the entries to change in polynomial time, as there is a simple one-to-one correspondence between these entries and the edges of perfect matchings in a certain bipartite graph, which we can find using Kőnig's algorithm [20].

Corollary 9. *For $n \leq 2$ and any $n \times n$ ASM A, there exists a 2-regular ASM A' that can be obtained from A by changing some of the 0 entries of A to $+1$ and some of its -1 entries to 0s.*

Proof. Lemma 3 implies that we only need to show that for any $n \times n$ ASM A, for every $r \times s$ submatrix of A that contains only $+1$s, $r + s \leq n$. We prove this by contradiction.

Suppose indirectly that there exists an $n \times n$ ASM A that has an $r \times s$ submatrix that consists of only $+1$ entries, and $r + s > n$. This implies that either $r > \frac{n}{2}$ or $s > \frac{n}{2}$. By symmetry, we can assume that $r > \frac{n}{2}$. Observe that a in row or column of an alternating sign matrix the number of $+1$ entries is at most $\left\lceil \frac{n}{2} \right\rceil$, thus $r > \frac{n}{2}$ implies that n is odd and $r = \left\lceil \frac{n}{2} \right\rceil < \frac{n}{2} + 1$. Note that a column with $\left\lceil \frac{n}{2} \right\rceil$ $+1$ entries has a $+1$ entry in the first and last rows, thus it is the only column of A in which the number of $+1$ entries is greater than $\frac{n}{2}$. This means $s = 1$. From these we get

$$n < r + s = \left\lceil \frac{n}{2} \right\rceil + 1 < \frac{n}{2} + 1 + 1.$$

By rearrangements, we get $n < 4$. One can easily check the $n \times n$ ASMs when $2 \leq n < 4$ and see that they do not have such submatrices either, giving us the desired contradiction. Actually, one only needs to check the 3×3 "diamond" ASM shown in Figure 5, as all other $n \times n$ ASMs with $n < 4$ are permutation matrices, for which the statement clearly holds. ☐

5 Extensions and related problems

Heuer and Striker [18] introduced the notion of *partial alternating sign matrices* (PASMs for short), defined as $(0, \pm 1)$-valued $m \times n$ matrices in which the sum of entries in every prefix is either 0 or +1, which are of course prefix bounded matrices, and thus similar theorems can be formulated as those we showed in Section 3. *Higher spin alternating sign matrices* introduced by Behrend and Knight [3] and *sign matrices* introduced by Aval [1] are also prefix bounded matrices.

Brualdi and Dahl introduced the notion of *alternating sign hypermatrices* (ASHMs for short) as a generalization of ASMs in three dimensions [5]. An $n \times n \times n$ hypermatrix is an ASHM if each of its planes obtained by fixing one of the three indices is an ASM. Naturally, one could ask whether our results can be extended for ASHMs. The answer is negative, as it is NP-complete to decide whether there exists an ASHM between bounds f and g. The proof is by reduction from the *3 dimensional matching problem*.

A natural step forward could be having bounds on the *suffixes* (the complements of prefixes) as well as the prefixes and the entries, but it is NP-complete to decide if there exists a prefix bounded matrix between lower and upper bounds on its entries such that it satisfies the bounds on the suffixes as well. The proof is by reduction from the NP-complete *simultaneous matching problem* [13]. We do not discuss the details here, but we plan to show them in a future work.

This result immediately implies that it is NP-complete to decide if there is an integer matrix satisfying lower and upper bounds on its intervals. In a special case, we only have bounds for the intervals $\{(i,j), \ldots, (i, \min\{n, j+d-1\})\}$ and $\{(i,j), \ldots, (\min\{m, i+d-1\}, j)\}$ for every $i \in [m]$ and $j \in [n]$. It turns out that it is still NP-complete to decide whether there exists an $m \times n$ integer matrix between lower and upper bounds, satisfying the lower and upper bounds on these intervals. The proof of this is by reduction form the *d-distance matching problem* [23].

Note that it is not completely hopeless to have bounds on the intervals of the matrix, as the case when the lower and upper bounds on the horizontal and the vertical intervals both form a so-called *weak pair*, the approach based on g-polymatroids shown in Section 2.4 still works and the problem is algorithmically tractable [16].

6 Future plans

The circulation model we use for investigating the ASMs with constraints on their entries can only handle those that can be described as $f_{i,j} \leq a_{i,j} \leq g_{i,j}$, that is, every constraint except the one, where we require an entry to be non-zero. A question strongly related to this constraint, is deciding whether there exists an ASM with prescribed number of non-zero entries in each row and column. Brualdi et al. provided necessary conditions for the number of non-zeros in the rows and columns of an ASM [9]. We hope to extend their results and provide a characterization for these.

Brualdi and Kim investigated completion questions regarding symmetric ASMs [10]. As we answered completion questions in the general case, it might be interesting to try and answer the same questions with the additional constraint that the desired ASM must be symmetric. The answer to these questions probably requires a new model. If we just add the symmetry constraints as additional inequalities to our system, the coefficient matrix will no longer be TU. The same is true if instead of adding extra constraints, we have one common variable for $a_{i,j}$ and $a_{j,i}$.

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Alulírott Takács Tamás nyilatkozom, hogy szakdolgozatom elkészítése során az alább felsorolt feladatok elvégzésére a megadott MI alapú eszközöket alkalmaztam:

Feladat	**Felhasznált eszköz**	**Felhasználás helye**	**Megjegyzés**
Nyelvhelyesség ellenőrzése	Writefull	Teljes dolgozat	

A felsoroltakon túl más MI alapú eszközt nem használtam.