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Contents

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2 Introduction

Measurable combinatorics mostly investigates graph theoretic problems in measure preserving graphs on standard probability spaces. It is closely related to sparse graph limit theory [24], the ergodic theory of infinite groups [16], and probability theory on groups and graphs [27].

In this thesis we examine two topics in measurable combinatorics: local-global convergence of sparse graph sequences, and the weak containment topology on the space of actions of a fixed countable group Γ. They will turn out to be two sides of the same coin.

The motivation for this work is to lay the foundations for future research on the limit theory of submodular set functions, including matroid rank functions. This line of research has recently been initiated by Lovász [25], and has developed rapidly [26, 5, 6, 14, 7, 8]. In [6] the authors introduce a convergence notion reminiscent of local-global convergence, and prove compactness of the corresponding space. We plan to use the ultraproduct method – presented and shown to be completely parallel to local-global convergence in section 5.3 – to reprove and hopefully strengthen this result.

In Sections 3 we are going to list some notations and tools that we will use later. Then in Section 4 we introduce the notions of local- and local-global convergence. We also introduce the corresponding limit objects, namely unimodular random graphs and graphings. The main theorem of Section 4 is Theorem 4.29, which claims that every local-global sequence of graphs has a limit. Then we turn to a somewhat different setting, and examine the space of probability measure preserving actions of a fixed countable group Γ. The main goal is to define a suitable space for the actions of Γ, in which isomorphic actions are not distinguished. The space of actions is going to be a compact metric space with the so called partition metric. We prove this in Theorem 5.2, and the result we get is paralell to Theorem 4.29. In Section 5.2 we explain the connection between local- and local-global convergence of graphs and the convergence of actions in the partition metric.

3 Preliminaries

In this section we list some well-known results that we will use later in the thesis. We will not include most of the theorems's proofs, but we will give references to them. We assume that the reader is familiar with basic notions and definitions from topology, measure theory, graph theory and group theory.

Definition 3.1 (Independence ratio of a finite graph)**.** Let G be a finite graph. Then the *independence ratio* of G, denoted by $i(G)$, is the size of the largest independent set in G divided by $|V(G)|$.

Definition 3.2 (Zero-dimesional topological space)**.** We say that a topological space is *zero dimensional*, if it admits a base consisting of clopen subsets.

Definition 3.3 (Weak convergence of measures)**.** Let X be a Polish space. We denote by $P(X)$ the set of Borel probability measures on X. We say that a sequence of probability measures $\mu_1, \mu_2 \ldots \in P(X)$ *weakly converges* to $\mu \in P(X)$, if for all bounded continuous functions $f \in C_b(X)$

$$\int_X f d\mu_n \to \int_X f d\mu \text{ as } n \to \infty.$$

The Portmanteau theorem (see, e.g. [20] Theorem 17.20) provides several equivalent conditions for weak convergence. We present an equivalent form for zero-dimensional Polish spaces.

Lemma 3.4. *Let X be a zero-dimensional Polish space. Then $\mu_n \to \mu$ weakly if and only if $\lim_{n\to\infty} \mu_n(B) = \mu(B)$ for all clopen sets B (or for all $B \in \mathcal{B}$ for a base $\mathcal{B}$ consisting of clopen sets).*

Proof. The only if part follows immediately from [20, Theorem 17.20 v)]. For the if part we are going to prove that for every open set U we have $\mu(U) \leq \liminf_{n\to\infty} \mu_n(U)$, which is equivalent to weak convergence by the Portmanteau theorem. We can write U as a countable disjoint union of clopen sets $U = \bigsqcup_{k=1}^{\infty} U_k$. Then

$$\mu(U) = \sum_{k=1}^{\infty} \mu(U_k) = \sum_{k=1}^{\infty} \liminf_{n\to\infty} \mu_n(U_k) \leq \liminf_{n\to\infty} \sum_{k=1}^{\infty} \mu_n(U_k) = \liminf_{n\to\infty} \mu_n(U),$$

where we used the assumption that $\lim_{n\to\infty} \mu_n(U_k) = \mu(U_k)$ for all $k \in \mathbb{N}$ in the second step, and Fatou's lemma in the third step. □

We are going to use the following theorem (for the proof see [20] Theorem 17.22).

Theorem 3.5. *If X is a compact metrizable space, so is $P(X)$ with the weak convergence.*

In Section 5 we are going to examine measure preserving actions of groups on a standard probability space.

Notation 3.6 (Standard Borel space)**.** A measurable space $(X, \mathcal{B})$ is said to be a *standard Borel space* if there exists a metric on X that makes it a complete separable metric space such that $\mathcal{B}$ is exactly the σ-algebra of the Borel sets of the metric topology.

Notation 3.7. A *standard probability measure space* $(X, \mathcal{B}, \mu)$, or standard probability space for short, is a standard Borel space $(X, \mathcal{B})$ equipped with a continuous probability measure μ, that is, singletons have measure 0.

Remark 3.8. In fact there is a unique standard probability space up to isomorphism. That is, for any standard probability spaces $(X, \mathcal{B}, \mu)$ there is a measure preserving Borel bijection with the space $(2^{\mathbb{N}}, \mathcal{B}(2^{\mathbb{N}}), \lambda)$, where λ denotes the Lebesgue measure, and $\mathcal{B}(2^{\mathbb{N}})$ denotes the Borel σ-algebra of $2^{\mathbb{N}}$. See e.g. [20, Theorem 17.41].

Notation 3.9. There is a natural topology on the measurable subsets of a standard probability space $(X, \mathcal{B}, \mu)$, induced by the distance function $d_\mu(A, B) := \mu(A \triangle B)$. We call this metric space the measure algebra of $(X, \mathcal{B}, \mu)$ and denote it with MALG_μ. We do not distinguish sets which differ only by a μ-null set. Let us denote the set of measure preserving Borel bijections of the standard Borel probability space $(X, \mathcal{B}, \mu)$ by $\mathrm{Aut}(X, \mathcal{B}, \mu)$.

The space $\mathrm{Aut}(X, \mathcal{B}, \mu)$ is usually equipped with the weak or the uniform topology (see Chapter 1 in [21]). We are going to use the the former, so we recall its definition. As usual, we identify two automorphisms $f, g \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ if they only differ by a μ-null set.

Definition 3.10. We call the *weak topology* on $\mathrm{Aut}(X, \mathcal{B}, \mu)$ the smallest topology in which the functions $\varphi_A : \mathrm{Aut}(X, \mathcal{B}, \mu) \to \mathrm{MALG}_\mu$, $\varphi_A(T) = T(A)$ are continuous for all $A \in \mathrm{MALG}_\mu$.

We recall the definition of a group action. Throughout the thesis we are going to work with left actions.

Definition 3.11 (Group action)**.** Let G be group and X be an arbitrary set. Then (left) group action a of G on X is a function $a : G \times X \to X$ such that

1. $a(1_G, x) = x$ for all $x \in X$
2. $a(gh, x) = a(g, a(h, x))$ for all $g, h \in G$ and $x \in X$.

We will denote an action a of G on X by $G \overset{a}{\curvearrowright} X$. Notice that for fixed $g \in G$ the function $a(g, .) : X \to X$ is a bijection of X, and we are going to denote it by g^a.

Definition 3.12 (Measure preserving action of a group)**.** Let $(X, \mathcal{B}, \mu)$ be an arbitrary measurable space. We say that an action a of a group G on X is measure preserving, if for all $g \in G$ the bijection $g^a : X \to X$ is measure preserving, that is $\mu(g^a(B)) = \mu(B)$ for all $B \in \mathcal{B}$.

We can encode group actions by their so called Schreier graphs.

Definition 3.13 (Schreier graph)**.** Let Γ be a group generated by the finite set S, and let a be a transitive action of Γ on the countable set X. We define the *Schreier graph* of this action as follows. The vertex set is X, and for every $s \in S$ and $x \in X$ we draw an s-labelled directed edge from x to $s^a(x)$. We say that the Schreier graph is *rooted*, if X has a distinguished point x, which we call the root of the Schreier graph.

3.1 Ultraproducts of probability spaces

In Section 5.3 we are going to use an ultraproduct construction. So in this chapter we recall the notions and results related to ultraproducts that we will use later. Our primary source is [13]. First we recall the definition of an ultrafilter.

Definition 3.14. An *ultrafilter* $\mathcal{U}$ on a set X is a subset of $\mathcal{P}(X)$ with the following properties:

1. $\emptyset \notin \mathcal{U}$
2. if $A, B \in \mathcal{U}$ then $A \cap B \in \mathcal{U}$
3. if $A \in \mathcal{U}$ and $B \supseteq A$ then $B \in \mathcal{U}$
4. for every $A \subseteq X$ either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.

We call an ultrafilter on X non-principal if for every $x \in X$ the set $\{x\} \notin \mathcal{U}$.

Remark 3.15. Throughout the thesis we will take ultrafilters on the set $\mathbb{N}$. A benefit of working with ultrafilters is that we can take the *ultralimit* of any bounded sequence, which can be useful if the sequence is not convergent in the usual sense. For a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers (or even elements from a topological space) let a be the limit of $(a_n)_{n\in\mathbb{N}}$ along $\mathcal{U}$ (denoted $a = \lim_{n\in\mathcal{U}} a_n$) if for all neighbourhoods V of a the set $\{n : a_n \in V\} \in \mathcal{U}$. It is easy to see that bounded sequences of real numbers always admit an ultralimit.

Now we define the ultraproduct of a sequence of sets $\{X_n\}_{n\in\mathbb{N}}$.

Definition 3.16. Let $X_1, X_2 \dots$ be a sequence of sets and let X denote their product, that is $X = \prod_{n\in\mathbb{N}} X_n$. Let us fix an ultrafilter $\mathcal{U}$ on $\mathbb{N}$. We introduce a relation $\sim_{\mathcal{U}}$ on X:

$$(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} \iff \{n : x_n = y_n\} \in \mathcal{U}.$$

It is easy to see from the ultrafilter axioms that this is an equivalence relation. We define the *ultraproduct* $X_{\mathcal{U}}$ of the sequence $\{X_n\}_{n\in\mathbb{N}}$ as the quotient $X_{\mathcal{U}} := X/\sim_{\mathcal{U}}$.

As usual, we are going to use the notation $[x_n]_{\mathcal{U}}$ to denote the equivalence class of a sequence $(x_n)_{n\in\mathbb{N}}$. Similarly, for a sequence of subsets $A_n \subseteq X_n$ we are going to denote by $[A_n]_{n\in\mathcal{U}}$ (or simply by $[A_n]_{\mathcal{U}}$ if there is no danger of confusion) the set $\{[x_n]_{\mathcal{U}} : \{n : x_n \in A_n\} \in \mathcal{U}\}$. Notice that $[A_n]_{\mathcal{U}} \cap [B_n]_{\mathcal{U}} = [A_n \cap B_n]_{\mathcal{U}}$, and similarly for taking unions and complements. Usually we are going to denote by $A_{\mathcal{U}}, x_{\mathcal{U}}$ etc. the subsets and respectively, elements of $X_{\mathcal{U}}$.

In Section 5.3 we are going to work with ultraproducts of measure spaces, which we define below. Let $(X_n, \mathcal{B}_n, \mu_n)_{n\in\mathbb{N}}$ be a family of probability spaces. Let $X_{\mathcal{U}}$ denote the ultraproduct of the sets X_n along $\mathcal{U}$. The ultraproduct measure will be defined on $X_{\mathcal{U}}$, and it will arise as a measure associated to a set function. Let θ_0 be the following set funtion on the algebra $\{[A_n]_{n\in\mathcal{U}} : A_n \subseteq X_n \forall n \in \mathbb{N}\}$:

$$\theta_0([A_n]_{\mathcal{U}}) := \lim_{n\in\mathcal{U}} \mu_n(A_n).$$

Now we define the following associated outer measure $\theta : \mathcal{P}(X_{\mathcal{U}}) \to [0, \infty]$ on every subset $A_{\mathcal{U}} \subseteq X_{\mathcal{U}}$:

$$\theta(A_{\mathcal{U}}) := \inf\{\sum_{i=1}^{\infty} \theta_0([A_n^i]_{n\in\mathcal{U}}) : A_{\mathcal{U}} \subseteq \bigcup_{i\in\mathbb{N}} [A_n^i]_{n\in\mathcal{U}}, A_n^i \in \mathcal{B}_n \ \forall n, i \in \mathbb{N}\}.$$

The usual calculations show that this is indeed an outer measure, that is, monotone, σ-subadditive and $\theta(\emptyset) = 0$. According to the Carathéodory theorem, the family of subsets

$$\mathcal{B}_{\mathcal{U}} := \{A_{\mathcal{U}} \subset X_{\mathcal{U}} : \theta(B_{\mathcal{U}}) = \theta(B_{\mathcal{U}} \cap A_{\mathcal{U}}) + \theta(B_{\mathcal{U}} \setminus A_{\mathcal{U}})\}$$

is a σ-algebra and θ restricted to this family is a measure. Let us denote by $\mu_{\mathcal{U}}$ the restriction of θ to $\mathcal{B}_{\mathcal{U}}$ and let us call elements of $\mathcal{B}_{\mathcal{U}}$ the measurable sets. Notice that $X_{\mathcal{U}}$ is of course measurable, and its measure is 1.

Notation 3.17. For a family of probability spaces $\{(X_n, \mathcal{B}_n, \mu_n)\}_{n\in\mathbb{N}}$ we call the probability space $(X_{\mathcal{U}}, \mathcal{B}_{\mathcal{U}}, \mu_{\mathcal{U}})$ constructed above the *ultraproduct measure space* associated to the sequence $\{(X_n, \mathcal{B}_n, \mu_n)\}_{n\in\mathbb{N}}$.

In the following proposition we prove that the cylinder sets $[A_n]_{\mathcal{U}}$ are measurable, and conversely, every set is a cylinder set modulo a θ-null set.

Proposition 3.18. *Let $(X_n, \mathcal{B}_n, \mu_n)$ be a sequence of measurable spaces, and let $(X_{\mathcal{U}}, \mathcal{B}_{\mathcal{U}}, \mu_{\mathcal{U}})$ be the measure space constructed above. Then for every sequence $A_n \in \mathcal{B}_n$ the cylinder set $[A_n]_{\mathcal{U}} \in \mathcal{B}_{\mathcal{U}}$ and $\mu_{\mathcal{U}}([A_n]_{\mathcal{U}}) = \lim_{n\in\mathcal{U}} \mu_n(A_n)$. Moreover for every $A_{\mathcal{U}} \in \mathcal{B}_{\mathcal{U}}$ there is a sequence $\{B_n\}_{n\in\mathbb{N}}$ ($B_n \in \mathcal{B}_n \ \forall n \in \mathbb{N}$), such that $\mu_{\mathcal{U}}(A_{\mathcal{U}} \triangle [B_n]_{\mathcal{U}}) = 0$.*

In the proof of Proposition 3.18 the following lemma will be crucial.

Lemma 3.19. *Let $\{B_n^i\}_{i,n\in\mathbb{N}}$ be a family of subsets such that $B_n^i \in \mathcal{B}_n$ for all $i, n \in \mathbb{N}$. Then there exist a sequence of sets $C_n \in \mathcal{B}_n$ with the following properties:*

$$\lim_{n\in\mathcal{U}} \mu_n(C_n) = \lim_{i\to\infty} \lim_{n\in\mathcal{U}} \mu_n\left(\bigcup_{j=1}^{i} B_n^j\right) \text{ and } [C_n]_{\mathcal{U}} \supseteq \bigcup_{i=1}^{\infty} [B_n^i]_{\mathcal{U}}.$$

The proof of the lemma is a standard diagonalization argument, and can be found in [13] as Lemma 1.1.8. We have the following corollary of Lemma 3.19, which says that we can use one set to cover instead of the countable cover in the definition of θ.

Corollary 3.20. *For any set $A_{\mathcal{U}} \subseteq X_{\mathcal{U}}$ we have*

$$\theta(A_{\mathcal{U}}) = \inf\{\theta_0([A_n]_{\mathcal{U}}) : A_{\mathcal{U}} \subseteq [A_n]_{\mathcal{U}}, A_n \in \mathcal{B}_n \ \forall n \in \mathbb{N}\}.$$

Proof. Indeed, take any sequence of sets $A_n^i \in \mathcal{B}_n$ such that $A_{\mathcal{U}} \subseteq \bigcup_{i\in\mathbb{N}} [A_n^i]_{n\in\mathcal{U}}$. Take the measurable sets $A_n \in \mathcal{B}_n$ according to Lemma 3.19. Then on one hand $\bigcup_{i=1}^{\infty} [A_n^i]_{n\in\mathcal{U}} \subseteq [A_n]_{\mathcal{U}}$ and therefore $A_{\mathcal{U}} \subseteq [A_n]_{\mathcal{U}}$. On the other hand,

$$\theta_0([A_n]_{\mathcal{U}}) = \lim_{n\in\mathcal{U}} \mu_n(A_n) = \lim_{i\to\infty} \lim_{n\in\mathcal{U}} \mu_n\left(\bigcup_{j=1}^{i} A_n^j\right) \leq \lim_{i\to\infty} \sum_{j=1}^{i} \lim_{n\in\mathcal{U}} \mu_n\left(A_n^j\right) = \sum_{j=1}^{\infty} \theta_0([A_n^j]_{n\in\mathcal{U}}).$$

And since we are taking the infimum for such covers, we are done. □

Proof of Proposition 3.18. First we prove that $[A_n]_{\mathcal{U}} \in \mathcal{B}_{\mathcal{U}}$. Fix an arbitrary set $B_{\mathcal{U}} \subseteq X_{\mathcal{U}}$. From the fact that θ is an outer measure it follows that

$$\theta(B_{\mathcal{U}}) \leq \theta(B_{\mathcal{U}} \cap [A_n]_{\mathcal{U}}) + \theta(B_{\mathcal{U}} \setminus [A_n]_{\mathcal{U}}).$$

For the other direction fix an arbitrary $\varepsilon > 0$. Then using Corollary 3.20 there are measurable sets $C_n \in \mathcal{B}_n$, such that $\theta_0([C_n]_{\mathcal{U}}) \leq \theta(B_{\mathcal{U}}) + \varepsilon$ and $B_{\mathcal{U}} \subseteq [C_n]_{\mathcal{U}}$. Thus

$$\theta(B_{\mathcal{U}} \cap [A_n]_{\mathcal{U}}) + \theta(B_{\mathcal{U}} \setminus [A_n]_{\mathcal{U}}) \leq \theta([C_n]_{\mathcal{U}} \cap [A_n]_{\mathcal{U}}) + \theta([C_n]_{\mathcal{U}} \setminus [A_n]_{\mathcal{U}}) \leq$$

$$\leq \theta_0([C_n \cap A_n]_{\mathcal{U}}) + \theta_0([C_n \setminus A_n]_{\mathcal{U}}) = \theta_0([C_n]_{\mathcal{U}}) \leq \theta(B_{\mathcal{U}}) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this proves that $[A_n]_{\mathcal{U}} \in \mathcal{B}_{\mathcal{U}}$. Now we calculate the $\mu_{\mathcal{U}}([A_n]_{\mathcal{U}})$. Using Corollary 3.20 again, we can suppose that there is a sequence $C_n \in \mathcal{B}_n$ such that $\theta_0([C_n]_{\mathcal{U}}) \leq \theta([A_n]_{\mathcal{U}}) + \varepsilon$ and $[A_n]_{\mathcal{U}} \subseteq [C_n]_{\mathcal{U}}$. But the second expression implies that for ultrafilter many n $A_n \subseteq C_n$ and therefore $\theta_0([A_n]_{\mathcal{U}}) \leq \theta_0([C_n]_{\mathcal{U}}) \leq \theta([A_n]_{\mathcal{U}}) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we can conclude that $\theta_0([A_n]_{\mathcal{U}}) \leq \theta([A_n]_{\mathcal{U}})$. The other direction follows directly from the trivial cover $[A_n]_{\mathcal{U}} \subseteq [A_n]_{\mathcal{U}}$.

For the second part of the proposition, take an arbitrary measurable set $A_{\mathcal{U}} \in \mathcal{B}_{\mathcal{U}}$. Using Corollary 3.20 and the first part of the proposition, we can take measurable sets $B_n^i \in \mathcal{B}_n$ such that for every $i \in \mathbb{N}$

$$A_{\mathcal{U}} \subseteq [B_n^i]_{n\in\mathcal{U}} \text{ and } \mu_{\mathcal{U}}([B_n^i]_{n\in\mathcal{U}}) \leq \mu_{\mathcal{U}}(A_{\mathcal{U}}) + \frac{1}{2^i}.$$

Now using Lemma 3.19 for the sequence $[X_n \setminus B_n^i]_{n\in\mathcal{U}}$, we get measurable sets $C_n \in \mathcal{B}_n$ such that $[C_n]_{\mathcal{U}} \supseteq \bigcup_{i=1}^{\infty}[X_n \setminus B_n^i]$ and

$$\mu_{\mathcal{U}}([C_n]_{\mathcal{U}}) = \lim_{n\in\mathcal{U}} \mu_n(C_n) = \lim_{i\to\infty} \lim_{n\in\mathcal{U}} \mu_n(\bigcup_{j=1}^{i}(X_n \setminus B_n^j)) = \lim_{i\to\infty} \mu_{\mathcal{U}}([\bigcup_{j=1}^{i}(X_n \setminus B_n^j)]_{n\in\mathcal{U}}).$$

Now using the fact that $[\bigcup_{j=1}^{i} X_n \setminus B_n^j]_{n\in\mathcal{U}} = X_{\mathcal{U}} \setminus \bigcap_{j=1}^{i}[B_n^j]_{n\in\mathcal{U}} \subseteq X_{\mathcal{U}} \setminus A_{\mathcal{U}}$ we get by the continuity of measures that $\mu_{\mathcal{U}}([C_n]_{\mathcal{U}}) = 1 - \lim_{i\to\infty} \mu_{\mathcal{U}}(\bigcap_{j=1}^{i}[B_n^j]_{n\in\mathcal{U}}) = 1 - \mu_{\mathcal{U}}(A_{\mathcal{U}})$. Therefore $\mu_{\mathcal{U}}([X_n \setminus C_n]_{\mathcal{U}}) = \mu_{\mathcal{U}}(A_{\mathcal{U}})$, on the other hand both $A_{\mathcal{U}}$ and $[X_n \setminus C_n]_{\mathcal{U}}$ are contained in the measurable set $\bigcap_{j=1}^{\infty}[B_n^j]_{n\in\mathcal{U}}$ which has the same measure. From this we can derive that $\mu_{\mathcal{U}}(A_{\mathcal{U}} \triangle [X_n \setminus C_n]_{\mathcal{U}}) = 0$, which concludes the proof. □

4 Local and local-global convergence of graphs

The limit theory of graphs is a very active field. Interestingly, there is no unified theory of graph convergence. Instead, there are various notions which work well in different situations. The limit theory of dense graphs, i.e. when the number of edges is quadratic in the number of vertices, is well understood field. It turned out that understanding the limit object, called *graphons*, helps to understand the behaviuor of finite dense graphs. In this thesis we focus to the other end of the spectrum, i.e. very sparse graphs, which have degrees bounded by a fixed constant d. Our primary source for this section is [17] and [2].

Unless stated otherwise, in this section every graph and graphing is supposed to have all degrees bounded by some constant d.

We begin by defining the notion of local and local-global convergence, and we examine some fundamental statements related to them.

We say that a pair (G, o) is a *rooted graph* if o is a distinguished vertex (the *root*) of the graph G. We say that (G, o) and (G, o') are *isomorphic* if there is a graph isomorphism $\varphi : G \to G'$ for which $\varphi(o) = o'$. The *radius* of a rooted graph is the distance of the farthest vertex in G to o (in the graph distance). Let us denote by U^r the set of (isomorphism classes of) rooted graphs with radius at most r. Since the degree is bounded by a fixed constant d, obviously $|U^r| < \infty$. For a finite graph G we can associate a natural probability measure on U^r denoted by $P_{G,r}$ as follows.

For a finite (not necessary connected) graph G let us choose a vertex $v \in V(G)$ uniformly at random, and for a chosen vertex v consider the r-neighbourhood (in the graph distance) of v in G (with the inherited graph structure) as rooted graph with root v. This associated rooted graph will be denoted by $N_{G,r}(v)$. Then $P_{G,r}$ denotes the distribution of $N_{G,r}(v)$ in U^r, that is

$$P_{G,r}((S,o)) = \frac{|\{v \in G : N_{G,r}(v) \text{ is isomorphic to } (S,o)\}|}{|V(G)|},$$

where (S,o) is an arbitrary element of U^r. With this data we can define *local convergence* of finite graphs.

Definition 4.1. A sequence of finite graphs $(G_n)_{n\in\mathbb{N}}$ is *locally convergent* (or Benjamini-Schram convergent) if for all fixed r, $P_{G_n,r}$ converges to a limit distribution as $n \to \infty$ for all r. That is, $P_{G_n,r}((S,o))$ converges for all $(S,o) \in U^r$ as $n \to \infty$.

Example 4.2. Consider a sequence of d-regular graphs G_n for which the size of the shortest cycle converges to infinity as $n \to \infty$. This sequence is locally convergent. A much stronger statement is also true, namely that if we choose a random d-regular graph on n vertices for every n, then with probability 1 the sequence we get converges locally. This can be derived from Theorem 2.5 in [28].

Remark 4.3. An important result proved in [9] is that the independece ratio of the sequence of sequence random d-regular graphs is bounded away from $\frac{1}{2}$ (with positive probabilty). On the other hand if we consider random d-regular bipartite graphs, their independence ratio is always $\frac{1}{2}$. However if we merge the two sequences, we still get a locally convergent graph sequence. We conclude the independece ratio is not countinuous with respect to the local convergence.

As we discussed in Remark 4.3, the local convergence is not strong enough to capture the global structure of a sequence of graphs. The following refinement of it, called local-global convergence, was introduced in [10]. The idea is that we would like to measure the distribution of colored r-neighbourhoods instead of just the r-neighbourhoods. Let us denote by $U^{r,k}$ the set of all triples (H,o,c), where $(H,o) \in U^r$ and c is a k-coloring of H. Notice that for a finite graph G, a vertex $v \in G$ and a k-coloring $c : G \to [k]$, the restriction of c to $N_{G,r}(v)$ is in $U^{r,k}$. Similarly as in the uncolored version, a fixed finite graph G and a fixed k-coloring $c : G \to [k]$ induce a distribution $P_{G,r}[c]$ on $U^{r,k}$ (we refer to these as *local statistics* of c). Now let us denote by $Q_{G,r,k}$ the set of all local statistics, that is:

$$Q_{G,r,k} := \{P_{G,r}[c] : c \text{ is a } k\text{-coloring of } G\} \subseteq [0,1]^{U_{r,k}}.$$

As there are only finitely many k-colorings for a fixed finite graph, $Q_{G,r,k}$ is a finite (and thus compact) subset of $[0,1]^{U_{r,k}}$. Now we are ready to define local-global convergence of graphs.

Definition 4.4. A sequence of finite graphs $(G_n)_{n\in\mathbb{N}}$ is *locally-globally convergent* if for all fixed $r, k \geq 1$ the sequence of sets $(Q_{G_n,r,k})_{n\in\mathbb{N}}$ converge in the Hausdorff distance in $\mathcal{K}([0,1]^{U_{r,k}})$.

Here $\mathcal{K}([0,1]^{U_{r,k}})$ denotes the compact sets of $[0,1]^{U_{r,k}}$, and we equip $[0,1]^{U_{r,k}}$ with the usual product topology. In particular we are going to use the $||.||_1$ metric on it. Notice that since $\mathcal{K}([0,1]^{U_{r,k}})$ is compact, we can pass to a subsequence $(m_n)_{n\in\mathbb{N}}$ of $(G_n)_{n\in\mathbb{N}}$ such that $(Q_{G_{m_n},r,k})_{n\in\mathbb{N}}$ is convergent for every $r, k \in \mathbb{N}$, and therefore we can conclude that every graph sequence admits a local-global convergent subsequence.

Remark 4.5. Notice that local convergence is implied by *local-global convergence*, since it just means the convergence of the sequences $(Q_{G_n,r,1})_{n\in\mathbb{N}}$ for all r.

The following remark shows that local-global convergence is a strictly stronger notion than local convergence.

Remark 4.6. In Remark 4.3 we have seen that the independece ratio is not continuous with respect to the local convergence. We show that for the local-global convergence it is. Indeed, $i(G)$ can be decoded from $Q_{G,1,2}$. It is the maximal value of sum of those coordinates in elements of $Q_{G,1,2}$, in which the root has color 0 and all other vertices have color 1. The convergence of $Q_{G_n,1,2}$ in the Hausdorff distance implies that the maximum of this sum converges. Therefore the graph sequence from Example 4.2 converges locally, but does not converge locally-globally.

Remark 4.7. There is a family of graphs (the so called hyperfinite graphs) for which local and local-global convergence coincide. This was shown in Theorem 8 in [15].

Remark 4.8. It is natural to ask if we obtain a different notion of convergence if instead of vertex coloring we examine the local statistics of more complicated locally definable structures. However it turns out that vertex colorings already capture these, here for the sake of simplicity we show that edge colorings are encoded by vertex colorings. We claim that every edge k-coloring c of a graph (with degrees bounded by d) can be encoded by a 2^{5d^2k}-coloring of its vertices. Indeed, first modify c to a $5d^2k$ edge coloring c' by setting $c'(e) = (c(e), l)$, where $l \in [5d^2]$, and if two edges e_1 and e_2 have distance at most two then $c'(e_1) \neq c'(e_2)$. Obviously c' encodes c. Now let $d(v)$ encode the subset of the $5d^2k$ many colors which are used by c' on an edge that contains v. This way d encodes c', since the color of an edge $e = (v, w)$ is just the unique common element of $d(v)$ and $d(w)$.

4.1 The limit object of the convergence

In Definition 4.1 and 4.4 we gave two notions of convergence of finite graphs, but we did not define what they converge towards.

In the case of local convergence the natural limit objects are *unimodular random graphs*, while in the local-global case *graphings* serve this purpose. We introduce these notions in the following Section 4.1.1 and 4.1.2.

4.1.1 Unimodular random graphs

The notion of unimodular random graph is due to Benjamini and Schram [4]. Let $\mathfrak{G}$ be the space of (isomorphism classes of) rooted connected graphs, in which every degree is at most d. Notice that the size of such graphs is possibly infinite, but due to the connectivity assumption, it is at most countable. We put a topology on $\mathfrak{G}$ as follows. For a fixed finite rooted graph (S, o) with radius at most r we denote by $\mathfrak{G}_r(S, o)$ those elements (G, σ) in $\mathfrak{G}$ for which $N_{G,r}(\sigma)$ is isomorphic to (S, o). Let us consider the the topology on $\mathfrak{G}$ where we declare the family

$$\mathcal{B} := \{\mathfrak{G}_r(S, o) : r \in \mathbb{N}, (S, o) \text{ is a finite rooted graph with radius at most } r\}$$

to be basic open sets.

Lemma 4.9. *With the topology described above $\mathfrak{G}$ is a compact, zero-dimensional, second countable and Hausdorff space, in which finite graphs form a countable discrete set.*

Proof. First of all let us notice that $\mathcal{B}$ indeed satisfies the basis property (since every pair of sets in $\mathcal{B}$ is either disjoint or contain eachother), thus it is the basis of the topology generated by it. Therefore the second countable property follows immeditaley from the fact that $\mathcal{B}$ is countable. Notice, that every element of $\mathcal{B}$ is clopen, thus $\mathfrak{G}$ is zero-dimensoinal. It is trivial to see that $\mathfrak{G}$ is Hausdorff. It is also easy to see that for every finite graph (G, σ) the one element set $\{(G, \sigma)\}$ is open, since if the radius of (G, σ) is r, then $\mathfrak{G}_k(G, \sigma) = \{(G, \sigma)\}$ if $k > r$.

Now we prove compactness. It is well known that sequential compactness and compactness are equivalent notions in second countable spaces. So let $(G_n, \sigma_n) \in \mathfrak{G}$ be an arbitrary sequence of graphs. Then by a standard diagonalization argument we can find a convergent subsequence as follows. For every r the set U^r is finite, so for every r let $(G_{k_n^r}, \sigma_{k_n^r})$ be a subsequence of $(G_{k_n^{r-1}}, \sigma_{k_n^{r-1}})$ such that $(N_{G,r}(o), o)$ is constant for every $(G, o) \in \{(G_{k_n^r}, \sigma_{k_n^r}) : n \in \mathbb{N}\}$. (Here we set $k_n^0 = n$). Then the subsequence $(G_{k_r^r}, \sigma_{k_r^r})$ of the original sequence is indeed convergent. □

Actually, $\mathfrak{G}$ is homeomorphic to the Cantor set plus a dense (in $\mathfrak{G}$) set of discrete points, and so it is a standard Borel space.

In Definition 4.10 we are going to need so called *payoff functions*. For this we need to introduce the space of double rooted, connected, bounded degree graphs, that is, graphs with an ordered pair of distinguished vertices. We are going to denote this space by $\mathfrak{G}_2$. The topology on $\mathfrak{G}_2$ is defined analoguously to the topology on $\mathfrak{G}$, i.e. declare those subsets open, for which the r-neighbourhood of the first root is isomorphic to a fixed double rooted finite graph. Now we are ready to define the notion of unimodular random graphs. There are many versions of this definition, we follow Definition 2.1 in [2].

Definition 4.10. We call a Borel probability measure ν on $\mathfrak{G}$ a *unimodular random graph* (hereafter referred to as URG), if it satisfies the mass-transport principle, that is for all Borel function $f : \mathfrak{G}_2 \to [0, \infty]$ the following holds:

$$\int \sum_{x \in V(G)} f(G, o, x) d\nu(G, o) = \int \sum_{x \in V(G)} f(G, x, o) d\nu(G, o). \tag{4.1}$$

Remark 4.11. To be very precise one should check that $\sum_{x \in V(G)} f(G, o, x)$ is a Borel $\mathfrak{G} \to [0, \infty]$ function in (G, o). This is an elementary calculation that only uses the definitions, and therefore we leave it to the reader.

The mass-transport principle (4.1) intuitively says that the expected income of the payoff function f is the same as the expected out pay. This definition often turns out to be useful, as an appropriately choosen payoff function often makes proofs more convenient. To illustrate this, we present here a short proof sketch. For a countable graph G we define the number of *ends* in G as the supremum of the number of infinite connected components of the graph $G|_{V(G) \setminus F}$ where $F \subseteq V(G)$ is finite.

Example 4.12. A uniform random graph ν has 0, 1, 2 or infinitely many ends with probability 1. We show this using the payoff function described as follows. Vertices of graphs with $e \leq 2$ or $e = \infty$ many ends do not pay anything. Let G be a graph with e many ends where $3 \leq e < \infty$. Then every vertex in G pays 1 to every vertex which is the element of a finite set $F \subseteq V(G)$ for which $G|_{V(G) \setminus F}$ has e many infinite connected components and F has the smallest possible size and diameter. It is easy to see that the expected out pay is finite, while the expected income is infinite if $\nu(\{(G, o) :$ G has finitely many but more than 2 ends$\}) > 0$, which would contradict the mass-transport principle.

In the following we give an other definition to URGs, which is often easier to check. We follow Section 3 in [17].

Definition 4.13. Let $\tilde{\mathfrak{G}}$ denote the space of the graphs in $\mathfrak{G}$, with a distinguished edge incident to the root. The topology on $\tilde{\mathfrak{G}}$ is defined analogously to the topology on $\mathfrak{G}$, that is, for a finite rooted graph (with radius at most r) (G, o, e) with the edge e incident to o we denote by $\tilde{\mathfrak{G}}_r(G, o, e)$ those elements of $\tilde{\mathfrak{G}}$, for which their r-neighbourhood is isomorphic to (G, o, e). Then it is easy to see that the function $\alpha : \tilde{\mathfrak{G}} \to \tilde{\mathfrak{G}}$, which moves the root along the selected edge, is continuous. For every measure μ on $\mathfrak{G}$ let us denote with $\mu^\star$ the probability measure for which the Radon-Nikodym derivative $d\mu^\star/d\mu$ is proportional to the degree of the root in the graph. That is,

$$\mu^\star(\mathfrak{G}_r(S, o)) = \frac{\deg_S(o)}{c}\mu(\mathfrak{G}_r(S, o)),$$

where $c = \int \deg_G(o) d\mu(G, o)$. Let us denote by $\tilde{\mu}$ the following measure on $\mathfrak{G}$: first we pick a $\mu^\star$-random rooted graph, then we distinguish a uniform random edge incident to the root of the picked graph. By formula:

$$\tilde{\mu}(\tilde{\mathfrak{G}}_r(S, o, e)) = \frac{1}{\deg_G(o)}\mu^\star(\mathfrak{G}_r(S, o)) = \frac{1}{c}\mu(\mathfrak{G}_r(S, o)). \tag{4.2}$$

We say that μ is *involution-invariant*, if $\tilde{\mu}$ is α invariant, that is the pushforward of $\tilde{\mu}$ by the map α is $\tilde{\mu}$.

We show that Definition 4.10 and 4.13 describe the same notion.

Proposition 4.14. *Let μ be a measure on $\mathfrak{G}$. Then μ is a unimodular random graph if and only if it is involution-invariant.*

Sketch of the proof. Let μ be a unimodular random graph. To show, that it is involution-invariant it is enough to prove that

$$\tilde{\mu}(\tilde{\mathfrak{G}}_r(S, o, e)) = \tilde{\mu}(\alpha(\tilde{\mathfrak{G}}_r(S, o, e))) \tag{4.3}$$

for every basic clopen set $\tilde{\mathfrak{G}}_r(S, o, e)$. Set the payoff function f to be 1 on (G, o, x) if $(G, o, (o, x)) \in \tilde{\mathfrak{G}}_r(S, o, e)$, and 0 otherwise. It is easy to see that the mass-transport principle implies 4.3 directly. For the converse it is enough to consider payoff functions that are continuous and have finite range. Moreover we can assume by [2, Proposition 2.2] that f is supported on (G, o, x) where o and x are neighbours. Therefore it is enough to consider payoff functions of form χ_B where B is a basic clopen set. For these functions mass-transport principle is directly implied by involution-invariance. □

For a sequence of locally convergent graphs $(G_n)_{n\in\mathbb{N}}$ we can associate the following natural URG ν. We prove that it is indeed unimodular in Proposition 4.16.

$$\nu(\mathfrak{G}_r(S,o)) := \lim_{n\to\infty} P_{G_n,r}(S,o). \tag{4.4}$$

Example 4.15. We can turn a finite graph into a URG by picking the root uniformly and randomly. That is for a finite graph G

$$\nu_G = \frac{1}{|G|}\sum_{o\in V(G)} \chi_{(G_o,o)},$$

where G_o denotes the connected component of G which contains o. In this case (4.1) is just a double counting. It says that for every non-negative function f on the pairs of vertices of G the following holds:

$$\frac{1}{|G|}\sum_{o\in V(G)}\sum_{x\in V(G)} f(G,o,x) = \frac{1}{|G|}\sum_{o\in V(G)}\sum_{x\in V(G)} f(G,x,o).$$

Proposition 4.16. *The measure defined in (4.4) is a URG.*

Proposition 4.16 will be an easy corollary of the following lemma, which says that URGs are closed under weak convergence (see Definition 3.3).

Lemma 4.17. *The set of uniform random graphs on $\mathfrak{G}$ is closed under weak convergence.*

Proof. We are going to use involution-invariance of URGs described in Definition 4.13. So let μ_n be a weakly converging sequence of URGs, with $\lim_{n\to\infty}\mu_n = \mu$. Let us fix a finite rooted graph (with radius at most r) (G,σ,e) with a distinguished edge e incident to the root σ. It is enough to show that the basic clopen sets are α invariant, that is

$$\tilde{\mu}(\tilde{\mathfrak{G}}_r(G,\sigma,e)) = \tilde{\mu}(\alpha(\tilde{\mathfrak{G}}_r(G,\sigma,e))). \tag{4.5}$$

Notice that $\tilde{\mathfrak{G}}_r(G,\sigma,e)$ is a basic clopen set and $\alpha(\tilde{\mathfrak{G}}_r(G,\sigma,e))$ can be written as a finite disjoint union of basic clopen sets $\bigsqcup_{m=1}^k \mathfrak{G}_{r_m}(S_m,o_m,e_m)$. Let $c_n = \int \deg_G(o)d\mu_n(G,o)$, and $c = \int \deg_G(o)d\mu(G,o)$, then by the definition of weak convergence and the fact that $\deg_G(o)$ is bounded and continuous, we get that $\lim_{n\to\infty} c_n = c$. Using this, the formula (4.2) and Lemma 3.4 we get that

$$\tilde{\mu}(\tilde{\mathfrak{G}}_r(G,\sigma,e)) = \frac{1}{c}\mu(\mathfrak{G}_r(G,\sigma)) = \lim_{n\to\infty}\frac{1}{c_n}\mu_n(\mathfrak{G}_r(G,\sigma))$$

and

$$\tilde{\mu}(\alpha(\tilde{\mathfrak{G}}_r(G,\sigma,e))) = \sum_{m=1}^k \frac{1}{c}\mu(\mathfrak{G}_{r_m}(G_m,o_m)) = \sum_{m=1}^k \lim_{n\to\infty}\frac{1}{c_n}\mu_n(\mathfrak{G}_{r_m}(G_m,o_m)).$$

But using the involution-invariance of the measures μ_n we know that $\frac{1}{c_n}\mu_n(\mathfrak{G}_r(G,\sigma)) = \sum_{m=1}^{k}\frac{1}{c_n}\mu_n(\mathfrak{G}_{r_m}(G_m,o_m))$, and thus (4.5) holds. □

Proof of Proposition 4.16. Let μ_n denote the URG associated to the finite graph G_n as in Example 4.15. Then $\mu_n(\mathfrak{G}_r(S,o)) = P_{G_n,r}(S,o)$, and therefore using Lemma 3.4 we can conclude that (4.4) exactly says that ν is defined as the weak limit of the measures μ_n. Here we used the fact that $\mathfrak{G}$ is zero-dimensional. Therefore by Lemma 4.17 we get that ν is a URG too. □

Remark 4.18. The converse of Proposition 4.16 (whether every URG arises as the local limit of finite graphs) is called the Aldous-Lyons conjecture (see Question 10.1 in [2]). It is considered to be one of the most important problem of the area and it was a central open question in the last two decades. The conjecture was answered negatively very recently in [11] and [12].

Notation 4.19. To close this section, we introduce the space of decorated rooted graphs, which we will use in the next Section 4.1.2. Let C be a compact Polish space. Then we denote by $\mathfrak{G}(C)$ the space of (isomorphism classes of) rooted connected graphs, in which every degree is at most d and every vertex is labelled with an element of C. That is, the points of $\mathfrak{G}(C)$ are triples (G,o,c) where $(G,o)\in\mathfrak{G}$ and c is a $V(G)\to C$ function. The topology in $\mathfrak{G}(C)$ is defined analogously as in the case of $\mathfrak{G}$. For a fixed finite rooted graph (S,o) of radius at most r and a collection of open sets $\mathcal{V}$ of C indexed by $V(S)$, let us denote by $\mathfrak{G}_r(S,o,\mathcal{V})$ those elements (G,σ,c) in $\mathfrak{G}(C)$ for which there is a rooted graph isomorphism $\varphi : (N_{G,r}(\sigma),\sigma)\to(S,o)$ such that $c(\varphi^{-1}(v))\in\mathcal{V}(v)$ for every $v\in V(S)$. An analogous proof as of Lemma 4.9 shows that $\mathfrak{G}(C)$ is a compact Polish space.

4.1.2 Graphing

In this section we introduce the limit object of a locally-globally convergent sequence of finite graphs, which are the so called *graphings*. This notion was already present before it became natural to look at it as the limit of finite graphs, namely the original definition comes from Borel equivalence relations. We are going to consider both points of view.

Borel equivalence relations and Borel combinatorics have been extensively studied, for an introduction see e.g. [19] or [22]. In particular, countable Borel equivalence relations (that is, Borel equivalence relations in which all classes are countable) turn out to be a highly active research area. One of the earliest papers in the field and at the same time a brief introduction, is [18]. Borel graphs form a core concept in Borel combinatorics. We begin with their definition.

Definition 4.20. Let X be a standard Borel space. We say that $\mathcal{G}$ is a *Borel* (undirected) *graph* on X if the vertex set of $\mathcal{G}$ (hereinafter referred to as $V(\mathcal{G})$) is X and the edge set (hereinafter referred to as $E(\mathcal{G})$) is a symmetric Borel subset of $X \times X \setminus \{(x,x) : x \in X\}$.

We are going consider a special kind of Borel graphs. The idea behind the notion of *graphing* is to mimic double counting in finite graphs. The following definition is from [17].

Definition 4.21. Let X be a standard Borel space and μ be a Borel probablity measure. We say that a Borel graph $\mathcal{G}$ on X is a *graphing* if all degrees are at most d and the following property holds for every μ-measurable set $A, B \subseteq X$:

$$\int_A e(x,B)d\mu(x) = \int_B e(x,A)d\mu(x), \tag{4.6}$$

where $e(x,S)$ denotes the number of edges from $x \in X$ to $S \subseteq X$.

However, the original definition came from a somewhat different setting. First we define Borel equivalence relations.

Definition 4.22. Let X be a standard Borel space. We say that $\mathcal{E} \subseteq X \times X$ is a *Borel equivalence relation* on X, if it is an equivalence relation and it is a Borel subset of $X \times X$.

We define measure preserving equivalence relations. For an equivalence relation $\mathcal{E}$, let $[[\mathcal{E}]]$ denote its *full groupoid*, i.e. the set of partial isomorphisms $\theta : A \to B$ such that $(x, \theta(x)) \in \mathcal{E}$, for all $x \in A$.

Definition 4.23. Let X be a standard Borel space, and μ be a Borel measure on it. Let $\mathcal{E}$ be a Borel equivalence relation with countable classes on X. We say that $\mathcal{E}$ is a *probability measure preserving Borel equivalence relation* (hereinafter referred as p.m.p. equivalence relation), if for every $f \in [[\mathcal{E}]]$ which is a partial bijection between A and B, we have $\mu(A) = \mu(B)$.

The connection between p.m.p. equivalence relations and graphings is as follows. For every Borel graph $\mathcal{G}$ (with at most countable degrees) we can associate a Borel equivalence relation $\mathcal{E}_{\mathcal{G}}$, whose classes are the connected components of $\mathcal{G}$.

Proposition 4.24. *A Borel graph $\mathcal{G}$ on a standard measure space (X,μ) is a graphing if and only if the equivalence relation $\mathcal{E}_{\mathcal{G}}$ is measure preserving.*

Proof. First let $\mathcal{G}$ be an arbitrary Borel graph with all degrees bounded by d. Then there is a proper Borel coloring of its edges by $2d-1$ colors. Indeed, let us take the Borel graph on the edge set $E(\mathcal{G})$, and connect two of them if they are incident in $\mathcal{G}$. This is again a Borel graph, with degrees bounded by $2d-2$, thus it admits a proper Borel $2d-1$ coloring (see e.g. Proposition 4.6 in [23]). Swapping the vertices connected by edges of color i gives an element of the full groupoid $f_i \in [[\mathcal{E}]]$.

Now for the 'if' part, let us fix such a $2d-1$ coloring of the edges. Let us denote by A_i the set $\{x \in A : f_i(x) \in B\}$, and let us define B_i analogously. Notice that $f_i(A_i) = B_i$. Then $\int_A \deg(x, B) = \sum_{i=1}^{2d-1} \mu(A_i) = \sum_{i=1}^{2d-1} \mu(B_i) = \int_A \deg(x, B)$, where the second equality follows from $\mathcal{E}$ being measure preserving.

For the 'only if' part let $f : A \to B$ be an element of the full groupoid. For every $a \in A$ there is a shortest path to $f(x)$ which goes along the edges of $\mathcal{G}$. Let us partition $A = \sqcup_{s \in [2d-1]^{<\omega}} A_s$ according to the sequence of colors of the edges on this shortest path (if there are more, fix one in a Borel way, e.g. the lexicographically smallest sequence). Clearly, it is enough to prove that $\mu(A_s) = \mu(f(A_s))$ for every $s \in [2d-1]^{<\omega}$. Since $f(A_s) = f_{s_{|s|}} \circ \ldots \circ f_{s_1}(A_s)$, it suffices to show that $\mu(f_i(H)) = \mu(H)$ for any measurable set H. Lemma 18.19 in [24] says that every subgraph of a graphing is a graphing, and the proof turns out to be surprisingly nontrivial. We can apply this to the subgraph consisting of all edges with color i. Then the graphing property gives that $\mu(H) = \int_H e(x, f_i(H)) d\mu(x) = \int_{f_i(H)} e(x, H) d\mu(x) = \mu(f_i(H))$. □

A natural way to obtain graphings is through group actions.

Example 4.25. Let a be a p.m.p. action of a finitely generated group Γ on a standard probability space $(X, \mathcal{B}, \mu)$. Let us fix a symmetric generator set $|S| < \infty$. We can associate the following graphing $\mathcal{G}_a$ to the action: $(x, y) \in E(\mathcal{G}_a) \iff x = \gamma^a(y)$ for some $\gamma \in S$. It is easy to show using Definition 4.21 that this is indeed a graphing.

Notice that it makes sense to define the probability distribution $P_{\mathcal{G},r}$ on U^r for a graphing $\mathcal{G}$ (with invariant measure μ) in the following way:

$$P_{\mathcal{G},r}((S, o)) = \mu(\{v \in \mathcal{G} : N_{\mathcal{G},r}(v) \text{ is isomorphic to } (S, o)\}).$$

We will also need the local statistics of a graphing, which is also analogous to the finite case, but we require the coloring to be measurable. Let $c : \mathcal{G} \to [k]$ be a μ-measurable (or by changing the value on a null set we can always suppose that it is Borel measurable) coloring. Then $P_{\mathcal{G},r}[c]$ denotes the natural distribution on $U^{r,k}$. Thus $Q_{\mathcal{G},r,k}$ can be also defined the following way:

$$Q_{\mathcal{G},r,k} := \{P_{\mathcal{G},r}[c] : c \text{ is a measurable } k\text{-coloring of } \mathcal{G}\}.$$

Again, as in the finite case, we look at $Q_{\mathcal{G},r,k}$ as a subset of $[0,1]^{U_{r,k}}$. Notice that this set is usually not closed.

Definition 4.26. For two graphings $(\mathcal{G}_1, X_1, \mu_1)$ and $(\mathcal{G}_2, X_2, \mu_2)$ we say that they are *locally-globally equivalent* if $\overline{Q_{\mathcal{G}_1,r,k}} = \overline{Q_{\mathcal{G}_1,r,k}}$ for all $r, k \in \mathbb{N}$. We say that they are *isomorphic*, if there is a measure preserving bijection $\varphi : X_1 \to X_2$ such that for every $x, y \in X_1$ $(x, y) \in E(\mathcal{G}_1) \iff (\varphi(x), \varphi(y)) \in E(\mathcal{G}_2)$.

Of course isomorphism implies local-global equivalence. We are ready to define the limit of a locally-globally convergent graph sequence.

Definition 4.27. We say that a graphing $\mathcal{G}$ is the *local-global limit* of the locally-globally convergent graph sequence $(G_n)_{n\in\mathbb{N}}$ (or the graphing sequence $(\mathcal{G}_n)_{n\in\mathbb{N}}$), if $Q_{G_n,r,k}$ (or respectively $\overline{Q_{\mathcal{G}_n,r,k}}$) converges to $\overline{Q_{\mathcal{G},r,k}}$ in the Hausdorff distance for every $r, k \in \mathbb{N}$.

Remark 4.28. The limit object in case of the local convergence was a well defined measure on the space of rooted graphs $\mathfrak{G}$. In case of the local-global convergence the limit object is a graphing, which has the advantage of being the same type of object as the converging ones, namely a graph. On the other hand in this case we lose uniqueness, since it is not hard to construct graphings $\mathcal{G}_1, \mathcal{G}_2$ which are local-global equivalent but not isomorphic.

The following theorem (which is the main result of [17]) states that we can always find a limit graphing of a locally-globally converging sequence of graphs.

Theorem 4.29. *For every $(G_n)_{n\in\mathbb{N}}$ locally-globally convergent sequence of finite graphs with degree bounded by d there exists a graphing $\mathcal{G}$ which is the local-global limit of the sequence.*

The following "regularity" lemma plays a key role in the proof.

Lemma 4.30. *Fix positive integers k, r and n. Then there exists a constant denoted by $t_{r,k,n}$ such that for every finite graph G (with all degree at most d) there is a $t_{r,k,n}$-coloring q of G with the following properties:*

- *for every pair of vertices $u \neq v$ in G if $q(u) = q(v)$ then the graph-distance of u and v is at least r,*
- *every k-coloring g of G it is an almost factor of q, that is, there is a function $\alpha : [t_{r,k,n}] \to [k]$ such that*
$$||P_{G,r}[g] - P_{G,r}[\alpha \circ q]||_1 < \frac{1}{n}.$$

Proof. The space $[0,1]^{U^{r,k}}$ with the usual $||.||_1$ distance is compact, so we can fix a $\frac{1}{2n}$-net N in it. Notice that the size of N depends only on r, k, n and d. Now fix a finite graph

G. Then let us denote by N_G those elements $x \in N$ for which there is a k-coloring c of G such that $||P_{G,r}[c] - x||_1 < \frac{1}{2n}$. Let us fix k-colorings $g_1, g_2 \dots g_{|N_G|}$ of G which are inducing distributions $\frac{1}{2n}$ close to the points of N_G. Let f be the coloring which we get by combining $g_1, g_2 \dots g_{|G_N|}$, that is, $f(v) := (g_1(v)), g_2(v), \dots g_{|G_N|(v)})$. Notice that f uses at most $k^{|N_G|} \leq k^{|N|}$-many colors, which only depends on r, k, n and d.

To satisfy the first condition, take the following graph: we connect every pair of distinct vertices in G which are at most r-close in the graph distance. In this graph every degree is bounded by d^{r+1}. It is well known that such a graph always admits a $d^{k+1} + 1$ coloring f'. Combining f and f' as before into a coloring q, where $q(v) = (f(v), f'(v))$ concludes the proof, since the arity of q (which is going to be $t_{r,k,n}$) only depends on r, k, n and d, and for every k-coloring g there is i such that $||P_{G,r}[g] - P_{G,r}[g_i]||_1 < \frac{1}{n}$ and g_i is a factor of q. □

Proof of Theorem 4.29. We are ready to introduce the base space of the graphing which will be the local-global limit of the sequence $(G_i)_{i\in\mathbb{N}}$. Let us denote by C be the compact metric space $\prod_{r,k,n\in\mathbb{N}}[t_{r,k,n}]$, where $t_{r,k,n}$ comes from Lemma 4.30. The vertex set of the limit graphing will be $X := \mathfrak{G}(C)$ (see Notation 4.19). We put a measure on it as follows. Let us fix colorings $q^i_{r,k,n} : V(G_i) \to [t_{r,k,n}]$ for every graph G_i guarateed by Lemma 4.30. Let us denote by c_i the C-decoration of $V(G_i)$ which we get by $c_i(v) := (q^i_{r,k,n}(v))$. Analogously to for the uncolored version in Example 4.15 we can associate a probability measure $\mu_i \in P(X)$ to (G_i, c_i) by choosing a random vertex of G_i to be the root, and considering the restriction of c_i to its connected component. Using Theorem 3.5 and the fact that X is a compact metrizable space we can take a weakly convergent subsequence of $(\mu_i)_{i\in\mathbb{N}}$. Since obviously the local-global limit of a subsequence of a local-global convergent sequence is the limit of the original one, we may assume that $(\mu_i)_{i\in\mathbb{N}}$ was already weakly convergent. Let the weak limit be $\mu \in P(X)$. Now we put the following graph structure $\mathcal{G}$ on X. Let (G_1, o_1, c_1) and (G_2, o_2, c_2) be connected if $G_1 = G_2 = G, c_1 = c_2$ moreover o_1 and o_2 are neighbours in G. Notice that loop edges can arise this way, but we show that only on a set with μ measure 0. Indeed, μ a.e. point (G, o, c) has the property that c is injective on G, since the following set is open:

$$B_K := \{(G, o, c) \in X : c|_{\prod_{r,k,n\leq 2K}[t_{r,k,n}]} \text{ is not injective on the } K \text{ neighbourhood of } o\},$$

moreover $\mu_i(B_K) = 0$ by the first property of the coloring $q^i_{2K,k,n}$, and $\bigcup_{K\in\mathbb{N}} B_K$ covers every point (G, o, c) for which c is not injective on G. Thus, using weak convergence, from [23, Theorem 17.20 iv)] it follows that $\mu(\bigcup_{K\in\mathbb{N}} B_K) = 0$.

We will show that $\mathcal{G}$ together with the measure μ is a graphing. Since the σ-algebra generated by the clopen sets of X is the σ-algebra of the Borel sets, it is enough to prove that clopen sets $A, B \subseteq X$ satisfy (4.6). For this notice that both of the the functions $f^\star(x) = \chi_A(x) \cdot e(x,B)$ and $f_\star(x) = \chi_B(x) \cdot e(x,A)$ are continuous and bounded, and therefore by the definition of the weak convergence $\int_X f^\star(x) d\mu(x) = \lim_{n\to\infty} \int_X f^\star(x) d\mu_n(x)$ and $\int f_\star(x) d\mu(x) = \lim_{n\to\infty} \int_X f_\star(x) d\mu_n(x)$. On the other hand $\int_X f_\star(x) d\mu_n(x) = \int_X f^\star(x) d\mu_n(x)$, since both sides expresses the edges between the finite sets $\{v : ((G_n)_v, v, c_n) \in A\}$ and $\{v : ((G_n)_v, v, c_n) \in B\}$ divided by $|V(G_n)|$. Thus $\mathcal{G}$ together with the probability measure μ is indeed a graphing.

We introduce a C-decoration of the vertices of the graphing $\mathcal{G}$. Let $q(G, o, c) = c(o)$, that is, the decoration of the root. Of course q is a measurable function, in fact it is countinuous. Let us denote by $q_{r,k,n}$ the (r, k, n)-th coordinate function of q. The key observation for completing the proof of the theorem is the following. Recall that we interpret both $P_{G,r}[c]$ and $P_{\mathcal{G},r}[c]$ as an element of $[0, 1]^{U^{r,k}}$.

Lemma 4.31. *The probability distributions $P_{G_i,r}[q^i_{r,k,n}]$ converge to $P_{\mathcal{G},r}[q_{r,k,n}]$.*

Proof of Lemma 4.31. It is enough to show the statement coordinatewise, that is

$$P_{G_i,r}[q^i_{r,k,n}](S, o, c) = P_{\mathcal{G},r}[q_{r,k,n}](S, o, c)$$

for all $(S, o, c) \in U^{r,t_{r,k,n}}$. By definition, $P_{G_i,r}[q^i_{r,k,n}](S, o, c)$ is the proprtion of vertices v in G_i for which the r-neighbourhood of v is isomorphic with (S, o, c). Notice that $\mu_i(\mathfrak{G}_{r(S)}(S, o, \mathcal{V}))$ encodes the same expression (see Notation 4.19), where $\mathcal{V}(v)$ denotes the open set $\{x \in C : x(r, k, n) = c(v)\}$ inside C. The set $\mathfrak{G}_{r(S)}(S, o, \mathcal{V})$ is a basic clopen set in X, so by Lemma 3.4 we know that $\lim_{i\to\infty} \mu_i(\mathfrak{G}(S, o, \mathcal{V})) = \mu((\mathfrak{G}_{r(S)}(S, o, \mathcal{V}))$, which by the definition of $q_{r,k,n}$ equals $P_{\mathcal{G},r}[q_{r,k,n}]$ and this concludes the proof of the lemma. □

To finish the proof of the theorem, we have to show that $Q_{G_i,r,k}$ converges to the compact set $\overline{Q_{\mathcal{G},r,k}}$ in the Hausdorff distance, i.e. $(\mathcal{G}, \mu)$ is the local-global limit of $(G_i)_{i\in\mathbb{N}}$. This is exactly the statement of the following two claims.

Claim 1. For every $r, k \in \mathbb{N}$ and $\varepsilon > 0$ there is i_0 such that for every $i \geq i_0$ and a k-coloring $c : V(G_i) \to [k]$ there is a measurable k-coloring c' of X such that $||P_{G_i,r}[c], P_{\mathcal{G},r}[c']||_1 \leq \varepsilon$.
Proof. Let $n \geq \frac{2}{\varepsilon}$. By Lemma 4.31 there is an index i_0 such that

$$||P_{G_i,r}[q^i_{r,k,n}] - P_{\mathcal{G},r}[q_{r,k,n}]||_1 \leq \frac{\varepsilon}{2}$$

for every $i \geq i_0$. Let us fix $i \geq i_0$ and a k-coloring c of G_i. By Lemma 4.30 there is a map $\alpha : [t_{r,k,n}] \to [k]$ such that

$$||P_{G_i,r}[c] - P_{G_i,r}[\alpha \circ q^i_{r,k,n}]||_1 \leq \frac{1}{n} \leq \frac{\varepsilon}{2}.$$

It is easy to see that $||P_{G_i,r}[\alpha \circ q^i_{r,k,n}] - P_{\mathcal{G},r}[\alpha \circ q_{r,k,n}]||_1 \leq ||P_{G_i,r}[q^i_{r,k,n}] - P_{\mathcal{G},r}[q_{r,k,n}]||_1$, and therefore combining the above two equations we get that $c' = \alpha \circ q_{r,k,n}$ satisfies the required condition of the claim.

Claim 2. For every $r, k \in \mathbb{N}$, $\varepsilon > 0$ and for every measurable k-coloring c of X there is i_0 such that for every $i \geq i_0$ there is a k-coloring $c' : V(G_i) \to [k]$ such that $||P_{G_i,r}[c'], P_{\mathcal{G},r}[c]||_1 \leq \varepsilon$.

Proof. Let $c : X \to [k]$ be a measurable k-coloring. Then for every $\delta > 0$ there a continuous k-coloring $c_\delta : X \to [k]$ such that $\mu(c^{-1}(a) \triangle c_\delta^{-1}(a)) < \delta$ for every $a \in [k]$. By choosing δ sufficiently small (e.g. smaller than $\frac{1}{|U^{r,k}|} \cdot \frac{\varepsilon}{2d^{r+1}}$) we can make sure that

$$||P_{\mathcal{G},r}[c_\delta], P_{\mathcal{G},r}[c]||_1 \leq \frac{\varepsilon}{2}.$$

Fix some $i \in \mathbb{N}$. Let us create the following k-coloring c' of G_i: for every vertex v let $c'(v) := c_\delta((G_i)_v, v, q^i)$. We claim that if i is large enough, then $P_{\mathcal{G},r}[c_\delta]$ and $P_{G_i,r}[c']$ are closer than $\frac{\varepsilon}{2}$ in the $||.||_1$ distance. Indeed, we can take a clopen partition of X such that on every piece the $P_{\mathcal{G},r}[c_\delta]$ value is constant, and therefore the μ-measure of the pieces determines $P_{\mathcal{G},r}[c_\delta]$. On one hand the μ_i measures of the same pieces converge to the μ measure of them, on the other hand $P_{G_i,r}[c']$ is also determined by the μ_i measure of the pieces which concludes the proof of the claim and with that the proof of the theorem. □

5 Measure preserving group actions

In this section we present a language that closely parallels the notions introduced in Section 4. We are going to examine measure preserving actions of a countable group on a fixed standard Borel space. Theorem 5.2 turns out to be the perfect analog of Theorem 4.29 in this setting. In Section 5.2 we discuss the connections between the two languages in detail. We mainly follow [13], [1] and [21] throughout the section.

For the rest of the section let us fix a countable group Γ and a standard Borel probability space $(X_0, \mathcal{B}, \lambda)$. For a fixed measure space $(X, \mathcal{B}, \mu)$ let $\mathrm{Aut}(X, \mathcal{B}, \mu)$ denote the *group of automorphisms*, that is, the group of measure preserving Borel bijections of $(X, \mathcal{B}, \mu)$ equipped with composition as group operation. As usual, we endow the space $\mathrm{Aut}(X, \mathcal{B}, \mu)$ with the weak topology (see Definition 3.10). The set of measure preserving actions (see Definition 3.12) of the group Γ on X can be embedded into the space $\mathrm{Aut}(X, \mathcal{B}, \mu)^\Gamma$, by sending an action a to the element $(\gamma^a)_{\gamma \in \Gamma}$. We will denote this subspace by Act(Γ, X), or when the fixed measure space is $(X_0, \mathcal{B}, \lambda)$, then simply by $\mathrm{Act}(\Gamma)$. The elements of Act(Γ, X) will usually be denoted by a, b, c etc.

Working directly with the space $\mathrm{Act}(\Gamma)$ comes with a disadvantage. It is natural not to distinguish actions $a, b \in \mathrm{Act}(\Gamma)$ that are *isomorphic* in the sense that the structure of the two action is the same. That is, there is an automorphism $f \in \mathrm{Aut}(X_0, \mathcal{B}, \lambda)$ such that $\gamma^a \circ f = f \circ \gamma^b$ for all $\gamma \in \Gamma$. For isomorphic actions a and b we will use the notation $a \cong b$. We denote by $C(a)$ the actions that are isomorphic to a. The problem is that conjugacy classes can have infimal distance zero. Thus taking the quotient by the conjugacy classes does not yield a nice topological space. So we introduce weak containment and weak equivalence for two action $a, b \in \mathrm{Act}(\Gamma)$. Here we follow [1].

Definition 5.1. For two actions $a, b \in \mathrm{Act}(\Gamma)$ we say that a *weakly contains* b if $b \in \overline{C(a)}$ (where the closure is in the weak topology, see Observation 5.3). We denote weak containment by $\prec a$. We say that a and b are *weakly equivalent* if $a \prec b$ and $b \prec a$, and denote this by $a \sim b$.

We are going to put a pseudometric on the space $\mathrm{Act}(\Gamma)$ called *partition metric*, which, following [1], will be denoted by pd. The main result of this section is the following theorem (see Theorem 1 in [1] or Theorem 1.2.22 in [13]).

Theorem 5.2. *The zero classes of* pd *are exactly the weak equivalence classes of the space* $\mathrm{Act}(\Gamma)$*, moreover the space we get by identifying the zero classes of* pd *is compact with respect to the inherited metric.*

5.1 The space $(\mathrm{Act}(\Gamma), \mathrm{pd})$

In this section we are going to define the metric pd on $\mathrm{Act}(\Gamma)$, and prove the first part of Theorem 5.2, namely that pd zero-classes in $\mathrm{Act}(\Gamma)$ are exactly weak equivalence classes.

We begin by analyzing the topology on $\mathrm{Act}(\Gamma)$. As we wrote in the introduction of Section 5, a natural topology on it is the subspace topology, that $\mathrm{Act}(\Gamma)$ inherits from the Polish space $\mathrm{Aut}(X_0, \mathcal{B}, \lambda)^\Gamma$.

Observation 5.3. Notice that $\mathrm{Act}(\Gamma)$ is a closed subspace in $\mathrm{Aut}(X_0, \mathcal{B}, \lambda)^\Gamma$, so it is Polish as well. Let us call this topology the *weak topology* on $\mathrm{Act}(\Gamma)$.

It will be more convenient to work with another description of the weak topology. First we will introduce some notations, most of them are based on Section 1.2 in [13]. For a measure space $(X, \mathcal{B}, \mu)$ we will denote by $\mathrm{Part}_{fin}(X)$ the set of finite partitions of X, and respectively, $\mathrm{Part}_k(X)$ denotes the set of partitions with k atoms. When we say partition, it will always mean a partition with measurable atoms and we neglect mistakes on a measure zero set (e.g. atoms can overlap on a set of measure 0). For $\alpha \in \mathrm{Part}_{fin}(X)$ we will denote by $|\alpha|$ the number of atoms in the partition. For a given action $a \in \mathrm{Act}(\Gamma, X)$, a finite set $F \subset \Gamma$ and a partition $\alpha \in \mathrm{Part}_{fin}(X), \alpha = (A_1, \ldots, A_{|\alpha|})$ we introduce the following vector:

$$\mathbf{c}(a, F, \alpha) := (\mu(A_i \cap \gamma^a A_j))_{i,j \leq |\alpha|, \gamma \in F}.$$

As usual, for two actions a, b, partitions $\alpha = \{A_1, \ldots, A_k\}, \beta = \{B_1, \ldots, B_k\} \in \mathrm{Part}_k(X)$ and a finite set $F \subset \Gamma$ we are going to denote by $||\mathbf{c}(a, F, \alpha) - \mathbf{c}(b, F, \beta)||_1$ the following expression:

$$\sum_{i,j \leq k} \sum_{\gamma \in F} |\mu(A_i \cap f A_j) - \mu(B_i \cap \gamma B_j)|.$$

This allows us to give an alternative description of the weak topology on $\mathrm{Act}(\Gamma)$.

Lemma 5.4. *The sets of the form*

$$\{b \in \mathrm{Act}(\Gamma) : ||\mathbf{c}(a, F, \alpha) - \mathbf{c}(b, F, \alpha)||_1 < \varepsilon\}$$

for all $\alpha \in \mathrm{Part}_{fin}(X_0), F \subset \Gamma$ *finite and* $\varepsilon > 0$ *form an open base for the weak topology.*

The proof is an easy calculation and only uses the definitions, so we leave it to the reader.

We are ready to define the pseudometric pd. It will be the sum of a countable family of pseudometrics described in the following definition.

Definition 5.5. For given actions $a, b \in \mathrm{Act}(\Gamma)$, a finite subset $F \subset \Gamma$ and $k \in \mathbb{N}$ we set:

$$d_{F,\alpha}(a, b) := \inf_{\beta \in \mathrm{Part}_{|\alpha|}(X_0)} ||\mathbf{c}(a, F, \alpha) - \mathbf{c}(b, F, \beta)||_1 \text{ for every } \alpha \in \mathrm{Part}_k(X_0), \text{ and}$$

$$d_{F,k}(a, b) := \sup_{\alpha \in \mathrm{Part}_k(X_0)} d_{F,\alpha}(a, b).$$

The goal is to produce a pseudometric and $d_{F,k}$ is not symmetric, so let us define $\overline{d}_{F,k}(a, b) := d_{F,k}(a, b) + d_{F,k}(b, a)$. In the following lemma we prove that $\overline{d}_{F,k}$ is a pseudometric for every fixed $F \subset \Gamma$ finite and $k \in \mathbb{N}$.

Lemma 5.6. *For every three actions* $a, b, c \in \mathrm{Act}(\Gamma)$ *and partition* $\alpha \in \mathrm{Part}_k(X_0)$ *we have*

$$d_{F,\alpha}(a, c) \leq d_{F,\alpha}(a, b) + d_{F,k}(b, c). \tag{5.1}$$

Moreover the function $\overline{d}_{F,k}$ *is a pseudometric on* $\mathrm{Act}(\Gamma)$.

Proof. The first part is a straightforward calculation which can be found as Proposition 1.2.8 in [13]. For the second statement notice that after taking the supremum in α

on both sides it follows from (5.1) that $d_{F,k}(a,c) \leq d_{F,k}(a,b) + d_{F,k}(b,c)$. Therefore $\overline{d}_{F,k}(a,c) = d_{F,k}(c,a)+d_{F,k}(a,c) \leq d_{F,k}(c,b)+d_{F,k}(b,a)+d_{F,k}(a,b)+d_{F,k}(b,c) = \overline{d}_{F,k}(a,b)+\overline{d}_{F,k}(b,c)$. □

Now we create the pseudometric pd as follows: we take a weighted sum of the pseudometrics $\overline{d}_{F,k}$ for all $F \subset \Gamma$ finite and $k \in \mathbb{N}$ in a way that the sum is convergent. Notice that as long as the sum always converges, these pseudometrics define the same zero-classes and the same topology. In particular, for two actions $a, b \in \mathrm{Act}(\Gamma)$ their pd distance is 0, if and only if $\overline{d}_{F,k}(a,b) = 0$ for all $F \subset \Gamma$ finite and $k \in \mathbb{N}$. In the following we prove the first part of Theorem 5.2, which is the following (see Proposition 10.1 in [21]).

Theorem 5.7. *Let $a, b \in \mathrm{Act}(\Gamma)$ be given actions. Then*

$$a \prec b \iff d_{F,k}(a,b) = 0 \text{ for all } F \subset \Gamma \text{ finite and } k \in \mathbb{N}.$$

In particular $a \sim b \iff \mathrm{pd}(a,b) = 0$.

Proof. By definition of $\sim$ and pd it is enough to prove the first statement. First we prove from left to right. By the definition of $d_{F,k}$ it is enough to show that for every $\varepsilon > 0$, $\alpha = \{A_1, A_2 \dots A_k\} \in \mathrm{Part}_k(X_0)$ and $F \subset \Gamma$ finite $d_{F,\alpha}(a,b) < \varepsilon$ holds. Using that $a \prec b$ and Lemma 5.4 we can find an action $c \in \mathrm{Act}(\Gamma)$ for which $c \cong b$ moreover $||\mathbf{c}(a,F,\alpha) - \mathbf{c}(c,F,\alpha)||_1 < \varepsilon$. Let φ be the measure preserwing bijection of X that witnesses the isomorphism of b and c (that is, $\varphi \circ \gamma^b = \gamma^c \circ \varphi$ for all $\gamma \in \Gamma$), and let us define $B_i := \varphi(A_i)$. Then we get that (up to an error of measure 0) $\{B_1, \dots, B_k\}$ is a partition and let us denote it by β. Then clearly

$$\lambda(A_i \cap \gamma^c(A_j)) = \lambda(\varphi(A_i \cap \gamma^c(A_j))) = \lambda(B_i \cap \gamma^b(B_j)),$$

and therefore $\lambda(A_i \cap \gamma^a(A_j)) - \lambda(A_i \cap \gamma^c(A_j)) = \lambda(A_i \cap \gamma^a(A_j)) - \lambda(B_i \cap \gamma^b(B_j))$ for all $\gamma \in F$ and $i, j \leq k$. Thus $||\mathbf{c}(a,F,\alpha) - \mathbf{c}(b,F,\beta)||_1 = ||\mathbf{c}(a,F,\alpha) - \mathbf{c}(c,F,\alpha)||_1 < \varepsilon$, which proves that $d_{F,\alpha}(a,b) < \varepsilon$.

Now we prove the other direction. By Lemma 5.4, we have to show that for every $F \subset \Gamma$ finite and $\alpha \in \mathrm{Part}_{fin}(X_0)$ the basic open neighbourhood

$$U := \{c \in \mathrm{Act}(\Gamma) : ||\mathbf{c}(a,F,\alpha) - \mathbf{c}(c,F,\alpha)||_1 < \varepsilon\}$$

of a conatains an action that is isomorphic to b. We may assume that the identity element of Γ is in F. From the fact that $d_{F,\alpha}(a,b) < \frac{\varepsilon}{4|\alpha|^2|F|}$, there is a partition β such that $||\mathbf{c}(a,F,\alpha) - \mathbf{c}(b,F,\beta)||_1 < \frac{\varepsilon}{4|\alpha|^2|F|}$. Since $1_\Gamma \in F$, we know that $|\lambda(A_i) - \lambda(B_i)| < \frac{\varepsilon}{4|\alpha|^2|F|}$. Now we prove the following.

Claim. There is a partition $\{C_1, \dots C_{|\alpha|}\} \in \mathrm{Part}_{|\alpha|}(X_0)$ such that $\lambda(A_i \triangle C_i) < \frac{\varepsilon}{4|\alpha|^2|F|}$ for every $i \leq |\alpha|$ with the property, that $\lambda(C_i) = \lambda(B_i)$.

Proof. Indeed, we can suppose without loss of generality that there is $1 \leq m \leq |\alpha|$ such that $\lambda(B_i) \leq \lambda(A_i)$ for all $i \leq m$ and $\lambda(B_i) \geq \lambda(A_i)$ for all $i > m$. Then by the isomorphism theorem of continuous standard Borel measures (see Remark 3.8) there is a subset C_i of A_i such that $\lambda(C_i) = \lambda(B_i)$ for all $i \leq m$. Using the same theorem, let us construct C_j recursively for every $j > m$ such that it is a superset of A_j, $C_j \setminus A_j \subset \bigcup_{i \leq m} A_i \setminus C_i$ and $\lambda(C_j) = \lambda(B_j)$. Since $\sum_{i=1}^{|\alpha|} \lambda(A_i) = \sum_{i=1}^{|\alpha|} \lambda(B_i)$, we can do this for every $j \leq |\alpha|$. Notice that the symmetric difference $A_i \triangle C_i$ is either $A_i \setminus C_i$ or $C_i \setminus A_i$, and therefore by the assumptoin $|\lambda(A_i) - \lambda(C_i)| = |\lambda(A_i) - \lambda(B_i)| < \frac{\varepsilon}{4|\alpha|^2|F|}$ we can conclude the claim.

To finish the proof of the theorem, $\{C_1, \dots C_{|\alpha|}\} \in \mathrm{Part}_{|\alpha|}(X)$ be a partition with the properties ensured by the claim. Then using the isomorphism theorem again, there is a measure preserving bijection φ on X such that $\varphi(B_i) = C_i$ for every $i \leq |\alpha|$. We claim that the action $\varphi \circ b \circ \varphi^{-1}$ (which is of course isomorphic to b by definition) is in U. Indeed, for fixed $i, j \leq |\alpha|$ and $\gamma \in F$ we have the following:

$$|\lambda(A_i \cap \gamma^a(A_j)) - \lambda(A_i \cap \gamma^c(A_j))| \leq |\lambda(A_i \cap \gamma^a(A_j)) - \lambda(C_i \cap \gamma^c(C_j))| + 2 \cdot \frac{\varepsilon}{4|\alpha|^2|F|},$$

moreover

$$\lambda(C_i \cap \gamma^c(C_j)) = \lambda(\varphi^{-1}(C_i \cap \gamma^c(C_j))) = \lambda(B_i \cap \gamma^b(B_j)).$$

Thus

$$|\lambda(A_i \cap \gamma^a(A_j)) - \lambda(A_i \cap \gamma^c(A_j))| \leq |\lambda(A_i \cap \gamma^a(A_j)) - \lambda(B_i \cap \gamma^b(B_j))| + \frac{\varepsilon}{2|\alpha|^2|F|} \leq \frac{3\varepsilon}{4|\alpha|^2|F|}.$$

Taking the sum for all $i, j \leq |\alpha|$ and $\gamma \in F$ we get that $c \in U$, and this concludes the proof of the theorem. □

Now we introduce the following space of actions, in which we do not distinguish actions that are weakly equivalent.

Notation 5.8. Let us denote by $\mathrm{Act}(\Gamma) / \sim$ the factor of $\mathrm{Act}(\Gamma)$ by the zero-classes of pd. By a little abuse of notation we are going to denote by pd the metric on $\mathrm{Act}(\Gamma) / \sim$ that is inherited from the pseudometric pd on $\mathrm{Act}(\Gamma)$. We say that a sequence of actions $\{a_n\}_{n \in \mathbb{N}}$ *converges in the partition distance* to a, if $\lim_{n \to \infty} \mathrm{pd}(a_n, a) = 0$.

Remark 5.9. There are two very natural topologies on the space $\mathrm{Act}(\Gamma) / \sim$. The first is the factor topology inherited from $\mathrm{Act}(\Gamma)$, the other one is the metric topology we get from pd (that becomes a metric instead of a pseudometric when factoring with the pd-zero classes). However, these *do not* coincide. In fact the factor topology is not T_1. Indeed, in Theorem 10.7 in [21] the authors claim that there is a dense conjugacy class of $\mathrm{Act}(\Gamma)$ with respect to the weak topology. Let $c \in \mathrm{Act}(\Gamma) / \sim$ be the image of this conjugacy class. Then $(\mathrm{Act}(\Gamma) / \sim) \setminus \{c\}$ is not open, since its preimage does not intersect a dense set, namely the dense conjugacy class belonging to c. On the other hand a metric topology is always T_1, which proves that the two topologies differ. However, an easy calculation shows that the metric topology contains the factor topology.

Remark 5.10. Notice that so far weak equivalence was interpreted for actions on the same standard Borel space X_0. Theorem 5.7 allows us to extend this definition. Notice that for p.m.p. actions $\Gamma \overset{a}{\curvearrowright} (Y, \mathcal{M}, \nu)$ and $\Gamma \overset{b}{\curvearrowright} (Z, \mathcal{N}, \vartheta)$ the definition of $d_{F,\alpha}(a, b)$ makes sense with the following minor change:

$$d_{F,\alpha}(a, b) := \inf_{\beta \in \mathrm{Part}_{|\alpha|}(Z)} ||\mathbf{c}(a, F, \alpha) - \mathbf{c}(b, F, \beta)||_1 \text{ for } \alpha \in \mathrm{Part}_{|\alpha|}(Y).$$

Then $d_{F,k}(a, b)$ and $\overline{d}_{F,k}(a, b)$ can be defined the same way as before. So we say that a is *weakly contained* in b, and denote it by $a \prec b$ if $d_{F,k}(a, b) = 0$ for all F and k. Respectively, we say that a and b are *weakly equivalent*, and denote it by $a \sim b$ if $a \prec b$ and $b \prec a$, that is $\overline{d}_{F,k}(a, b) = 0$ for all F and k. By Theorem 5.7 this extends our previously defined weak containment and equivalence.

5.2 Connections with local and local global convergence

In this section we explain the connection between the language of graphs and actions of a countable group Γ. Throughout this section we are going to suppose that Γ is finitely generated by a finite generating set S. In this case an action of Γ is always an action of the free group freely generated by S (which we will denote by F_S) and thus we are going to focus on actions of F_S. Our main source for this section is [1].

5.2.1 The local structure

First we take a look at the *local* structure of the actions of F_S. Let $\mathrm{SC}(S)$ denote the set of (isomorphism classes of) rooted Schreier graphs (see Definition 3.13) of the group F_S. Notice that for a given $2|S|$-regular rooted graph (G, o) we can associate a Schreier graph as follows. The well-known 2-factor theorem says that we can take $|S|$ many disjoint 2-factors (that is, 2-regular subgraphs) covering $E(G)$. Then if we direct each 2-factor in a way that each vertex x has one edge leaving x and one arriving to x, and label each 2-factor by an element of S, we get an action of F_S on $V(G)$, and the correspondig Schreier graph will be exactly the described $2|S|$-regular directed edge-labelled graph. We can put a similar topology on SC(S) as we did to $\mathfrak{G}$ in the beginning on Section 4.1.1, that is, we endow it with the rooted neighbourhood topology. An analogous version of the proof of Lemma 4.9 shows that this space is compact and totally disconnected. The free group F_S acts on SC(S) by moving the root along the corresponding edges. The notion corresponding to a unimodular random graph (see Definition 4.10) is an F_S-invariant Borel probability measure μ on SC(S), that is, $s^\star\mu = \mu$ for every $s \in S$. We will denote the set of these measures by $U(S)$. Notice that an advantage of this setting is that defining F_S-invariance was more convenient than defining involution-invariance in Definition 4.13. Another way to look at $U(S)$ is via the so called *invariant random subgroups*.

Definition 5.11 (Invariant random subgroup)**.** Let Sub$(F_S) \subseteq 2^{F_S}$ denote the space of subgroups. It is easy to see that this is a closed subset of 2^{F_S}, and therefore a standard Borel space. The group F_S acts naturally on Sub(F_S) by conjugation. We say that a Borel probability measure μ on Sub(F_S) is an *invariant random subgroup* (or IRS for short), if it is invariant under the action of F_S. That is, $c(s)^\star\mu = \mu$ for all $s \in S$, where $c(s)$ denotes conjugation by s.

It turns out that there is a translation between IRSs and $U(S)$. First, subgroups of the free group F_S and rooted Schreier graphs are in a one-to-one correspondence. Indeed, for a rooted Schreier graph (G, o) we can take the stabilzer subgroup of o, that is Stab$_{F_S}(o) = \{g \in F_S : g(o) = o\}$. To construct the inverse of this correspondence, for every subgroup $H \leq F_S$ we can take the Schreier graph associated to the action of F_S on the left cosets of H, acting by left multiplication, rooted at H. By the fact that $\mathrm{Stab}_{F_S}(s(o)) = s\,\mathrm{Stab}_{F_S}(o)s^{-1}$ we get that translating the root along the edge s becomes conjugation in the this correspondence. Therefore this map is F_S-equivariant and thus induces a bijection between elements of $U(S)$ and IRSs.

5.2.2 The local-global structure

Now we analyze the *local-global* structure of an action. The aim of this section is to argue that local-global convergence of graphs (or graphings) and weak convergence of actions are genuienly the same phenomenon. First of all, notice that by Example 4.25 we can associate a graphing $\mathcal{G}_a$ to any p.m.p action a of F_S. Notice that a graphing that is associated to an action of F_S is automatically equipped with an edge labeling. It makes sense to define the notion of local-global convergence for edge-decorated graphs the same way as we did for undecorated graphs. So what one can expect is a correspondence between the pd convergence of actions and the edge labelled local-global convergence of the associated graphings. The following turnes out to be true.

Proposition 5.12. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of actions. Then the corresponding sequence of graphings $\mathcal{G}_{a_n}$ locally-globally converges (as edge labelled graphings) if and only if the sequence $\{a_n\}_{n\in\mathbb{N}}$ pd-converges. Moreover if $a_n \to a$ in the pd metric, then the local-global limit of $\mathcal{G}_{a_n}$ is $\mathcal{G}_a$.*

Proof. The 'only if' part is easy to see. For the other part of the statement let us fix $\varepsilon > 0, r, k_0 \in \mathbb{N}$. It is enough to show the following. There is $\eta > 0$ such that for any two actions $a, b \in \mathrm{Act}(\Gamma)$ with $\mathrm{pd}(a,b) \leq \eta$ we have that $\overline{Q_{\mathcal{G}_a,r,k_0}} \subseteq B(\overline{Q_{\mathcal{G}_b,r,k_0}}, \varepsilon)$ for the corresponding graphings $\mathcal{G}_a$ and $\mathcal{G}_b$. Here $B(K,\varepsilon)$ denotes the ε neighbourhood of the compact set K, that is $\{x \in [0,1]^{U^{k_0,r}} : \exists y \in K \; |x-y| < \varepsilon\}$.

For notational simplicity, let us assume that a acts on (X,μ) and b on (Y,λ), thus $V(\mathcal{G}_a) = X$, while $V(\mathcal{G}_b) = Y$. Let us fix a coloring $c : X \to [k_0]$, then we would like to produce a coloring $d : Y \to [k_0]$ such that $||P_{\mathcal{G}_a,r,k_0}[c] - P_{\mathcal{G}_b,r,k_0}[d]||_1 < \varepsilon$. First, we detail the construction of d and later prove examine its properties. We start by modifying c to a k-coloring c_0 such that c_0 encodes c, moreover whenever the graph distance of $x, y \in X$ is at most $2r$, we have that $c_0(x) \neq c_0(y)$. Now we create the coloring c_1, such that $c_1(x)$ encodes the isomorphism type of the colored and edge labelled r-neighbourhood of x. Now let us denote the partition of X according to the c_1 colors of the points by $\alpha = \{A_U : U \in U_l^{r,k}\}$, where $U_l^{r,k}$ denotes the isomorphism classes of the k-colored and edge labelled rooted graphs with radius at most r. Let F denote those words in F_S which have lenght at most $2r$. By $\mathrm{pd}(a,b) \leq \eta$ we know that there is a partition $\beta = \{B_U : U \in U_l^{r,k}\}$ such that

$$||\mathbf{c}(a,F,\alpha) - \mathbf{c}(b,F,\beta)||_1 < \delta, \tag{5.2}$$

where δ will be choosen later. Now we define $d_0(x)$ to be the c_0 color of the root of U, where $x \in B_U$. We show that by choosing δ sufficiently small we get that $||P_{\mathcal{G}_a,r,k}[c_0] - P_{\mathcal{G}_b,r,k}[d_0]||_1 < \varepsilon$.

First we prove that there is a set $Y' \subseteq Y$ such that $d_0(y) \neq d_0(\gamma^b(y))$ for every $y \in Y'$ and $\gamma \in F$, moreover $\lambda(Y') > 1-\delta$. Indeed, let $y \in Y$ be a point for which $d_0(y) = d_0(\gamma^b(y))$ for some $\gamma \in F$. By definition, there is $U, V \in U_l^{r,k}$ for which the root of U and the root of V have the same color, moreover $y \in B_U$ and $\gamma^b(y) \in B_V$. Then $y \in B_U \cap (\gamma^{-1})^b(B_V)$. By the property of the coloring c_0 that close points have different colors we know that $A_U \cap (\gamma^{-1})^a(A_V) = \emptyset$, and therefore $\mu(A_U \cap (\gamma^{-1})^a(A_V)) = 0$. By (5.2) we get that the set $\bigcup_{U,V\in U_l^{r,k},\gamma\in F}(B_U \cap (\gamma^{-1})^b(B_V))$ has measure smaller than δ.

Now we claim that the modulo a set of small measure, the β-label of the points in Y' is the same as the actual d_0-colored r-neighbourhood of the points. In particular, we show that

$$\lambda\left(\{y \in Y' : \exists U \; y \in B_U \text{ and } N_{\mathcal{G}_b,r,d_0}(y) \cong U\}\right) > 1 - 5\delta. \tag{5.3}$$

Indeed, fix $y \in Y'$ from the complement. We are going to denote by r_U the root of U, and use the notation $\gamma(r_U)$ for the corresponding vertices in U. Then either there is $\gamma \in F$ such that $\gamma^b(y) = y$ but $\gamma(r_U) \neq r_U$, or there is $\gamma \in F$ such that $\gamma^b(y) \neq y$ but $\gamma(r_U) = r_U$, or there is U such that $y \in B_U$, but $\gamma^b(y)$ does not have the same d_0-color as $\gamma(r_U)$. In the first case we know that the color of r_U and $\gamma(r_U)$ are different. In particular $A_U \cap (\gamma^{-1})^a A_U = \emptyset$, and therefore $\mu(A_U \cap (\gamma^{-1})^a(A_U)) = 0$. On the other hand $B_U \cap (\gamma^{-1})^b(B_U)$. By taking the union for all possible U we get that these points have measure smaller than δ.

In the second case by the fact that $y \in Y'$ we know that $d_0(\gamma^b(y)) \neq d_0(y)$. Therefore if $\gamma^b(y) \in B_V$, then $U \not\cong V$. Therefore $y \in B_U \setminus (\gamma^{-1})^b(B_U)$. On the other hand $A_U \cap (\gamma^{-1})^a(A_U) = A_U$, and thus $\mu(A_U \cap (\gamma^{-1})^a(A_U)) = \mu(A_U)$. This implies that $\sum_{U:\gamma(r_U)=r_U} |\mu(B_U \cap (\gamma^{-1})^b(B_U)) - \mu(B_U)| < 2\delta$, and thus $\mu(\bigcup_{U:\gamma(r_U)=r_U} B_U \setminus (\gamma^{-1})^b(B_U)) < 2\delta$.

In the third case there is $V \in U_l^{r,k}$ such that $\gamma^b(y) \in B_V$, and the color of the root in V does not have the same color as $\gamma(r_U)$ in U. Then $y \in B_U \cap (\gamma^{-1})^a B_V$. By definition $A_U \cap (\gamma^{-1})^a A_V = \emptyset$, thus $\mu(A_U \cap (\gamma^{-1})^a A_V) = 0$. We get again that the set of these points has measure smaller than δ, which proves (5.3).

Now to finish the proof let us denote by $Y'' = \{y \in Y' : \exists U \; y \in B_U \text{ and } N_{\mathcal{G}_b,r,d_0}(y) \cong U\}$. Then for every U we have that $P_{\mathcal{G}_a,r,k}[c_0](U) = \mu(A_U)$. On the other hand from (5.3) we have $|P_{\mathcal{G}_b,r,k}[d_0](U) - \mu(B_U)| < 5\delta$. Taking this for all $U \in U^{r,k}$ we can conclude that $||P_{\mathcal{G}_a,r,k}[c_0] - P_{\mathcal{G}_b,r,k}[d_0]||_1 < \varepsilon$ if δ is sufficiently small. If $c = f \circ c_0$, then for $d = f \circ d_0$ we have that $||P_{\mathcal{G}_a,r,k_0}[c] - P_{\mathcal{G}_b,r,k_0}[d]||_1 < \varepsilon$. □

5.3 The limit of actions

Finally, in this section we are going to prove the remaining part of Theorem 5.2, namely that after identifying the pd zero-classes in $\mathrm{Act}(\Gamma)$, we get a compact space. The main idea is that for a fixed sequence of actions $(a_n)_{n\in\mathbb{N}}$ we build an ultraproduct measure space (which will not be a standard probability space) and a measure preserving action on it. After that we find an action a on X_0 that is weakly equivalent to the ultraproduct action, and this a will be the limit of a subsequence of $(a_n)_{n\in\mathbb{N}}$. First let us build the ultralimit space. In Section 3.1 we have constructed the *ultraproduct measure space* associated to a countable sequence of probability spaces.

Notation 5.13. We fix the following notation for the rest of Section 5.3. Let $(X_{\mathcal{U}}, \mathcal{B}_{\mathcal{U}}, \mu_{\mathcal{U}})$ denote the ultraproduct measure space associated to the sequence $(X_n, \mathcal{B}_n, \mu_n)$, where each element denotes the standard probability space $(X_0, \mathcal{B}, \lambda)$ that we fixed in the beginning of Section 5. Moreover let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of actions in $\mathrm{Act}(\Gamma)$, where we look at a_n as a p.m.p. action of $(X_n, \mathcal{B}_n, \mu_n)$. The following is the natural action $a_{\mathcal{U}}$ of Γ on $(X_{\mathcal{U}}, \mathcal{B}_{\mathcal{U}}, \mu_{\mathcal{U}})$associated to the sequence $\{a_n\}_{n\in\mathbb{N}}$:

$$\gamma^{a_{\mathcal{U}}}([x_n]_{\mathcal{U}}) := [\gamma^{a_n}(x_n)]_{\mathcal{U}}.$$

Notice that of course for different sequences we get different actions $a_{\mathcal{U}}$. For notational simplicity, we omit specifying the sequence in the notation to which $a_{\mathcal{U}}$ belongs. When we talk about $a_{\mathcal{U}}$ without specifying the original sequence, we always refer to an action $a_{\mathcal{U}}$ associated to an arbitrary sequence of actions from $\mathrm{Act}(\Gamma)$.

The action $a_{\mathcal{U}}$ is p.m.p. Indeed, by Proposition 3.18 it is enough to check that for every $\gamma \in \Gamma$ and $A_n \in \mathcal{B}_n$ we have that $\mu_{\mathcal{U}}(\gamma^{a_{\mathcal{U}}}([A_n]_{\mathcal{U}})) = \mu_{\mathcal{U}}([A_n]_{\mathcal{U}})$, which is clear by the following calculation:

$$\mu_{\mathcal{U}}(\gamma^{a_{\mathcal{U}}}([A_n]_{\mathcal{U}})) = \mu_{\mathcal{U}}([\gamma^{a_n}(A_n)]_{\mathcal{U}}) = \lim_{n\in\mathcal{U}} \mu_n(\gamma^{a_n}(A_n)) = \lim_{n\in\mathcal{U}} \mu_n(A_n) = \mu_{\mathcal{U}}([A_n]_{\mathcal{U}}).$$

As mentioned in the introduction, we will now prove that there is a p.m.p. action of the group Γ on a standard Borel space which is weakly equivalent to the action $a_{\mathcal{U}}$ on $(X_{\mathcal{U}}, \mathcal{B}_{\mathcal{U}}, \mu_{\mathcal{U}})$. (Originally, we defined weak equivalence for actions on the same space, for clarification see Remark 5.10.) First let us notice in the following lemma that the ultraproduct space is indeed not standard, so it makes sense to prove Theorem 5.15.

Lemma 5.14. *The space $(X_{\mathcal{U}}, \mathcal{B}_{\mathcal{U}}, \mu_{\mathcal{U}})$ is not a standard probability space.*

Proof. It is well known that in the probability space $(X_0, \mathcal{B}, \lambda)$ there are measurable sets $A_1, \ldots, A_n \ldots$ such that $\lambda(A_i \triangle A_j) = \frac{1}{2}$ (in fact, this follows from Remark 3.8 and the observation that in $2^{\mathbb{N}}$ the sequence $A_n := \{x \in 2^{\mathbb{N}} : x(n) = 1\}$ possesses this property). Then let us consider the following family of measurable subsets of the ultraproduct:

$$\mathcal{A} := \{[A_{f(n)}]_{\mathcal{U}} : \text{ where } f \text{is a } \mathbb{N} \to \mathbb{N} \text{ function for which } f(k+1) \in \{2k, 2k+1\}\ \forall k \in \mathbb{N}\}.$$

Notice that there are continuum many different such functions, and if for $g, h : \mathbb{N} \to \mathbb{N}$ with the above property there is n such that $g(n) \neq h(n)$, then for every $m \geq n$ we have $g(m) \neq h(m)$. Thus for two measurable sets $[A_{g(n)}]_{\mathcal{U}}, [A_{h(n)}]_{\mathcal{U}} \in \mathcal{A}$ with $g \neq h$ we have that

$$\mu_{\mathcal{U}}([A_{g(n)}]_{\mathcal{U}} \triangle [A_{h(n)}]_{\mathcal{U}}) = \mu_{\mathcal{U}}([A_{g(n)} \triangle A_{h(n)}]_{\mathcal{U}}) = \lim_{n\in\mathcal{U}} \mu_n(A_{g(n)} \triangle A_{h(n)}) = \frac{1}{2}.$$

But it is also well known that for a standard probability measure space its measure algebra $MALG_\mu$ is separable when equipped with the usual distance mentioned in Notation 3.9 (see e.g. Exercise 17.43 in [20]), therefore it cannot contain continuum many set which have pairwise distance $\frac{1}{2}$. Consequently, the ultraproduct measure space is not standard. □

Theorem 5.15. *There is an action a of Γ on the standard Borel space $(X_0, \mathcal{B}, \lambda)$ which is weakly equivalent to the action $a_\mathcal{U}$ of Γ on $(X_\mathcal{U}, \mathcal{B}_\mathcal{U}, \mu_\mathcal{U})$.*

Before we prove the theorem, let us introduce the following definition of a factor of an action.

Definition 5.16. Let a be a p.m.p. action of Γ on the space $(Y, \mathcal{M}, \nu)$. We say that an action b on $(Z, \mathcal{N}, \vartheta)$ is a *factor* of a, if there is a measurable, measure preserving map $\pi : Y \to Z$ which is Γ-equivariant, that is, $\pi \circ \gamma^a = \gamma^b \circ \pi$ for every $\gamma \in \Gamma$.

Observation 5.17. Notice that if b is a factor of a, then $b \prec a$. By definition, we have to show that $d_{F,k}(b, a) = 0$ for every $F \subset \Gamma$ finite and $k \in \mathbb{N}$. If a acts on $(Y, \mathcal{M}, \nu)$, b on $(Z, \mathcal{N}, \vartheta)$ and the factor function is π, then for any partition β of Z we can take the pullback of β by π, that is $\alpha := \{\pi^{-1}(B_i) : B_i \in \beta\}$. It is easy to see that for all $F \subset \Gamma$ finite we have that $||\mathbf{c}(a, F, \alpha) - \mathbf{c}(b, F, \beta)||_1 = 0$, and thus $d_{F,k}(b, a) = 0$ for every $F \subset \Gamma$ finite and $k \in \mathbb{N}$.

Proof of Theorem 5.15. We are going to construct a standard factor of $a_\mathcal{U}$ that is going to contain $a_\mathcal{U}$ weakly. Notice that it suffices to show this, since by Observation 5.17 $a_\mathcal{U}$ automatically contains the factor weakly. We begin with the construction.

For every fixed $k \in \mathbb{N}$, $F \subset \Gamma$ finite, and a partition $\alpha \in \mathrm{Part}_k(X_\mathcal{U})$ we can think of the vector $\mathbf{c}(a_\mathcal{U}, F, \alpha)$ as an element of the space $[0, 1]^{k^2|F|}$. Let us consider the following subset of $[0, 1]^{k^2|F|}$:

$$S_{F,k} := \{\mathbf{c}(a_\mathcal{U}, F, \alpha) : \alpha \in \mathrm{Part}_k(X_\mathcal{U})\}.$$

Then for every k and F let us fix a countable collection of partitions of $X_\mathcal{U}$, denoted by $\mathcal{A}_{F,k}$ such that $\{\mathbf{c}(a_\mathcal{U}, F, \alpha) : \alpha \in \mathcal{A}_{F,k}\}$ is a dense subset of $S_{F,k}$. Moreover let

$$\mathcal{A} := \{\gamma(\alpha) : \gamma \in \Gamma,\ \alpha \in \mathcal{A}_{F,k} \text{ for some } F \subset \Gamma \text{ finite, } k \in \mathbb{N}\},$$

where $\gamma(\alpha)$ denotes the pushforward of α by γ, that is, if $\alpha = \{A_1, \ldots, A_k\}$, then $\gamma(\alpha) := \{\gamma(A_1), \ldots, \gamma(A_k)\}$. Clearly, $\mathcal{A}$ is a countable family of finite partitions, thus the following set is countable:

$$I = \{A \in \mathcal{B}_\mathcal{U} : A \in \alpha \text{ for some } \alpha \in \mathcal{A}\}.$$

We are going to define the factor action on the set 2^I, which is a standard Borel space with the usual topology. So let $\pi : X_{\mathcal{U}} \to 2^I$ be defined as follows:

$$\pi(x)(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Moreover let us define μ on 2^I to be the pushforward measure $\pi^\star \mu_{\mathcal{U}}$ (restricted to the Borel subsets of 2^I). Clearly μ is a continuous Borel probability measure on 2^I. Finally we define the action a of Γ on 2^I as $\gamma^a(y)(A) = y((\gamma^{-1})^{a_{\mathcal{U}}}(A))$. One can easily check that this is indeed a μ-measure preserving action. Now we show that for this action a the function π is indeed a factor map. Since the measure preserving property follows by the definition of the pushforward, we just have to justify the Γ-equivariance. This is shown by the following calculation:

$$\gamma^a(\pi(x))(A) = 1 \iff x \in (\gamma^{-1})^{a_{\mathcal{U}}}(A) \iff \gamma^{a_{\mathcal{U}}}(x) \in A \iff \pi(\gamma^{a_{\mathcal{U}}}(x))(A) = 1,$$

thus $\gamma^a \circ \pi = \pi \circ \gamma^{a_{\mathcal{U}}}$. Now we prove that $a_{\mathcal{U}} \prec a$. By definition, we have to show that $d_{F,\alpha}(a_{\mathcal{U}}, a) = 0$ for every $F \subset \Gamma$ finite and every $\alpha \in \mathrm{Part}_k(X_{\mathcal{U}})$. Let us fix $\varepsilon > 0$ arbitrarily. By the definition of $\mathcal{A}_{F,k}$, there is a partition $\{B_1, \ldots B_k\} = \beta \in \mathcal{A}_{F,k}$ such that $||\mathbf{c}(a_{\mathcal{U}}, F, \alpha) - \mathbf{c}(a_{\mathcal{U}}, F, \beta)||_1 < \varepsilon$. Consider the sets $C_i = \{y \in 2^I : y(B_i) = 1\}$ for all $1 \leq i \leq k$. First of all noice that $\eta = \{C_1, \ldots, C_k\}$ is a partition of 2^I, since $\mu(C_i \cap C_j) = \mu_{\mathcal{U}}(\pi^{-1}(C_i) \cap \pi^{-1}(C_j)) = \mu_{\mathcal{U}}(B_i \cap B_j) = 0$. Now we claim that $\mathbf{c}(a_{\mathcal{U}}, F, \beta) = \mathbf{c}(a, F, \eta)$. Indeed, for any $\gamma \in F$ we have $\gamma^a(C_j) = \{y \in 2^I : y(\gamma^{a_{\mathcal{U}}}(B_j)) = 1\}$, and therefore

$$\pi^{-1}(\gamma^a(C_j)) = \gamma^{a_{\mathcal{U}}}(B_j).$$

Using this we get that

$$\mu(C_i \cap \gamma^a(C_j)) = \mu_{\mathcal{U}}(\pi^{-1}(C_i) \cap \pi^{-1}(\gamma^a(C_j))) = \mu_{\mathcal{U}}(B_i \cap \gamma^{a_{\mathcal{U}}}(B_j)),$$

and therefore $||\mathbf{c}(a_{\mathcal{U}}, F, \alpha) - \mathbf{c}(a, F, \eta)||_1 = ||\mathbf{c}(a_{\mathcal{U}}, F, \alpha) - \mathbf{c}(a_{\mathcal{U}}, F, \beta)||_1 < \varepsilon$, from which $d_{F,\alpha}(a, a_{\mathcal{U}}) = 0$ follows. To conclude Theorem 5.15 we refer to the isomorphism theorem of continuous standard Borel measures (Remark 3.8), and thus we can replace the action a on $(2^I, \mathcal{B}(2^I), \mu)$ by an isomorphic action a' on $(X_0, \mathcal{B}, \lambda)$. □

Remark 5.18. The following is another equivalent definition of factors. Let a be a p.m.p action of Γ on $(Y, \mathcal{M}, \nu)$. An action b of Γ on $(Z, \mathcal{N}, \vartheta)$ is a *factor* of a if if there is a Γ-equivariant embedding $\mathrm{MALG}_\vartheta \hookrightarrow \mathrm{MALG}_\nu$. It is easy to see that the former definition of factor implies this notion. Notice that in the proof of Theorem 5.15 we have proved the converse.

Finally we show that for a fixed sequence of actions $a_n \in \mathrm{Act}(\Gamma)$ the sequence pd-converges along $\mathcal{U}$ to the associated action on the ultraproduct in the following sense. From Proposition 5.19 we will get Theorem 5.2 as a corollary.

Proposition 5.19. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of actions in $\mathrm{Act}(\Gamma)$, $\mathcal{U}$ be the fixed ultrafilter and $a_{\mathcal{U}}$ be the action associated to the sequence $\{a_n\}_{n\in\mathbb{N}}$. Then for every $k \in \mathbb{N}$ and $F \subset \Gamma$ finite both $\lim_{n\in\mathcal{U}} d_{k,F}(a_{\mathcal{U}}, a_n) = 0$ and $\lim_{n\in\mathcal{U}} d_{k,F}(a_n, a_{\mathcal{U}}) = 0$ hold.*

Proof. Let us fix k and F. For calculating the first limit, we have to show that $\{n \in \mathbb{N} : d_{F,k}(a_{\mathcal{U}}, a_n) < \varepsilon\} \in \mathcal{U}$ for every $\varepsilon > 0$. In the following claim we prove that it suffices to show this for the functions $d_{F,\alpha}$.

Claim. It is enough to show that $\{n \in \mathbb{N} : d_{F,\alpha}(a_{\mathcal{U}}, a_n) < \varepsilon\} \in \mathcal{U}$, where α is from $\mathrm{Part}_k(X_{\mathcal{U}})$.

Proof. Indeed, let P denote the set $\{\mathbf{c}(a_{\mathcal{U}}, F, \alpha) \in [0,1]^{|F|k^2} : \alpha \in \mathrm{Part}_k(X_{\mathcal{U}})\}$. Using that the space $[0,1]^{|F|k^2}$ is compact, we can take a $\frac{\varepsilon}{2}$-net $\{x_1, \dots, x_m\} \subseteq P$, where $x_i = \mathbf{c}(a_{\mathcal{U}}, F, \alpha_i)$. Now from the fact that for every $\mathbf{c}(a_{\mathcal{U}}, F, \alpha) \in P$ there is j such that $||\mathbf{c}(a_{\mathcal{U}}, F, \alpha) - \mathbf{c}(a_{\mathcal{U}}, F, \alpha_j)||_1 < \frac{\varepsilon}{2}$, it follows that

$$\bigcap_{j=1}^{m} \{n \in \mathbb{N} : d_{F,\alpha_j}(a_{\mathcal{U}}, a_n) < \frac{\varepsilon}{2}\} \subseteq \{n \in \mathbb{N} : d_{F,k}(a_{\mathcal{U}}, a_n) < \varepsilon\}.$$

But the former set is a finite intersection, so if all the sets $\{n \in \mathbb{N} : d_{F,\alpha_j}(a_{\mathcal{U}}, a_n) < \frac{\varepsilon}{2}\}$ are in $\mathcal{U}$, so is $\{n \in \mathbb{N} : d_{F,k}(a_{\mathcal{U}}, a_n) < \varepsilon\}$. This concludes the proof of the claim.

So let us fix $\alpha \in \mathrm{Part}_k(X_{\mathcal{U}})$ too. From Proposition 3.18 we may suppose that α is of the form $\{[A_n^1]_{n\in\mathcal{U}}, \dots [A_n^k]_{n\in\mathcal{U}}\}$. By definition for every $i, j \leq k$ and $\gamma \in \Gamma$ we have

$$\mu_{\mathcal{U}}([A_n^i]_{n\in\mathcal{U}} \cap \gamma^{a_{\mathcal{U}}}([A_n^j]_{n\in\mathcal{U}})) = \lim_{n\in\mathcal{U}} \mu_n(A_n^i \cap \gamma^{a_n}(A_n^j)). \tag{5.4}$$

Let us denote the partition $\{A_n^1, \dots A_n^k\} \in \mathrm{Part}_k(X_n)$ by α_n, then from (5.4) we have that

$$U_{i,j} := \left\{ n \in \mathbb{N} : |\mu_n(A_n^i \cap \gamma^{a_n}(A_n^j)) - \mu_{\mathcal{U}}([A_n^i]_{n\in\mathcal{U}} \cap \gamma^{a_{\mathcal{U}}}([A_n^j]_{n\in\mathcal{U}}))| < \frac{\varepsilon}{|F|k^2} \right\} \in \mathcal{U}.$$

Since $\bigcap_{i,j\leq k} U_{i,j} \subseteq \{n \in \mathbb{N} : \mathbf{c}(a_n, F, \alpha_n) - \mathbf{c}(a_{\mathcal{U}}, F, \alpha) < \varepsilon\}$, the latter is in $\mathcal{U}$ as well, which concludes the proof of $d_{F,k}(a_{\mathcal{U}}, a_n) = 0$.

For the other part let us suppose to the contrary that $\lim_{n\in\mathcal{U}} d_{F,k}(a_n, a_{\mathcal{U}}) > \varepsilon$ for some $\varepsilon > 0$ (notice that $d_{F,k}(a_n, a_{\mathcal{U}})$ is bounded, thus the limit exist). Then there is $U \in \mathcal{U}$ such that for every $n \in U$ there is a partition $\alpha_n = \{A_n^1, \dots A_n^k\}$ for which we have $d_{F,\alpha_n}(a_n, a_{\mathcal{U}}) > \varepsilon$. Therefore specifically for the partition $\alpha = \{[A_n^1]_{n\in\mathcal{U}}, \dots [A_n^k]_{n\in\mathcal{U}}\}$ of $X_{\mathcal{U}}$ we have $||\mathbf{c}(a_n, F, \alpha_n) - \mathbf{c}(a_{\mathcal{U}}, F, \alpha)||_1 \geq d_{F,\alpha_n}(a_n, a_{\mathcal{U}}) > \varepsilon$ for every $n \in U$. Thus we can conclude that $\lim_{n\in\mathcal{U}} ||\mathbf{c}(a_n, F, \alpha_n) - \mathbf{c}(a_{\mathcal{U}}, F, \alpha)||_1 \geq \varepsilon$. On the other hand this contradicts the fact that $\mu_{\mathcal{U}}([A_n^i]_{n\in\mathcal{U}} \cap \gamma^{a_{\mathcal{U}}}([A_n^j]_{n\in\mathcal{U}})) = \lim_{n\in\mathcal{U}} \mu_n(A_n^i \cap \gamma^{a_n}(A_n^j))$ for every $i, j \leq k$. $\square$

Now we are ready to prove the remaining part of Theorem 5.2.

Proof of Theorem 5.2. The first part was proved in Theorem 5.7. So we have to show that $\mathrm{Act}(\Gamma) / \sim$ (see Notation 5.8) is a compact space when we equip it with the metric topology we get from the metric pd. It is well known that sequential compactness and compactness are equivalent in metric spaces and thus it is enough to show the latter. So let us fix a sequence of actions $\{a_n\}_{n\in\mathbb{N}} \in \mathrm{Act}(\Gamma) / \sim$. Let us pick an element $\widetilde{a_n} \in \mathrm{Act}(\Gamma)$ from the preimage of a_n for every n, moreover let $a_{\mathcal{U}}$ denote the ultraproduct action associated to the sequence $\{\widetilde{a_n}\}_{n\in\mathbb{N}}$. Using Theorem 5.15 we get an action $\tilde{a} \in \mathrm{Act}(\Gamma)$ such that $\overline{d}_{k,F}(\tilde{a}, a_{\mathcal{U}}) = 0$ for every $F \subset \Gamma$ finite and $k \in \mathbb{N}$. Let a be the $\sim$-equivalence class of $\tilde{a}$. By Proposition 5.19 we know that $\lim_{n\in\mathcal{U}} d_{k,F}(a_{\mathcal{U}}, a_n) = 0$ and $\lim_{n\in\mathcal{U}} d_{k,F}(a_n, a_{\mathcal{U}}) = 0$, and therefore (using the extended version of Lemma 5.6) we get that $\lim_{n\in\mathcal{U}} \overline{d}_{k,F}(a, a_n) = 0$. This exactly means that for every $\varepsilon > 0$ there is $U \in \mathcal{U}$ such that $\overline{d}_{F,k}(a, a_n) < \varepsilon$ for every $n \in U$. But since $\mathcal{U}$ is a non-principal ultrafilter, for every $\varepsilon > 0$ there are infinitely many elements in the ε-neighbourhood of a, and thus we can choose a subsequence of a_n which converges to a. $\square$

6 Outlook

As mentioned in the introduction, this thesis is a stepping stone towards future research. Recently Lovász in [25] initiated the study of matroids from an analytical point of view. One can capture the matroids through their rank function, which leads to the study of submodular functions. Motivated by the definition of local-global convergence of bounded degree graphs the authors in [6] introduced the quotient-convergence of submodular functions. They prove that the space of increasing submodular functions under the quotient-convergence is complete. This result is analogous to Theorem 4.29. As we highlighted earlier, the statement of Theorem 5.2 is paralell to Theorem 4.29, while the proof relies on a different technique, namely on the ultraproduct method. We hope that by understanding the concepts and proofs of this thesis, we will also be able to approach the following question.

Question 6.1. Is it possible to conclude the completeness of the quotient-convergence of increasing submodular functions from the ultraproduct method?

We also mention a couple of other questions we intend to pursue that arose during the preparation of this thesis.

Question 6.2. Is there a graphing $\mathcal{G}$ on a standard probability space for which the sets $Q_{\mathcal{G},r,k}$ are closed?

The ultraproduct technique produces such graphings on non-standard spaces, but it is not clear if this property can be inherited by a standard factor.

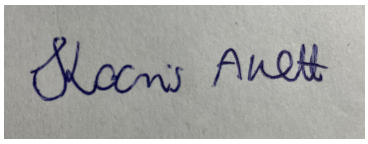
Question 6.3. Does a more careful version of the proof of Proposition 5.12 also show that local-global equivalence is already witnessed by the colored 1-neighborhoods? This is an unpublished result of Szegedy, see [3, Theorem 9.1]

Alulírott Kocsis Anett nyilatkozom, hogy szakdolgozatom elkészítése során az alább felsorolt feladatok elvégzésére a megadott MI alapú eszközöket alkalmaztam:

Feladat	**Felhasznált eszköz**	**Felhasználás helye**	**Megjegyzés**
Nyelvtani helyeség ellenőrzése	ChatGPT	A teljes dokumentum	-

A felsoroltakon túl más MI alapú eszközt nem használtam.

Budapest, 2025. június 1.



Aláírás

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