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Drinfeld's Lemma for Perfectoid Spaces

Thesis

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Contents

Introduction

0.1. **Global Langlands Correspondence.**

The motivation for most of modern number theory originates from the Langland's correspondence. Let K be a global field and $\mathbb{A}_K$ be it's ring of adéles. Let G be the split reductive group over K (eg: GL_n, GSp_{2g}, ..., etc.). In 1970 Robert Langlands proposed the following conjecture.

Conjecture 1 (Global Langlands Correspondence)**.** *There is a functorial bijection*

$$\left\{ \begin{array}{c} \textit{cuspidal automorphic} \\ \textit{representations of } G(\mathbf{A}_K) \end{array} \right\} /\simeq \quad \overset{LC}{\longleftrightarrow} \quad \left\{ \begin{array}{c} \textit{irreducible Galois} \\ \textit{representations of } G_K \textit{ in} \\ {}^L G(\bar{\mathbf{Q}}_\ell) \end{array} \right\} /\simeq$$

such that Hecke eigensystems on the left hand side correspond to the Frobenius eigensystems on the right hand side. Here ${}^L G$ is the Langland's dual group of G (eg: For GL_n, it is it's self Langlands dual, ${}^L GSp_{2g} = {}^L GSp_{2g+1}$ and so on...)

In other words, Langlands conjecture suggests that one can read the information related to it's automorphic L-functions (depending entirely on the Hecke eigensystems on the objects on the left-hand side) from Artin L-functions (depending entirely on the Frobenius eigensystems on the objects on the right-hand side).

In case of $G = \mathrm{GL}_1$ this conjecture is well known as this is the case of class field theory. However very little is known in case when K is any integer and G is a higher rank group. In comparison to the case of number fields, surprisingly much more is known for the case when K is a function field. The results are due to the work of V.G Drinfeld, L.Lafforgue and V. Lafforgue which we summarise as follows

- Around 1980 - Drinfeld proved the case for GL_2.
- Around 2002 - L. Lafforgue extended the methods of Drinfeld and proved it for the case of GL_n.
- Around 2018 - V. Lafforgue established the result for reductive groups

The idea is to replicate the methods for function field in the case of number field. We discuss more concretely what happens in the function field case.

0.2. **The Case of Characteristic p.**

We fix a rational prime number p. Let X be a smooth projective curve over $\mathbb{F}_p$ and let K be the function field of X. We write $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$. There are two main ingredients in the recipe of Global Langlands correspondence: namely

- Drinfeld's Lemma
- Drinfeld's Shtukas

Drinfeld's Lemma. Let $U \subset X$ be an open dense subset. Then the étale fundamental group under the quotient of G_K. Let $m \in \mathbb{N}$ and we consider $U^m \subset X^m$. Drinfeld's lemma establishes a relation of the étale fundamental group $\pi_1(U^m)$ to the fundamental group $\pi_1(U)$.

We first discuss what do we mean by partial Frobenii on X^m. First we recall that X is of characteristic p and it admits a Frobenius

$$\begin{aligned} \mathrm{Frob} : X &\to X \\ x &\mapsto x^p \end{aligned}$$

Then for $i = 1, 2, 3, \ldots m$ we have the partial Frobenii given by

$$\begin{aligned} \mathrm{Fr}_i : \mathrm{id}_X \times \ldots \mathrm{Frob}_X \times \ldots \mathrm{id}_X \\ x \mapsto (x, x, \ldots, x^p, x, \ldots, x) \end{aligned}$$

where we have m-tuples but the i^{th} component is the Frobenius. We define $(U^m/\text{partial Frobenii})_{\text{fét}}$ to be the category of finite étale morphisms $Y \to U^m$ equipped with commuting homomorphisms $\beta_i : Y \xrightarrow{\sim} \mathrm{Fr}_i^* Y$ such that $\beta_m \circ \beta_{m-1} \circ \ldots \beta_1$ is the relative Frobenius

$$\mathrm{Frob}_{Y/U^m} \xrightarrow{\sim} (\mathrm{Frob}_{X^m|U^m})Y$$

Lemma 1 (Drinfeld)**.** *We have a natural isomorphism*

$$\pi_1(U^m/\textit{partial Frobeni}) \simeq (\pi_1(U))^m$$

If l is a prime number different from p and $\mathbb{L}$ is a $\bar{\mathbb{Q}}_l$ local system of U^m equipped with commuting homomorphisms $\mathrm{Fr}_i^* \mathbb{L} \simeq \mathbb{L}$, then Drinfeld's Lemma says that $\mathbb{L}$ determines a representation of $\pi_1(U^m)$. Since $\pi_1(U^m)$ is the quotient of G_K- the absolute Galois group, L determines a representation of $(G_K)^m$.

Drinfeld Shtukas.

The notion of shtukas was first developed by Drinfeld for GL_2, which was generalised to GL_n by L. Lafforgue and later to reductive groups by V. Lafforgue.

Definition 1. *A **shutka** of rank n over an $\mathbb{F}_p$-scheme S is a pair $(\varepsilon, \varphi_\varepsilon)$ such that*

- *ε is a rank n vector bundle on $S \times_{\mathbb{F}_l} X$*
- *$\varphi_\varepsilon : Frob_S^* \varepsilon \dashrightarrow \varepsilon$ is a meromorphic isomorphism defined on an open subset $U \subset S \times_{\mathbb{F}_l} X$, which is fibrewise-dense in X*

where $Frob_S$ stands for the product of the Frobenius morphism on S and the identity morphism on X.

Let k be a closed field of characteristic p, and consider $S = \mathrm{Spec}(k)$. There are several information that can be encoded into a shtuka $(\varepsilon, \varphi_\varepsilon)$ over S:

- The collection of points $x_1, x_2, \ldots, x_n \in X(k)$ where φ_ε is undefined. These points are referred as the legs of the shtuka.
- For each $i = 1, 2, \ldots m$, a conjugacy characters μ_i of cocharacters $\mathbb{G}_m \to \mathrm{GL}_n$, encoding the behavior of φ_ε near to x_i.
- Level structures depending on choices of effective divisors.

By fixing m an ordered collection of conjugacy classes of cocharacters $\{\mu_1, \mu_2, \ldots \mu_n\}$ and a level structure N, we can define a moduli space $\mathrm{Sht}_{\mathrm{GL}_n, \{\mu_1, \mu_2, \ldots \mu_n\}, N}$, whose k-points classifies tuples $(\varepsilon, \varphi_\varepsilon, \{x_i\}, \psi_N)$, where we have

- $(\varepsilon, \varphi_\varepsilon)$ is shtuka of rank n over k with legs $x_1, x_2, \ldots x_m$
- the relative position of $\varepsilon^\wedge_{x_i}$ and $(\mathrm{Frob}_S^* \varepsilon)^\wedge_{x_i}$ for each x_i bounded by the cocharacter μ_i
- ψ_N is an N-level structure on $(\varepsilon, \varphi_\varepsilon)$.

Consequently, we have a forgetful morphism

$$f_N : \mathrm{GL}_{n, \{\mu_1, \mu_2, \ldots \mu_n\}, N} \to X^m, \quad (\varepsilon, \varphi_\varepsilon, \{x_i\}, \psi_N) \to \{x_1, x_2, \ldots, x_n\} \tag{1}$$

A few important characteristics of this

- The moduli space $\mathrm{Sht}_{\mathrm{GL}_{n, \{\mu_1, \mu_2, \ldots \mu_n\}, N}}$ is a Deligne-Mumford stack. It lives in a nice category where one can study it's étale site and it's étale cohomology.
- Namely, there are two actions on $\mathrm{Sht}_{\mathrm{GL}_{n, \{\mu_1, \mu_2, \ldots \mu_n\}, N}}$
  - An action of $\mathrm{GL}_n(\mathcal{O}_n)$ depending on the effective divisor N. Here the algebraic group $\mathrm{GL}_n(\mathcal{O}_n)$ appears since we are considering the vector bundles ε of rank n.
  - Partial Frobenii's F_i.

Let $d = \dim(\mathrm{Sht}_{\mathrm{GL}_{n, \{\mu_1, \mu_2, \ldots \mu_n\}, N}}) - m$, be the relative dimension of $\mathrm{Sht}_{\mathrm{GL}_{n, \{\mu_1, \mu_2, \ldots \mu_n\}, N}}$ and let $l \neq p$ be a prime number. We consider the middle degree cohomology $R^d f_{N,!} \bar{\mathbb{Q}}_l$, which is an étale $\bar{\mathbb{Q}}_l$-sheaf on X^m. Combining the two actions along with Drinfeld's lemma and Langlands correspondence we get (atleast expect to get) a decomposition of $\mathrm{GL}_n(\mathbb{A}_K) \times G_K^m$-representations

$$\varinjlim_{\text{level structure } N} R^d f_{N,!} \bar{\mathbb{Q}}_l = \bigoplus_{\pi : \text{cuspidal automorphic representations of } \mathrm{GL}_n(\mathbb{A}_K)} \pi \oplus (r_1 \circ \mathrm{LC}(\pi) \oplus \ldots r_m \circ \mathrm{LC}(\pi)) \tag{2}$$

where

- $\mathrm{LC}(\pi)$ is the Galois representation associated to π in the Langland's correspondence.
- r_i's are some algebraic representation depending on μ_i's respectively.

Theorem 1 (Drinfeld, V.Lafforgue, L.Lafforgue)**.** *The decomposition (2) holds.*

0.3. **The mixed characteristic case.**

Motivated by the case of function fields, one immediately sees that it would be helpful to establish the theory of moduli space of shtukas in the number field case. We can consider $\operatorname{Spec}\mathbb{Z}$, which is of dimension 1. In doing so, we can see the modulii space of shtukas should lie over somewhere in $\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}\times\cdots\times\operatorname{Spec}\mathbb{Z}$, where the product is over $\mathbb{F}_1$. This setting looks a bit impractical but for some kinds of p-adic specific incarnations this indeed makes sense. We will find later in the thesis where we establish an equivalence of categories

$$\left\{ \text{perfectoid spaces over } \mathbb{Q}_p^{\text{cycl}} \right\} \overset{\text{tilting}}{\longleftrightarrow} \left\{ \begin{array}{c} \text{perfectoid spaces over} \\ \mathbb{F}_p((t^{\frac{1}{p^\infty}})) \end{array} \right\}$$

In particular using this process we can move from characteristic 0 to characteristic p, which then provides us the actions of Frobenii. We recall the fact that the moduli space of Shtukas are Deligne-Mumford stacks, which are geometric object modelled by on schemes. In the mixed characteristic case we would naturally want mixed characteristic shtukas to be some geometric objects modelled over perfectoid spaces. It turns out that that notion of "diamonds" works out for the best in this case.

Throughout the thesis we will develop the notion of shtukas and diamonds. We will also be discussing further the notion of "Fargues-Fontaine" curve and developing Langland's correspondence over this.

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1. Preliminaries

In this chapter we discuss some preliminaries we need to establish later the results presented in this thesis.

1.1. **The valuation spectrum.**

Let R be a local ring and we denote m_R to be it's unique maximal ideal.

Definition 2. *If $A \subset B$ are local rings we say that B dominates A if $m_A \subset m_B$, equivalently $m_A = A \cap m_B$.*

One observes that for a field K, the domination is basically an order relation between local subrings of K.

Proposition 1. *Let K be a field and $R \subset K$ be a subring. Then the following are equivalent*

1. *R is local and it is maximal for the relation of domination(among local subrings of K)*
2. *for every $x \in K^{\times}$, we have $x \in R$ or $x^{-1} \in R$.*
3. *Frac$(R) = K$, and the set of principal ideals of R is totally ordered for inclusion.*
4. *Frac$(R) = K$, and the set of ideals of R is totally ordered for inclusion.*

Proof. (1) $\implies$ (2): Let $x \notin R$. Then we consider $R' = R[x] =$ smallest subring generated by R and $x \subset K$. Since R is local, R has a unique maximal ideal, say m_R.

Claim 1. *There is no prime lying above m_R in R'.*

Proof. We note that $R \subset R'$ is an integral extension. In that case we have a a surjective homomorphism $\varphi : \mathrm{Spec}(R') \to \mathrm{Spec}(R)$. Let $\phi : R \to R'$ be a ring homomorphism. Assume for the sake of contradiction that P is a prime lying above m_R. Then we must have $\phi^{-1}(P) = m_R \implies P \cap R' = m_R$. But in that case we consider the localisation of R' at the prime P, namely R'_P. Since localisation at a prime ideal is a local ring, then we have $R \subset R'_P$, and $m_R = PR'_P \cap R$ which contradicts the fact that R is maximal in the domnination relation. □

Since m_R is maximal and no prime contains it, hence must have $V(mR') = \varnothing \implies mR' = R'$. But then we can write $1 = \sum_{i=1}^{n} t_i x^i$ for some $t_i \in m_R$. Then we can re-write $(1 - t_0)(x^{-1})^n - \sum t^i (x^{-1})^i = 0 \implies x^{-1}$ is integral over R and hence $x^{-1} \in R$(since from Claim 1 it also follows that R must be integrally closed).

(2) $\implies$ (3): Let R be a ring such that for every $x \in K^{\times}$ either $x \in R$ or $x^{-1} \in R$. Since K is a field the inclusion $\mathrm{Frac}(R) \subset K$ is clear. To prove the other inclusion let $x \in K$. If $x = 0$, then $x \in R \subset \mathrm{Frac}(R)$, trivially. If $x \neq 0$, then $x \in K^{\times}$ and then by condition either $x \in R$ or $x^{-1} \in R \implies x \in \mathrm{Frac}(R)$ or $x^{-1} \in \mathrm{Frac}(R)$. But in either cases we have $x \in \mathrm{Frac}(R)$, hence proving the other inclusion. To prove the other part we choose the ideal generated by two elements a and b. We have $ab^{-1} \in R$ or $a^{-1}b \in R$ by assumption. Without Loss of Generality let's say $ab^{-1} = x \in R$, then this implies $a = bx \implies a \in (b) \implies (a) \subset (b)$.

(3) $\implies$ (4): Suppose I and J are ideals of R and $I \notin J$. We choose an element $a \in I \backslash J$. Let $b \in J$. Then we have $a \notin J \implies (a) \notin J \implies (a) \notin (b)$, then by hypothesis we have $(b) \subset (a) \subset (I)$ and hence $J \subset I$.

(4) $\implies$ (1): We have an inclusion ordering on the ideals of R which is bounded above the ring itself. Hence by Zorn's Lemma we get an unique maximal element, which is the unique maximal ideal of R. Hence R is local.

Claim 2. *Every finitely generated ideal of R is principal.*

Proof. It suffices to show that any ideal generated by two elements is principal. Let $I = (f, g)$ for some $f, g \in R$. But since by hypothesis $(f) \subset (g)$ or $(g) \subset (f)$ then accordingly $I = (g)$ or $I = (f)$. □

Claim 3. *For every element $r \in K^\times$ either $r \in K$ or $r^{-1} \in K$.*

Proof. Suppose that $a, b \in R$ and let m_R be the unique maximal ideal of R. Consider another ideal $I = (a, b)$. By previous claim I is principal. Then $I/m_R I$ is a one-dimensional vector space over the field $k = R/m_R$, hence the images of a and b are linearly dependent $\implies \exists\, u, v \in R$ such that $ua + vb \subset m_R I$. We also know $\exists\, x, y \in m_R \ni ax + by = mx + ny \implies a(u - x) = b(y - v)$. We may choose u to be unit and hence $u - x$ is also unit and hence $a/b = (y - v)/(u - x) \in R$ □

Finally we conclude the proof by contradiction argument. Suppose R lies within another local ring R' under the domination. We may choose R' to be the maximal local subring of K under domination. Let $a \notin R$ but $a \in R'$. But then by above claim one observes that $a^{-1} \in R \implies a^{-1} \in R'$. But in that case a^{-1} maps to a unit in R' but not in R, and hence R' cannot dominate R, and hence a contradiction. □

Definition 3. *If the conditions of the above proposition are satisfied then we call R a valuation subring of K.*

Definition 4. *An ordered abelian group is an abelian group $(\Gamma, +)$ with an order relation $\leq$ such that $\forall a, b, c \in \Gamma$ we have $a \leq b \implies a + c \leq b + c$.*

We will only be interested in totally ordered abelian groups.

Example 1.
- $(\mathbb{R}, +)$ with the usual order relation.
- $(\mathbb{R}_{>0}, \times)$ with usual order, which is isomorphic to the previous example by logarithm map, as an ordered subgroup.

Proposition 2. *Let $(\Gamma, +)$ be a totally ordered abelian group. Then*

(1) Γ *is torsion free.*
(2) *For every $\gamma \in \Gamma\ \exists\ \gamma' \in \Gamma \ni \gamma' < \gamma$.*

Proof. (1) If $\gamma \in \Gamma$ is a torsion element, then there is some $n \in \mathbb{N}$ such that $\gamma^n = 1$. If $\gamma \geq 1$, then we have $1 \leq \gamma \leq \gamma^n = 1 \implies \gamma = 1$. Also if $\gamma \leq 1$ such that $1 \geq \gamma \geq \gamma^n = 1 \implies \gamma = 1$.

(2) If $\gamma > 1$ then we have $\gamma^{-1} < \gamma$, if $\gamma < 1$ then $\gamma^2 < \gamma$ and if $\gamma = 1$ then we consider some element $\delta \in \Gamma \backslash \{1\}$ such that either $\delta < 1$ or $\delta^{-1} < 1$.

□

Definition 5. *Let R be a ring. A valuation (or more precisely multiplicative valuation) on R is a map $|.| : R \to \Gamma \cup \{0\}$ such that $(\Gamma, \times)$ is a totally ordered abelian group satisfying the following conditions:*

- $|0| = 0$, $|1| = 1$
- $\forall\, x, y \in R$, $|xy| = |x||y|$
- $\forall\, x, y \in R$, $|x + y| \leq max(|x|, |y|)$

Here 1 is the unit element of Γ and we extend the group Γ to an element $\{0\}$ such that it is the least of all elements in Γ and gives itself when multiplied with any element. The value group of $|.|$ is the subgroup of Γ generated by $\Gamma \cap |R|$. The kernel(or support) of $|.|$ is $Ker(|.|) = \{x \in R \,|\, |x| = 0\}$, which is a prime ideal of R.

Definition 6. *Let R be a ring and $|.|_1, |.|_2$ be two valuations on R. We denote by Γ_1 and Γ_2 be their respective value groups. We say the two valuations are equivalent if there exists a isomorphism $\varphi : \Gamma_1 \xrightarrow{\sim} \Gamma_2$ of ordered groups such that the following diagram*

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\varphi} & \Gamma_2 \\ & {\scriptstyle |.|_1}\nwarrow \quad \nearrow{\scriptstyle |.|_2} & \\ & R & \end{array}$$

commutes.(with $\varphi(0) = 0$)

Proposition 3. *Let K be a field.*

(i) If $v : K \to \Gamma \cup \{0\}$, then $R = \{x \in K \,|\, |x| \leq 1\}$ is a valuation subring of K, and it's maximal ideal is given by $m_R = \{x \in K \,|\, |x| < 1\}$.

(ii) Let R be a valuation subring of K. Let $\Gamma = K^\times / R^\times$; for $a, b \in K^\times$, we write $aR^\times \leq bR^\times$ if $ab^{-1} \in R$. Then this makes Γ into a totally ordered abelian group, and the map $|.| : K \to \Gamma \cup \{0\}$ defined by $|0| = 0$ and $|a| = aR^\times$ if $a \neq 0$ is a valuation on K.

(iii) The constructions of (i) and (ii) induce inverse bijections between the set of valuation subrings of K and the set of equivalence classes of valuations of K.

Proof. (i) We first observe that if $x \in K^\times$ then we have $|1| = |xx^{-1}| = |x||x^{-1}|$ hence if either $|x| \leq 1$ then we have $x \in R$ or $|x| > 1 \implies |x^{-1}| \leq 1$ and hence $x^{-1} \in R$. Hence by Proposition 1 we have R is a valuation subring. Then we know again from Proposition 1 that R is local and hence it has a unique maximal ideal given by the non-unit elements in R. We just observe the fact that x is unit in R then we have $|x| = 1$. This can be established by the fact that $|1| \leq |x.1| = |x|$. We also have that since x is unit then there is a y such that $xy = 1$, and hence $1 = |xy| \geq |x|$, and hence $|x| = 1$. We observe that if $x \in R \implies x \in K$. But since K is a field we have $x^{-1} \in K$. Then we have $|x^{-1}| = |x|^{-1} = 1$ and hence $x^{-1} \in R$. Hence we conclude that precisely the non-unit elements have norm less than 1. Hence by condition of local rings, we have clearly that m_R is maximal ideal.

(ii) If R is a valuation subring then we have for any $x \in K^\times$ wither $x \in R$ or $x^{-1} \in R$. Since $a, b \in K^\times$ we have $ab^{-1} \in K^\times$, hence either $ab^{-1} \in R$ or $ba^{-1} \in R$ and by definition we get for any $aR^\times, bR^\times \in \Gamma$ either we have $aR^\times \leq bR^\times$ or $bR^\times \leq aR^\times$, and hence Γ is totaly ordered. To check for valuation we have $|0| = 0$ and $|1| = R^\times$ which is unit in Γ. Also $|ab| = abR^\times = aR^\times bR^\times = |a||b|$. Finally we have, for $e \in K^\times$ where e is the identity element, then we clearly have $ae^{-1} \in R \implies a \in R$ and hence we have $aR^\times \leq R^\times$ for all $a \in K^\times$. Now we know from definition that $|a+b| = (a+b)R^\times$. Now if $ab^{-1} \in R \implies aR^\times \leq bR^\times \implies bR^\times = max\{aR^\times, bR^\times\} = max\{|a|, |b|\}$ and also since $ab^{-1} \in R$ and $e \in R$ (since $e \in K^\times$) and hence we have $(a+b)b^{-1} \in R \implies (a+b)R^\times \leq bR^\times$ and as a result $|a+b| \leq aR^\times$. Similarly when $ba^{-1} \in R$ we have by same procedure $|a+b| \leq aR^\times$. Hence combining we have $|a+b| \leq \max\{aR^\times, bR^\times\} = \max\{|a|, |b|\}$, which clearly makes this a valuation.

(iii) We give a map defined in the following way,

$$\left\{ \text{ Valuation Subrings of } K \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{equivalence classes of} \\ \text{valuations on } K \end{array} \right\}$$

$$V \longmapsto (|.|_V : K \to K^\times/V^\times \cup \{0\})$$

$$\{x \in K \mid |x| \leq 1\} \longleftarrow |.|$$

The conclusion follows readily from (i) and (ii). □

1.1.1. *Important properties of valuations and valuation rings.*

Lemma 2. *For elements $a, b \in R$, we have either $a|b$ or $b|a$ for a valuation ring R.*

Proof. From Proposition 1, we get that principal ideals of R are ordered with inclusion. Hence for $a, b \in R$ we have either $(a) \subset (b)$ or $(b) \subset (a)$ and in either cases $b|a$ or $a|b$. □

Proposition 4. *Let K be a field and $R \in K$ is a valuation subring. Then the following holds*

- *R is integrally closed in K.*
- *Every finitely generated ideal of R is principal. (in other words R is a Bézout Domain)*
- *If $I \subset R$ be a finitely generated ideal, then $\wp = \sqrt{I}$ is a prime ideal of R and is minimal amongst prime ideals of R containing I.*

Proof. The first two results follow from previous claims proved in the previous proposition. To prove the last claim, let I be a finitely generated ideal $\implies I$ is principal. Let $I = (x)$. Now we see $ab \in \wp \implies \exists\, n \in \mathbb{N} \ni (ab)^n \in I$. Without loss of generality let's assume that $b|a \implies \exists c \in R \ni a = bc$. As a result $(ab)^n = a^n b^n = b^{2n}c^n$ and by condition $b^{2n}c^n = xr$ for some $r \in R$. But the we have $b^{2n}|xr \implies b^{2n} \in (x) = I$. And as a result $\wp$ is a prime ideal. The rest follows from definition of radical of an ideal. □

Proposition 5. *Let K be a field and A be a subring of K.*

(i) Let P be a prime ideal of A. Then there exists a valuation subring $R \supset A$ such that $P = A \cap m_R$.
(ii) Let B be the integral closure of A in K. Then B is the intersection of all valuation subrings of K containing A.

Proof. (i) We may replace A by A_P(localisation of A at prime ideal P) and assume without loss of generality that A is a local ring. Now we define

$$\mathfrak{F} = \{B \subset K | B \supset A \text{ and } 1 \notin PB\}$$

We clearly observe that $\mathfrak{F}$ is nonempty as $A \in \mathfrak{F}$. If $\mathfrak{L} \subset \mathfrak{F}$ is a subset totally ordered by inclusion then the union of all elements of $\mathfrak{L}$ is again an element of $\mathfrak{F}$ and hence by applying Zorn's Lemma, there exists an $R \in \mathfrak{F}$ which is maximal for inclusion. Since $PR \neq R$ there is a maximal ideal m_R or R containing PR. Then $R \in R_{m_R} \in \mathfrak{F}$, so that $R = R_{m_R}$ and hence R is local. Also we have $P \subset m_R$ and P is maximal ideal of A and hence we have that $P = A \cap m_R$. We now just have to prove that R is valuation ring. For that suppose on contrary that $x \in K^\times$ and $x \notin R$ and $x^{-1} \notin R$. Consider the ring $R[x]$, generated by R and x and consider that $1 \in PR[x]$. So there is a relation of the form,

$$1 = a_0 + a_1 x + \ldots a_n x^n, \;\; a_i \in PR$$

Taking $b_i = (1 - a_0)^{-1} a_i$ we get that $b_i \in PR \subset m_R$ and

$$1 = b_1 x + b_2 x^2 + \ldots b_n x^n$$

We choose n mnimimal over all such relations.

Now we choose $R[x^{-1}]$ and hence we can find similar relations

$$1 = c_1 x^{-1} + \cdots + c_n x^{-m}$$

$c_i \in m_R$, and again choose the minimal m. If $n \geq m$ we multiply the second relation by $b_n x^n$ and subtract from the first relation and get a new relation similar to the first with degree smaller than n which is a contradiction. If $n < m$ then we just replace the places of x and x^{-1} and get similar contradiction.

(ii) Let

$$B' = \bigcap \{R \subset K \mid R \text{ is a valuation subring of } K, \text{ containing } A\}$$

Then by (i) we have clearly that $B' \supset B$. To prove the opposite inclusion we just need to show that $x \in K$ which is not integral over A, there is a valuation subring of K containing A but not containing x. Let $y = x^{-1}$. The ideal $yA[y]$

of $A[y]$ does not contain 1, indeed if it does then we have $1 = a_1 y + a_2 y^2 + \dots a_n y^n$ with $a_i \in A$, then x is integral over A contradicting the assumption. Therefore there is a maximal ideal P or $A[y]$ which contains $yA[y]$ abd again following (i) we are done.

□

1.1.2. *Rank of a valuation.*

We first define the notion of height in a totally ordered Abelian group.

Definition 7. *Let Γ be totally ordered abelian group. A convex subgroup (or isolated subgroup) of Γ is a subgroup Δ of Γ such that, $\forall a, b, c \in \Gamma$ if $a \leq b \leq c$ and $a, c \in \Delta$, then $b \in \Delta$.*

Example 2. (i) $\{0\}$ and Γ are the trivial convex subgroups of Γ.
(ii) If $\Gamma = \mathbb{R} \times \mathbb{R}$ with the lexicographic ordering then $\{0\} \times \mathbb{R}$ is a convex subgroup.

Proposition 6. *The set of all convex subgroups of Γ, ordered by inclusion, is a well ordered set*

Proof. Let $A, B \in \Gamma$ be convex subgroups. Assume that $A \notin B$ and $B \notin A$. Choose an element $b \in B \backslash A$ and $a \in A \backslash B$. Since $a, b \in \Gamma$, and Γ is totally ordered, so without loss of generality we assume $0 \leq a \leq b$ but then by convexity of B we have that $a \in B$ which is a contradiction. Hence set of convex subgroups is totally ordered by inclusion. The rest follows from the fact well-ordering principle is equivalent to Axiom of Choice. □

Definition 8. *The ordinal of the set of all convex subgroups of Γ ordered by inclusion is called the height of Γ and is denoted by $ht(\Gamma)$.*

Example 3.
- If $\mathrm{ht}(\Gamma) = 0$, then we have the ordinal ordered by inclusion is level 0 and this implies that Γ has only one convex subgroups but from previous example it can be observed that $\Gamma = \{0\}$ the trivial group.
- The condition that $\mathrm{ht}(\Gamma) = 1$ means that Γ is non-trivial and that the only convex subgroups of Γ are the trivial group and Γ itself.
- $\Gamma = \mathbb{R}^n$ has order n.

Proposition 7. *Let Γ be a nontrivial totally ordered abelian group. Then the following are equivalent*

(i) $ht(\Gamma) = 1$
(ii) $\forall a, b, \in \Gamma$ *such that* $a \geq 0$ *and* $b \geq 0$, $\exists n \ni b \leq na$.
(iii) *There exists an injective homomorphism of ordered groups* $\Gamma \to \mathbb{R}$

Proof. (i) $\implies$ (ii): If $\mathrm{ht}(\Gamma) = 1$ then the only subgroup are Γ and $\{0\}$. Let $a, b \in \Gamma$. Then we have $a, b \in \Gamma$, $0 < a \leq b$ or $0 \leq b < a$. In the second case we trivially have for all

$n > 1 \in \mathbb{N}$ such that $0 \le b < a < na$. Considering the second case $0 < a \le b$ and assume the contradiction that $b > na$ for all $n \in \mathbb{N}$. We consider the subgroup

$$\Delta = \{x \in \Gamma \mid -na \le x \le na \quad \text{for some } n \in \mathbb{N}\}$$

Claim 4. *Δ is a convex subgroup of Γ.*

Proof. Firstly Δ is non-empty since $a \in \Delta$. Now consider $x, y, z \in \Gamma$ such that $x \le y \le z$ and $x, z \in \Delta$. But then we have $-na \le x \le y \le z \le ma$ for some $m, n \in \mathbb{N}$. Consider $k = \max\{m, n\}$. Then clearly $-ka \le y \le ka$, and hence Δ is convex. □

By assumption $\beta \notin \Delta$ and it is a non-trivial convex subgroup of Γ which is a contradiction.

(ii) $\implies$ (iii): Let $\phi : \Gamma \to \mathbb{R}$ be a map defined in this way: Fix some $a \in \Gamma$ such that $a > 0$. Let $b \in \Gamma$, and let $n_0 = \min\{n \in \mathbb{Z} \mid na \le b\}$. We define sequences $\{n_i\}_i$ of $\mathbb{N}$ and $\{b_i\}_{i \ge 1}$ by induction:

Suppose we have defined $b_1, b_2, \ldots, b_i$ and $n_1, n_2, \ldots, n_{i-1}$, for some $i \ge 1$. Then set $n_i = \min\{n \in \mathbb{N} \mid n_i a \le 10 b_i\}$ and $b_{i+1} = 10 b_i - n_i a$. Then $\varphi(b) = n_0 + \sum_{i \ge 1} n_i 10^{-i}$. □

Definition 9. *The rank of a valuation is the height of it's absolute value group.*

1.2. **Valuation Topology and Microbial Valuation.**

Definition 10. *Let R be a ring and $|.| : R \to \Gamma \cup \{0\}$ be the valuation on R. The valuation topology on R associated to $|.|$ is the topology given by the base of open subsets $B(a, \gamma) = \{x \in R \mid |x - a| < \gamma\}$. for $a \in R$ and $\gamma \in \Gamma$.*

This makes R a topological ring and the map is continuous if we put the discrete topology on $\Gamma \cup \{0\}$.

Proposition 8. *(i) The valuation topology is Hausdorff if and only if Ker$(|.|) = \{0\}$.*
(ii) If the value group is trivial, then the valuation topology is the discrete topology.
(iii) For every $a \in R$, and every $\gamma \in \Gamma$, let $\bar{B}(a, \gamma) = \{x \in R \mid |x - a| \le \gamma\}$. This is an open subset in the valuation topology.

Proof. (i) $\implies$: If the topology is Hausdorff, then for every x we must have an open basis around x and 0 such that they are disjoint. But then if $x \ne 0 \in \mathrm{Ker}(|.|)$ then $|x - 0| = |x| = 0$, hence there are no open basis separating x and 0, which is a contradiction.

$\impliedby$: If Ker$(|.|)$ is trivial then let $\gamma_1 = |x - 0|$ and $\gamma_2 = |y - 0|$. Now we have $|x - y| \le \max\{|x - 0|, |y - 0|\} = \max\{\gamma_1, \gamma_2\}$ and neither of them are 0. So we can get some open basis around both with radii say $\frac{\max\{\gamma_2, \gamma_1\}}{3}$.

(ii) The value group generated by $\Gamma \cap R$ is trivial $\implies |R|$ is trivial $\implies |R| = 0 \implies \forall x, y \in R$ we have $|x-y| = 0$, and hence for everything $B(a,\gamma) = \{x \in R||a-x| < \gamma\}$ but then this is 0 for ever element in R, which means the only elements in $B(a,\gamma)$ are a itself and hence they are just singletons. The topology generated by singletons is the discrete topology.

(iii) Let $x \in \bar{B}(a,\gamma)$. Then indeed we have $B(x,\gamma) \subset \bar{B}(a,\gamma)$, and hence by definition of topology generated by a basis, we have the desired conclusion.

□

Definition 11. *Let R be a topological ring. An element $x \in R$ is called topologically nilpotent if* 0 *is a limit of the sequence* $(x^n)_{n\geq 0}$.

Theorem 2. *Let K be a field, and let $|.| : K \to \Gamma \cup \{0\}$ be a valuation, and R be the corresponding valuation ring. We put the valuation topology on K. Then the following are equivalent:*

(i) The topology on K coincides with the valuation topology defined by rank 1 *valuation on K.*
(ii) There exists a topologically nilpotent element in K.
(iii) R has a prime ideal of height 1.

Proof. (i) $\implies$ (ii): Let $\varpi \in K$ be such that $|\varpi| < 1$. Now since the valuation is of rank 1, then we let $a_n = \varpi^n$. We clearly have $|a_n| > 0$ for all n, and we also have $|a_{n-1}| < |a_n|$ and hence since rank 1 value groups are order isomorphic to $\mathbb{R}$ we have that limits of a_n is 0.

(i) $\implies$ (iii): Using the relation we have rank of R = Krull dimension of R. Hene Krull dimension of $R = 1$. And hence R must have a prime ideal of rank 1.

(ii) $\implies$ (i):

□

Definition 12. *If the conditions of the above proposition are satisfied, then we say the valuation on R is microbial.*

1.3. **The Riemann-Zariski space of a field.**

In this entire section we let K to be a field and $A \subset K$ be a subring.

Definition 13. *We say that a valuation subring $R \subset K$ has a center in A if $A \subset R$; in that case, the center of R in A is the prime ideal $A \cap m_R$ of A.*

Definition 14. *The Riemann-Zariski space of K over A is the set $RZ(K,A)$ of valuation subrings $R \supset A$ of K. We put the topology on it with the following base of open subsets: the sets*

$$U(x_1, x_2, \dots, x_n) = RZ(K, A[x_1, x_2, \dots, x_n]) = \{R \in RZ(K,A)|x_1, x_2, \dots, R\}$$

for $x_1, x_2, \ldots, x_n \in K$ *(If* $|.|$ *is a valuation on* K *corresponding to* R*, the condition that* $R \supset A$ *becomes* $|a| \leq 1$ *for every* $a \in A$*, and the condition that* $R \in RZ(K, A)$ *becomes* $|x_i| \leq 1$ *for* $1 \leq i \leq n$*).*

If A is the image of $\mathbb{Z}$, we write $RZ(K, \mathbb{Z}) = RZ(K)$ and we call it the Riemann-Zariski space of K; this is the set of all valuation subrings of K.

Remark 1. This definition does define a topology on $RZ(K, A)$

Proof. We have $U(x_1, x_2, \ldots, x_n) \cap U(y_1, y_2, \ldots, y_n) = U(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$ which is trivially true since if $R \in U(x_1, x_2, \ldots, x_n)$ and $R \in U(y_1, y_2, \ldots, y_n)$ contains everything and this is true for every such R and hence the result. □

Example 4. Let k be a field, X/k be a smooth projective geometrically connected curve and K be the function field of X. Then $RZ(K, k)$ is canonically isomorphic to X as a topological space.

More generally, if K/k is a finitely generated field extension, then $RZ(K, k)$ is isomorphic as a topological space to the inverse limit of all projective integral k-schemes with function field K.

Remark 2. Let $R, R' \in RZ(K, A)$. Then R is a specialization of R' in $RZ(K, A)$(i.e R is in the closure of $\{R'\}$) $\iff R \subset R'$.

Proof. We have trivially R is a specialization of $R' \iff$ every open set of $RZ(K, A)$ that contains R also contains $R' \implies \forall\ a_1, a_2, \ldots, a_n \in R$, we must also have $a_1, a_2, \ldots, a_n \in K$, we must also have $a_1, a_2, \ldots, a_n \in R'$, and hence $R \subset R'$. □

1.4. **The valuation spectrum of a ring.**

Definition 15. *Let* A *be a commutative ring. The valuation spectrum* $Spv(A)$ *of* A *is the set of equivalence classes of valuations on* A*, equipped for the topology generated by the subsets*

$$U(\frac{f_1, f_2, \ldots, f_n}{g}) = \{|.| \in Spv(A) \,|\, |f_1|, |f_2|, \ldots, |f_n| \leq |g| \neq 0\}$$

for all $f_1, f_2, \ldots, f_n, g \in A$*.*

We see that

$$U(\frac{f_1, f_2, \ldots, f_n}{g}) \cap U(\frac{f_1', f_2', \ldots, f_m'}{g'}) = U(\frac{f_1 g', f_2 g', \ldots, f_n g', f_1' g, f_2' g, \ldots, f_m' g}{gg'})$$

In particular, we can use the subsets $U(\frac{f}{g}), f, g \in A$ to generate the topology of $\mathrm{Spv}(A)$.

Remark 3. Let $X = \{(\wp, R) | \wp \text{is a prime ideal of} A \text{and} R \text{is a valuation subring of} \mathrm{Frac}(A/\wp)\}$. Then $\mathrm{Spv}(A)$ and X are in natural bijection.

Proof. Let $(\wp, R) \in X$, then we let v_1 to be the valuation on $\mathrm{Frac}(A/\wp)$ following Proposition 3. We define the valuation on A by $v_2 = \pi \circ v_1$, where π is the projection from $\pi : A \to A/\wp$. Conversely, if $|.|$ is a valuation on A, then $\wp = \mathrm{Ker}(|.|)$ is a prime ideal of A, and $|.|$ defines a valuation on the domain $A/\wp$ and hence also on it's fraction field, we take R the valuation subring of that valuation and follow Proposition 3 to construct the bijection. □

For $f_1, f_2, \dots, f_n, g \in A$, the image of the open subset $U(\frac{f_1, f_2, \dots, f_n}{g})$ by this bijection is

$$\{(\wp, R) \in X \,|g \notin \wp \text{ and } \forall\, i \in \{1, 2, 3, \dots, n\}, (f_i + \wp)(g + \wp)^{-1} \in R\},$$

where $(f_i + \wp)(g + \wp)^{-1} \in \mathrm{Frac}(A/\wp)$.

In particular we have a canonical map $\mathrm{supp} : \mathrm{Spv}(A) \to \mathrm{Spec}(A)$ that sends a valuation to it's kernel or support.

Definition 16. *If $x \in Spv(A)$ corresponds to a pair $(\wp, R)$ we write $\wp_x = \wp, R_x = R, \Gamma_x = \Gamma_R, K(x) = Frac(A/\wp_x)$. We denote by $|.|_x : A \to \Gamma \cup \{0\}$ the composition of $\pi : A \to A/\wp_x$ and of the valuation corresponding to R_x on $K(x)$. For $f \in A$, we often write $f(x)$ for the image of f in $A/\wp_x$, and $|f(x)|$ for the image of $f(x)$ in Γ_x.*

$$\overset{|.|_x}{\overbrace{A \xrightarrow{\ \pi\ } A/\wp_x \xrightarrow{\ v_{R_x}\ } \Gamma_x \cup \{0\}}}$$

Proposition 9. *Let A be a commutative ring.*

(i) If A is a field, then $Spv(A) = RZ(A)$ as topological spaces.
(ii) In general, the map $supp : Spv(A) \to Spec(A)$ is continuous and surjective. For every $\wp \in Spec(A)$, the fiber of this map over $\wp$ is isomorphic to $RZ(Frac(A/\wp))$

Proof. We first prove (ii). For every $\wp \in \mathrm{Spec}(A)$, we have $(\wp, \mathrm{Frac}(A/\wp)) \in \mathrm{supp}^{-1}(\wp)$, so supp is surjective. Now for every $f \in A$, we have

$$\mathrm{supp}^{-1}(D(f)) = \{(\wp, R) \in \mathrm{Spv}(A) \mid f \notin \wp\} = U(\frac{f}{f})$$

(where $D(f) = \{\wp \in \mathrm{Spec}(A) \mid f \notin \wp\}$), so supp is continuous. Finally let $\wp \in \mathrm{Spec}(A)$. Then we have $\mathrm{supp}^{-1}(\wp) = \{(\wp, R) \mid R \in \mathrm{Frac}(A/\wp)\}$, is canonically in bijection with $RZ(\mathrm{Frac}(A/\wp))$. Morover if $f_1, f_2, \dots, f_n, g \in A$ then

$$(3)\quad U(\frac{f_1, f_2, \dots, f_n}{g}) \cap \mathrm{supp}^{-1}(\wp) = \begin{cases} \phi & \text{if } g \in \wp \\ \{(\wp, R) \in \mathrm{supp}(\rho) \mid (f_1 + \wp)(g + \wp)^{-1} \in R\} & \text{if } g \notin \wp \end{cases}$$

which corresponds by bijection $\mathrm{supp}^{-1}(\wp) \simeq RZ(\mathrm{Frac}(A/\wp))$ to the open subset $U(\frac{(f_1+\wp),(f_2+\wp)\dots,(f_n+\wp)}{(g+\wp)})$. So the bijection is a homeomorphism. □

Definition 17. *Let $\varphi : A \to B$ be a morphism of rings. We denote by $Spv(\varphi)$ the map $Spv(B) \to Spv(A), |.| \to |.| \circ \varphi$.*

The map is continuous since

$$\mathrm{Spv}(\varphi)^{-1}(U(\frac{f_1, f_2, \dots, f_n}{g})) = U(\frac{\varphi(f_1), \varphi(f_2), \dots, \varphi(f_n)}{\varphi(g)})$$

We observe this since if $|f_i| \leq |g|$ and applying $\mathrm{Spv}(\varphi)$ on both sides we get $|\varphi(f_i)| \leq |\varphi(g)|$. Conversely, if $|\varphi(f_i)| \leq |\varphi(g)|$ then applying $\mathrm{Spv}(\varphi)^{-1}$ on both sides we get $|f_i| \leq |g|$. This shows the map is continuous.

We get the commutative square

$$\begin{array}{ccc} \mathrm{Spv}(B) & \xrightarrow{\mathrm{Spv}(\varphi)} & \mathrm{Spv}(A) \\ \downarrow{\scriptstyle \mathrm{supp}_B} & & \downarrow{\scriptstyle \mathrm{supp}_A} \\ \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec(A)} \end{array}$$

where the bottom comes from the morphism induced by φ.

1.4.1. *Topological notions.* We fix the following setting: X is a topological space and $\mathscr{U}$ be a collection of open subsets of X. We say that $\mathscr{U}$ is a base of the topology of X if every open open subspace of X is a union of elements of $\mathscr{U}$; we say $\mathscr{U}$ generates the topology of X or that $\mathscr{U}$ is a subbase of the topology of X if the collection of finite intersections of $\mathscr{U}$ is a base of the topology.

Definition 18. *(i) We say that a topological space X is quasicompact if every open covering of X has a finite refinement.*

(ii) We say that a continuous mao $f : X \to Y$ is quasicompact if the inverse image of any quasicompact open subset of Y is quasicompact.

(iii) We say that X is quasicompact if the diagonal embedding $X \to X \times X$ is quasicompact. This means that, for any quasicompact open subspaces U and V of X, the intersection $U \cap V$ is still quasicompact.

Definition 19. *Let X be a topological space with topology τ*

(a) We say that X is irreducible if, whenever $X = Y \cup Z$ with Y and Z closed subsets of X, then we have $Y = X$ or $Z = X$.

(b) For $x, y \in X$ we say that X is a specialization of y (or y is a generalization of x) if $x \in \overline{\{y\}}$.

(c) We say that a point $x \in X$ is closed if $\{x\}$ is closed.

(d) We say that a point $x \in X$ is generic if $\{x\}$ is dense in X.

(e) We say that X is a T_0 space if $\forall x, y \in X, x \neq y \; \exists \, U \in \tau \ni (x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)$.

(f) We say that X is sober if every irreducible closed subset of X has a unique generic point.

Proposition 10. *A topological space X is T_0 $\Leftrightarrow$ every irreducible closed subset of X has atmost one generic point.*

Proof. X is $T_0 \Leftrightarrow \forall x, y \in X \ni x \neq y$ there exists a closed subset of X, that contains exactly one of x or $y \Leftrightarrow$ either $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$. If X is T_0 then let Z be an irreducible closed subset of X and let $x, y \in X$ be two generic points. Then $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$ and hence $x = y$. Conversely if every irreducible subset of X has atmost one generic point, and let $x, y \in X \ni x \neq y$. As $\overline{\{x\}}$ and $\overline{\{y\}}$ are irreducible closed and have distinct generic points so we have $\overline{\{x\}} \neq \overline{\{y\}}$ and as a result we have $\overline{\{x\}} \notin \overline{\{y\}}$ and hence $x \notin \overline{\{y\}}$ or $\overline{\{y\}} \notin \overline{\{x\}}$ and hence $y \notin \overline{\{x\}}$. □

Definition 20. *A topological space X is called spectral if it satisfies the following conditions*

(i) X is quasi-compact and quasi-seperated.
(ii) the topology of X has a base of quasi-compact open subsets.
(iii) X is sober.

X is locally spectral if it has an open covering by spectral spaces.

For example any affine scheme is a spectral space, and any scheme is a locally spectral space.

Proposition 11. *Let X be a topological space. Then*

(i) If X is sober then X is T_0.
(ii) If X is locally spectral then X is sober.
(iii) X is locally spectral $\Longleftrightarrow$ X is quasi-compact and quasi-separated.
(iv) X is sober $\Longrightarrow$ every locally closed subspace of X is sober.
(v) X be a spectral space. Then every quasi-compact subspace of X is spectral.
(vi) Let X be a locally spectral space. Then every open subspace of X is locally spectral, and the topology of X has a base consisting of spectral open subspaces.

Proof. (i) Let X be sober $\Longrightarrow$ every irreducible closed subset of X has at most one generic point. Let $x, y \in X$ and $x \neq y$. As $\overline{\{x\}}$ and $\overline{\{y\}}$ are irreducible closed subspaces and have distinct generic points then $\overline{\{x\}} \neq \overline{\{y\}}$. So we have $\overline{\{x\}} \notin \overline{\{y\}}$ which means $x \notin \overline{\{y\}}$ or we have $\overline{\{y\}} \notin \overline{\{x\}}$ which means $y \notin \overline{\{x\}}$.

(ii) Let X be a topological space such that $X = \bigcup_{i \in I} X_i$ where all X_i are T_0 spaces.

Claim 5. *X is T_0.*

Proof. Let $x, y \in X$ such that $\overline{\{x\}} = \overline{\{y\}}$. There exists some $i \in I$ with $x \in X_i$. There exists an open subset $U \subset X$ such that X_i is a closed subset of U. If $y \notin U$ then we have $x \in \overline{\{x\}} \cap U = \overline{\{y\}} \cap U = \phi$(since $y \notin U$), but this is a contradiction. Hence $y \in U$. It follows that $y \in \overline{\{y\}} \cap U = \overline{\{x\}} \cap U \subset X_i$. And hence $y \in X_i$, and as a result $\overline{\{x\}} \cap X_i = \overline{\{y\}} \cap X_i$ and since X_i is T_0 and hence $x = y$, and hence X is T_0. □

Since X is T_0, then we must have $\forall x, y \in X$ such that $x \neq y$, $\exists U \in X$ where U is closed in X, and U contains exactly one of x or y. This implies $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$. Let Z be an irreducible closed subset of X and let $x, y \in Z$ be two generic points. Then clearly $x \in \overline{\{y\}} = Z$ and also $y \in \overline{\{x\}} = Z$ and hence $x = y$.

(iii) From (ii) if X is locally spectral then X is sober. Then from definition it is spectral if and only if it is quasi-compact and quasi separated.

(iv) Let $Y \subset X$ be a locally closed subspace. Let X be sober and hence T_0. We just have to show that Y has a generic point. We already have that X has a generic point. It just suffices to consider the cases of Y is either closed or open. Let Y be closed and Z be an irreducible closed subset of Y. Then Z is an irreducible closed subset of X. Hence there exists $x \in Z$ such that $\overline{\{x\}} = Z$ and as a result $\overline{\{x\}} \cap Y = Z$, and hence it is generic. In case of Y being open, let Z be an irreducible closed subset of Y. Then $\overline{Z}$ is an irreducible closed subset of X and similarly as above. Then let $x, y \in Y$ with $x \neq y$. Then $\overline{\overline{\{x\}} \cap Y} = \overline{\{x\}} \neq \overline{\{y\}} = \overline{\overline{\{y\}} \cap Y}$ and hence Y is also T_0, and hence Y is sober.

(v) Let

□

1.5. **The constructible topology.**

Definition 21. *Let X be a quasi-compact and quasi-seperated topological space and $Y \subset X$.*

(i) We say that Y is constructible if it is a finite union of subsets of the form $U \cap (X \backslash V)$ with U and V quasi-compact open subsets of X.

(ii) We say that Y is ind-contructible (resp. pro-constructible) if it is a union (resp. intersection) of constructible subsets of X.

Remark 4. Any finite union of quasi compact open sets is quasicompact and if X is quasi-seperated so is any finite intersection of quasicompact open subsets.

Proof. Let $\{U\}_{i \in I}$ be a collection of quasi-compact subsets of X, where I is finite. Consider the space $U = \bigcup_{i \in I} U_i = U_1 \cup U_2 \cup \ldots U_n$. We have to show that U is quasi-compact. Let $\{Y\}_{i \in J}$ is a covering for $U \implies$ it is also a covering for each U_i and since U_i's are quasi-compact and hence admits a finite subcover. And union of all those are finite and hence the statement. Similarly for the other case. □

This implies the following lemma

Lemma 3. *Let X be a quasi-compact and quasi-seperated topological space. The collection of constructible subsets of X is closed under finite unions, finite intersections and taking complements.*

Definition 22. *Let X be quasi-compact and quasi-seperated topological space. The constructible topology on X is the topology with base the collection of the constructible subsets of X. Equivalently, it is the topology generated by the quasi-compact open subsets of X and their complements.*

Proposition 12. *Let X be a finite T_0 space. Then X is spectral and every subset of X is constructible.*

Proof. Since X is finite, then every subset of X is quasi-compact. By the given condition we have X is T_0 and by a lemma above we just have to show that if Z is an irreducible closed subset of X then Z has atleast one generic point. Now since Z is a subset of a finite set $\implies Z$ has finite elements and hence $Z = \bigcup_{i=1}^{n} \overline{\{z\}}$. But Z is irreducible then $Z = \overline{\{z\}}$ for some $z \in Z$. To prove that every subset of X is constructible it suffices to show that singletons are constructible. Let $x \in X$. As X is T_0, hence $\forall y \in X \backslash \{x\}$ we can choose some Y_y open or closed and constructible such that $x \in Y_y$ and $y \notin Y_y$ and as a result we have $\{x\} = \bigcap_{y \in X \backslash \{x\}} Y_y$ is constructible. □

We end this section with an important lemma.

Lemma 4. *Let X be a quasi-compact T_0 space, and suppose that it's topology has a basis concerning of quasi-compact open subsets which is stable under finite intersections. Let X' be the topological space with the same underlying set as X, and whose topology is generated by the quasi-compact open subsets of X and their complements. The following are equivalent*

(i) X is spectral.
(ii) X' is Hausdorff and quasi-compact, and it's topology has a basis consisting of open and closed subsets.
(iii) X' is quasi-compact.

If these are conditions are satisfied we additionally have $X' = X_{cons}$

Proof.
- (i) $\implies$ (ii): If X is spectral, then it is quasi-compact and quasi-seperated, so $X' = X_{\text{cons}}$ by definition of constructible topology and as a result X' is Hausdorff and quasi-compact. It is also by the fact that it is generated by constructible subsets of X, which are open and closed subsets of X'.
- (ii) $\implies$ (iii): There is nothing to prove as the hypothesis itself implies that this is quasi-compact.
- (iii) $\implies$ (i): We already have that X is quasi-seperated and has a basis of quasi-compact open subsets. So we just need to show that X is sober. Let Z be an irreducible closed subset of X. As X is T_0, this subset has atmost one generic point, and we want to show that it has atleast one. Let Z' be the set with the subspace topology of X'. As X' is quasi-compact and Z' is closed in X', we have Z' is also quasi-compact. Suppose that Z has no generic point, hence

for every $z \in Z$ we have $\overline{\{z\}} \neq Z$, and hence by Hausdorffness there is an open quasi-compact subset U_z such that $U_z \cap \overline{\{z\}} = \phi$. We have $Z = \bigcup_{z \in Z}(Z \backslash U_z)$, and each $Z \backslash U_z$ is open in Z' by definition of topology of X'. As Z' is quasi-compact we have $z_1, z_2, \ldots, z_n \in Z$ such that $Z = \bigcup_{i=1}^{n}(Z \backslash U_{z_i})$; but then Z is reducible and is a contradiction.

□

1.6. **Spectrality of Spv**(A)**.**

We want to prove a criterion for a topological space to be spectral, due to Hochester.

Proposition 13. *Let $X' = (X, \tau')$ be a quasi-compact topological space, let $\mathscr{U} \subset \tau'$ be a collection of open and closed subsets of X', let τ be the topology on X_0 generated by $\mathscr{U}$, and let $X = (X_0, \tau)$.*

If X is T_0, then it is spectral, every element of $\mathscr{U}$ is a quasi-compact open subset of X and $X' = X_{cons}$, where the later denotes the constructible topology on X.

Proof. We may assume without loss of generality that $\mathscr{U}$ is stable by finite intersections. Since we have by the hypothesis of the condition that $\mathscr{U} \subset \tau'$ and the topology on X_0 is generated by $\mathscr{U}$, we clearly have the topology of X is coarser than the topology of X' and hence this implies every quasi-compact subset of X' is also a quasi-compact subset of X and hence as a result X itself is quasi-compact. As elements of $\mathscr{U}$ are closed in X', and hence the previous argument applies similarly and hence they are all quasi-compact subsets of X. Hence the topology of X has a basis of quasi-compact open subsets which is stable under finite intersections. Let τ'' be the topology on X generated by quasi-compact open subsets and their complements. By Lemma 4 above it just suffices to show that $\tau'' = \tau'$.

We first observe that τ'' is coarser than τ'.

Claim 6. *τ'' is Hausdorff*

Proof. Let $x, y \in X$ such that $x \neq y$; as X is T_0 we assume that $\exists U \in \mathscr{U}$ such that $x \in U$ and $y \in X \backslash U$ and U and $X \backslash U$ are both open in τ'' by definition of the same. So the identity map from X' to $X'' = (X_0, \tau'')$ is a continuous map from a quasi-compact space from a Hausdorff space, and hence it is a homeomorphism. □

□

Theorem 3. *Let A be a commutative ring. Then $Spv(A)$ is spectral. The open subsets*

$$U\left(\frac{f_1, f_2, \ldots, f_n}{g}\right) = \{|\,.\,| \in Spv(A) \mid |f_1|, |f_2|, \ldots, |f_n| \leq |g| \neq 0\}$$

for $f_1, \dots, f_n, g \in A$ are quasi-compact, and they and their complements generate the constructible topology of Spv(A).

Moreover, if $\varphi : A \to B$ is a morphism of rings, then the induced map Spv(φ) : Spv$(B) \to$ Spv(A) is spectral.

Proof. Let $X = \mathrm{Spv}(A)$, and let τ be the topology on X. Our main idea is to apply the Hochester's condition for this case in the family of subsets $U(\frac{f_1, f_2, \dots, f_n}{g})$ for $f_1, f_2, \dots, f_n, g \in A$.

(A) For every $x \in X$, if (ρ_x, R_x) be the pair corresponding to x, we define the relation $|_x$ on A by : $f|_x g$ if there exists an element a of R_x such that $a(f + \rho_x) = g + \rho_x$. Hence we have a map $\rho : X \to \{0,1\}^{A \times A}$ of relations on A. It follows then $\forall x \in X$, we have
$$\mathrm{supp}(x) = \{f \in A \mid 0|_x f\}$$
and hence supp(x) is completely determined by $\rho(x)$.

(B) The map ρ is injective.

Proof. Let $x, y \in X$ such that $\rho(x) = \rho(y)$ and we let the corresponding pairs be $(\wp_x, R_x)$ and $(\wp_y, R_y)$ respectively, and trivially have $\wp_x = \wp_y$. Let $K = \mathrm{Frac}(A \backslash \wp_x)$. Let $a \in K$. Then $\exists f, g \in A$ such that $a = (f + \wp_x)(g + \wp_x)^{-1}$ and we have
$$a \in R_x \Leftrightarrow g|_x f \Leftrightarrow g|_y f \Leftrightarrow a \in R_y$$
and hence $R_x = R_y$. ☐

(C) The image of ρ is the set of relations $|$ on A which satisfies the following conditions

(a) either $f|g$ or $g|f$
(b) If $f|g$ and $g|h$ then $f|h$
(c) if $f|g$ and $f|h$ we have $f|(g+h)$
(d) if $f|g$, then $fh|gh$
(e) if $fh|gh$ and $0 \not| h$, then $f|g$
(f) $0 \not| 1$

Proof. Let $f, g, h \in A$. We make the following claim

Claim 7. *If $|.|_x$ is a valuation in the equivalence class corresponding to x, we have $f|_x g$ iff $|f|_x \geq |g|_x$*

Proof. If $f|_x g$ then we have $a(f + \rho_x) = g + \rho_x$, for some $a \in R_x$, and as a result we clearly have the result following from previous remark. ☐

Now as a result of the claim we clearly have either $|f|_x \leq |g|_x$ or $|g|_x \leq |f|_x$ and as a result (a) follows. Also we have if $|f|_x \leq |g|_x$ and $|g|_x \leq |h|_x$ then $|f|_x \leq |h|_x$ and as a result (b) follows. (c), (d) and (e) follows similarly by claim.

Conversely, let $|$ be a relation on A satisfying all of the above. We clearly have $f|0$ for every $f \in R$ by condition (d). Let $\rho = \{f \in A \mid 0|f\}$. Then by conditions (c) and (d) this is an ideal of A and it does note contain 1 by (f) and is a prime ideal by condition (e). Now $\forall f, f' \in A$ such that $f + \rho = f' + \rho$ we clearly have $f|f'$ and $f'|f$. Now let $f, f', g, g' \in A$ such that $f = f', g - g' \in \rho$, we just proved that $f|f'$ and $g|g'$ and as a result apply (b) we get $f|g \implies f'|g'$. Now we let

$$R = \{a \in \mathrm{Frac}(A \backslash \rho) \mid a = (f + \rho)(g + \rho)^{-1},\ \ f, g \in A,\ \text{ and } g|f\}$$

using conditions (d) and (e) this condition is independent of the choice of f and g. Thus we observe that R is a subring of $\mathrm{Frac}(A/\rho)$ and it is a vaulation subring by (a). Hence the pair (ρ, R) defines a point $x \in X$, and hence the image is same as definition. $\square$

(D) We induce $\{0,1\}$ with the discrete topology and the product topology on $\{0,1\}^{A\times A}$. By Tychonoff's theorem the space $\{0,1\}^{A\times A}$ is compact. Let τ' be the topology on X induced by the topology of $\{0,1\}^{A\times A}$ via ρ, and let $X' = (X, \tau')$ and as a result X' is also compact. For $f, g, h \in A$ fixed, each of the conditions (a)-(f) of (X) defines a closed subset of $\{0,1\}^{A\times A}$ and hence by (C) we clearly have $\rho(X)$ is a closed subset of $\{0,1\}^{A\times A}$.

(E) Let $f_1, f_2, \dots, f_n, g \in A$. Then an element $|$ of $\rho(X)$ is in $\rho(U(\frac{f_1, f_2, \dots, f_n}{g}))$ if and only if $g|f_i$ for all i and $0 \not| \ g$. Each of these conditions defines an open and closed subset on $\{0,1\}^{A\times A}$, so $U(\frac{f_1, f_2, \dots, f_n}{g})$ is open and closed in topology τ'. Also the space X is T_0. Now we are in a position to apply Lemma 4. We get that X is spectral and that $U(\frac{f_1, f_2, \dots, f_n}{g})$ form a basis of quasi-compact open subsets of X and $X' = X_{\mathrm{cons}}$.

(F) Let $\varphi : A \to B$ be an isomorphism of rings. We then have to show that the continuous map $\mathrm{Spv}(\varphi) : \mathrm{Spv}(B) \to \mathrm{Spv}(A)$ is spectral, it suffices to show that it is quasi-compact, but this follows from the fact that

$$\mathrm{Spv}(\varphi)^{-1}(U(\frac{f_1, f_2, \dots, f_n}{g})) = U(\frac{\varphi(f_1), \varphi(f_2), \dots, \varphi(f_n)}{\varphi(g)})$$

$\square$

1.7. **The specialization relation in Spv(A).**

Let X be a topological space, and let $x, y \in X$. We say y specializes to x if $x \in \overline{\{y\}}$. In that case we say $y \rightsquigarrow x$.

Lemma 5. *The specialization is an order relation for a T_0 space.*

Proof. We observe that id is reflexive and transitive from definition. We just need to check anti-symmetricity. Let $x, y \in X$ such that $x \rightsquigarrow y$ and $y \rightsquigarrow x$. Then $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$, so x and y are both generic points of the irreducible closed subset $\overline{\{x\}}$ and as a result $x = y$. $\square$

1.7.1. *The Vertical Specialization.*

We fix a commutative ring A.

Remark 5. Let $\wp, \wp' \in \mathrm{Spec}(A)$. Then $\wp$ is a specialization of $\wp'$ if anf only if $\wp \supset \wp'$. Let $x, y \in \mathrm{Spv}(A)$. If x is a specialization of y, then $\mathrm{supp}(x) \supset \mathrm{supp}(y)$.

Definition 23. *Let $x, y \in Spv(A)$. We say that x is a vertical specialization of y if x is a specialization of y and $supp(x) = supp(y)$.*

1.7.2. *The Horizontal Specialization.*

Definition 24. *Let $x \in Spv(A)$. The characteristic group of x is the convex subgroup $c\Gamma_x$ generated by $\Gamma_{x,\geq 1} \cap |A|_x$.*

Definition 25. *Let $x \in Spv(A)$ and let $H \leq \Gamma_x$. We define a map $|.|_{x|_H} : A \to \Gamma_x \cup \{0\}$ by*

$$|f|_{x|_H} = \begin{cases} |f|_x & \text{if } |f|_x \in H \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

If the map is a valuation on A, we let the corresponding point if $Spv(A)$ by $x_{|_H}$.

Proposition 14. *Let $x \in Spv(A)$ and let H be a convex subgroup of Γ_x. We have the natural projection map $\pi : A \to A/\wp_x \subset Frac(A/\wp_x)$. The following are equivalent:*

(i) $|.|_{x|_H}$ *is a valuation on A.*
(ii) $H \supset c\Gamma_x$
(iii) *Let $\wp_h = \{a \in R_x \,|\, \forall\, \delta \in H, |a|_x < \delta\}$ be the corresponding prime ideal of R_x. Then $\pi(A) = A/\wp_x \subset R_{x,\wp_H}$.*

Proof.

- (i) $\implies$ (ii) : We have to show $H \supset c\Gamma_x \iff H \supset \Gamma_{x,\geq 1} \cap |A|_x$. Let $a \in A$ be such that $|a|_x \geq 1$. If $|a|_x = 1$ then clearly $a \in H$ and we are done. If $|a|_x > 1 \implies |a+1|_x = \max(|a|_x, 1) = |a|_x$. If $|a|_x \notin H$, then $0 = |a+1|_{x|_H} = \max(|a|_{x|_H}, 1) = 1$ a contradiction. And as a result $|a|_x \in H$.
- (ii) $\implies$ (iii) : Note that $R_{x,\wp_H}$ is the valuation subring of $\mathrm{Frac}(A/\wp_x)$ corresponding to the composition

$$\mathrm{Frac}(A/\wp_x) \xrightarrow{\quad |.|_x \quad} \Gamma_x \cup \{0\} \xrightarrow{\quad \pi \quad} (\Gamma_x/H) \cup \{0\}$$

Hence an element of $a \in \mathrm{Frac}(A/\wp_x)$ is in $R_{x,\wp_H}$ if and only if there exists $\gamma \in H \ni |a|_x \leq \gamma$. Now we have $H \supset c\Gamma_x$, and let $a \in A$. All we have to show is that there exists $\gamma \in H$ such that $|a|_x \leq \gamma$. If $|a|_x \leq 1$, then we choose $\gamma = 1$. Otherwise if $|a|_x > 1$, then $|a|_x \in c\Gamma_x \subset H$, hence the choice is $\gamma = |a|_x$.
- (iii) $\implies$ (ii) : From the first part of the above discussion, we prove (ii). We want to check $|.|_{x|_H}$ is a valuation. We clearly have $|0|_{x|_H} = 0$ since $0 \in H$ and also since $1 \in H$ and hence $|1|_{x|_H} = 1$. Let $a, b \in A$. We have to show that $|ab|_{x|_H} = |a|_{x|_H} |b|_{x|_H}$. Suppose this is not the case. Then the only way we can have such a possibility is that $|ab|_{x|_H} \in H$ but $|a|_{x|_H}, |b|_{x|_H} \notin H \cup \{0\}$. Hence we must have $|a|_{x|_H}, |b|_{x|_H} \notin H \cup \{0\}$. But then this implies that $|a|_x, |b|_x < 1$ and as a result $|ab|_x \leq |a|_x < 1$ but then $|ab|_x \notin H$ and hence a contradiction. Finally we have to show that $|a+b|_{x|_H} \leq \max(|a|_{x|_H}, |a|_{x|_H})$. Now this case is trivial if

$|a+b|_x \notin H$ as everything is equal to 0. Let $|a+b|_x \in H$. Without loss of generality let's assume that $|a|_x \leq |b|_x$. If $|b|_x \in H$ then we are done. If $|b|_x \notin H$ then we have $|b|_x < 1$ but then $|a+b|_x < 1$ and hence not in H a contradiction.

□

2. Adic Spaces

2.1. **Topological Rings and Continuous Valuations.**

Definition 26. *Let A be a topological ring.*

- *We say A is non-archimedean if 0 has a basis of neighborhoods consisting of subgroups of the underlying additive group of A.*
- *We say that A is adic if there exists an ideal I of A such that $\{I^n\}_{n\geq 0}$ is a fundamental system of neighbourhoods of 0 in A. In this case we say the topology on A is the I-adic topology and I is called ideal of definition.*
  If M is an A-module, the topology on M for which $\{I^n M\}_{n\geq 0}$ is a fundamental system of neighbourhoods of 0 is also called the I-adic topology on M.
- *We say that A is an f-adic ring(or a Huber ring) if there exists an open subring A_0 of A and a finitely generated ideal I of A_0 such that $\{I^n\}_{n\geq 0}$ is a fundamental system of neighbourhoods around 0 in A_0. In that case, we say that A_0 is the ring of definition and I is the ideal of definition.*
- *We call A is a Tate-ring if it is a f-adic ring and has a topologically nilpotent unit.*

Example 5.
- The ring $R = \mathbb{Z}_p$ equipped with p-adic topology is an adic ring with ideal of definition $I = p\mathbb{Z}_p$.
- Let A be any ring equipped with discrete topology, then the I-adic topology is when $I = 0$.

The motivation for "adic spaces" comes from two classical notions: formal schemes and rigid analytic spaces, which we will briefly discuss below.

2.1.1. *Formal Schemes.* Let A be an adic ring with respect to an ideal of definition I which is finitely generated. Then the formal spectrum Spf(A) is defined

$$\mathrm{Spf}(A) = \{p \in \mathrm{Spec}(A) \mid P \supset I\}$$

that is those prime ideals which are open in I-adic topology. In particular we have the well-known correspondence of two sets

$$\{\ \text{Ideals of } A \text{ containing } I\ \} \longleftrightarrow \{\ \text{Ideals of } A/I\ \}$$

and as a result we have from this correspondence $\mathrm{Spf}(A) = \mathrm{Spec}(A/I)$. The set Spf$(A)$ can be equipped with Zariski topology as in scheme theory. More precisely, for any

$f \in A$ we define a distinguished set as follows

$$D(f) = \{p \in \mathrm{Spf}(A) \mid f \notin p\}$$

Moreover we define

$$\mathscr{O}_{\mathrm{Spf}(A)}(D(f)) = I - \text{adic completion of } A[f^{-1}]$$

Such an assignment turns out is a gluing ito a sheaf of $\mathrm{Spf}(A)$ with respect to the Zariski Topology.

Definition 27. *A formal scheme is a locally ringed space such that it is locally isomorphic to Spf(A) for some adic ring A with finitely generated ideal of definition.*

The reason why we only work with finitely generated ideal is due to the following result.

Theorem 4. *If A is an adic ring with finitely generated ideal of definition I and M is an A-module, then the natural morphism $M \to \hat{M}$ induces an isomorphism $M/IM \tilde{\to} \hat{M}/I\hat{M}$.*

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & \hat{M} \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \pi_2} \\ I/IM & \xrightarrow{\quad \simeq \quad} & \hat{M}/I\hat{M} \end{array}$$

Proof. Since I is finitely generated this implies I^n is finitely generated. Let the generating set be $I^n = (f_1, f_2, \dots, f_r)$. Then we have a surjection from $(f_1, f_2, \dots, f_r) : M^{\oplus r} \to I^n M$. Now from the property of completion if $M \to N$ is a surjection then $\hat{M} \to \hat{N}$ is also a surjection. Using this in the previous map we have

$$(\hat{M})^{\oplus r} \xrightarrow{(f_1, f_2, \dots, f_r)} (\widehat{I^n M}) = \lim_{m \geq n} I^n M / I^m M = \mathrm{Ker}(\hat{M} \to M/I^n M)$$

On the otherhand the image of the map $(f_1, f_2, \dots, f_r) : M^{\oplus r} \to \hat{M}$ is $I^n \hat{M}$. And as a result we have the result. □

This property in general does not hold for non-finitely generated ideal I. For example take the ring $A = R[x_1, x_2, \dots]$ and the ideal $I = (x_1, x_2, , \dots)$.

The second motivation is Rigid Analytic Spaces

2.1.2. *Rigid Analytic Spaces.* Let K be a complete non-archimedean field. For any $n \in \mathbb{N}$, we consider the Tate-Algebra

$$K\langle T_1, T_2, \dots, T_n \rangle = \{ \sum_{i_1, i_2, \dots, i_n \in \mathbb{Z}_{\geq 0}} a_{i_1, i_2, \dots, i_n} T_1^{i_1} T_2^{i_2} \dots T_n^{i_n} \in K[[T_1, T_2, \dots, T_n]] : a_{i_1, i_2, \dots, i_n} \to 0, \sum_{j=1}^{n} i_j \to \infty \}$$

Equivalently this is also a p-adic completion of the polynomial ring $K[T_1, T_2, \dots, T_n]$.

Definition 28. *Let K be a non-archimedean field. A K-algebra A is called an affinoid algebra if it is isomorphic to a quotient of the Tate Algebra $K\langle T_1, T_2, \ldots, T_r\rangle$ for some $r \geq 0$.*

Definition 29. *Let $A = K\langle T_1, T_2, \ldots, T_n\rangle/(f_1, f_2, \ldots, f_s)$. Then we define*

$$Spf(A) \coloneqq Max(A) = \{x \in \mathbb{D}^r(\overline{K}) \mid f_i(x) = 0\}/Gal(\overline{K}/K)$$

We equip with the so called admissible topology which is the G-topology.

Definition 30. *The G-topology or the admissible topology is defined as the topology generated by the subsets*

$$X(f) = \{x \in Spf(A) \mid |f(x)| \leq 1\}, \quad f \in A$$

Let A be a a K-affinoid algebra and let $x \in \text{MaxSpec}(A)$. We define

$$\text{MaxSpec}(A) \coloneqq \{ \subset A \mid \text{ is a maximal ideal of } A\}$$

We define a so called rational domain much alike as previous

$$U(\frac{f_1, f_2, \ldots, f_n}{g}) \coloneqq \{x \in \text{MaxSpec}(A) \mid |f_i(x)| \leq |g(x)| \ \forall\, i \in \{1, 2, 3, \ldots, n\}\}$$

Here for any element $f \in A$ and for any $x \in \text{MaxSpec}(A)$ we denote by $|f(x)|$ the value of the element $f(x) \in k_x$ under the natural norm on k_x.

By similar strategy we define the G-topology on the MaxSpec(A) by the rational domains as open sets. We also let

$$\mathscr{O}_{\text{MaxSpec}(A)}(U(\frac{f_1, f_2, \ldots, f_n}{g})) = p-\text{adic completion of } A[T_1, T_2, \ldots, T_n]/(gT_i - f_i)$$

we can define a structure sheaf on MaxSpec(A). The pair $(A, \mathscr{O}_{\text{MaxSpec}(A)})$ is called a K-affinoid space.

Definition 31. *A rigid analytic space K is a G-topological space equipped with a sheaf of K-algebras which is locally isomorphic to a K-affinoid space.*

Our main idea of this chapter is to define the category of "adic-spaces" which will then contain the category of "rigid analytic spaces" and "formal schemes" as full subcategories.

2.2. **Huber Rings, Huber pairs and Continuous Valuations.**

Definition 32. *Let A be a topological ring. A subset E of A is called bounded in A, if for every neighborhood U of 0 in A $\exists$ an open neighborhood V of 0 such that $ax \in U$ for every $a \in E$ and $x \in V$.*

Remark 6. Let A be a topological ring and A_0 be an open-adic subring of A. Then A_0 is bounded in A.

Proof. Let I be an ideal of A_0 such that the topology on A_0 is the I-adic topology. Since by given hypothesis A_0 is open in A, the family $\{I^n\}_{n\geq 0}$ is a fundamental system of neighborhoods of 0 in A. Let U be an open subset of A such that $0 \in U$. Then $\exists\, n \geq 0 \ni I^n \subset U$, so we just choose $V = I^n$, then $ax \in U \quad \forall\ a \in A_0$ and $x \in V$. □

Proposition 15. *Let A be a topological ring. Then the following are equivalent:*

(i) A is an f-adic ring.
(ii) There exists an additive subgroup U of A and a finite subset T of U such that $\{U^n\}_{n\geq 0}$ is a fundamental system of neighborhoods of 0 in A such that $T.U = U^2 \subset U$.

Proof. We first prove an important lemma

Lemma 6. *Let A be a topological ring and A_0 be a subring of A equipped with subspace topology. The following are equivalent :*

(i) A is f-adic and A_0 is a ring of definition.
(ii) A is f-adic, and A_0 is open in A and adic.
(iii) A satisfies (ii) of the proposition and A_0 is open in A and bounded.

Proof. We begin the proof

(i) $\Longrightarrow$ (ii) : Trivially follows from definition.
(ii) $\Longrightarrow$ (iii) : By the above remark
(iii) $\Longrightarrow$ (i) : We choose U and T respecting the condition of the proposition and also suppose A_0 is open and bounded in A. Then $\exists\, r \in \mathbb{N}$ such that $U^r \subset A_0$, and we have $T(r) \subset U^r \subset A_0$. Let I be the ideal of A_0 generated by $T(r)$. Since $T(r)$ is finite $\Longrightarrow$ I is finitely generated. For every $n \geq 1$ we have,

$$I^n = T(nr).A_0 \supset T(nr)U^r = U^{r+nr}$$

so I^n is an open neighborhood of 0 in A_0. Let U be any open neighborhood of 0 in A_0. Since A_0 is bounded and $(U^m)_{m\geq 1}$ is a system of fundamental neighborhoods of 0 in A_0 $\Longrightarrow$ the topology on A_0 is I-adic and it is open in A so we complete. □

We use this lemma now.

(i) $\Longrightarrow$ (ii) : Take $U = I$ and choose T a finite system of generators of I.
(ii) $\Longrightarrow$ (i) : Let A satisfies (ii). Let $A_0 = \mathbb{Z} + U$. This is open in A since U is, and it is a subring because $U^n \subset U \ \forall\, n \geq 1$. We just have to show that A_0 is bounded and then Lemma 6 does the job for us. Let V be a neighborhood of 0 in A. By assumption there exists a positive integer m such that $U^m \subset V$. As U^m is open and $A_0.U^m = U^m + U^{m+1} \subset U^m \subset V$, we are done.

□

We prove the following important corollary

Corollary 1. *Let A be an f-adic ring.*

(i) If A_0 and A_1 are two rings of definition of A, then so are $A_0 \cap A_1$ and $A_0.A_1$.
(ii) Every open subring of A is f-adic.
(iii) If $B \subset C$ are subrings of A with B bounded and C open, then there exists rings of definition A_0 of A such that $B \subset A_0 \subset C$.
(iv) A is adic $\iff$ A is bounded in itself.

Proof. We begin the proof

(i) It is clear since A_0 and A_1 is open and bounded $\implies$ $A_0 \cap A_1$ and $A_0.A_1$ are also open and bounded.
(ii) Let B be an open subring of A, and let (A_0, I) be a couple of definition. Then $\exists\, n \in \mathbb{N}$ such that $I^n \subset B$, and $(B \cap A_0, I^n)$ is a couple of definition in B.
(iii) Without loos of generality let $C = A$. Let A_0 be a ring of definition of A. Then $A_0.B$ is open and bounded, and hence is ring of definition by the lemma.
(iv) If A is bounded, then it is ring of definition of itself and it is adic. Conversely, if A is adic, then it is bounded by the remark.

□

Definition 33. *Let A be a topological ring. We call a subset $E \subset A$ is power-bounded if the set $\bigcup_{n\geq 1} E(n)$ is bounded, where $E(n) = \{e_1, e_2, e_3, \ldots, e_n \mid e_i \in E\}$. We say E is topologically nilpotent if for every neighborhood W of 0 in A, $\exists\, N \in \mathbb{N} \ni E(n) \subset W \ \forall\, n \geq N$.*

If A is a topological ring. We denote A^0 the subset of it's power-bounded elements, and by A^{00} the subset of it's topologcally nilpotent elements.

Lemma 7. *Let A be an adic ring. If $x \in A$ then the following are equivalent:*

(i) x is topologically nilpotent.
(ii) $\exists\, I$ an ideal of definition such that image of x in A/I is nilpotent.
(iii) $\exists\, I$ an ideal of definition such that $x \in I$.

In particular A^{00} is an open radical ideal of A and it is the union of all ideals of definition. Moreover, A^{00} itself is an ideal of definition $\iff$ $\exists\, I$ an ideal of definition $\ni$ the nilradical of A/I is nilpotent.

Proof. We begin the proof,

(iii) $\implies$ (i) : Follows trivially from definition since I is ideal of definition and as a result forms a neighborhood around 0 since A is I-adic. Since $x \in I$ we have $\{x\} \subset I$, so we just have $n = N = 1$ and $W = I$.

(i) $\implies$ (ii) : Let I be an ideal of definition of A. As I is neighborhood of 0, hence as a result $\exists\, N \in \mathbb{N}$ such that $x^n \in I,\ \forall\, n \geq N$ and as a result image of x in A/I is nilpotent (since $x \mapsto x + I = I$ and trivially since $x^n \in I$ it follows).

(ii) $\implies$ (iii) : Let I be an ideal of definition such that $x + I$ is nilpotent in A/I. Let $n \in \mathbb{N} \ni x^n \in I$, and let $J = I + xA$. Then J is an open ideal of A, $I \subset J$, and $J^n \subset I$, so the I-adic and J-adic topologies on I conincide, and hence J is an ideal of definition.

We now prove the remaining part of the lemma. Since $(i) \iff (iii)$ we have $A^{00} = \bigcup_{n\geq 1} I$, where I is ideal of definition, and since I is open so is A^{00} and as a result we have A^{00} is a radical. If A^{00} is an ideal of definition, then there exists an ideal of definition I such that the nilradical of A/I is nilpotent, for this we just have to choose $I = A^{00}$ and we see that nilradical of A/I is just (0). Conversely, let's suppose that there exists an ideal of definition I such that the nilradical of A/I is nilpotent. Then $\exists\, n \in \mathbb{N} \ni (A^{00})^n \subset I$, so the I-adic and the A^{00}-adic topologies on A coincide and hence A^{00} is an ideal of definition. ☐

Proposition 16. *Let A be an f-adic ring. Then the set of power-bounded elements A^0 is an open and integrally closed subring of A, and it is the union of all rings of definition of A. Also A^{00} is a radical ideal of A^0.*

Proof. By (iii) of the Corollary 1 in this chapter, every bounded subring of A is contained in a ring of definition and by (ii) of lemma 6 in this chapter implies that every power bounded element of A is contained in a ring of definition, so A^0 is contained in the union of all the rings of definition of A. Conversely if A_0 is a ring of definition of A, then it's bounded and as a result all of it's elements are power-bounded and so $A_0 \subset A^0$. This proves that A^0 is the union of all rings of definition of A, so it is a subring of A; as rings of definition are open, A^0 is open.

We also show that A^0 is integrally closed in A. Let $a \in A$ be integral over A^0. Then there exits an ideal of definition A_0 such that a is integral over A_0, and hence A_0 is bounded. So $\exists\, n \in \mathbb{N} \ni A_0[a] = A_0 + A_0 a + \cdots + A_0 a^n$ and hence $A_0[a]$ is bounded, which implies a is power bounded hence $a \in A^0$.

Finally we show that A^{00} is a radical ideal of A^0. Let $a, a' \in A^{00}$ and $b \in A^0$. We prove that $a + a', ab \in A^{00}$. Let U be a neighborhood of 0 in A. Let $V \subset U$ be a neighborhood of 0 such that $b^n V \subset U$ for every $n \geq 1$, and let $N \in \mathbb{N}$ such that $a^n, (a')^n \in V\ \forall\, n \geq N$. Then we clearly have using binomial theorem, $(a + a')^n \in U\ \forall n \geq 2N$, and $(ab)^n = b^n a^n \in b^n V \subset U\ \forall\, n \geq N$. This proves it is an ideal. Let $a \in A^0$, and suppose that we have $a^r \in A^{00}$ for some $r \in \mathbb{N}$ Let U be a neighborhood of 0 in A, and let V be a neighborhood of 0 such that $a^n V \subset U$ for every $n \geq 1$. Choose a positive integer $N \ni (a^r)^n \in V\ \forall\, n \geq N$.

Then we have $a^n \in U$ for every $n \geq rN$ and hence $a \in A^{00}$. Finally we show it is radical ideal. □

Lemma 8. *Let A be a non-archimedean topological ring.*

(i) Let T be a subset of A, and let T' be the subgroup generated by T. Then T' is bounded $\iff$ T is bounded.
(ii) Let T be a subset of A. Then T is power-bounded $\iff$ subring generated by T is bounded.

Proof. We begin the proof

(i) " $\Longleftarrow$ " Suppose that T is bounded. Let U be a neighborhood of 0, and let V be a neighborhood of $0 \ni ax \in U$ for every $a \in T$ and $x \in V$. As A is non-archimdedean, we may assume that U and V are additive subgroups of A, and as a result we have $T'.V \subset U$. So T' is bounded.
" $\Longrightarrow$ " This is clear since if subgroup generated by T is bounded, then T is definitely bounded.
(ii) Let B be a subring generated by T. Then B is the subgroup generated $\{1\} \cup \bigcup_{n\geq 1} T(n)$, so by (i) it is bounded if and only if $\{1\} \cup \bigcup_{n\geq 1} T(n)$ is bounded equivalent to the fact that $\bigcup_{n\geq 1} T(n)$ is bounded.

□

2.2.1. *Bounded Sets and Continuous Maps.*

Definition 34. *Let A and B be f-adic rings. A morphism of rings $f : A \to B$ is called adic if there exists a couple of definition (A_0, I) of A and a ring of definition B_0 of B such that $f(A_0) \subset B_0$ such that $f(I)B_0$ is an ideal of definition of B.*

The aim is of this section is to give a criterion for a continuous map on f-adic rings to map bounded sets to bounded sets. In general this is not true.

Proposition 17. *Let A and B be f-adic rings and $f : A \to B$ be an adic morphism of rings. Then*

(i) f is continuous.
(ii) If A_0 and B_0 are rings of definition of A and B such that $f(A_0) \subset B_0$ then for every ideal of definition I of A_0, the ideal $f(I)B_0$ is an ideal of definition of B.
(iii) For every bounded subset E of A, the set $f(E)$ is bounded in B.

Proof. Let (A_0, I) and B_0 be as in definition and consider $J = f(I)B_0$.

(i) For every $n \geq 1$, we have $f^{-1}(J^n) = f^{-1}(f(I^n)B_0) \supset I^n$. So f is continuous.

(ii) Let (A'_0, I') be a couple of definition of A and B'_0 be a ring of definition of B such that $f(A'_0) \subset B'_0$. We want to show that $J := f(I')B'_0$ is an ideal of definition.

(iii) Let U be a neighborhood of 0 in B. We may assume that $U = J^n$ for some $n \geq 1$. Let V be a neighborhood of 0 in A such that $ax \in I^n$ for every $ax \in I^n$ for every $a \in E$ and every $x \in V$; we may assume that $V = I^m$ for some $m \geq 1$. Then $f(E).f(I)^m = f(E.I^m) \subset f(I^n) \subset J^n$, so $f(E)J^m \subset J^n$.

□

Proposition 18. *Let A and B be f-adic rings and $f : A \to B$ be a continuous isomorphism of rings. Suppose that A is Tate ring. Then B is a Tate ring, f is adic, and, for every ring of definition B_0 of B, we have $f(A).B_0 = B$.*

Proof. We use the following important lemma.

Lemma 9. *Let A be a Huber ring. If there exists a finite, faithfully flat morphism $A \to B$ such that B is Tate(resp. analytic) under it's natural topology as an A-module, then A is Tate.*

Proof. Let A'_0 and B_0 be rings of definition of A and B. Then $f^{-1}(B_0)$ is an open subring of A and $A'_0 \cap f^{-1}(B_0)$ is a bounded subring, so by Corollary 1 (iii) we have a ring of definition A_0 of A such that $A'_0 \cap f^{-1}(B_0) \subset A_0 \subset f^{-1}(B_0)$, and we clearly have $f(A_0) \subset B_0$. □

Let A'_0 and B_0 be rings of definition of A and B. Then $f^{-1}(B_0)$ is an open subring of A and $A'_0 \cap f^{-1}(B_0)$ is a bounded subring, so by corollary in the previous chapter, we have a ring of definition A_0 of A such that $A'_0 \cap f^{-1}(B_0) \subset A_0 \subset f^{-1}(B_0)$, and we clearly have $f(B_0) \subset A_0$. □

Now let us explain continuity of valuations.

Definition 35. *Let A be a topological ring. A continuous valuation on A is a map $|.| : A \to \Gamma \cup \{0\}$, where Γ is a totally ordered abelian group, such that*

- $|ab| = |a||b|$
- $|a+b| \leq max(|a|, |b|)$
- $|0| = 0$ *and* $|1| = 1$
- *(Continuity)* $\forall := \gamma \in \Gamma$ *we must have* $U_\gamma = \{a \in A \mid |a| < \gamma\}$ *is open in A.*

We discuss more about continuous valuations in the later subsections.

2.3. Appendix : A few important lemmas and solutions.

We give a few important results.

Claim 8. *(Kedlaya [Ked+19] Exercise 1.1.6) Let A be a Huber ring. If there exists a finite, faithfully flat morphism $A \to B$ such that B is Tate under the natural topology as an A-module, then A is Tate*

Proof. By given assumption, we have B has a topologically nilpotent element. We just need to show that A does as well. Now since we have an extension of rings, we have a multiplicative norm map $N : A \to B$ which is continuous. Then we have $N(0) = 0$. The map being continuous the topologically nilpotent elements map to itself and units map to itself. And hence the result. □

2.4. **Adic Spaces.**

We return to our notion of continuous valuations.

Definition 36. *Two continuous valuations $|.| : A \to \Gamma \cup \{0\}$ and $|.|' : A' \to \Gamma' \cup \{0\}$ are equivalent if $\forall\, a, b \in A$ we have $|a| \geq |b| \iff |a|' \geq |b|'$. In that case we write $|.| \sim |.|'$.*

Claim 9. *The kernel of the valuation map is a prime ideal of A.*

Proof. We clearly check that if $|ab| = 0 \iff |a|.|b| = 0 \iff |a| = 0$ or $|b| = 0$. □

The kernel just doesn't depends on the equivalence class of the valuation map. We define

$$\mathrm{Cont}(A) = \{|.| : A \to \Gamma \cup \{0\} \,|\, |.| \text{ is a continuous valuation}\}/\sim$$

We prove an important claim which suggests that whether or not a valuation is continuous depends only on the valuation upto equivalence.

Claim 10. *Let $|.|$ and $|.|'$ be equivalent valuations on a topological ring. Then we have $|.|$ is continuous $\iff |.|'$ is continuous.*

Proof. We just prove one side as the statement is symmetric. Let $|.|$ be continous, then we have $U_\gamma = \{a \in A \,|\, |a| < \gamma\}$ is open for all $\gamma \in \Gamma$. But then that implies $U'_\gamma = \{a \in A \,|\, |a|' < \gamma\}$ is open for all $\gamma \in \Gamma$ which is the condition for equivalence of valuations and as a result $|.|'$ is continuous. □

We work with the adic spectrum of rings. Let A be a Huber ring. We focus on

$$\mathrm{Spa}(A, A^+) = \{x \in \mathrm{Cont}(A) \,|\, |f(x)| \leq 1 \;\; \forall\, f \in A^+\}$$

The ring A^+ is the ring of integral elements, which means $A^+ \subseteq A^0$ and A^+ is open and integrally closed in A. A pair (A, A^+) is called a Huber pair. We make $X = \mathrm{Spa}(A, A^+)$ a topological space via the inclusion $X \subseteq \mathrm{Cont}(A)$. We equip X with the topology generated by rational subsets defined by

$$U(\frac{f}{g}) := \{x \in \mathrm{Spa}(A, A^+) \,|\, |f(x)| \leq |g(x)| \neq 0\}$$

for some $f, g \in A$.

Proposition 19. *Subsets of form*

$$\{x \in Spa(A, A^+) \,|\, |f(x)| \neq 0\}$$

and

$$\{x \in Spa(A, A^+) \,|\, |f(x)| \leq 1\}$$

are open.

Proof. It is easy to check that the second one forms a rational subset directly from definition. For the first one it is the union of two rational subsets, namely

$$\{x \in \mathrm{Spa}(A, A^+) \,|\, |f(x)| \neq 0\} = \{x \in \mathrm{Spa}(A, A^+) \,|\, |f(x)| < 0\} \cup \{x \in \mathrm{Spa}(A, A^+) \,|\, |f(x)| > 0\}$$

and this proves the statement. □

2.4.1. *A Few examples.*

1. Let $R = \mathbb{Z}$ be equipped with discrete topology. Then we have from definition

$$\mathrm{Spa}(\mathbb{Z}, \mathbb{Z}) = \{x \in \mathrm{Cont}(\mathbb{Z}) \,|\, |f(x)| \leq 1 \ \forall f \in \mathbb{Z}\}$$

Since the kernel entirely depends on the equivalence classes we analyze the kernel of the valuation. We must have (roughly speaking) the kernel can be (0) or a prime ideal of $\mathbb{Z}$ which are typically of the form $p\mathbb{Z}$ for which we have $V = \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}_p$ which is basically $\mathbb{F}_p$. Let's consider the case of the kernel being (0). In that case we have that $|x| = 1$, $\forall x \in \mathbb{Z}\backslash\{0\}$. Let's consider the case when the kernel is $p\mathbb{Z}_p$. Then by condition if $x \notin p\mathbb{Z}_p$ then $|x| \leq 1$ and we must also have $|x^{-1}| \leq 1$ which forces $|x| = 1$. Finally for the last kind of valuation we can have for the kernel as 0 is the usual p-adic valuation. Hence to summarise we have three kinds of points

- The valuation

$$\eta : \mathbb{Z} \to \{0, 1\} \ , \ x \mapsto \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

- For any prime number p we have the valuation

$$s_p : \mathbb{Z} \to \mathbb{F}_p \to \{0, 1\} \ \ , \ \ x \mapsto \bar{x} = x \bmod p \mapsto \begin{cases} 1 & \text{if } \ \bar{x} \neq 0 \\ 0 & \text{if } \ \bar{x} = 0 \end{cases}$$

- For any prime number p we have the third kind of valuation

$$\eta_p : \mathbb{Z} \to \mathbb{Z}_p \to p^{\mathbb{Z}}$$

which is the usual p-adic valuation.

2. Let us consider the same thing as above for $\mathbb{Z}_p$ which is the ring of p-adic integers. Notice that p is an uniformiser of $\mathbb{Z}_p$. So when we analyse the kernel we can just check the valuation of p. Again we have two possibilities for the kernel, that is, kernel is either (0) or $p\mathbb{Z}_p$. Let's start with the (0) kernel case. In this case we can have either $|p| < 1$ or $|p| = 1$. If $|p| < 1$ then we have $|x| = 1$ when $x \in \mathbb{Z}_p^*$. So we have the first kind of valuation as above. And similarly this time for the second case we have the second kind of valuation as above.
3. Let's proceed forward to a more general example. Let k be any non-archimdean field and we fix the norm $|.| : k \to \mathbb{R}^{\geq 0}$ which is unique upto passing from $|.|$ to $|.|^t$ for some $t \in \mathbb{R}^{>0}$. We have $k^0 = \{c \in k \,|\, |c| \leq 1\}$ and $k^{00} = \{c \in k \,|\, |c| < 1\}$. We let $|k^\times|$ be the value group. For $x \in k$ and $r \in \mathbb{R}^{\geq 0}$ we set $D(x,r) := \{y \in k \mid |y - x| \leq r\}$. Such things are called discs. We observe that for any $x' \in D(x,r)$ we have $D(x',r) = D(x,r)$. We also set $D^0(x,r) := \{y \in k \mid |y - x| < r\}$. We observe that trivially $k^{00} = D^0(0,1)$ is the maximal ideal of k^0 and let $\kappa = \frac{k^0}{k^{00}}$ the residue field. Let k be algebraically closed and complete. This implies κ is also algebraically closed.

   Let $A = k\langle t\rangle$ be the ring of convergent power series in one variable. Then we have $A^0 = A^+ = k^0\langle t\rangle$. Consider the space $X = \mathrm{Spa}(A, A^0)$. We have five different kinds of points from analysis. We visualise it via a tree,
   - The classical end points: Let $x \in k^0 \implies x \in D(0,1)$. Then for any $f \in k\langle t\rangle$ we can evaluate f at x to get a map $k\langle t\rangle \to k, f \mapsto f(x)$. Composing with the norm on k we have the valuation $f \mapsto |f(x)|$ on $k\langle t\rangle \to k$, which is continuous, and ≤ 1 for all $f \in R^+$. These classical points correspond to the maximal ideals of A: If $\subset A$ is a maximal ideal, then $A/ = k$ and is of the form $(t - x)$ for a unique $x \in D(0,1)$. These are end points of the branches. They are typically called Type 1 points.
   - Points on limbs : Let $0 \leq r \leq 1$ be some real number and $x \in k^0$. Then

     $$x_r : f = \sum_n a_n(t - x)^n \mapsto \sup_n |a_n| r^n = \sup_{y \in k^0, |y - x| \leq r} |f(y)|$$

     is a point of X. It depends only on $D(x,r)$. For $r = 0$ it agrees with the classical point defined by x, for $r = 1$, the disc $D(x,1)$ is independent of x and we obtain the Gauss norm $\sum a_n t^n \mapsto \sup_n |a_n|$ as the root of the tree. If $r \in |k^\times|$, then the point x_r corresponding to $D(x,r)$ is a branching point. These are points of type 2 and rest are points of type 3.
   - Dead End points : Let us assume we have nested discs $D_1 \supset D_2 \supset D_3 \supset \ldots$, then we define a valuation

     $$f \mapsto \inf_i \ \sup_{y \in D_i} \ |f(y)|$$

     is a continuous valuation. Then we have three cases
     * The intersection $\bigcap_{i \geq 0} D_i$ maybe a single point, say α. Then $D = D_r(\alpha)$
     * The intersection $\bigcap_{i \geq 0} D_i$ maybe another disc.
     * The intersection $\bigcap_{i \geq 0} D_i$ might be empty ! (Happens in cases like $\mathbb{C}_p$).
   - Finally there are some valuations of height 2. Let $x \in k^0$ and fix an $r \in \mathbb{R} \ni 0 < r \leq 1$ and let $\Gamma_{<r}$ be the abelian group $\mathbb{R}^{>0} \times \gamma^{\mathbb{Z}}$ endowed with the unique

total order such that $r' < \gamma < r$, $\forall\, r' < r$. Then

$$x_{>r} : f = \sum_n a_n(t-x)^n \mapsto \max_n |a_n|\gamma^n \in \Gamma_{<r}$$

is a point of X which depends only on $D^0(x,r)$. For $0 < r < 1$ let $\Gamma_{>r}$ be the abelian group $\mathbb{R}^{>0} \times \gamma^{\mathbb{Z}}$ endowed with the unique total order such that $r' > \gamma > r$, $\forall\, r' > r$. Then

$$x_{>r} : f = \sum_n a_n(t-x)^n \mapsto \max_n |a_n|\gamma^n \in \Gamma_{>r}$$

is a point of X and depends only on $D(x,r)$. If $r \notin |k^\times|$, then $x_{<r} = x_{>r} = x_r$. But for each point x_r of type 2, this gives one additional point for each ray starting from x_r. The points $x_{<r}$ correspond to rays towards the classical points, and the point $x_{>r}$ corresponds to the ray towards Gauss points.

4.

For now we fix a complete Huber pair (A, A^+) (i.e A is complete) and we set $X := \mathrm{Spa}(A, A^+)$. The goal is to mimic arguments from scheme theory to to define structure sheaves on X. In fact, one thinks of the rational subsets, in the following definition as an analogue of distinguished points in scheme theory.

Definition 37. *We fix $s \in A$ and a finite subset $T \subset A$ such that $TA \subset A$ is an open ideal. Then define*

$$U(\frac{T}{s}) := \{x \in X \mid |t(x)| \le |s(x)| \ne 0, \forall\, t \in T\}$$

A rational subset of U is a subset of the form $U = U(\frac{T}{s})$.

We have clearly from definition that

$$U(\frac{T}{s}) = \bigcap_{k=1}^{n} U(\frac{t_k}{s})$$

where $T = \{t_1, t_2, \ldots, t_k\}$. And hence any rational subset is open.

Proposition 20. *The collection of rational subsets form a basis for the topology of X, which is stable under finite intersections.*

Proof. We give a routine check. Let $U_1 = \{x \in X \mid |t(x)| \le |s(x)| \ne 0\}$ and $U_2 = \{x \in X \mid |t'(x)| < |s'(x)| \ne 0\}$. Then we clearly have $U_1 \cap U_2 = \{x \in X \mid |tt'(x)| < |ss'(x)| \ne 0\}$ which is a rational subset given only that tt' generate an open ideal of A. By hypothesis, there exists an ideal of definition I such that $I \subset TA$ and $I \subset T'A$. Then ideal generated by $tt' \supset I^2$, and hence is open. □

Theorem 5. *Let $U \subset X$ be a rational subset. Then there exists a unique complete Huber pair $(\mathscr{O}_X(U), \mathscr{O}_X^+(U))$ equipped with a morphism of Huber pairs $(A, A^+) \to (\mathscr{O}_X(U), \mathscr{O}_X^+(U))$ such that the corresponding map*

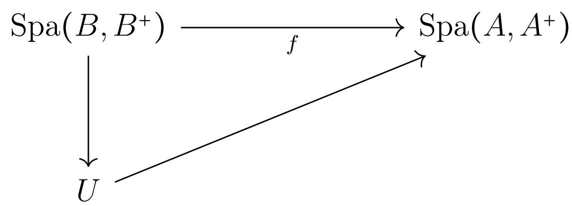
$$Spa(\mathscr{O}_X(U), \mathscr{O}_X^+(U)) \to Spa(A, A^+)$$

factors through U and it is universal with respect to this property. Morover this map induces a homeomorphism of U with Spa$(\mathscr{O}_X(U), \mathscr{O}_X^+(U))$.

Sketch of proof. Let $A_0 \subset A$ be a ring of definition and $I \subset A_0$ be an ideal of definition. Let $s \in A$ be a finite subset with $TA \subset A$ being open such that $U = U(\frac{T}{s})$. Let $\varphi : (A, A^+) \to (B, B^+)$ be a morphism of complete Huber pairs such that the corresponding maps

$$f : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$$
$$y = |\,|_y \mapsto (\varphi(y) = |\,.|_{\varphi(y)} : A \ni f \mapsto |\varphi(f)|_y)$$

factors through U.



We observe a few things

- For any $y \in |.|_y \in \mathrm{Spa}(B, B^+)$, we have $|\varphi(y)|_s = |s|_{f(y)} \neq 0$, by a theorem of Huber we have $\varphi(s) \in B^+$. And hence φ factors as $A \to A[1/s] \to B$.
- For $y \in \mathrm{Spa}(B, B^+)$, we have $|\varphi(t)|_y \leq |\varphi(s)|_y$ for all $t \in T$ since $|t|_{f(y)} \leq |s|_{f(y)}$. Since by above we have $\varphi(s) \in B^+$, we have
$$|\frac{\varphi(t)}{\varphi(s)}|_y \leq 1$$
for any $y \in \mathrm{Spa}(B, B^+)$. By the same theorem of Huber we have $\varphi(t)/\varphi(s) \in B^+$ for all $t \in T$.

These two points imply that ring homomorphism $(A, A^+) \to (B.B^+)$ factors as

$$(A, A^+) \to (A[1/s], A^+[t/s : t \in T]) \to (B, B^+)$$

We equip with $A[1/s]$ the topology that $A_0[t/s : t \in T]$ is a ring of definition with ideal of definition $IA_0[t/s : t \in T]$. With respect to this topology we define

$$\begin{aligned}\mathrm{Spa}(\mathscr{O}_X(U), \mathscr{O}_X^+(U)) &:= (\text{completion of } A[1/s], \text{integral closure of completion } A^+[t/s : t \in T]) \\ &=: (A\langle T/s\rangle, A^+\langle T/s\rangle)\end{aligned}$$

□

We now proceed to define the structure presheaf $\mathscr{O}_X$ on X.

Definition 38. *The structure presheaf is defined as follows*

- *If $U \subset X$ is a rational subset, then $\mathscr{O}_X$ send to U to $\mathscr{O}_X(U)$ as in the above theorem.*
- *If $W \subset X$ is a general open subset then we define*

$$\mathscr{O}_X(W) := \varprojlim_{U \subset W \,:\, U \text{ is rational subset of } W} \mathscr{O}_X(U)$$

Remark 7. As differed to actual scheme theory, the structure presheaves, $\mathscr{O}_X$ and $\mathscr{O}^+(X)$ are not sheaves. If they are sheaves then (A, A^+) is called sheafy. There are some criterion which I will state without proof.

Theorem 6. *Let (A, A^+) is a complete Huber pair. Then (A, A^+) is sheafy if it is one of the following*

- *A is discrete.*
- *A is finitely generated over a ring of definition which is noetherian.*
- *A is Tate and $A\langle T_1, T_2, \dots, T_n\rangle$ are noetherian for all $n \in \mathbb{N}$.*

However in this thesis, I avoid the complications and for me all Huber pairs are sheafy. I do not establish, prove or discuss the criterion for non-sheafy Huber pairs to be sheafy.

We come to main part of the chapter, to define adic spaces.

Definition 39. *We define a category (V) as follows:*

- *Objects : $(X, \mathscr{O}_X, (|\cdot|_x)_{x \in X})$, where*
  - *X is a topological space*
  - *$\mathscr{O}_X$ is a sheaf of topological rings*
  - *for each $x \in X$, $|.|_x$ is an equivalence class of continuous valuations on $\mathscr{O}_{X,x}$.*
- *Morphisms : Morphisms in (V) are morphisms of ringed topological spaces $f : X \to Y$ such that for any $x \in X$, the following diagram is commutative upto equivalence of classes of continuous valuations*

$$\begin{array}{ccc} \mathscr{O}_{Y,f(x)} & \longrightarrow & \mathscr{O}_{x,X} \\ \downarrow |.|_{f(x)} & & \downarrow |.|_x \\ \Gamma_{f(x)} \cup \{0\} & \longrightarrow & \Gamma_x \cup \{0\} \end{array}$$

We now define adic spaces as some special objects of this category

Definition 40. • *An adic space is an object of category (V) i.e it looks like $(X, \mathscr{O}_X, (|\cdot|_x)_{x \in X})$ that admits a covering $\{U_i : i \in I\}$ such that, for each $i \in I$, $(U_i, \mathscr{O}_{X|U_i}, (|\cdot|_x)_{x \in U_i})$ is isomorphic to Spa(A_i, A_i^+) for some sheafy Huber pair (A_i, A_i^+).*

- *For a sheafy Huber pair (A, A^+), the topological space $X = Spa(A, A^+)$, with the structure sheaf and the continuous valuations is an adic space, called an affinoid adic space.*

Just in the same way as of schemes, adic spaces can be formed by gluing affinoid adic spaces.

Lemma 10. *Suppose we are given*

- *an index set I*
- *for every $i \in I$, an adic space U_i*
- *for every $i, j \in I$ open subspaces $U_{ij} \subseteq U_i$ and $U_{ji} \subseteq U_j$ and an isomorphism $\varphi_{ij} : U_{ij} \to U_{ji}$*

satisfying $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap Uik$ and the cocycle condition: For $i, j, k \in I$ the diagram

$$\begin{array}{ccc} U_{ij} \cap U_{ik} & \xrightarrow{\varphi_{ik}} & U_{ki} \cap U_{kj} \\ & {\searrow}^{\varphi_{ij}} \quad {\nearrow}_{\varphi_{jk}} & \\ & U_{ji} \cap U_{jk} & \end{array}$$

commutes. Then there is an adic space X and open immersions $\varphi_i : U_i \to X$ satisfying

- *$X = \bigcup_i \varphi(U_i)$*
- *φ_i induce isomorphisms $U_{ij} \xrightarrow{\sim} \varphi_i(U_i) \cap \varphi_j(U_j)$*
- *the diagram*

$$\begin{array}{ccc} U_{ij} & \xrightarrow{\varphi_{ij}} & U_{ji} \\ & {\searrow}^{\varphi_i} \quad {\nearrow}_{\varphi_j} & \\ & X & \end{array}$$

  commutes.

Proof. The fact that the underlying topological space of X can be glued from the U_i is clear. As all morphisms are morphisms of locally v-ringed spaces, they respect the structure sheaf and the valuations. We thus obtain a structure sheaf on X and also a set of valuations $\{v_x\}_{x \in X}$. Since each U_i is locally isomorphic to an affinoid adic space, the same holds for X. □

We discuss some few but very important examples for adic spaces

Example 6. (The affine line) Let k be a non-archimedean field with pseudouniformizer $\varpi \in k$. For $n \in \mathbb{N}$ the closed adic disc of radius $|\varpi^{-n}|$ is defined as

$$\mathbb{D}_k(0, |\varpi^{-n}|) := \mathrm{Spa}(k\langle \varpi^n T \rangle, k^0\langle \varpi^n T \rangle)$$

where we treat the $\varpi^n T$ as a variable. So precisely the discs $\mathbb{D}_k(0,|\varpi^{-n}|)$ are all isomorphic. Let's look at all the classical points. For $\alpha \in k$ the valuation

$$x_\alpha : k[T] \xrightarrow{T\mapsto\alpha} k \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0}$$

extends to a continuous valuation on $k\langle\varpi^n T\rangle$ precisely if

$$|\varpi^n T(x_\alpha)| \leq 1 \Longleftrightarrow |\alpha| \leq \varpi^{-n}$$

For $m \geq n$ we have natural homomorphisms

$$\begin{array}{ccc} (k\langle\varpi^m T\rangle, k^0\langle\varpi^m T\rangle) & \to & (k\langle\varpi^n T\rangle, k^0\langle\varpi^n T\rangle) \\ \varpi^m T \mapsto \varpi^{m-n} T & & \end{array}$$

The corresponding morphism of discs

$$\mathbb{D}_k(0,|\varpi^{-n}|) \to \mathbb{D}_k(0,|\varpi^{-m}|)$$

is precisely the inclusion of the disc of radius $|\varpi^{-n}|$ into the disc of radius $|\varpi^{-m}|$. We can in fact identify the disc $\mathbb{D}_k(0,|\varpi^{-n}|)$ with the rational subset

$$\{x \in \mathbb{D}_k(0,|\varpi^{-m}|) \mid |T(x)| \leq \varpi^{-n}\}$$

As a result we have increasing sequence of inclusion of discs whose radii become arbitrarily large. Since the inclusions are open immersions, we now get the affine line by gluing all the discs

$$\mathbb{A}^1_k = \bigcup_n \mathbb{D}_k(0,|\varpi^{-n}|)$$

As an adic space, the affine line is not affinoid, since the discs $\mathbb{D}_k(0,|\varpi^{-n}|)$ form a covering of $\mathbb{A}^1_k$, but does not admit a finite sub-covering.

Let us compute the global sections of the structure sheaf of $\mathbb{A}^1_k$. The sheaf property gives us an exact sequence

$$\mathscr{O}_{\mathbb{A}^1_k}(\mathbb{A}^1_k) \to \prod_{m\in\mathbb{N}} \mathscr{O}_{\mathbb{A}^1_k}(\mathbb{D}_k(0,|\varpi^{-m}|)) \to \prod_{m,n\in\mathbb{N}} \mathscr{O}_{\mathbb{A}^1_k}(\mathbb{D}_k(0,|\varpi^{-m}|) \cap \mathbb{D}_k(0,|\varpi^{-n}|))$$

Example 7. (The closed unit adic disc over $\mathbb{Q}_p$) Consider

$$X = \mathrm{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle) \to \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$$

We observe that X admits the following property: Let (R, R^+) be a sheafy $(\mathbb{Q}_p,\mathbb{Z}_p)$-algebra, then

$$\begin{array}{rcl} X(\mathrm{Spa}(R,R^+)) & = & \{\text{morphisms } \mathrm{Spa}(R,R^+) \to \mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)\} \\ & = & \{\text{morphisms } (\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle) \to (R,R^+)\} \\ & = & \{\text{morphisms } \mathbb{Z}_p\langle T\rangle \to R^+\} \\ & = & R^+ \end{array}$$

Example 8. (The open unit adic disk over $\mathbb{Z}_p$) We consider $X = \mathrm{Spa}(\mathbb{Z}_p[[T]], \mathbb{Z}_p[[T]])$ over $\mathbb{Z}_p$, where $\mathbb{Z}_p[[T]]$ is endowed with (p,T)-adic topology. We study the underlying topological space $|X|$ of X. We observe just as in previous examples, that we have three obvious points

- The point $x_{\mathbb{F}_p} \in X$ with valuation
$$|\,.|_{x_{\mathbb{F}_p}} : \mathbb{Z}_p[[T]] \xrightarrow{\text{mod } (p,T)} \mathbb{F}_p \to \{0,1\}$$
where the second map is as follows
$$|\,.|_{x_{\mathbb{F}_p}} = \begin{cases} 1 \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$
It is clear that the kernel of the first map is (p,T), which is an open ideal of $\mathbb{Z}_p[[T]]$. Moreover the point $x_{\mathbb{F}_p}$ is the unique point, for which the kernel is open. This is because for an ideal to be open it must contain some power of the ideal (p,T).
- The point $x_{\mathbb{Q}_p} \in X$ with the map
$$|\,.|_{x_{\mathbb{Q}_p}} : \mathbb{Z}_p[[T]] \xrightarrow{\text{mod } (T)} \mathbb{Z}_p \to \mathbb{R}_{\geq 0}$$
where the second map is the usual p-adic valuation map. We say that $x_{\mathbb{Q}_p}$ is the only point in X with this property. In some sense we have $|T| = 0$ and $|p| \neq 0$.
- The point $x_{\mathbb{F}_p((T))} \in X$ with the valuation
$$|\,.| : \mathbb{Z}_p[[T]] \xrightarrow{\text{mod } p} \mathbb{F}_p[[T]]$$
Similarly, we have that this is the only unique point in X satisfying this. Roughly speaking this property is just $|T| \neq 0$ and $|p| = 0$.

Now that, we have the three points well known to us, we must investigate the rest of the adic spectrum.

Claim 11. *We have an unique point with open kernel.*

Proof. From the first point we have an existence of a point with open kernel with $|T| = |p| = 0$. Conversely let x be a point with open kernel. Then $\rho = \text{supp}(x) = \ker|\,.|_x$ is an open ideal and hence $I^n \subseteq \rho$ for some $n >> 0$ and hence it implies $\sqrt{I} \subseteq \rho \implies \sqrt{I} = \rho$. □

Hence we must now investigate points whose kernel is not open. That gives us the rest of the points of the adic spectrum. We call such points as "analytic" points.

Let (A, A^+) be a complete Huber pair, and let $X = \text{Spa}(A, A^+)$. Let A_0 be the ring of definition of A and let $I \subset A_0$ be an ideal of definition. Let $x \in X$ be an analytic point with the valuation $|\,.|_x : A \to \Gamma \cup \{0\}$. Since the kernel is not open, we have for some $f \in I$ that $|f(x)| = |f|_x = \gamma \in \Gamma(\text{say})$. We will prove that for any $\gamma' \in \gamma$, $\exists\, n \in \mathbb{N} \ni \gamma^n < \gamma'$. Since the valuation $|\,.|_x$ is continuous then we have the set
$$U = \{x \in A \,|\, |g(x)| = |g|_x < \gamma\}$$
is open. Also since $0 \in V$, we have that V is non-empty and hence there exists an $n \in \mathbb{Z}$ such that the above holds.

Lemma 11. *If Γ is a totally ordered abelian group, with an element $\gamma < 1$, such that for all $\gamma' \in \Gamma$ there exists a sufficiently large positive integer n satisfying $\gamma^n < \gamma'$, then there is a unique order preserving homomorphism $\Gamma \to \mathbb{R}_{>0}$ such that γ is mapped to $\frac{1}{2}$.*

Proof. Since γ is fixed, we take the following homomorphism

$$\theta : \Gamma \to \mathbb{R}_{>0}$$
$$\theta(\gamma) = \inf\{2^{-\frac{a}{b}} \mid a \in \mathbb{N}_0, b \in \mathbb{N} \ni \gamma^a > \gamma'^b\}$$

Clearly this is a homomorphism and we clearly have $\theta(\gamma) = \frac{1}{2}$. □

This above lemma implies that we can define a point $\tilde{x} \in \mathrm{Spa}(A, A^+)$ by the valuation

$$|\,.|_{\tilde{x}} : A \xrightarrow{|.|_x} \Gamma \cup \{0\} \to \mathbb{R}_{\geq 0}$$

where the last morphism is due to the preceeding lemma.

We get back to discussion of $X = \mathrm{Spa}(\mathbb{Z}_p[[T]], \mathbb{Z}_p[[T]])$. Let $\mathcal{Y} = X/\{x_{\mathbb{F}_p}\}$. Then this space, as we mentioned consists entirely of analytic points.

Proposition 21. *There exists an unique and continuous map*

$$\kappa : \mathcal{Y} \to [0, \infty]$$

such that

$$\kappa(x) = r$$

iff

- $\forall \frac{m}{n} \geq r$, *we must have* $|T(x)|^m \geq |p(x)|^n$
- $\forall \frac{m}{n} \leq r$, *we must have* $|T(x)|^m \leq |p(x)|^n$

Proof. We define $\kappa(x) = \frac{\log|T(\tilde{x})|}{\log|p(\tilde{x})|}$ where $\tilde{x}$ is the maximal generalization of x.

Well defined

The map is well defined for two reasons.

i) Since the points in the adic spectrum are analytic each of them has a unique maximal generalisation and hence as result it holds.

ii) The reason why we need unique maximal generalisation is because in the partial order of generalisations, the maximal is the one with rank 1 valuation and conversely if it is a rank 1 valuation it is maximal. (Refer: [Hub96]). Rank 1 valuations are typically the ones that can be embedded into reals and hence the log only makes sense if we take rank 1 valuations and hence it has to be maximal.

Uniqueness

The map is defined such that $\kappa(x) = r$ if and only if $\forall \frac{m}{n} > r$ we have $|T(x)|^m \geq |p(x)|^n$ and $\forall \frac{m}{n} < r$ we have $|T(x)|^m \leq |p(x)|^n$. The idea of the map is to define the map such that we have $|T(x)| \geq |p(x)|^r$ and if we define the set $\{r \in [0,\infty] \mid |T(x)| \geq |p(x)|^r\}$, this set necessarily has a supremum(or infimum) as this is a bounded set of real numbers and the supremum(or infimum) is defined as $\kappa(x)$ by surjectivity.

Continuity

I have map $\kappa(x) : \mathcal{Y} \to [0,\infty]$. I just have to show that inverse image of open sets are open. From definition of the map one can easily observe that $\kappa^{-1}((m/n,\infty]) = \bigcup_{a/b>m/n}\{x \in \mathcal{Y} \| T(x)|^b \leq |p(x)|^a\}$ and $\kappa^{-1}([0,m/n)) = \bigcup_{0\leq a/b<m/n}\{x \in \mathcal{Y} \| T(x)|^b \geq |p(x)|^a\}$ and they are rational subsets which are precisely the open sets in $\mathcal{Y}$.

Surjectivity

Let $\kappa(x) = r \implies \frac{\log|T(\tilde{x})|}{\log|p(\tilde{x})|} = r \implies |T(\tilde{x})| = |p(\tilde{x})|^r$. Now say if $r \leq m/n$ then clearly we have $|T(\tilde{x})| = |p(\tilde{x})|^r \geq |p(\tilde{x})|^{m/n} \implies |T(\tilde{x})|^n \geq |p(\tilde{x})|^m$ (since valuations in adic spectrum valuation less than 1) and by property of generlisations (refer : [Mor]) this implies $|T(x)|^n \geq |p(x)|^m$ (the implications holds in both directions), and hence there is a point $x \in \mathcal{Y}$ such that for the valuation induced by x, $|.|_x$ is such that $|T(x)|^n \geq |p(x)|^m$. (since there exists some valuation for which we can choose it in this way such that the relation holds, you just take the supremum valuation and you are done)

□

Finally we discuss the locus of $\mathcal{Y}_{(0,\infty]} = \kappa^{-1}((0,\infty])$. Since we have $(0,\infty]$ is not compact in $[0,\infty]$ and as a result we have $\mathcal{Y}_{(0,\infty]} = \kappa^{-1}((0,\infty])$ is not quasi-compact. Thus, $\mathcal{Y}_{(0,\infty]}$ cannot be an adic spectrum of some Huber pair. However this is an adic space since

$$\begin{aligned}
\mathcal{Y}_{(0,\infty]} &= \kappa^{-1}((0,\infty]) \\
&= \{x \in X \mid |p(x)| \neq 0\} \\
&= \bigcup_{n\in\mathbb{Z}_{>0}} \{x \in X \mid |T(x)|^n \leq |p(x)| \neq 0\} \\
&= \bigcup_{n\in\mathbb{Z}_{>0}} \{x \in X \mid |T(x)|^n \leq |p(x)| \neq 0, \text{ and } |p(x)| \leq |p(x)| \neq 0\} \\
&= \bigcup_{n\in\mathbb{Z}_{>0}} \mathrm{Spa}(\mathbb{Q}_p\langle \frac{T^n}{p}, T\rangle, \mathbb{Z}_p\langle \frac{T^n}{p}, T\rangle)
\end{aligned}$$

As each $\mathrm{Spa}(\mathbb{Q}_p\langle \frac{T^n}{p}, T\rangle, \mathbb{Z}_p\langle \frac{T^n}{p}, T\rangle)$ gives the ball of radius $|p|^{\frac{1}{n}}$ and $\lim_{n\to\infty} |p|^{\frac{1}{n}} = 1$, we see that it is the union of open disc

$$D_{\mathbb{Q}_p} = \{|T| < 1\}$$

exhausted by the ascending affinoid spaces

$$\mathrm{Spa}(\mathbb{Q}_p\langle\frac{T^n}{p},T\rangle,\mathbb{Z}_p\langle\frac{T^n}{p},T\rangle)=\{|T|\leq|p|^{\frac{1}{n}}\}$$

3. Perfectoid Spaces

Throughout this chapter we fix a prime number p. We present the following definition relevant to the whole chapter

Definition 41. *A non-archimedean topological ring A is uniform if A^0 is bounded.*

3.1. Perfectoid Rings.

3.1.1. *Preliminaries on Strict p-rings.* We begin by defining and obtaining some preliminaries on Strict p-rings.

Lemma 12. *For any ring R and any non-negative integer n, the map $x\mapsto x^{p^n}$ induces a well-defined multiplicative monoid $\theta_n : R/(p)\to R/(p^{n+1})$*

Proof. If $x\equiv y\,(\mathrm{mod}\ p^m)$ for some $m\in\mathbb{N}$, then $x^p-y^p=(x-y)(x^{p-1}+x^{p-2}y+\cdots+y^{p-1})\equiv kp^m.px^{p-1}k_1p^m\equiv 0\,(\mathrm{mod}\ p^{m+1})$, and as a result θ_n is well-defined. And this is clear from definition that this is multiplicative. □

Definition 42. *A ring R of characteristic p is perfect if the Frobenius homomorphism $x\mapsto x^p$ is a bijection. A strict p-ring is a p-torsion free, p-adically complete ring R for which $R/(p)$ is perfect, regarded as a topological ring using the p-adic topology.*

Example 9. The ring $\mathbb{Z}_p$ is a strict p-ring. This is since

- We see that $\mathbb{Z}_p$ is p-torsion free since $px=0\implies x=0$.
- We also know that
$$\mathbb{Z}_p\cong\varprojlim\mathbb{Z}/p\mathbb{Z}$$
and hence is p-adically complete.
- $\mathbb{Z}_p/(p)\cong\mathbb{F}_p$ and finite fields are perfect.

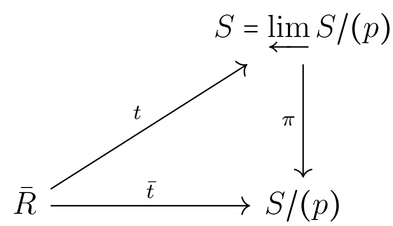
Example 10. Let X be a set(possibly infinite). We denote $\mathbb{Z}[X]$ for the polynomial ring over $\mathbb{Z}$ generated by X. We define $\mathbb{Z}[X^{p^{-\infty}}]:=\bigcup_{n=0}^{\infty}\mathbb{Z}[X^{p^{-n}}]$. And consider the ring $R=\widehat{\mathbb{Z}[X^{p^{-\infty}}]}$. We claim that this is also a strict p-ring.

- Trivially again R is p-torsion free as polynomial ring over $\mathbb{Z}$ is p-torsion free.
- R is definitely p-adically complete from definition.
- Finally we have
$$R/(p)\cong\mathbb{F}_p[X^{p^{-\infty}}]$$
noting that completion doesn't change residue fields.

We prove the following important lemma.

Lemma 13. *Let $\bar{R}$ be a perfect ring of characteristic p, and let S be a p-adically complete ring, and let $\pi : S \to S/(p)$ be the natural projection map. Let $\bar{t} : \bar{R} \to S/(p)$ be a ring homomorphism. Then there exists a unique multiplication map $t : \bar{R} \to S$ with $\pi \circ t = \bar{t}$. In fact we have $t(\bar{x}) \equiv x_n^{p^n} \pmod{p^{n+1}}$ for any non-negative integer n and $x_n \in S$ lifting $\bar{t}(\bar{x}^{p^{-n}})$.*

Proof. The lemma actually implies the following commutative diagram



This can easily be seen using the previous lemma. We just have to check the compatibility of the two maps as per definition. □

Definition 43. *Let R be a strict p-ring. By the case $R = S$ of the previous lemma, the projection $R \to R/(p)$ admits a unique multiplicative section $[\bullet] : R/(p) \to R$, called the Teichmuller map. Each $x \in R$ admits a unique representation as a p-adically convergent sum $\sum_{n=0}^{\infty} p^n [\bar{x}_n]$ for some elements $\bar{x}_n \in R/(p)$, called the Teichmuller corrdinates of x.*

Example 11. Let $R = \mathbb{Z}_p$. Then we have $R/(p) \cong \mathbb{F}_p$. Let's investigate the Teichmuller lift from $[\bullet] : \mathbb{F}_p \to \mathbb{Z}_p$. Firstly from the lemma above, the Teichmuller co-ordinates are supposed to be multiplicative. And hence as a result

$$[x]^{p-1} = [x^{p-1}] = [1] = 1$$

and as a result it consists of 0 along with $(p-1)^{th}$ roots of unity in $\mathbb{Z}_p$.

Lemma 14. *Let R be a strict p-ring, and let S be a p-adically complete ring, amd let $\pi : S \to S/(p)$ be the natural projection. Let $t : R/(p) \to S$ be a multiplicative map such that $\bar{t} = \pi \circ t$ is a ring homomorphism. Then the formula*

$$T\Big(\sum_{n=0}^{\infty} p^n [\bar{x}_n]\Big) = \sum_{n=0}^{\infty} p^n t(\bar{x}_n)$$

defines a p-adically continuous homomorphism $T : R \to S$ such that $T \circ [\bullet] = t$.

Proof. We check directly by induction on n. Our claim is that T induces an additive map $R/(p^n) \to S/(p^n)$.

Base Case: This holds for $n = 1$ since then the map is $R/(p) \to S/(p)$ is basically the map $\pi \circ t$ and both are homomorphisms and as a result the composition is a homomorphism.

Induction Hypothesis: Assume this holds for some $n > 0$.

Inductive Step: Let $x = [x] + px_1$, $y = [y] + py_1$ and $z = [z] + pz_1 \in R$ with $x + y = z$. Applying the previous lemma we get

$$\begin{array}{rcl}[\bar{z}] & \equiv & ([\bar{x^{p^{-n}}}] + [\bar{y^{p^{-n}}}])^{p^n} \pmod{p^{n+1}} \\ t(\bar{z}) & \equiv & (t(\bar{x^{p^{-n}}}) + t(\bar{y^{p^{-n}}}))^{p^n} \pmod{p^{n+1}}\end{array}$$

In particular by the definition of T as in the hypothesis we get

$$T([\bar{z}]) - T([\bar{x}]) - T([\bar{y}]) \equiv \sum_{i=1}^{p^n-1} \binom{p^n}{i} t(\bar{x}^{ip^{-n}} \bar{y}^{1-ip^{-n}}) \pmod{p^{n+1}}$$

From a property of binomial coefficients we know that p divides $\binom{p^n}{i}$ and as a result we may write

$$z_1 - x_1 - y_1 = \frac{[\bar{x}] + [\bar{y}] - [\bar{z}]}{p} \equiv - \sum_{i=1}^{p^n-1} \frac{1}{p} \binom{p^n}{i} [\bar{x}^{ip^{-n}} \bar{y}^{1-ip^{-n}}] \pmod{p^n}$$

We apply T on both sides and multiply by p. Applying induction hypothesis, we get

$$pT(z_1) - pT(x_1) - pT(y_1) \equiv - \sum_{i=1}^{p^n-1} \binom{p^n}{i} t(\bar{x}^{ip^{-n}} \bar{y}^{1-ip^{-n}}) \pmod{p^{n+1}}$$

Since $T(x) = T([\bar{x}]) + pT(x_1)$, $T(y) = T([\bar{y}]) + pT(y_1)$ and $T(z) = T([\bar{z}]) + pT(z_1)$ we add the previous equation with the last equation to get

$$T(z) - T(x) - T(y) \equiv 0 \pmod{p^{n+1}}$$

and we complete the induction.

Hence we deduce that T is additive and also T is p-adically continuous. Now let $x = \sum_{n=0}^{\infty} p^n [\bar{x}_n]$ and $y = \sum_{n=0}^{\infty} p^n [\bar{y}_n]$. Then we compute

$$\begin{array}{rcl}T(x)T(y) & = & \displaystyle\sum_{m,n=0}^{\infty} p^{m+n} t(\bar{x}_m) t(\bar{y}_n) \\ & = & \displaystyle\sum_{m,n=0}^{\infty} p^{m+n} t(\bar{x}_m \bar{y}_n) \\ & = & \displaystyle T(\sum_{m,n=0}^{\infty} p^{m+n} [\bar{x}_m \bar{y}_n]) \\ & = & T(xy)\end{array}$$

and hence T is a ring homomorphism as claimed. □

Theorem 7. *The functor $R \rightsquigarrow R/(p)$ from strict p-rings to perfect rings of characteristic p is an equivalence of categories.*

Proof. By previous lemma we observe that the functor is full and faithful. We will have to show essential surjectivity. Let $\bar{R}$ be a perfect ring of characteristic p, and we choose a surjection $\psi : \mathbb{F}_p[X^{p^{-\infty}}] \to \bar{R}$ for some set X. Let $\bar{I} = \ker(\psi)$. Let $R_0 = \widehat{\mathbb{Z}[X^{p^{-\infty}}]}$, by a previous example this is indeed a strict p-ring. Following the example we put $I = \{\sum_{n=0}^{\infty} p^n[\bar{x}_n] \in R_0 : \bar{x}_0, \bar{x}_1, \dots \in I\}$; this forms an ideal in R_0. Then $R = R_0/I$ is a strict p-ring with $R/(p) \cong \bar{R}$. □

Definition 44. *For $\bar{R}$ a perfect ring of characteristic p, we write $W(\bar{R})$ for the unique (by the previous theorem) to be the unique strict p-ring with $W(\bar{R})/(p) \cong \bar{R}$. This $W(\bar{R})$ is called as the "ring of p-typical Witt vectors over $\bar{R}$".*

3.1.2. *Witt Vectors.* Let p be a prime number and $(X_1, X_2, \dots, X_n, \dots)$ a sequence of indeterminates, and consider the following polynomials

$$\begin{aligned} W_0 &= X_0 \\ W_1 &= X_0^p + pX_1 \\ &\vdots \\ W_n &= X_0^{p^n} + pX_1^{p^{n-1}} + \dots + p^n X_n \\ &\vdots \end{aligned}$$

These are called "Witt Polynomials". If $\mathbb{Z}'$ denotes the ring $\mathbb{Z}[p^{-1}]$, it is clear that X_i can be expressed as polynomials with respect to the W_i with coefficients in $\mathbb{Z}'$:

$X_0 = W_0,\ X_1 = p^{-1}W_1 - W_0^p, \dots,$ etc.

Let $(Y_0, \dots, Y_n, \dots)$ be another sequence of indeterminates

Theorem 8. *For every $\Phi \in \mathbb{Z}[X, Y]$, there exists a unique sequence $(\varphi_0, \varphi_1, \dots, \varphi_n, \dots)$ of elements of $\mathbb{Z}[X_0, \dots, X_n, Y_0, \dots, Y_n, \dots]$ such that*

$$W_n(\varphi_0, \dots, \varphi_n, \dots) = \Phi(W_0(X_0, \dots), \dots, W_n(X_0, \dots))$$

Proof. [Ser79] □

We apply this theorem to $\Phi(X, Y) = X + Y$ and $\Phi(X, Y) = XY$ and we get sequences of polynomials.

If A is an arbitrary commutative ring, and if $\mathfrak{a} = (a_0, a_1, \dots, a_n, \dots)$, $b = (b_0, b_1, \dots, b_n, \dots)$ are the elements of $A^{\mathbb{N}}$ called "Witt Vectors with coefficients in A". We set

$$\begin{aligned} \mathfrak{a} + \mathfrak{b} &= (S_0(a, b), S_1(a, b), \dots, S_n(a, b), \dots) \\ \mathfrak{a}.\mathfrak{b} &= (P_0(a, b), \dots, P_n(a, b), \dots) \end{aligned}$$

This motivates us to do the following

Theorem 9. *The laws of composition defined above make $A^{\mathbb{N}}$ into a commutative unitary ring called the ring of Witt vectors with coefficients in A and we denote them by $W(A)$.*

Proof. Refer [Ser79] □

Example 12. We compute some small values of these polynomials

$$\begin{aligned} S_0(a,b) &= a_0 + b_0 \\ S_1(a,b) &= a_1 + b_1 + \frac{a_0^p + b_0^p - (a_0+b_0)^p}{p} \end{aligned}$$

and as far as product is concerned

$$\begin{aligned} P_0(a,b) &= a.b \\ P_1(a,b) &= b_0^p a_1 + b_1 a_0^p + p a_1 b_1 \end{aligned}$$

We denote by W_* the map $W(A) \to A^{\mathbb{N}}$ as

$$\begin{aligned} W_* : W(A) &\to A^{\mathbb{N}} \\ a &\mapsto (W_0(a), W_1(a) \dots) \end{aligned}$$

This map can be observed to be an isomorphism of rings if p is invertible in A.

The ring morphism $W_m : W(A) \to A$ is the called the m^{th} ghost component map.

Example 13. Let's recall definition 44 and try to compute for $\bar{R} = \mathbb{F}_p$. Clearly it is perfect and we need a strict p-ring $W(\bar{R})$ such that the definition holds. In other words we need a ring such that $W(\bar{R})/(p) \cong \mathbb{F}_p$. But we already have that $\mathbb{Z}_p/(p) \cong \mathbb{F}_p$ and $\mathbb{Z}_p$ is indeed a strict p-ring. Since by theorem 7, the ring is unique we have $W(\mathbb{F}_p) = \mathbb{Z}_p$.

Again when we do the truncation of any n-components we have $W_n(\mathbb{F}_p) = \mathbb{Z}/p\mathbb{Z}$.

Example 14. Let's do the same for $R = \mathbb{F}_q$ where $q = p^r$. Then again following definition 44 we get that $R \cong W(R)/(p) \implies \mathbb{F}_{p^r} \cong W(R)/(p)$. Now we know one thing from algebraic number theory that if $K/\mathbb{Q}_p$ is an unramified finite extension of degree r, then we have the residue field $\mathcal{O}_K/(p) \cong \mathbb{F}_{p^r}$ and hence $W(R) = \mathcal{O}_K$.

Example 15. Following the above example if $R = \overline{\mathbb{F}_p}$ then $W(R) = \mathcal{O}_{\widehat{Q_p^{\text{unr}}}}$

Example 16. Let $R = \mathbb{F}_p[T^{1/p^\infty}]$. We know that since completion doesn't change base field we have $\mathbb{Z}\widehat{[T^{1/p^\infty}]} = \mathbb{Z}_p[T^{1/p^\infty}]$ and $\mathbb{Z}_p[T^{1/p^\infty}]/(p) \cong \mathbb{F}_p[T^{1/p^\infty}]$ and we are done from here since this suggests $W(R) = \mathbb{Z}_p[T^{1/p^\infty}]$

3.1.3. *Perfectoid Tate Rings.*

Definition 45. *Let A be a Tate ring. We say that A is perfectoid if A is complete and uniform, and if there exists a pseudo-uniformizer(topologically nilpotent unit)* $\varpi \in A$ *such that*

(a) ϖ^l *divides p in* A^0
(b) the frobenius map $Frob : A^0/\varpi \to A^0/\varpi^l, a \to a^l$ *is a bijection.*

If A is perfectoid of characteristic 0, then p is topologically nilpotent (since it is a multiple in A^0 of some uniformizer), so, if p is invertible in A, then the topology on A^0 is the p-adic topology and $A = A^0[\frac{1}{p}]$.

Proposition 22. *Let A be a Tate ring, and let ϖ be a pseudo-uniformizer of A such that ϖ^p divides p in A^0. Then we have*

(i) *The Frobenius map is injective.*
(ii) *If A is complete and uniform, then the following conditions are equivalent*
   (a) *every element of $A^0/p\varpi A^0$ is a p^{th} power.*
   (b) *every element of A^0/pA^0 is a p^{th} power.*
   (c) *ever element of $A^0/\varpi^p A^0$ is a p^{th} power.*

Moreover if these conditions holds then we have units $u, v \in A^0$ such that $u\varpi$ and vp admit compatible systems of p-power roots in A^0.

Proof. (i) Let $a \in A^0$ such that $a^p \in \varpi^p A^0$. Then $(a\varpi^{-1})^l \in A^0$ and hence $a\varpi^{-1} \in A^0 \implies a \in \varpi A^0$.

(ii) From the hypothesis we get ϖ^p divides $p$and also p divides $p\varpi \in A^0$. Then clearly we have the following inclusions $A^0/\varpi^p A^0 \subset A^0/pA_0 \subset A^0/p\varpi A^0$ and hence we clearly have the forward implications (a) to (b) and (b) to (c). We are just left to show that (c) $\implies$ (a). Let $a \in A^0$. We prove the following important lemma

Lemma 15. *Suppose that every element of $A^0/\varpi^p A^0$ is a p^{th} power, and let $p \in A^0$. Then there exists a sequence $(a_n)_{n\geq 0}$ of elements of A_0 such that for every $n \in \mathbb{N}$, we have*

$$a - \sum_{i-0}^{n} a_i^p \varpi^{ip} \in \varpi^{p(n+1)} A^0$$

Proof. We construct the elements a_n by induction on n. Then our induction assumption implies that there exists $a_0 \in A^0$ such that $a - a_0^p \in \varpi^p A^0$. Suppose that $n \geq 1$ and that we have found $a_0, a_1, \ldots, a_{n-1}$ such that $a - \sum_{i=0}^{n-1} a_i^{p_i} \varpi^{p_i} \in \varpi^{p_n} A^0$. Let $b \in A^0$ such that the hypothesis holds, and we choose $a_n \in A^0$ such that $b - a_n^p \in \varpi^p A^0$. Then $a - \sum_{i=0}^{n} a_i^p \varpi^{p_i} \in \varpi^{p(n+1)} A^0$ □

Using the above lemma we have a sequence $\{a_n\}_{n\geq 0}$ of elements of A^0 such that $a = \sum_{n\geq 0} a_n^p \varpi^{p_n}$. We now use another important lemma

Lemma 16. $\forall\, a, b \in A^0$*, we have* $(a + \varpi b)^p - a^p - (\varpi b)^p \in \varpi p A^0$

Proof. Using the binomial theorem we see that

$$\begin{aligned}(a + \varpi b)^p - a^p - (\varpi b)^p &= \sum_{k=0}^{p} \binom{p}{k} a^k (\varpi b)^{p-k} - a^p - (\varpi b)^p \\ &= \sum_{k=1}^{p-1} \binom{p}{k} a^k (\varpi b)^{p-k}\end{aligned}$$

Now we have clearly $a \in A^0$ and $\varpi b \in \varpi A^0$ and as a result their product lies in ϖA^0. But we have $p | \binom{p}{k}$ and hence as a result the who product lies in $\varpi p A^0$ □

Using this lemma directly gives (a).

To establish the last statement we need the following important lemma

Lemma 17. *Let S be a ring and $\varpi \in S$. Suppose that S is ϖ-adically complete and Hausdorff and $\varpi \mid l$ in S. The canonical map*

$$\varprojlim_{a\mapsto a^l} S \to \varprojlim_{a\mapsto a^l} S/\varpi S$$

is an isomorphism of topological monoids.

Proof. Let $S_1 = \varprojlim_{a\mapsto a^l} S$ and $S_2 = \varprojlim_{a\mapsto a^l} S/\varpi S$. We construct a continuous multiplicative inverse of the canonical map $S_1 \to S_2$. First, we construct a multiplicative map $\alpha : S_2 \to S$ such that $\alpha((\bar{s}_n)_{n\geq 0}) = \bar{s}_0 \bmod \pi S$. Let $(\bar{s}_n)_{n\geq 0} \in S_2$. Choose representatives $s_n \in S$ of the $\bar{s}_n$.

Claim 12. *We claim that:*
(i) $\lim_{n\to+\infty} s_n^{l_n}$ *exists.*
(ii) The limit above is independent on the choice of representatives s_n.

Proof. To prove (i) we observe that $\forall\, n \in \mathbb{N}$, we have $s_{n+1}^l - s_n \in \varpi S$. Applying the previous lemms inductively and using ϖ divides l we get $s_n^{l^{n+1}} - s_n^{l^n} \in \varpi S$ and hence we see that $(s_n^{l^n})_{n\geq 0}$ is Cauchy in S, and hence it is convergent.

To prove (ii), we choose some other lifts s_n' of the $\bar{s}_n$. Then $\forall\, n \in \mathbb{N}$, we have $s_n' - s_n \in \varpi S$, so as before we get $(s_n')^{l^n} - s_n^{l^n} \in \varpi^{n+1} S$. And as a result we have $\lim_{n\to+\infty} s_n^{l^n} = \lim_{n\to+\infty} (s_n')^{l^n}$ □

Using this claim, we can define a map $\alpha : S_2 \to S$ by sending $(\bar{s}_n)_{n\geq 0} \mapsto \lim_{n\to+\infty} s_n^{l^n}$, where the $s_n \in S$ are any lifts of $\bar{s}_n$. Clearly α is multiplicative and also continuous. We also observe that if $\bar{s} = (\bar{s}_n)_{n\geq 0} \in S_2$, then it has a canonical n^{th} root, which is it's shift $\bar{s}' = (\bar{s}_{n+1})_{n\geq 0}$, and we have $\alpha(\bar{s}')^p = \alpha(\bar{s})$ by definition of α.

Hence we get the desired map $S_2 \to S_1$ by sending $(\bar{s}_n)_{n\geq 0}$ to the sequence $(\alpha((\bar{s}_{r+n})_{n\geq 0})_{r\geq 0}$ □

which is in S_1.

Applying the above lemma to A^0 and to $A^0/p\pi A^0$, the canonical map is an isomorphism. In particular, we can find $\omega = (\omega^{(n)}) \in \varprojlim_{a\to a^l} A^0$ such that $\omega^{(0)} \equiv \varpi \bmod \varpi p A^0$. In other words, there exists $a \in A^0$ such that $\omega^{(0)} = \varpi(1+pa)$. THe claim now follows since $\forall\, a \in A^0, 1+\omega a$ and $1+la$ are units in A^0 □

To prove that a Tate ring is perfectoid it is much easier to use the following criterion which is an immediate consequence of the above proposition.

Corollary 2. *Let A be a complete uniform Tate ring. Then the following are equivalent*

(i) A is perfectoid
(ii) every element of A^0/pA^0 is a p^{th} power, and A has a pseudouniformizer ϖ such that ω^p divides p in A_0.

Proof. To prove (i) $\implies$ (ii) : Since A is perfectoid, we have from definition that the Frobenius is injective and also a pseudouniformizer ϖ such that ϖ^p divides p in

A^0. Also since by hypothesis A is complete and uniform we also have from previous proposition every element is a p^{th} power in A^0/pA^0.

To prove (ii) $\implies$ (i) : All we are left to prove is that the Frobenius is bijective. By previous proposition (i) we have Frobenius is injective. The surjection is clear since by (ii) of the proposition we have every element is p^{th} power in $A^0/\varpi^p A^0$, and as a result we also have existence of compatible system of p^{th} power roots. □

We just want to bring attention to the following important fact

Proposition 23. *Let A be a perfectoid ring. Then A is reduced.*

Proof. Since we have $A = A_0[\frac{1}{\varpi}]$ it suffices to show that A_0 is reduced. We choose an $a \in A^0$ such that $a^N = 0$ for some large enough N. Then $(a/\varpi^n)^N = 0$ for any $n \geq 1$ and $N >> 1$, and so a/ϖ^n is power-bounded for all $n \geq 1$. Then $a \in \bigcap \varpi^n A^0$, which is 0 since ϖ is topologically nilpotent. □

We begin to look at two particular cases : Perfectoid fields and perfectoid rings of characteristic p.

Proposition 24. *Let A be a Tate ring of characteristic p. Then the following are equivalent*

(i) A is perfectoid.
(ii) A is complete and perfect.

Proof. To prove (i) $\implies$ (ii) : Let A be perfectoid $\implies$ A is complete. Since we know $A = A^0[\frac{1}{\varpi}]$ it suffices to show that A^0 is perfect. As $p = 0$ in A, we have $A^0/pA^0 = A^0$ and hence the conclusion follows.

To prove (ii) $\implies$ (i) : Let A be complete and perfect. Then it is also uniform. So by the corollary 2 it just suffices to show that every element of A^0 is a p^{th} power, but this follows clearly from the fact that A is perfect. □

We the ring is a field and satisfies definition 44, we call it a perfectoid field. We state the following important result of Kedlaya without proof

Theorem 10. *(Kedlaya) Let A be a perfectoid Tate ring that is also a field. Then the topology on A is given by a rank 1 valuation; hence A is a complete non-archimedean field.*

Proposition 25. *Let K be a complete topological field. Then the following conditions are equivalent:*

(i) K is a perfectoid field.

(ii) the topology on K is given by rank 1 valuation $|.|$ satisfying the following conditions
(a) $|.|$ is not discrete
(b) $|p| < 1$
and the p^{th} power map is on K^0/pK^0 is surjective.

Proof. To prove (i) $\implies$ (ii): Since K is perfectoid, then we have that the it's topology is given by rank 1 valuation by the theorem of Kedlaya. By corollary 2 we have that the p^{th} power map is surjective, and K has a topologically nilpotent unit ϖ such that ϖ^p divides p in K^0. In particular, p is topologically nilpotent in K so $|p| < 1$.

To prove (ii) $\implies$ (i): Since K satisfies (ii), we have that $K^0 = \{a \in K \,|\, |a| \leq 1\}$, so K^0 is bounded in K and hence K is uniform. Now since every element of A^0/pA^0 is a p^{th} power by assumption we just need to check that ϖ^p divides p in K^0. But this is simple since $|\varpi| < 1$, and also $|p\varpi^{-p}| \leq 1$, we have $p\varpi^{-p} \in K^0$, and hence that implies. □

3.1.4. *Examples of perfectoid rings and fields.*

Example 17. The field $\mathbb{Q}_p$ is not perfectoid since the topology is not given by a discrete valuation.

Example 18.

Claim 13. *Any algebraically closed, complete non-archimedean field is perfectoid.*

Proof. The complete non-archimedean field condition is satisfied. We just have to check that the Frobenius is bijective. But this is clear since the field is algebraically closed then $x^p - a$ always has a solution which shows surjectivity and also we see that since it is complete. To see that is uniform we have to see the fact from previous chapter that A^0 is integrally closed in A and hence is bounded. Since this complete, uniform and Tate then the map $R^0/\varpi R^0 \to R^0/\varpi^p R^0$ is an injection by the Frobenius and we have the result. □

This claim suggests that $\mathbb{C}_p$ is perfectoid.

Example 19. Let $\mathbb{Q}_p^{\text{cycl}} = \widehat{\mathbb{Q}_p(\mu_p^{1/p^\infty})}$ where $\mathbb{Q}_p(\mu_p^{1/p^\infty}) := \bigcup_{n\geq 0} \mathbb{Q}_p(\mu_p^{1/p^n})$ for the unique valuation $|.|$ extending to the p-adic valuation on $\mathbb{Q}_p$. We let $\mathbb{Z}_p^{\text{cycl}} = (\mathbb{Q}_p^{\text{cycl}})^0$. Then $\mathbb{Q}_p^{\text{cycl}}$ is perfectoid.

Proof. Clearly this is complete and non-Archimedean. Now $\forall\, r \geq 1$, the cyclotomic extension $\mathbb{Q}_p(\mu_{1/p^r})/\mathbb{Q}_p$ is of degree $p^{r-1}(p-1)$, and, if ω is a primitive p^r-th root of unity, then $N_{\mathbb{Q}_p(\omega)/\mathbb{Q}_p}(1-\omega) = p$, hence $|\omega|^{p^{r-1}p} = |p|$ and hence we show that $|.|$ is not a discrete valuation. We obviously have $|p| < 1$. Finally, let $\bar{a} \in \mathbb{Z}_p^{\text{cycl}}/p\mathbb{Z}_p^{\text{cycl}}$. We can find a lift a of $\bar{a}$ in the ring of integers of $\mathbb{Q}_p(\mu_{1/p^n})$ for some $n \geq 0$. Pick a primitive p^{n+1}-th root of unity ω. Then we can write $a = \sum_{i=1}^{p^n-1} a_i\omega^i$, with $a_i \in \mathbb{Z}_p$. Then, if $b = \sum_{i=1}^{p^n-1} a_i\omega^i$, then $b^p = a$ modulo $p\mathbb{Z}_p^{\text{cycl}}$. □

Example 20. Motivated by the previous example we can construct a new example. Let $L = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ where $\mathbb{Q}_p(p^{1/p^\infty}) := \bigcup_{n\geq 0}\mathbb{Q}_p(p^{1/p^n})$ for the unique valuation $|.|$ extending the p-adic valuation on $\mathbb{Q}_p$ is a perfectoid field.

Proof. By definition, L is complete and non-Archimedean. and the valuation group of $|.|$ is not isomorphic to $\mathbb{Z}$ because it's element $|p| < 1$ is divisible by p^r for every $r \in \mathbb{N}$. Also, we have $L^0/pL^0 = \mathbb{F}_p$, so every element of L^0/pL^0 is a p^{th} power. And hence by Corollary 2 we are done. □

Example 21. Let S be an absolutely integrally closed field. Then we have $\widehat{S}^p$(meaning p-adic completion) is perfectoid.

Proof. Since S is absolutely integrally closed then for any element a we have $x^p - a = 0$ has solutions and the Frobenius is surjective on $S/(p)$.

Since completion doesn't change the residue so completion doesn't affect and hence it is perfectoid. □

3.2. **Tilting.**

Proposition 26. *Let A be a perfectoid Tate ring. We consider the set*

$$A^\flat = \varprojlim_{a\to a^p} A := \{(a^{(n)}) \in A^{\mathbb{N}} \mid \forall\, n \in \mathbb{N}, (a^{(n+1)})^p = a^{(n)}\}$$

with the projective limit topology, we define the pointwise multiplication and addition defined by $(a^{(n)}) + (b^{(n)}) = (c^{(\ltimes)})$, with

$$c^{(n)} = \lim_{r\to+\infty}(a^{(n+r)} + b^{(n+r)})^{p^r}$$

We denote the map $A^\flat \to A$, $(a^{(n)}) \to a^{(0)}$ by $f \to f^\sharp$.

Then:

(i) The addition is well defined and makes $A^\flat$ into a perfectoid Tate ring of characteristic p.
(ii) The subring $A^{\flat 0}$ of power bounded elements in $A^\flat$ is given by

$$A^{\flat 0} = \varprojlim_{a\to a^p} A^0$$

If ϖ is a pseudo-uniformizer of A that divides p in A^0, then the canonical map

$$A^{\flat 0} \to \varprojlim_{a\to a^p} A^0/\varpi$$

is an isomorphism of topological rings.

(iii) *There exists a pseudo-uniformizer* $\varpi \in A$ *such that* ϖ^p *divides* p *in* A^0 *and that* ϖ *is in the image map* $(.)^\sharp : A^\flat \to A$. *Moreover if* $\varpi^\flat$ *is an element of* $A^\flat$ *such that* $\varpi = (\varpi^\flat)^\sharp$, *then* $\varpi^\flat$ *is a pseudo-uniformizer of* $A^\flat$, *the map* $f \to f^\sharp$ *induces a ring isomorphism* $A^{\flat 0}/\varpi^\flat \cong A^0/\varpi$, *and* $A^\flat = A^{\flat 0}[\frac{1}{\varpi^\flat}]$

Proof. □

Definition 46. *The perfectoid Tate ring* $A^\flat$ *in the previous proposition is called the tilt of* A.

we begin with an important remark.

Remark 8. Let A be a perfectoid Tate ring. The map $(\,.)^\sharp : A^{\flat 0} \to A^0$ induces an isomorphism of rings $A^{\flat 0}/A^{\flat 00} \xrightarrow{\simeq} A^0/A^{00}$

Proof. We first note that A^{00} is an ideal of A^0 and same for the respective tilts and hence the statement makes sense. By the previous proposition we can choose a pseudouniformizer $\varpi^\flat$ of $A^\flat$ such that $\varpi := (\varpi^\flat)^\sharp$ is a pseudo-uniformizer of A, and then $(\,.)^\sharp$ induces an isomorphism $A^{\flat 0}/\varpi A^{\flat 0} \xrightarrow{\simeq} A^0/\varpi A^0$. We have $A^{00} \supset \varpi A^0$, and hence $A^{00}/\varpi A^0$ is the nilradical of $A^0/\varpi A^0$. Similar analogy holds for the tilt and hence the result. □

We state the following proposition without proof

Proposition 27. *Let* R *be a perfectoid ring. Then there exists bijection between the two sets*

$$\left\{ \begin{array}{c} R^+ \subset R^0 \text{ : ring of integral} \\ \text{elements} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} R^{\flat,+} \subset R^{\flat 0} \text{ : ring of integral} \\ \text{elements.} \end{array} \right\}$$

$$R^+ \mapsto \varprojlim_{x \to x^p} R^+$$

In particular we have

$$\varprojlim_{x \to x^p} R^0 = R^{\flat 0}$$

Definition 47. *If* A *is a perfectoid Tate ring and* $A^+ \subset A$ *is a ring of integral elements, we say that* (A, A^+) *is a perfectoid Huber pair. We call* $(A^\flat, A^{+\flat})$ *to be it's tilt.*

We state a proposition without proof which is important

Theorem 11. *Let* (R, R^+) *be a perfectoid Huber pair with tilt* $(R^\flat, R^{\flat +})$. *Then there exists a homeomorphism between topological spaces*

$$\begin{array}{rcl} X = Spa(R, R^+) & \xrightarrow{\simeq} & X^\flat = Spa(R^\flat, R^{\flat +}) \\ x & \mapsto & (x^\flat : f \mapsto |f^\sharp(x)|) \end{array}$$

The proof relies on Proposition V.1.2.7 of [Mor].

We give some examples of tilting of some important fields.

Example 22. Let K be a perfectoid field of characteristic p with residue field k. Then $K \supset \widehat{k((t^{\frac{1}{p^\infty}}))}$, where t is any element of K with $0 < |t| < 1$, i.e t is pseudo-uniformizer of K:

(i) $\mathbb{F}_p \hookrightarrow K^\flat$, (the inclusion map being the Teichmuller map) since $\mathrm{char}(K^\flat) = p$.
(ii) $K^\flat$ is non-archimedean field of characteristic p, so $\mathbb{F}_p((t)) \hookrightarrow K^\flat$.
(iii) $K^\flat$ is perfect, and hence it contains the perfect closure of $\mathbb{F}_p((t))$ which is $\mathbb{F}_p((t^{\frac{1}{p^\infty}}))$
(iv) $K^\flat$ is t-adically complete so it contain the t-adic completion of $\mathbb{F}_p((t^{\frac{1}{p^\infty}}))$.

Example 23. Let $K = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$. We have seen in the previous section that K is perfectoid. Let's compute it's tilt. From definition $K^\flat = \varprojlim_{x \to x^p} K$. We observe that $K^\flat$ contains the element $t := (p, p^{1/p}, p^{1/p^2}, \dots)$ and $|t|_{K^\flat} = |p|_K$ so t is a pseudo-uniformizer of $K^\flat$. Since $K^\flat$ is perfectoid of characteristic p and residue field $\mathbb{F}_p$(since $\kappa_K = \kappa_{K^\flat}$), so K contains $\widehat{\mathbb{F}_p((t^{1/p^\infty}))}$.

Claim 14. $K^\flat = \widehat{\mathbb{F}_p((t^{1/p^\infty}))}$

Proof. We know that

$$K^{\flat 0} = \varprojlim_{x \to x^p} K^0/(p), \quad K^\flat = K^{\flat 0}[\frac{1}{t}]$$

We have

$$\begin{aligned} K^{\flat 0} &= \varprojlim_{x \to x^p} K^0/(p) \\ &\cong \varprojlim_{x \to x^p} \mathbb{F}_p[t^{1/p^\infty}]/(t) \\ &\cong \widehat{\mathbb{F}_p[[t^{1/p^\infty}]]} \end{aligned}$$

and hence we have $K^\flat = \widehat{\mathbb{F}_p((t^{1/p^\infty}))}$. The reason for the last isomorphism is since this is perfect ring of characteristic p then the inverse limit is just the t-adic completion. $\square$

Example 24. Let $K = \widehat{\mathbb{Q}_p(\mu_{p^\infty})}$. We have already seen in the previous chapter this is perfectoid. Let's compute it's tilt. Before we do that let's evaluate the important expression

$$
\begin{aligned}
K^0/(p) &= \widehat{\mathbb{Z}_p[\mu_{p^\infty}]}/(p) \\
&= \mathbb{Z}_p[\mu_{p^\infty}]/(p) \\
&= \mathbb{Z}_p[\zeta_p, \zeta_{p^2}, \zeta_{p^3} \dots]/(p) \\
&\cong \frac{\mathbb{Z}_p[x_1, x_2, x_3, \dots]/ < x_1^{p-1} + x_1^{p-1} + \dots + x_1 + 1, x_2^p - x_1, x_3^p - x_2, \dots >}{(p)} \\
&\cong \mathbb{F}_p[x, x^{1/p}, x^{1/p^2}, \dots]/ < x^{p-1} + x^{p-2} + \dots + x^2 + x + 1 > \\
&= \mathbb{F}_p[x^{1/p^\infty}]/ < x^{p-1} + x^{p-2} + \dots + x^2 + x + 1 > \\
&= \mathbb{F}_p[x^{1/p^\infty}]/\frac{x^p - 1}{x - 1} \\
&= \mathbb{F}_p[x^{1/p^\infty}]/\frac{(x-1)^p}{x - 1} \\
&= \mathbb{F}_p[x^{1/p^\infty}]/(x-1)^{p-1} \\
&= \mathbb{F}_p[(1+t)^{1/p^\infty}]/t^{p-1} \\
&= \mathbb{F}_p[(1+t^{1/p^\infty})]/t^{p-1} \\
&= \mathbb{F}_p[t^{1/p^\infty}]/t^{p-1}
\end{aligned}
$$

Now it will be easier to compute the tilt.

$$
K^{\flat 0} = \varprojlim_{x \to x^p} K^o/(p) \cong \varprojlim_{x \to x^p} \mathbb{F}_p[t^{1/p^\infty}]/t^{p-1} = \mathbb{F}_p\widehat{[[t^{1/p^\infty}]]} \implies K^\flat = \mathbb{F}_p\widehat{((t^{1/p^\infty}))}
$$

3.2.1. *Appendix : General definition of Tilting.*

In this chapter we have introduced tilting for just perfectoid tate rings. In fact we can define it for rings which are not perfectoid Tate. We follow [Ked15].

Definition 48. *A field K is called analytic if it is complete with respect to a multiplicative non-archimedean norm $|.|$. For K an analytic field we denote $\mathfrak{o}_K = \{x \in K \,|\, |x| \leq 1\}$; then this is a local ring with maximal ideal $\mathfrak{m}_K = \{x \in K \,|\, |x| < 1\}$. We say K has mixed characteristics if $|p| = p^{-1}$ and the residue field $\kappa_K = \mathfrak{o}_K/\mathfrak{m}_K$ of K has characteristic p.*

Let K be an analytic field of mixed characteristic. We denote

$$
\mathfrak{o}_{K'} = \varprojlim_{a \to a^p} \mathfrak{o}_K/(p) = \{(\bar{x}_n) \in \prod_{n=0}^{\infty} \mathfrak{o}_K/(p) : \bar{x}_{n+1}^p = \bar{x}_n\}
$$

By construction, this is a perfect ring of characteristic p: the inverse of Frobenius is the shift map $(\bar{x}_n)_{n=0}^\infty \mapsto (\bar{x}_{n+1})_{n=0}^\infty$. Using the lemmas 13 and 14 to the homomorphism $\bar{\theta} : \mathfrak{o}_{K'} \to \mathfrak{o}_K/(p)$, we obtain a multiplicative map $\theta : \mathfrak{o}_{K'} \to \mathfrak{o}_K$ and a homomorphism $\Theta : W(\mathfrak{o}_{K'}) \to \mathfrak{o}_K$.

For $\bar{x} = (\bar{x}_0, \bar{x}_1, \dots) \in \mathfrak{o}_{K'}$, define $|\bar{x}|' = |\theta(x)|$. If we lift $\bar{x}_n \in \mathfrak{o}_K/(p)$ to $x_n \in \mathfrak{o}_K$, then the $|\bar{x}|' = |x_n|^{p^n}$ whenever $|x_n| > |p|$. Consequently, $|\,.|'$ is a multiplicative non-archimedean norm on $\mathfrak{o}_{K'}$ under which it is complete.

Lemma 18. *With the above notations, for $\bar{x}, \bar{y} \in \mathfrak{o}_{K'}$, $\bar{x}$ is divisible $\bar{y} \iff |\bar{x}|' \le |\bar{y}|'$.*

Proof. If $\bar{x}$ is divisible by $\bar{y}$, then $|\bar{x}|' \le |\bar{y}|'|\bar{x}/\bar{y}|' \le |\bar{y}|'$. Conversely if $|\bar{x}|' \le |\bar{y}|'$, then we write $\bar{x} = (\bar{x}_0, \bar{x}_1, \dots)$ and $\bar{y} = (\bar{y}_0, \bar{y}_1, \dots)$. We can assume that $x, y \ne 0$ since if they are 0 then there is nothing to prove. Now we choose lifts of each component $\bar{x}_n, \bar{y}_n$ to x_n, y_n to $\mathfrak{o}_K$. Since $\bar{y} \ne 0$, and we have $\bar{y}_{n+1}^p = \bar{y}_n$.

Claim 15. $\lim_{n\to\infty} |y_n|^{1/p^n} = 1$

Proof. Since $y_n \in \mathfrak{o}_K$ hence we have $|y_n| < 1$ and hence as a result if we let the desired limit to be L then we can re-write $L = e^{\frac{\ln a_n}{p^n}}$ and as $n \to \infty$ L goes to 1. □

By this claim we have an integer $n_0 \ge 0 \ni |y_n| \ge p^{-1+1/p}$ for $n = n_0$, and also for $n \ge n_0$. Then for $n \ge n_0$, the elements $z_n = x_n/y_n \in \mathfrak{o}_K$ have the property that $|z_n^p - z_n| \le p^{-1/p}$. By writing $z_{n+2}^{p^2} = (z_{n+1} + (z_{n+1}^p - z_{n+1}))^p$, and applying the inequality we get that $|z_{n+2}^{p^2} - z_{n+1}^p| \le p^{-1}$. Hence we constructed an element $\bar{z} = (\bar{z}_0, \bar{z}_1, \dots)$ with $\bar{x} = \bar{y}\bar{z}$ by taking $\bar{z}_n$ to be the reduction of z_{n+1}^p for $n \ge n_0 + 1$. □

By the above lemma we see that $\mathfrak{o}_{K'}$ a valuation ring. This also suggests that $\mathfrak{o}_{K'}$ is a valuation ring of the fraction field of $\mathfrak{o}_{K'}$. We denote it by K'.

Definition 49. *The field K' is called the tilt of K.*

Remark 9. The map θ is explicitly given as $\theta : \mathfrak{o}_{K'} \to \mathfrak{o}_K$ is given by $x \mapsto x^\sharp$ where $x^\sharp := \lim_{i\to\infty} x_i^{q^i} \in \mathfrak{o}_K$

In general the functor $K \rightsquigarrow K'$ is called the tilting functor. In general this is not faithful.

Example 25. Let $K = \mathbb{Q}_p$. Then $\mathfrak{o}_K = \mathbb{Z}_p$. Then we have

$$\mathfrak{o}_{K'} = \varprojlim_{a\to a^p} \mathfrak{o}_K/(p) = \varprojlim_{a\to a^p} \mathbb{Z}_p/(p) = \varprojlim_{a\to a^p} \mathbb{F}_p = \mathbb{F}_p$$

where the last equality follows since finite fields are perfect and Frobenius is already an isomorphism. And hence $K' = \mathbb{F}_p$.

Let $K = \mathbb{F}_p(t)$. Then we have

$$\mathfrak{o}_{K'} = \varprojlim_{a\to a^p} \mathfrak{o}_K/(p) = \varprojlim_{a\to a^p} \mathbb{F}_p(t)$$

To compute this inverse limit we observe that the Frobenius in this case is an inclusion and hence the inverse limit is $\bigcap \mathbb{F}_p(t^{1/p^n}) = \mathbb{F}_p$. And hence $K' = \mathbb{F}_p$.

The above example shows that the tilting functor is faithful. In general we have

Claim 16 (Exercise 1.3.1 of [Ked15]). *Let K be a discretely valued analytic field of mixed characteristics. Then K' is isomorphic to the maximal perfect subfield of κ_K*

Proof. Let π, be the uniformizer of K. We follow the same strategy. We get

$$\mathfrak{o}_{K'} = \varprojlim_{a \to a^p} \mathfrak{o}_K/(p) = \varprojlim_{a \to a^p} \mathfrak{o}_K/(\pi) = \varprojlim_{a \to a^p} \kappa_K$$

We have the following commutative diagram

$$\begin{array}{ccc} \kappa_K & \xrightarrow{(.)^p} & \kappa_K \\ {\scriptstyle x \mapsto x^p}\downarrow & & \downarrow{\scriptstyle y \mapsto y} \\ \kappa_K^p & \xrightarrow[a \mapsto a]{} & \kappa_K \end{array}$$

Since the top arrow is an isomorphism, so is the bottom arrow. And hence computing inverse limits of the above system results in just looking at the system below and we have the limit as $\bigcap \kappa_K^p$ and since it is a field it has the same fraction field which is the maximal perfect subfield of κ_K. □

But the point in this chapter dealing with perfectoid fields is that, when restricted to perfectoid fields this functor is indeed faithful and moreover an equivalence of categories of perfectoid analytic fields and perfect analytic fields of characteristic p. Notice when we are basically restrcting to analytic fields with a great deal of ramification.

We also want to prove another property of tilting and breifly prove it below

Claim 17. *(Exercise 1.2.7 of [Ked15]) The mapping $x \mapsto (\theta(x^{p^{-n}}))_{n=0}^{\infty}$ defines a multiplicative bijection from $\mathfrak{o}_{K'}$ to the inverse limit of multiplicative monoids*

$$\{(x_n) \in \prod_{n=0}^{\infty} \mathfrak{o}_K : x_{n+1}^p = x_n\}$$

This map extends to a multiplicative bijection from K' to the inverse limit of multiplication monoids

$$\{(x_n) \in \prod_{n=0}^{\infty} K : x_{n+1}^p = x_n\}$$

Proof. The asserted projection map is given by $\theta' : \mathfrak{o}_{K'} \to \varprojlim_{x \to x^p} \mathfrak{o}_K/(p)$, $\theta'((\dots, x_i, \dots, x_0)) = (\dots, x_i \bmod \pi\mathfrak{o}_K, \dots, x_0 \bmod \pi\mathfrak{o}_K)$. This map is clearly multiplicative. We just have to show that this is a bijection. Let $\bar{x} = (\bar{x}_0, \bar{x}_1, \dots) \in \mathfrak{o}_{K'}$. We choose liftings $x_n \in \mathfrak{o}_K$. We define the inverse map as $\bar{x} = (\bar{x}_0, \bar{x}_1, \dots) \mapsto (\bar{x}^\sharp, (\bar{x}^{1/p})^\sharp, \dots)$ where $(x^{1/p^i})^\sharp = \lim_{j\to\infty} x_{i+j}^{p^j}$. Now we check the compatibility with the definition

$$((\bar{x}^{1/p^i})^\sharp)^p = \lim_{j\to\infty} x_{i+j+1}^{q^{j+1}} = \lim_{j\to\infty} x_{i+j}^{q^j} = (\bar{x}^{1/p^i})^\sharp$$

Since $(\bar{x}^{1/p^i})^\sharp \bmod \pi\mathfrak{o}_K = x_i$, we see that it is definitely a right inverse of the projection map. It just remains to show that the projection map is injective. Let $\bar{x}' = (\bar{x}_0, \bar{x}_1, \dots)$ and $\bar{y}' = (\bar{y}_0, \bar{y}_1, \dots)$ be elements in $\varprojlim \mathfrak{o}_K$ such that $a_i \equiv b_i \bmod \pi\mathfrak{o}_K$ for any $i \geq 0$. Then we obtain that $a_i = a_{i+j}^{q^j} \equiv b_{i+j}^{q^j} = b_i$ and hence $a_i = b_i$.

The other part replies on the following claim.

Claim 18. *Let A be a strict p-ring. Then we have*

$$A/(p) \simeq \varprojlim_{x\to x^p} A$$

Proof. We define the following maps

$$\begin{array}{rcl} A/(p) & \to & \varprojlim_{x\to x^p} A \\ a & \longmapsto & (\theta(a), \theta(a^{1/p}), \theta(a^{1/p^2}), \dots) \\ (\bar{a}_0, \theta(\bar{a_0}^{1/p}), \theta(\bar{a_0}^{1/p^2}), \dots) & \longleftarrow & (a_0, a_1, a_2, \dots, a_n) \end{array}$$

Now it is easy to check by conditions that this map satisfies the properties of bijection, and hence the result. ☐

Then we clearly have $\varprojlim_{x\to x^p} A/(p) = \varprojlim_{x\to x^p} A/(p)$. And by definition of tilting we are done. ☐

By this claim we have established the fact that both definitions of tilting are basically equivalent.

We also prove some good results on perfectoid fields

Definition 50. *An analytic field K is perfectoid is K is of mixed characteristics, K is not discretely valued and p^{th} power endomorphism on $\mathfrak{o}_K/(p)$ is surjective.*

Claim 19. *(Exercise 1.3.7 of [Ked15]) Let K be an analytic field of mixed characteristics which is not discretely valued.*

- *(i) Assume there exists $\xi \in K$ with $p^{-1} \leq |\xi| < 1$ such that the Frobenius is surjective on $\mathfrak{o}_K/(\xi)$. Then K is perfectoid.*
- *(ii) Suppose that there exists an ideal $I \subseteq \mathfrak{m}_K$ such that the I-adic topology and the norm topology on $\mathfrak{o}_K$ and the Frobenius is surjective on $\mathfrak{o}_K/I$. Then K is perfectoid.*

Proof. (i) Since the value group is dense, there exists an element $\xi_1 \in K$ such that $|\xi|^{1/q} \leq |\xi_1| < 1$. Hence it follows that $\xi\mathfrak{o}_K \subseteq \xi_1\mathfrak{o}_K$ and hence $(\mathfrak{o}_K/\xi_1\mathfrak{o}_K)^q = \mathfrak{o}_K/\xi\mathfrak{o}_K$. Now let $a \in \mathfrak{o}_K$. Inductively using the relation we find sequences (b_n) and (c_n) such that

$$\begin{aligned} a &= b_0^q + \xi_1^q a_1 \\ a_1 &= b_1^q + \xi_1^q a_2 \\ &\vdots \\ a_n &= b_n^q + \xi_1^q a_{n+1} \\ &\vdots \end{aligned}$$

It follows that there exists elements $(c_n) \in \mathfrak{o}_K$ such that

$$a \equiv c_n^q + \xi^{q(n+1)} a_{n+1} \bmod p\mathfrak{o}_K$$

But then we have $|\xi_1^{q(n+1)}| \leq p^{-1}$ and hence $\xi_1^{q(n+1)}\mathfrak{o}_K \subseteq p\mathfrak{o}_K$ for sufficiently large n. Hence the conclusion follows.

(ii) Since the norm topology and the I-adic topology coincide, then K is complete under I-adic topology as well and hence $\bigcap_{n\in\mathbb{N}} I^n = \{0\}$. All we are left to show is that the Frobenius is surjective. Let's assume that it is not, i.e. $\exists\, \bar{a} \in \mathfrak{o}_K/(p)$, $\nexists\, \bar{b} \in \mathfrak{o}_K/(p) \ni \bar{b}^p = \bar{a}$. Now we choose liftings $a \in \mathfrak{o}_K$, and clearly by our assumption $\nexists\, b \in \mathfrak{o}_K \ni b^p \equiv a \bmod p\mathfrak{o}_K \implies \nexists\, b \in \mathfrak{o}_K \ni |b^p - a| < 1$. Now we use the equivalence, and observe that $\nexists\, b \in \mathfrak{o}_K \ni b^p - a \in I^n \implies \nexists\, b \in \mathfrak{o}_K \ni b^p \in a + I^n \implies \nexists\, b \in \mathfrak{o}_K \ni b^p \equiv a \bmod I^n$, for any $n \in \mathbb{N}$. But this is not true since by hypothesis the Frobenius is surjective on $\mathfrak{o}_K/I$ and we indeed have an existence of such a b.

□

Claim 20. *(Exercise 1.3.8 of [Ked15]) An analytic field K with mixed characteristics is perfectoid $\iff$ for every $x \in K$, there exists $y \in K$ with $|y^p - x| \leq p^{-1}|x|$.*

Proof. To prove $\implies$: Let K be perfectoid, then the Frobenius is surjective, hence $\forall x \in K\ \exists y \in K \ni y^p = x$. Hence this direction is trivial just choose the same p^th root.

To prove $\impliedby$: Let K be an analytic field of mixed characteristics satisfying $|y^p - x| \leq p^{-1}|x|$. Our aim is to show that the Frobenius is surjective on $\mathfrak{o}_K/(p) \implies$ that for every $x \in \mathfrak{o}_K/(p)$ we need a $y \in \mathfrak{o}_K/(p)$ such that $y^p = x \implies$ if we choose corresponding lifts $\bar{y}, \bar{x}$to $\mathfrak{o}_K$ we should have $\bar{y}^p \equiv \bar{x}\ (p)$. Now we directly follow the conditions, i.e let $x \in \mathfrak{o}_K$ be an element, and by condition we will have $y \in K$ such that the above condition holds. But then we have that

$$\begin{aligned} |y^p - x| &\leq p^{-1}|x| \\ \implies |y^p - x| &\leq |p| \\ \implies p \mid y^p - a & \\ \implies y^p \equiv a\ (p) & \end{aligned}$$

Now we are only left to show that the $y \in \mathfrak{o}_K$. But this is true since

$$\begin{aligned} |y|^p &= |y^p - x + x| \\ \Longrightarrow |y|^p &\leq \max(|y^p - x|, |x|) \\ \Longrightarrow |y|^p &\leq 1 \\ \Longrightarrow |y| &\leq 1 \end{aligned}$$

and hence we are done. □

3.3. **Untilting.**

We will explain Fontaine's strategy for untilting using Witt Vectors.

Definition 51. *Let (A, A^+) be perfectoid Huber pair of characteristic p. An ideal I of $W(A^+)$ is called primitive of degree 1 if it is generated by an element ξ of the form $\xi = p + [\varpi]\alpha$, where ϖ is pseudouniformizer of A and $\alpha \in W(A^+)$. We also say that the element ξ is primitive of degree 1.*

Lemma 19. *Any element $\xi \in W(A^+)$ that is primitive of degree 1 is torsionfree in $W(A^+)$.*

Proof. Let $\xi = p + [\varpi]\alpha$. Let $b \in W(A^+)$ such that $b\xi = 0$. We want to show that $b = 0$. We can write $b = \sum_{n\geq 0}[a_n]p^n$, with $a_n \in A^+$ uniquely determined, and it suffices to show that all $a_n = 0$. We show by induction on r that $a_n \in \varpi^r A^+$ for every $n, r \in \mathbb{N}$, which implies that $a_n = 0$ because A^+ is ϖ-adically seperated.

Base Case: For $r = 0$ this is true trivially.

Induction Hypothesis: Let's assume this holds for $r - 1 \geq 1$.

Inductive Step: From hypothesis we can write $a_n = \varpi^{r-1}a'_n$ with $a'_n \in A^+$. Let $b' = \sum_{n\geq 0}[a'_n]p^n \in W(R^+)$. Then $0 = b\xi = [\varpi]^{r-1}b'\xi$ and hence $b'\xi = 0$ because $[\varpi]$ is not a zero divisor in $W(A^+)$. Reducing the equality in $(p + [\varpi]\alpha)b' = 0$ modulo $[\varpi]$ gives $\sum_{n\geq 0}[a'_n]p^{n+1} \in [\varpi]W(A^+)$, hence $a'_n \in W(A^+)$ for every $n \in \mathbb{N}$. □

Theorem 12. *Let (A, A^+) be a perfectoid Huber pair, and let $(A^\flat, A^{\flat +})$ be it's tilt.*

(i) The map

$$\theta : \begin{cases} W(A^{\flat +}) \to A^+ \\ \sum_{n\geq 0}[a_n]p^n \longmapsto \sum_{n\geq 0} a_n^\sharp p^n \end{cases}$$

is a surjective morphism of rings.

(ii) The kernel θ is primitive of degree 1.

Also if $char(A) = p$(such that $A^\flat = A$), then $Ker(\theta) = (p)$.

Proof. □

Lemma 20. *Let R be a ring, let $\varpi \in R$, and let $\theta : M \to N$ be a morphism of R-modules. Suppose that M and N are ϖ-adically complete and Hausdorff, that N is ϖ-torsion free, and that θ induces an isomorphism $M/\varpi M \xrightarrow{\sim} N/\varpi N$. Then θ is an isomorphism.*

Proof. We know that θ is surjective by Lemma 0315 of [Stacks]. To show that it is injective it suffices to show that, for every $n \geq 1$, the morphism $M/\varpi^n M \to N/\varpi^n N$ induced by θ is injective. We proceed by induction on n.

Base Case: For $n = 1$ it is true by assumption.

Induction hypothesis: Let the statement be true for $n - 1 \geq 1$.

Inductive Step: We have the following commutative diagram

$$\begin{array}{ccccccccc} & & M/\varpi^{n-1}M & \xrightarrow{\varpi\times(.)} & M/\varpi^n M & \longrightarrow & M/\varpi^n M & \longrightarrow & 0 \\ & & \downarrow \theta & & \downarrow \theta & & \downarrow \theta & & \\ 0 & \longrightarrow & N/\varpi^n N & \xrightarrow{\varpi\times(.)} & N/\varpi^n N & \longrightarrow & N/\varpi^n N & \longrightarrow & 0 \end{array}$$

Indeed the map on $N/\varpi^{n-1}N \to N/\varpi^n N$ is injective since N is ϖ-torsionfree. By induction hypothesis the first and third vertical maps are injective. By five lemma the middle one is also injective. □

Corollary 3. *There is an equivalence of categories between:*

- *Perfectoid Huber Pairs (S, S^+);*
- *triples (R, R^+, I) where (R, R^+) is a perfectoid Huber pair of characteristic p and $I \subset W(R^+)$ is an ideal that is primitive of degree 1*

This equivalence is given by the functors that send a pair (S, S^+) to $(S, S^\flat +, Ker(\theta : W(S^{\flat +}) \to S^+))$, and a triple (R, R^+, I) to $((W(R^+)/I)[\frac{1}{\varpi}], (W(R^+)/I))$, where ϖ is any pseudo-uniformizer of R

Proof. This is Theorem 3.17 of [Sch22] □

Corollary 4. *Fix a Huber pair (A, A^+). Then there is an equivalence of categories between perfectoid Huber pairs over (A, A^+) and over $(A^\flat, A^{\flat +})$.*

The equivalence is given by the functor that send a morphism of perfectoid Huber pairs $(A, A^+) \to (B, B^+)$ to $(A^\flat, A^{\flat +}) \to (B^\flat, B^{\flat +})$. In the other direction the functor is the one that sends the morphism of perfectoid Huber pairs $(A^\flat, A^{\flat +}) \to (B^\flat, B^{\flat +})$ then we get a morphism $\alpha : W(A^{\flat +}) \to W(B^{\flat +})$. By definition of primitive ideal of degree 1, if $I = \mathrm{Ker}(\theta : W(A^{\flat +}) \to A^+)$, then $J := \alpha(I)$ is a primitive ideal of degree 1 of $W(B^{\flat +})$, so we get a morphism of perfectoid Huber pairs

$$(A, A^+) \simeq ((W(A^{\flat +})/I)[\frac{1}{\varpi^\flat}], (W(A^{\flat +})/I)) \to ((W(B^{\flat +})/J)[\frac{1}{\varpi^\flat}], (W(B^{\flat +})/J))$$

3.4. An Brief on Almost Mathematics.

The theory of almost mathematics was first introduced by Faltings and formalized methodically by Gabber and Romero (See [GR02]).

Let K be a perfectoid field, and let $\mathfrak{m}$ be a maximal ideal of K^o(the valuation ring of K). We can observe that $\mathfrak{m}^2 = \mathfrak{m} = \mathfrak{m} \oplus \mathfrak{m}$. Let us define $\Sigma = \{\mathfrak{m} - \text{torsion modules }\} \subseteq K^o - \text{mod}$. Then this is a thick abelian Serre subcategory, i.e it is closed under subobjects, quotients and extensions.

Definition 52. *A K^o-module is almost zero if it is in Σ, i.e it is killed by $\mathfrak{m}$.*

Example 26. Let K be a perfectoid field. Let $\mathfrak{o}_K = K^o$ be it's ring of integers and $\mathfrak{m}_K$ be the maximal ideal of $\mathfrak{o}_K$. Then $\kappa_K = \mathfrak{o}_K/\mathfrak{m}_K$ is almost zero. This is since we have $\mathfrak{m}_K = K^{oo}$, which is the ideal of topologically nilpotents in K^o.

Remark 10. If M and N are K^o-modules, with $M \subseteq N$ then we say M and N are almost equal if N/M is almost zero. By above example K^o and K^{oo} are almost equal.

We let K^{oa}-mod be be K^o-mod$/\Sigma$, the localization of the category K^o-mod by the Serre subcategory Σ. (This means the objects of the two categories are the same, but we change the hom-sets so that everything in Σ is isomorphic to 0).

The category K^{oa}-mod is precisely the one where do our formulations of "almost mathematics". As an example, $K^o/\mathfrak{m}$ is an almost zero module which becomes 0 in K^{oa}-mod. On the other hand K^o/π is not almost zero.

In general categorical localizations can be abstract and difficult to work with, but our situation here lets us actually compute things. The key fact is that the "almost" functor K^o-mod $\to K^{oa}$-mod(which we denote by $M \to M^a$) has a right adjoint $N \to N_*$ and a left adjoint $N \to N_!$. This means that we have

$$\mathrm{Hom}_{K^o}(M_!, N) = \mathrm{Hom}_{K^{oa}}(M^a, N^a) = \mathrm{Hom}_{K^o}(M, N_*)$$

The main question right now what exactly are these adjoint functors. If $M = T^a$ is an almost module then

$$(T^a)_* = \mathrm{Hom}_{K^o}(\mathfrak{m}, T), \quad (T^a)_! = m \otimes T$$

Here we call M_* the module of almost elements of M.

3.4.1. *Some almost commutative algebra.*

All of the above discussion is essentially category; we now want to translate commutative algebra over this category.

Definition 53. *Let A be a K^{oa}-algebra and M be an A-module.*

- *We say M is flat if $M \otimes_A$ is exact.*
- *We say M is almost projective if $Hom_A(M, -)$ is exact.*
- *If $M = N^a$ and $A = R^a$, we say M is almost finitely generated, if for every $\epsilon \in \mathfrak{m}$, there exists finitely generated R-module N_ϵ and a morphism $f_\epsilon : N_\epsilon \to N$ such that $ker(f)$ and $coker(f)$ are killed by ϵ.*
- *If M is almost finitely generated and the number of generators we need for N_ϵ is bounded independently of ϵ then we say it is uniformly almost finitely generated.*
- *There is a similar notion of almost finitely presented.*

We emphasize that all of the statements about exactness are computed in the abelian category A-mod of almost modules over the almost algebra A!

3.4.2. *Almost étale extensions.*

Our goal in this section is to define "almost étale extensions"; we've been building up the almost commutative algebra we need to. We now take a digression to the usual (non-almost) case of commutative algebra to review étale maps.

If $A \to B$ is a finite etale map of commutative rings then there's a closed and open embedding $\mathrm{Spec}(B) \hookrightarrow \mathrm{Spec}(B \otimes B)$. Then there is a unique diagonal idempotent $e \in B \otimes_A B$, which satisfies

- $e^2 = e$.
- $\mu(e) = 1$ for $\mu : B \otimes_A B \to B$ the multiplication map.
- $\ker(\mu).e = 0$

For the sake of example,

Example 27. If $A \to B$ is a Galois extension of fields with group G, then $B \otimes_A B \cong \prod_g B$ with $b_1 \otimes b_2 \mapsto (b_1.g(b_2))_{g \in G}$, then e is basically $(1, 0, \dots, 0)$.

An important fact is that if $e = \sum_{i=1}^N x_i \otimes y_i \in B \otimes B$ then :

- $\mathrm{tr}(e) = \sum \mathrm{tr}(x_i y_i) = \mathrm{tr}(B/A)$
- If we take the map $B \mapsto A^{\oplus n} \to B$ with
$$B \xmapsto{b \mapsto (\mathrm{tr}(b.x_i))} A^{\oplus n} \xrightarrow{(a_i) \mapsto \sum a_i y_i} B$$
the composition of the maps is identity.

Let K be a perfectoid field, $f : A \to B$ a map of K^{oa} -algebras.

Definition 54. *We say f is unramified if there exists an almost element $e \in (B \otimes_A B)_*$ such that $e^2 = e$, $\mu(e) = 1$, and $ker(\mu)e = 0$. We say f is étale if it's flat and unramified.*

We say f is finite étale if it's étale and B is an almost finitely presented projective A-module.

Let us give an example for this above scenario. Let K be a perfectoid field of characteristic p. We fix a non-zero element $t \in \mathfrak{m}$. Let A be a flat K^o-algebra, integrally closed in $A[1/t]$ and B' a finite etale $A[1/t]$-algebra. Take B to be the integral closure of A in B'.This map will not be étale. If we were in characteristic zero, the purity theorem in algebraic geometry would control how badly this fails, and say we could get it to be étale by base change. In characteristic p our "almost purity theorem" says that if A is perfect, then $A^a \to B^a$ is finite etale.

Proof. Let $e \in B' \otimes_A B'$ be a diagonal idempotent. Then $\exists N \in \mathbb{N}$ such that $t^N e \in B \otimes_A B$. Since A is perfect, B' is perfect and B is perfect. Hence the Frobenius is a surjection and as a result applying the Frobenius we get $(t^N)^{1/p^n} e \in B \otimes_A B$. Then we have the maps as before, and computing gives us the desired result. □

3.5. **Integral Perfectoid Rings.** There is a notion of integral perfectoid rings which are useful in proofs.

Proposition 28. *(Integral Perfectoid Rings) Let R be a ring and $\varpi \in R$. Suppose the following conditions hold*

(i) ϖ is a non-zero divisor.
(ii) R is ϖ-adically seperated complete.
(iii) $\varpi^p | p$ and Frob $: R/\varpi \xrightarrow{\sim} R/\varpi^p$

Then $R[\frac{1}{\varpi}] = \varinjlim_{\times\varpi} R$ equipped with the inductive limit topology is a perfectoid ring such that R is almost equal to $R[\frac{1}{\varpi}]^o$

Proof. By definition, R is a ring of definition of the Huber ring $R[\frac{1}{\varpi}]$. Let

$$\frac{a}{\varpi^k} \in R[\frac{1}{\varpi}]^o$$

with $a \in R$ and $k \geq 1$. There exists $N \geq 1$ such that for all $n \geq 0, (\frac{a}{\varpi})^n \in \varpi^{-N} R$ that is to say $a^n \in \varpi^{kn-N} R$. Replacing n by p^n we obtain for all $n \geq 0$,

$$a^{p^n} \in \varpi^{kn-N} R$$

Thus, for $n >> 0$,

$$a^{p^n} \in \varpi^{(k-1)p^n} R$$

The injectivity of the Frobenius implies that for any $x \in R$, $(\frac{x}{\varpi})^p \in R \implies \frac{x}{\varpi} \in R$. From this and applying induction we deduce that

$$\forall\, x \in R,\ \forall\, i, j \geq 0, (\frac{x}{\varpi^j})^p \in R \implies \frac{x}{\varpi^j} \in R$$

Applying this we deduce that $a \in \varpi^{k-1}R$ since $(\frac{a}{\varpi^{k-1}})^{p^n} \in R$. Thus we obtain that $R[\frac{1}{\varpi}]^o \subset \varpi^{-1}R$ and it is thus bounded. We state the following important lemma

Lemma 21. *For A a perfectoid ring, for any pseudo-uniformizer ϖ, there exists $\lambda \in (A^o)^\times$ such that $\lambda\varpi = x^\sharp$ for some $x \in A^{o,\flat}$*

Proof. Refer [Far] □

By the above lemma we can suppose up to multiplying ϖ by a unit in R, that there exists $\varpi^\flat \in R^\flat$ such that $(\varpi^\flat)^\sharp = \varpi$. The result is also true if we replace ϖ by $(\varpi^{\flat,1/p^n})$ for all n. And hence we deduce that R is almost equal to $R[\frac{1}{\varpi}]^o$.

Finally we have to check the fact that the Frobenius

$$\text{Frob} : R[\frac{1}{\varpi}]^o/\varpi' \to R[\frac{1}{\varpi}]^o/\varpi'^p$$

is almost surjective implies it is surjective. Suppose we write $\varpi = \varpi^{\flat,\sharp}$. Let $x \in R[\frac{1}{\varpi}]^o$. One has $\varpi x \in R$ and thus there exists $y \in R$ such that $\varpi x \equiv y^p \bmod \varpi^p R$. We deduce that $x \equiv z^p \bmod \varpi^{p-1}R$ with $z \in R[\frac{1}{\varpi}]^o$ since $z^p \in R$. We thus have $\varpi' = (\varpi^{\flat,\frac{p-1}{p}})^\sharp$ then the Frobenius is surjective and we are done. □

Example 28. Let (A, A^+) be a Huber pair with A perfectoid. Then for any pseudo-uniformizer ϖ satisfying $\varpi^p | p$, $\text{Frob} : A^+/\varpi \xrightarrow{\sim} A^+/\varpi^p$ and A^+ is an integral perfectoid ring.

Proof. To show injectivity we argue in this manner; if $x \in A^+$ satisfies $x^p \in A^+\varpi^p$ it then satisfies $x^p \in A^o\varpi^p \implies x = \lambda\varpi$ with $\lambda \in A^o$. But then we have $\lambda^p\varpi^p \in \varpi^p A^+$ which implies $\lambda^p \in A^+$ and thus $\lambda \in A^+$ since A^+ is integrally closed.

To show the surjectivity, we argue in this manner; for any $x \in A^+$ we can wrtie $x = a^p + \lambda\varpi^p$ with $a, \lambda \in A^o$. This implies $a^p \in A^+$ and thus $a \in A^+$. We deduce that the Frobenius from $A^+/\varpi \to A^+/\varpi^p A^o$ is surjective. Using that the ring A^o/A^+ is perfect we conclude. □

3.6. **Perfectoid Spaces.**

With perfectoid rings and algebras in hand we now define perfectoid spaces.

Definition 55. *A perfectoid space is an adic space which is locally isomorphic to Spa(A, A^+), for (A, A^+) a perfectoid Huber pair. We say X is affinoid perfectoid if $X = $ Spa(A, A^+) for some perfectoid Huber pair.*

In other words a perfectoid space is an adic space covered by (open) affnoid perfectoid spaces.

Theorem 13. *Every perfectoid space X has a tilt $X^\flat$ which is a perfectoid space over $\mathbb{F}_p$, with an isomorphism of topological spaces $(\ .)^\flat : X \to X^\flat$, such that for every affinoid perfectoid subspace $U \subset X$, if $U^\flat$ is the image of U in $X^\flat$ then tilt of the pair $(\mathscr{O}_X(U), \mathscr{O}_X^+(U))$ is $(\mathscr{O}_{X^\flat}(U^\flat), \mathscr{O}_{X^\flat}^+(U^\flat))$*

Proof. Refer [Sch11]. □

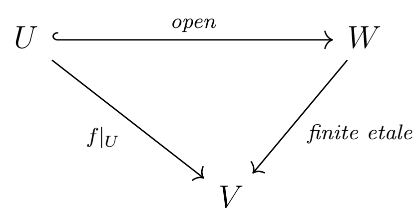
Unlike general adic spaces, perfectoid spaces does admit fiber products

Theorem 14. *Let $X \to Z$ and $Y \to Z$ be two morphism of perfectoid spaces. Then the fiber product $X \times_Z Y$ exists in the category of adic spaces and it is also a perfectoid space.*

Proof. Refer [Sch11]. □

3.7. Etale sites over Perfectoid Spaces.

Definition 56.
- *A morphism $f : X \to Y$ between perfectoid spaces is called finite etale if for all open affinoid perfectoid $Spa(B, B^+) \subset Y$, the fibered product $X \times_Y Spa(B, B^+) = Spa(A, A^+)$ is also affinoid perfectoid and such that A is finite etale over B.*
- *A morphism $f : X \to Y$ of perfectoid spaces is etale, if for any $x \in X$, there exists an open neighborhood $U \subset X$ of x and an open subset $V \subset Y$ containing $f(U)$ such that we have the following commutative diagram*



- *An etale cover is a family of etale morphisms $\{f_i : U_i \to U\}_i$ such that $U = \bigcup_i f_i(U_i)$*

We state the following important theorem.

Theorem 15. *Let X be a perfectoid space. Then the tilting functor yeilds an isomorphism of sites*

$$X_{et} \simeq X^\flat_{et}$$

Moreover if X is affinoid perfectoid then we have

$$H^i(X_{et}, \mathscr{O}_X^+) = \begin{cases} \mathscr{O}_X, & \text{if } i = 0 \\ \text{almost zero}, & \text{if } i > 0 \end{cases}$$

In the remaining of this chapter we devote to discussing the proof of this theorem in the first case. More precisely, let $f : X \to Y$ be a morphism of perfectoid spaces, we are interested to show the following equivalence

$$f : X \to Y \text{ is (finite) etale } \Longleftrightarrow f^\flat : X^\flat \to Y^\flat \text{ is (finite) etale}$$

We can reduce to the case where X and Y are affinoid perfectoid since finite etaleness is a local property.

The Dimension 0 **Case**: When X is dimension 0 then $X = \mathrm{Spa}(A, A^+)$ with A being perfectoid field we aim to prove the following

Theorem 16. *Let K be a perfectoid field with tilt $K^\flat$. Then*

(i) For any finite L of K, L is a perfectoid field.
(ii) The tilting functor gives an equivalence of categories

$$\left[\begin{array}{c}\textit{the category of}\\ \textit{finite extensions over } K\end{array}\right] \to \left[\begin{array}{c}\textit{the category of}\\ \textit{finite extensions over } K^\flat\end{array}\right], \quad L \mapsto L^\flat.$$

Proof. We use almost mathematics tools to prove this statement. We have the following diagram for which explanation will follow.

$$
\begin{array}{ccc}
\left[\begin{array}{c}\text{the category of}\\ \text{finite étale } K\text{-algebras}\end{array}\right] & \xrightarrow{\simeq} & \left[\begin{array}{c}\text{the category of}\\ \text{almost finite étale } \mathcal{O}_K\text{-algebras}\end{array}\right] \\
 & & \Big\downarrow \simeq \\
 & & \left[\begin{array}{c}\text{the category of}\\ \text{almost finite étale } (\mathcal{O}_K/\varpi)\text{-algebras}\end{array}\right] \\
 & & \Big\| \\
 & & \left[\begin{array}{c}\text{the category of}\\ \text{almost finite étale } (\mathcal{O}_K^\flat/\varpi^\flat)\text{-algebras}\end{array}\right] \\
 & & \simeq \Big\uparrow \\
\left[\begin{array}{c}\text{the category of}\\ \text{finite étale } K^\flat\text{-algebras}\end{array}\right] & \xrightarrow{\simeq} & \left[\begin{array}{c}\text{the category of}\\ \text{almost finite étale } \mathcal{O}_K^\flat\text{-algebras}\end{array}\right]
\end{array}
$$

We state some points of discussion

- The middle equality follows owing to the fact $\mathcal{O}_K/\varpi \cong \mathcal{O}_K^\flat/\varpi^\flat$.
- The equivalence at the top and bottom is because ϖ is believed to be 0 in almost mathematics.

$\square$

Example 29. The most obvious consequence of this theorem is that one has the isomorphism of the absolute Galois groups

$$G_K \cong G_{K^\flat}$$

Example 30. We have seen in a previous example that $L = \mathbb{C}_p$ is perfectoid. Let's try and compute it's tilt. We know two facts about L, which are

- $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}_p}}$
- $\mathbb{C}_p =$ completion of algebraic closure of $\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})} = \mathbb{Q}_p^{\mathrm{cycl}}$

We make use the second point. Then by the theorem and making of second point we can prove that $\mathbb{C}_p^\flat$ is basically the completion of the algebraic closure of $\mathbb{Q}_p^{\mathrm{cycl},\flat} = \widehat{\mathbb{F}_p((t^{\frac{1}{p^\infty}}))}$, which is itself equal to $\widehat{\overline{\mathbb{F}_p((t))}}$ and hence $\mathbb{C}_p^\flat = \widehat{\overline{\mathbb{F}_p((t))}}$.

Example 31. Using the example 24, let us use it for $K = \widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$, then we have from examples in tilting chapter $K^\flat = \widehat{\mathbb{F}_p((t^{\frac{1}{p^\infty}}))}$. Hence as a result we have

$$G_{\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}} = G_{\widehat{\mathbb{F}_p((t^{\frac{1}{p^\infty}}))}}$$

which is basically the celebrated theorem of Fontaine and Wintenberger.

The General Case: For the general case we have the following theorem

Theorem 17. *Let R be a perfectoid ring with tilt $R^\flat$ being it's tilt. Then*

(i) For any finite etale algebra S over R, S is also a perfectoid ring.
(ii) The tilting functor gives an equivalence of categories

$$\begin{bmatrix} \text{the category of} \\ \text{finite etale } R\text{-algebras} \end{bmatrix} \to \begin{bmatrix} \text{the category of} \\ \text{finite etale } R^\flat\text{-algebras} \end{bmatrix}, \quad S \mapsto S^\flat.$$

(iii) If S is a finite etale R-algebra, then S^o is almost finite etale over R^o.

Proof. The idea is to reduce the case to the previous case. To do so we should establish the fact that for any $x \in X = \mathrm{Spa}(R, R^+)$ with residue k_x, we have an equivalence of categories. For details we refer to [SW20]. $\square$

3.8. **Appendix: The Almost Purity Theorem.**

In this section we briefly discuss the almost purity theorem for the second part of almost purity theorem. The main reference for this section will be [Mor]. We will follow a simple discussion due to Kazuma Shimomoto. The main theorem of almost purity is due to Gerd Faltings using his tool of almost mathematics.

Theorem 18. *(Almost Purity) Let V be a perfectoid valuation ring with field of fractions K, and let W be the integral closure of V in a finite seperable extension of fields $K \to L$. Then*

(1) The Frobenius endormorphism on W/pW is surjective.
(2) $V \to W$ is faithfully flat and almost finite etale.

We recall from the section of almost mathematics that a ring map $f : A \to B$ is weakly etale if both $A \to B$ and the diagonal map $\mu_B : B \otimes_A B \to B$ are flat. We give a proof sketch. An important result from [GR02] is as follows

Lemma 22. *Let $f : A \to B$ be a ring map of $\mathbb{F}_p$-algebras. If the map f is weakly etale, then $F_{B/A} : A^{[F]} \otimes_A B \to B^{[F]}$ is an isomorphism.*

Lemma 23. *Let $f : A \to B$ be a ring homomorphism. Then f is surjective $\iff \exists$ a faithfully flat A-algebra C such that $C \to C \otimes_A B$ is surjective.*

Proof. Clearly if f is surjective we can take $C = A$. Conversely, if there exists a faithfully flat A-algebra C such that $C \to C \otimes_A B$ is surjective, then we have $C \otimes_A \operatorname{coker}(f) = 0$. But C is faithfully flat so we have $\operatorname{coker}(f) = 0$. □

Proposition 29. *Let R be a commutative ring and let $I = \bigcup_{n>0}(\varpi^{\frac{1}{p^n}})$ be an ideal of R for some regular element $\varpi \in V$ with a system of p^{th} power roots(so that (R, I) is a basic setup). Let M be an R-module which is ϖ-adically separated. If $M/\varpi M$ is an I-almost finitely generated $R/\varpi R$-generated module, then M is I-almost finitely generated V-module.*

Proof. Since by hypothesis we have $M/\varpi M$ is I-almost finitely generated, $\exists$ finitely generated submodule $M' \subseteq M/\varpi M \ni \varpi^{\frac{1}{p^n}}(M/\varpi M) \subseteq M'$ for any fixed $n > 0$. After writing $M' = \sum_{i=1}^{r}(R/\varpi R)\overline{m}_{n_i}$, we have

$$\varpi^{\frac{1}{p^n}} M \subseteq \sum_{n=1}^{r} Rm_{n_i} + \varpi M \subseteq \sum_{n=1}^{r} Rm_{n_i} + \bigcap_{k>0} \varpi^{k(\frac{p^n-1}{p^n})} M$$

where m_{n_i} is the lift of element $\overline{m}_{n_i}$ and the second inclusion follows from the first inclusion by applying it repeatedly. But since M is ϖ-adically separated $\bigcap_{k>0} \varpi^{k(\frac{p^n-1}{p^n})} M = 0$. So we have the conclusion as desired. □

Lemma 24. *Let (R, I) be a basic setup as in the previous proposition and let M be an almost finitely generated R-module. If M is an almost projective module then the following holds:*

$\phi : N \to M$ is a homomorphism of R-modules, such that coker(ϕ) is kiled by ϖ^α for a fixed $\alpha \in \mathbb{Q}_{>0} \implies \forall\, k > 0\ \exists\, g : M \to N \ni \phi \circ g = \varpi^{\frac{1}{p^k}+2\alpha} id_M$

Proof. Take the exact sequence $N \to M \to \mathrm{Coker}(\phi) \to 0$. Let M' be the image of ϕ. Then we obtain the short exact sequence $0 \to K \to N \to M' \to 0$, which in turn induces the long exact sequence

$$0 \to \mathrm{Hom}_R(M', K) \to \mathrm{Hom}_R(M', N) \to \mathrm{Hom}_R(M', M') \to \mathrm{Ext}^1_R(M', K) \to \dots$$

Further we take the exact sequence $0 \to M' \to M \to M/M' \to 0$ and it induces the long exact sequence

$$\dots \to \mathrm{Ext}^1_R(M, K) \to \mathrm{Ext}^1_R(M', K) \to \mathrm{Ext}^2_R(M/M', K) \to \dots$$

Now we follow the conditions; $\varpi^{1/p^k}\mathrm{Ext}^1_R(M, K) = 0$ for any $k > 0$ by assumption and $\varpi^\alpha \mathrm{Ext}^2_R(M/M', K) = 0$. So we obtain $\varpi^{\frac{1}{p^k}+2\alpha}\mathrm{Ext}^1_R(M', K) = 0$. Considering the first long exact sequence, we can find an element $f \in \mathrm{Hom}_R(M', N)$ such that $\phi \circ f = \varpi^{\frac{1}{p^k}+2\alpha}\mathrm{id}_M$. Finally just take $g = f \circ \varpi^\alpha$ and we are done. □

Proposition 30. *Let (R, I) be a basic setup as in above and let M be an almost finitely generated V-module such that $M/\varpi^n M$ is an almost projective $R/\varpi^n R$-module for any $n > 0$. Suppose that M and R are ϖ-adically complete. Then M is an almost projective R-module.*

We omit the proof of this proposition as this requires some investigation similar to above. Instead we head towards the proof of the important theorem. The proof is divided into two cases $\mathrm{char}(K) = p > 0$ and $\mathrm{char}(K) = 0$. The two cases are a consequence of the propositions mentioned above.

4. The Notion and Geometry of Diamonds

The main references for this section will be [Far],[Ked+19] and [Han16].

In the previous chapter we have seen the following fact: For a perfectoid space X, the tilting functor yeilds an isomorphism of sites $X_{et} \simeq X^\flat_{et}$. By this isomorphism we have one of the ultimate aim of this thesis i.e passing from characteristic 0 to characteristic p. Let's consider the case when X is defined over $\mathbb{Z}_p$. In this case the ideal scenario would be a morphism from $X^\flat \to \mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$, but in this case it does not admit such a morphism. Hence we realise we might loose information with this functor, the functor preserves the underlying topological spaces and further provides the Frobenius actions.

We would like to have similar things for analytic adic spaces. But the immediate problem is that analytic adic spaces are not usually perfectoid spaces.

The idea to resolve the problem is the following. Given an analytic adic space $X/\mathbb{Z}_p$, we can try to show that there is a perfectoid space $\tilde{X}$ with a pro-etale map (which will be defined and discussed subsequently) $\tilde{X} \to X$ such that

$$X = \mathrm{Coeq}(\tilde{X} \times_X \tilde{X} \overset{f}{\underset{g}{\rightrightarrows}} \tilde{X})$$

and $R := \tilde{X} \times_X \tilde{X}$ is also a perfectoid space. Since the tilting functor preserves topological information and we would like to have an object in characteristic p. The most natural functor that we can have is

$$X \to X^{\lozenge} = \mathrm{Coeq}(R^{\flat} \overset{f}{\underset{g}{\rightrightarrows}} \tilde{X}^{\flat})$$

4.1. **The Pro-Etale Morphism.**

4.1.1. *Pro-etale perfectoid covers before Scholze.*

Proposition 31. *Let A be a complete uniform Tate $\mathbb{Q}_p$-algebra. There exists a filtered system of finite etale uniform A-algebras $(A_i)_{i \in I}$ such that*

$$\widehat{\varprojlim_{i \in I} A_i}$$

is perfectoid.

Proof. Refer [Col02] □

Remark 11. The C-perfectoid rings A, $C|\mathbb{Q}_p$ complete algebraically closed, for which any element of $1 + A^{oo}$ has a p-power are called sympathetic by Colmez. Those are precisely the one for which the $\log : 1 + A^{oo} \to A$ is surjection. Another characterisation is that those are precisely those satisfying $H^1_{et}(\mathrm{Spa}(A, A^o), \mathbb{Q}_p) = 0$

Proposition 32. *Let A be a uniform Tate ring and B a finite etale A-algebra. Then B is a uniform Tate ring and B^o is the integral closure of A^o in B.*

Proof. Let R be the integral closure of A^o in B. We can find a finite etale B-algebra B' such that $A \to B'$ is Galois with Galois group G and $B \to B'$ is Galois with Galois group H. One can for example take

$$\mathrm{Spec}(B') = \underline{\mathrm{Isom}}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A) \times \{1, 2, \ldots, n\}, \mathrm{Spec}(B))$$

if the rank of B is constant equal to n, $G = \mathfrak{S}_n$ and $H = \mathfrak{S}_{n-1}$. For any $x \in R$,

$$\mathrm{Tr}_{B/A}(x) = \sum_{[\tau] \in G/H} \tau(x)$$

which is thus integral over A^o and thus in A^o since A^o is integrally closed in A. We thus have

$$\mathrm{Tr}_{B/A}(R) \subset A^o$$

Let M be a finite type sub-A^o-module of B such that $M[\frac{1}{\varpi}] = B$. Upto replacing M by $\varpi^k M$ with $k >> 0$, we can suppose that $M \subset R$. Now, if

$$M^\vee = \{x \in B \mid \forall\, m \in M, \mathrm{Tr}_{B/A}(xm) \in A^0\},$$

we have

$$R \subset M^\vee$$

Now, if we fix a diagram of morphisms of A-modules

$$A^d \underset{s}{\overset{\epsilon}{\rightleftarrows}} B$$

with $\epsilon s = \mathrm{Id}$ and $\epsilon((A^o)^d) = M$, we have $M^\vee \subset s^\vee((A^o)^d)$ where $s^\vee : A^d = (A^d)^* \xrightarrow{s^*} B^* \xrightarrow{\mathrm{Tr}_{B/A}^{-1}} B$. We deduce from thus that R is contained in a finite type sub-A^o-module of A. We have $B = R[\frac{1}{\varpi}]$ and B with it's canonical topology as a finite type A-module is a Tate ring with R as a ring of definition. Now, if $x \in B^o$, there exists $N \geq 0$ such that for all $k \geq 0$, $x^k \in \varpi^{-N} R$ is contained in a finite type A^o-module and thus $A^o[x]$-module too. And hence $x \in R$. □

We give an example of uniform Tate algebras where one can construct some more explicit pro-etale perfectoid covers.

Example 32. Let K be a characteristic 0 non-archimedean perfectoid field. Let X be a K-adic space locally of finite type. Suppose X is smooth. Then upto replacing X by an affinoid cover we can find an etale morphism

$$X \to \mathbb{T}^n = \mathrm{Spa}(K\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle, \mathcal{O}_K\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle)$$

that is an open rational subset inside a finite etale morphism toward an open rational subset of $\mathbb{T}_n$. Let $\mathbb{T}_n^\infty = \varprojlim_{k \geq 0} \mathbb{T}_n$ where the transition map between level k and l with $l \leq k$ is the Kummer map $t_i \mapsto t_i^{p^{k-l}}$, $1 \leq i \leq n$. Then this is the affinoid perfectoid space

$$\mathbb{T}_n^\infty = \mathrm{Spa}(K\langle T_1^{\pm 1/p^\infty}, \dots, T_n^{\pm^{1/p^\infty}}\rangle, \mathcal{O}_K\langle T_1^{\pm 1/p^\infty}, \dots, T_n^{\pm 1/p^\infty}\rangle)$$

Now applying the almost purity theorem (Subsection 3.8) we get

$$X \times_{\mathbb{T}_n} \mathbb{T}_n^\infty = \varprojlim_{k \geq 0} (X \times_{\mathbb{T}_n, t \to t^{p^k}} \mathbb{T}_\ltimes)$$

is a perfectoid space pro-etale over X.

We prove the following important result due to Gerd Faltings which will be useful

Proposition 33. *Let R be an integral normal p-torsion free noetherian $\mathbb{Z}_{(p)}$-algebra. Fix an algebraic closure $\overline{K}$ of $K = Frac(R)$ and let $L|K$ inside $\overline{K}$ be the union of all* *finite degree extensions K' of K such that $\overline{R[\frac{1}{p}]}^{K'}$ is an etale $R[\frac{1}{p}]$-algebra. This means that if $\bar{x}: Spec(\overline{K}) \to Spec(R[\frac{1}{p}])$,*

$$Gal(L|K) = \pi_1(Spec(R[\frac{1}{p}]), \bar{x})$$

Let $\overline{R}$ be the integral closure of R in L. Then, the p-adic completion $\widehat{\overline{R}}$ is integral perfectoid(refer 3.5) and thus $\widehat{\overline{R}}[\frac{1}{p}]$ with $\widehat{\overline{R}}[\frac{1}{p}]^o$ almost equal to $\widehat{\overline{R}}$.

Proof. It is clearly p-adic without p-torsion. So we can choose $p^{1/p}$, a p-root of p in the fixed algebraic closure of Frac(R), as a peudo-uniformizer. Since $\overline{R}$ is integrally closed in $\overline{R}[1/p]$, the Frobenius is injective. For the surjectivity let $a \in \overline{R}$. Consider the polynomial $P(X) = X^p + pX - q$. This satisfies $(P(X), P'(X)) = \overline{R}[\frac{1}{p}][X]$. Thus, $\overline{R}[X]/(P(X))$ is a finite $\overline{R}$-algebra that is etale outside p and we can find $x \in \overline{R}$ such that $P(x) = 0$. □

4.1.2. *The universal cover of a p-divisible group.*

We propose the following setting:

Let R be a complete Noetherian ring and $\mathfrak{m}$ be it's maximal ideal. We assume that the residue field $\kappa = R/\mathfrak{m}$ has characteristic $p > 0$. By a finite group scheme G over R, we mean an affine commutative group scheme $G = \text{Spec}A$, where A is a finite R-algebra and it is free of rank m as an R-module. m is called the order of the group.

It can be shown that if G has order m then, $G \xrightarrow{\times m} G$ is the trivial map. In other words, $\forall\, x \in G(S), mx = e$ for any R-scheme S.

We define the augmentation ideal $I = \ker(\epsilon : A \to R)$. The differential of G is given as $\Omega_{G/R} = I/I^2 \otimes_R A$.

Now for any commutative finite group scheme $G = \text{Spec}(A)$, one can define it's Cartier dual $G^\vee = \text{Spec}(A^\vee)$, where $A^\vee = \text{Hom}_{R\text{-mod}}(A, R)$ and it's multiplication is given by the dual $m^* : A \to A \otimes A$. The dual $G^\vee$ has a natural structure of group scheme, whose comultiplication comes from the multiplication of the ring of A, and whose inverse is induced by the inverse of G. The construction of dual group scheme is functorial in G and we have a natural isomorphism $G \cong (G^\vee)^\vee$.

An alternative characterization of $G^\vee$ is given by

$$G^\vee(S) = \text{Hom}_{S\text{-group}}(G \times S, \mathbb{G}_m/S)$$

for any R-scheme S.

Example 33. The dual of $\mu_{p^n} = \text{Spec}(R[T])/(T^{p^n} - 1)$ is $(\mathbb{Z}/p^n\mathbb{Z})_R$

A sequence $0 \to G' \xrightarrow{i} G \xrightarrow{j} G'' \to 0$ is called exact if G' is identified with the kernel of j and j is faithfully flat. If the order of G, G', G" are m, m', m" respectively then we have $m = m'm''$.

Example 34. (i) $0 \to \mu_n \to \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m \to 0$.

(ii) $0 \to (\mathbb{Z}/p^n\mathbb{Z})_R \to \mathbb{G}_a \xrightarrow{x \mapsto x^p - x} \mathbb{G}_a \to 0$ for $R = k$, $\mathrm{char}(k) = p > 0$.

(iii) $0 \to \alpha_p = \mathrm{Spec}(k[T])/(T^p) \to \mathbb{G}_a \xrightarrow{x \mapsto x^p - x} \mathbb{G}_a \to 0$ for $R = k$, $\mathrm{char}(k) = p > 0$.

Proposition 34. *Any (noncommutative) finite flat group scheme $G = Spec(A)$ over a henselian local ring A admits a canonical functorial connected- ́etale decomposition*

$$0 \to G^0 \to G \to G^{et} \to 0$$

where $G^0 = Spec(A^0)$ is the connected component of G and $G^{et} = Spec(A^{et})$ corresponds to the maximal ́etale subalgebra of A. (The exact sequence has a canonical splitting if $R = k$ is a perfect field of characteristic $p > 0$.)

As a corollary, there is an equivalence of category between the category of finite ́etale group schemes over R and the category of finite continuous $\mathrm{Gal}(\bar{k}/k)$-modules.

Definition 57. *Let p be a prime number and $h \geq 0$ an integer. A p-divisible group G over R of height h is an inductive system $G = (G_\nu, i_\nu), \nu \geq 0$ where G_ν is a finite group scheme over R of rank $p^{h\nu}$ an an exact sequence*

$$0 \to G_\nu \xrightarrow{i_\nu} G_{\nu+1} \xrightarrow{\times p^\nu} G_{\nu+1}$$

Definition 58. *A morphism between p-divisible groups is a collection of morphisms $(f_\nu : G_\nu \to H_\nu)$ on each level, compatible with the structure of p-divisible groups.*

We state a few properties apparent from the definition of p-divisible groups

- G_ν can be identified as the kernel of the map $p^\nu : G_{\mu+\nu} \to G_{\mu+\nu}$.
- The homomorphism $p^\nu : G_{\mu+\nu} \to G_{\mu+\nu}$ factors through G_ν since $G_{\mu+\nu}$ is killed by $p^{\nu+\mu}$.
- We have the following exact sequence

$$0 \to G_\mu \to G_{\mu+\nu} \to G_\nu \to 0$$

- The connected- ́etale decomposition of G_ν gives rise to a decomposition of p-divisible groups

$$0 \to G^0 \to G \to G^{\mathrm{et}} \to 0$$

Example 35. $\mathbb{G}_m(p) = (\mu_{p^\nu})$ is a p-divisible group of height 1. Let X be an abelian variety then $X(p) = (X[p^\nu])$ has a natural structure of a p-divisible group of height $2n$, where $\dim X = n$

Definition 59. *An n-dimensional formal Lie group over R is the formal power series ring $\mathcal{A} = R[[X_1, X_2, \ldots, X_n]]$ with a suitable comultiplication structure $m^* : \mathcal{A} \to \mathcal{A}\widehat{\otimes}\mathcal{A} = R[[Y_1, Y_2, \ldots, Y_n, Z_1, Z_2, \ldots, Z_n]]$, which is determined by $F(Y, Z) = (f_i(Y, Z))$, where f_i are the images of X_i. We require m^* to satisfy the following*

- $X = F(X,0) = F(0,X)$
- $F(X,F(Y,Z)) = F(F(X,Y),Z)$
- $F(X,Y) = F(Y,X)$

Let $\psi : \mathcal{A} \to \mathcal{A}$ be the map $x \mapsto p.x$. We say $\mathcal{A}$ is divisible if ψ is an isogeny.

We now talk about the universal cover of a p-divisible group.

Proposition 35. *Let $C|\mathbb{Q}_p$ be an algebraically closed complete non-archimedean field. Let $\mathscr{G}$ be a p-divisible formal group over $\mathcal{O}_C$. Let $\mathscr{G}_\eta$ be it's generic fiber as an adic space. Then*

$$\varprojlim_{\times p} \mathscr{G}_n$$

is perfectoid with tilting isomorphic to

$$\varprojlim_{\times p} \mathscr{G}_{k_C} \times_{k_C} Spa(C^\flat)$$

We explain the meaning the implications of this result. Here $\mathscr{G}$ is a formal Lie group

$$\mathscr{G} \simeq \mathrm{Spf}(\mathcal{O}_C[[x_1, x_2, \dots, x_d]])$$

for some integer d, the dimension of our formal group. Then we have $\mathscr{G}_\eta \simeq \overset{o}{\mathbb{B}}{}^d_C$, the d-dimensional open ball as an adic space over $\mathrm{Spa}(C)$. The ψ map is finite locally free of rank p^h where h is height of $\mathscr{G}$. This is an fppf torsor under the finite locally free group scheme $\mathscr{G}[p]$ over $\mathrm{Spec}(\mathcal{O}_C)$. In generic fiber, ψ is finite etale, a torsor under the finite etale group $\mathscr{G}[p]_\eta$.

The formal group $\mathscr{G}_{k_C}$ is isomorphic to $\mathrm{Spf}(k_C[[x_1, x_2, \dots, x_d]])$ and

$$\varprojlim_{x \mapsto x^p} \mathscr{G}_{k_C} \simeq (k_C[[x_1^{1/p^\infty}, x_2^{1/p^\infty}, \dots, x_d^{1/p^\infty}]])$$

is a formal $\mathbb{Q}_p$-vector space. In the statement we have fixed a section of the projection $\mathcal{O}_C/p \to k_C$ so that $k \hookrightarrow C^\flat$. Then,

$$\varprojlim_{x \mapsto x^p} \mathscr{G}_{k_C} \times_{k_C} \mathrm{Spa}(C^\flat) \simeq \overset{o}{\mathbb{B}}{}^{d,1/p^\infty}_{C^\flat}$$

a formal perfectoid open ball (refer [Sch] and [Bha]).

4.1.3. *Properties and Definition.*

Definition 60. *Let $f : X \to Y$ be a morphism of perfectoid spaces.*

1. *f is affinoid pro-etale if X and Y are affinoid perfectoid and $Y \to X$ is a cofiltered limit of etale affinoid perfectoid spaces over X.*
2. *f is pro-etale if locally on X and Y it is affinoid pro-etale.*

Example 36. For X perfectoid space and $x \in X$, $\mathrm{Spa}(K(x), K(x)^+) = X_x \to X$ is pro-etale, $X_x = \varprojlim_{x \in U} U$ with U varying through the set of affinoid neighborhoods of x.

Example 37. If $X = \mathrm{Spa}(X, X^+)$ and I is an ideal of A then $V(I) \hookrightarrow X$ is affinoid pro-etale.

Example 38. If X is a perfectoid space and P a locally profinite set then $X \times \underline{P} \to X$ is pro-etale. If $P = \varprojlim_i P_i$ finite and the limit is cofiltered then $X \times \underline{P} = \varprojlim_i (X \times P_i)$

Definition 61. *A family $Y_i \xrightarrow{f_i} X_i$ of pro-etale morphisms toward the perfectoid space X is a pro-etale covering if for any quasi-compact open subset U of X there exists $I' \subset I$ finite and for each $i \in I'$ a quasi-compact open subset $V_i \subset Y_i$ such that*

$$\bigcup_{i \in I'} f_i(V_i) = U_i$$

Example 39. For any perfectoid space X, the family of pro-etale morphisms $(\mathrm{Spec}(K(x), K(x)^+) \to X)_{x \in X}$ is surjective but not a covering.

Example 40. $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p}) \to \mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cycl}}, \mathbb{Z}_p^{\mathrm{cycl}})$ is a pro-etale covering.

4.1.4. *Affinoid pro-etale morphisms.*

Proposition 36. *Let $X_\infty = \varprojlim_{i \in I} X_i$ be a cofiltered limit of affinoid perfectoid spaces.*

(i) There is an equivalence

$$2 - \varprojlim_{i \in I} \{\textit{finite etale}/X_i\} \xrightarrow{\sim} \{\textit{finite etale } /X_\infty\}$$

(ii) There is an equivalence

$$2 - \varprojlim_{i \in I} \{\textit{qc qs etale}/X_i\} \xrightarrow{\sim} \{\textit{qc qs etale } /X_\infty\}$$

Proof. Refer [Far] □

Proposition 37. *Let X be an affinoid perfectoid space. The functor*

$$\varprojlim : \textit{Pro}\,(\textit{affinoid perfectoid, etale } /X) \to \{\textit{affinoid pro-etale } /X\}$$

Proof. Refer [Far] □

4.1.5. *Totally disconnected Perfectoid Spaces.*

Definition 62. *A perfectoid space X is totally disconnected if it is quasi compact, quasi seperates and $|X|$ is totally disconnected.*

Equivalently, a perfectoid space is totally disconnected $\iff$ for each connected component C of X there exists a point $c \in X$ such that

$$C = \mathrm{Spa}(K(c), K(c)^+) \hookrightarrow X$$

Lemma 25. *Any totally disconnected perfectoid space is affinoid perfectoid.*

Proof. Let X be totally disconnected. We can fix a finite open covering of X by affinoid perfectoid spaces $X = \bigcup_i U_i$. It splits, $X = \bigsqcup_i V_i$ with $V_i \subset U_i$. Since V_i is open (closed) in U_i affinoid perfectoid $\implies$ V_i is affinoid perfectoid. ☐

4.1.6. *Quasi-Proetale morphisms.* Let K be a perfectoid space of characteristic different from 2.

Proposition 38. *Let $\mathbb{B} = Spa(K\langle T^{1/p^\infty}\rangle, \mathcal{O}_K\langle T^{1/p^\infty}\rangle)$ be the one-dimensional closed perfectoid ball over K. Consider the Kummer morphism $\mathbb{B} \to \mathbb{B}$, $T^{1/p^k} \to T^{2/p^k}$ for all $k \geq 0$. There exists an explicit pro-etale covering $\tilde{\mathbb{B}}$ of $\mathbb{B}$ such that the pull back of the Kummer map via this pro-etale cover is pro-etale*

Proof. Refer [Far] ☐

4.2. Diamonds.

We denote $\mathrm{Perf}_{\mathbb{F}_p}$ for the category of $\mathbb{F}_p$ perfectoid spaces.

Definition 63. *A pro-etale sheaf X on $Perf_{\mathbb{F}_p}$ is a diamond if there exists a perfectoid space $\tilde{X}$ and an equivalence $R \subset \tilde{X} \times \tilde{X}$ represented by a perfectoid space such that*

- *both morphisms*
$$R \overset{f}{\underset{g}{\rightrightarrows}} \tilde{X}$$
are pro-etale.
- *$X \simeq \tilde{X}/R$ as pro-etale sheaves.*

4.2.1. *The Diamond $Spa(\mathbb{Q}_p)^\diamond$.* We denote the category of perfectoid space over $\mathrm{Spa}(\mathbb{Q}_p)$ as $\mathrm{Perf}_{\mathbb{Q}_p}$.

Definition 64. *We note that $Spa(\mathbb{Q}_p)$ is a functor on $Perf_{\mathbb{Q}_p}$ defined by the formulae*
$$Spa(\mathbb{Q}_p)^\diamond(S) = \{(S^\sharp, \iota) \mid S^\sharp \in Perf_{\mathbb{Q}_p},\ \iota : S \xrightarrow{\sim} S^{\sharp,\flat}\}/\sim$$

One of the first basic results is the following

Proposition 39. *The functor $Spa(\mathbb{Q}_p)^\diamond$ is a v-sheaf.*

Proof. We prove more generally that $\mathrm{Spa}(\mathbb{Z}_p)^\diamond$ is a v-sheaf. We have to prove that the functor on $\mathbb{F}_p$-affinoid perfectoid rings
$$(A, A^+) \mapsto \mathscr{D}_1(A, A^+)/W(A^+)^\times$$

is a v-sheaf. We can re-write this functor as

$$(A, A^+) \mapsto \varinjlim_{\varpi \in A^{oo} \cap A^+} ([\varpi]W(A^+) + p)/(1 + [\varpi]W(A^+))$$

where $\varpi \leq \varpi'$ if $\varpi\varpi^{-1} \in A^+$. Now the result trivially follows from the following facts

- $\mathcal{O}^+$ and consequently (A, A^+) is a v-sheaf.
- If (R, R^+) is a $\mathbb{F}_p$ perfectoid ring then

$$\varinjlim_{\varpi \in A^{oo} \cap A^+} H^1_v(\mathrm{Spa}(R, R^+), 1 + [\varpi]W(\mathcal{O}^+)) = 0$$

□

Proposition 40. *The v-sheaf Spa$(\mathbb{Q}_p)^\diamond$ is a diamond.*

Proof. The tilting equivalence defines a morphism on v-sheaves

$$\mathrm{Spa}(\mathbb{C}_p^\flat) \to \mathrm{Spa}(\mathbb{Q}_p)^\diamond$$

This is an epimorphism of pro-´etale sheaves. In fact, if $X^\sharp$ is an untilt of $X \in \mathrm{Perf}_{\mathbb{F}_p}$ over $\mathbb{Q}_p$, for any $K/\mathbb{Q}_p$ of finite degree contained in $\overline{\mathbb{Q}_p}$, according to the purity theorem 3.8, $X^\sharp \otimes_{\mathbb{Q}_p} K \to X$ is perfectoid with $X^\sharp \otimes_{\mathbb{Q}_p} K \to X$ is finite etale. Over pro-etale cover

$$(X^\sharp \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^\flat = \varprojlim_{K|\mathbb{Q}_p} (X^\sharp \otimes_{\mathbb{Q}_p} K)^\flat \to X$$

there is a morphism to Spa$(\mathbb{C}_p^\flat)$. We have thus proven that the pro-etale is surjective. We now move forward to computing the diagram

$$\mathrm{Spa}(\mathbb{C}_p)^\flat \times_{\mathrm{Spa}(\mathbb{Q}_p)^\diamond} \mathrm{Spa}(\mathbb{C}_p^\flat) \overset{f}{\underset{g}{\rightrightarrows}} \mathrm{Spa}(\mathbb{C}_p^\flat)$$

This is equivalent to computing

$$\mathrm{Spa}(\mathbb{C}_p) \times_{\mathrm{Spa}(\mathbb{Q}_p)} \mathrm{Spa}(\mathbb{C}_p) \overset{f}{\underset{g}{\rightrightarrows}} \mathrm{Spa}(\mathbb{C}_p)$$

where the fiber product is taken in the category of functors on $\mathrm{Perf}_{\mathbb{Q}_p}$. But as functors on $\mathrm{Perf}_{\mathbb{Q}_p}$ we have $\mathrm{Spa}(\mathbb{C}_p) = \varprojlim_{K|\mathbb{Q}_p} \mathrm{Spa}(K)$, where the degree of extension is finite. We then have

$$\mathrm{Spa}(\mathbb{C}_p) \times_{\mathrm{Spa}(\mathbb{Q}_p)} \mathrm{Spa}(\mathbb{C}_p) = \varprojlim_{K|\mathbb{Q}_p} \mathrm{Spa}(\mathbb{C}_p) \times_{\mathrm{Spa}(\mathbb{Q}_p)} \mathrm{Spa}(K)$$

□

Proposition 41. *There is an equivalence of categories*

$$Perf_{\mathbb{Q}_p} \xrightarrow{\sim} Perf_{\mathbb{F}_p}/Spa(\mathbb{Q}_p)^\diamond$$

This induces an isomorphism of the v-sheaves (respectively etale, pro-etale, topoi)

$$\widetilde{Perf_{\mathbb{Q}_p}} \xrightarrow{\sim} \widetilde{Perf_{\mathbb{F}_p}}/Spa(\mathbb{Q}_p)^\diamond$$

4.3. **The main result in the Geometry of Diamonds.**

We now proceed to prove the main result of this chapter

Theorem 19. *Let X be a $Spa(\mathbb{Q}_p)$-adic space. The functor*

$$\begin{array}{rcl} X^\diamond : Perf(\mathbb{F}_p) & \to & Sets \\ S & \longmapsto & \{(S^\sharp, \iota, f) \mid S^\sharp \in Perf_{\mathbb{Q}_p},\ \iota : S \xrightarrow{\sim} S^{\sharp,\flat},\ f : S^\sharp \to X\}/\sim \end{array}$$

is represented by a diamond.

Moreover if $K/\mathbb{Q}_p$ is a complete non-archimedean field, this defines a fully faithful functor

$$(-)^\diamond : \{\textit{locally finite type normal adic spaces } /Spa(K)\} \hookrightarrow \{\textit{diamonds}/Spa(K)^\diamond\}$$

Proof. We can reduce everything to the case $X = \mathrm{Spa}(A, A^+)$ with (A, A^+) a complete sheafy affinoid Tate ring. Let (B, B^+) be the completion of (A, A^+) with respect to the spectral norm associated to a choice of pseudo-uniformizer and some number in $]0,1[$. There is a morphism $(A, A^+) \to (B, B^+)$ inducing an isomorphism of the functors on $\mathrm{Perf}_{\mathbb{F}_p}$, $\mathrm{Spa}(B, B^+)^\diamond \xrightarrow{\sim} \mathrm{Spa}(A, A^+)^\diamond$. We can thus suppose that A is uniform. Then we are in a setting to apply proposition of Colmez described in the previous chapter to deduce the existence of filtered system of $(A_i, A_i^+)_{i\in I}$ of uniform affinoid finite etale (A, A^+)-algebras such that

$$(A_\infty, A_\infty^+) = \widehat{\varinjlim_{i\in I}(A, A^+)}$$

is perfectoid and for all indices $i \geq j$, $|\mathrm{Spa}(A_i, A_i^+)| \to |\mathrm{Spa}(A_j, A_j^+)|$ is surjective. Let $X_\infty = \mathrm{Spa}(A_\infty, A_\infty^+) \in \mathrm{Perf}_{\mathbb{Q}_p}$. Then there is a morphism $X_\infty^\flat \to X^\diamond$ □

5. Shtukas and Drinfeld's Lemma

The main reference for this section [Ked+19].

5.1. **Motivation.**

We go back to very first chapter. The Langlands correspondence describes a relationship between Galois representations and automorphic forms extending class field theory, appropriately formulated as a statement about the algebraic group $\mathbb{G}_m$ to more general algebraic groups. In the setting where the Galois group in question is that of a function field over a finite field, there is a geometric approach pioneered by Drinfeld and subsequently extended by L. Lafforgue (for the group GL_n) and V. Lafforgue (for more general groups).

5.2. **Fundamental Groups.**

Definition 65. *For X a scheme, we denote $\boldsymbol{FEt}(X)$ denote the category of finite ´etale coverings of X; for A a ring we denote by $\boldsymbol{FEt}(A)$ in place $\boldsymbol{FEt}(Spec(A))$. We note the following observations*

(a) *If $A = \varprojlim_i A_i$ in the category of rings, then the base extension functor from the 2-direct limit $\varinjlim_i \boldsymbol{FEt}(A_i)$ to the category of $\boldsymbol{FEt}(A)$ is an equivalence of categories (Refer : [Stacks] 01ZC). We also can say for any $B \in \boldsymbol{FEt}(A)$ is the base extension of some ´etale A_i-algebra, B_i, for some i. (Refer : [Stacks] 00U2). We may increase i to ensure that B_i is also finite and faithfully flat over A_i, hence the functor is essentially surjective (refer : [Stacks] 017O, 07RR).*
(b) *If $f : Y \to X$ is a proper surjective morphism of schemes, then the functor from $\boldsymbol{FEt}(X)$ to descent data with respect to f is an equivalence of categories.*

We take the stacks project approach of [Stacks], 0BMQ and 0MBY.

Definition 66. *(Galois Category) Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to Set$ be a covariant functor. We say that the pair $(\mathcal{C}, F)$ is a Galois category if the following holds*

- *The category $\mathcal{C}$ admits finite limits and co-limits.*
- *Every object of $\mathcal{C}$ is a (possibly empty) finite coproduct of connected objects.*
- *For every $X \in \mathcal{C}$, $F(X) < \infty$.*
- *The functor F is exact and reflects isomorphisms. We often refer to F in this context as a "fiber functor" by analogy with the primary example.*

A key property of this definition is its relationship with profinite groups. Let G be the automorphism group of the functor F; then G is a finite group and the action of G on F induces an equivalence of categories between $\mathcal{C}$ and the category finite G-sets. ([Stacks], 0BN4)

Definition 67. *For X a connected scheme, the category $\boldsymbol{Fet}(X)$ is a Galois category (by [Stacks] 0BN4). For $\bar{x}$ a geometric point of X, which can be thought of as a scheme over X of the form $Spec(k)$ for some k algebraically closed, the profinite fundamental group $\pi_1^{prof}(X, \bar{x})$ is the automorphism group of the functor $\boldsymbol{FEt}(X) \to Set$ taking Y to $|Y \times_X \bar{x}|$; the point $\bar{x}$ is called the base-point in the definition.*

From the construction, we obtain a natural functor from $\mathbf{FEt}(X)$ to the category of finite sets equipped with $\pi_1^{\text{prof}}(X, \bar{x})$-actions. Using properties of Galois categories, we see that $\pi_1^{\text{prof}}(X, \bar{x})$ is profinite with a neighborhood basis of open subgroups given by the point stabilizers in $|Y \times_X \bar{x}|$ for each $Y \in \mathbf{FEt}(X)$. Moreover, the previous functor defines an equivalence between $\mathbf{FEt}(X)$ and the category of finite $\pi_1^{\text{prof}}(X, \bar{x})$-sets for the profinite topology on $\pi_1^{\text{prof}}(X, \bar{x})$, i.e finite sets with discrete topology carrying continuous group actions.

Definition 68. *The profinite fundamental group of a scheme is often called the ´etale fundamental group and denoted by* $\pi_1^{et}(X, \bar{x})$

Remark 12. Let $Y \to X$ be a morphismn of connected schemes. Suppose that for every connected $Z \in \mathbf{FEt}(X)$, the scheme $Y \times_X Z$ is connected. Then for any geometric point $\bar{y} \in Y$, we have the map $\pi_1^{\text{prof}}(Y, \bar{y}) \to \pi_1^{\text{prof}}(X, \bar{y})$ is surjective.

We state the following lemma without proof.

Lemma 26. *Let* $k \to k'$ *be an extension of algebraically closed fields. Let* X *be a connected scheme over* k*. Then* $X_{k'}$ *is also connected*

Remark 13. We would like to think of the profinite fundamental group of a scheme as a "topological invariant", but this goal is hampered by a fundamental defect: it is not stable under base change. More precisely, if $k \to k'$ is an extension of algebraically closed fields and X is connected scheme over k, then $X_{k'}$ is connected by lemma above; now for any geometric point $\bar{x} \in X_{k'}$, the morphism $\pi_1^{\text{prof}}(Y, \bar{y}) \to \pi_1^{\text{prof}}(X, \bar{y})$ is surjective. However, it is easy to exhibit examples where this map fails to be injective.

Example 41. Let $k \to k'$ be an extension of algebraically closed fields of characteristic $p > 0$ and let $X = \text{Spec}(k[T])$. For any geometric point $\bar{x} \in X$, the Artin-Schrier construction provides an isomorphism

$$\text{Hom}_{\text{Top Grp.}}(\pi_1^{\text{prof}}(X, \bar{x}), \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{n>0, n \not\equiv 0 \bmod p} kT^i$$

This group is not invariant under enlarging k.

Example 42. Let $k \to k$ be an extension of algebraically closed fields of characteristic $p > 0$. Let X be a smooth projective, connected curve of genus g over k. Then for any geometric point $\bar{x} \in X_{k'}$, $\text{Hom}(\pi_1^{\text{prof}}(X, \bar{x}), \mathbb{Z}/p\mathbb{Z})$ is a finite free $\mathbb{Z}/p\mathbb{Z}$-module of rank equal to the p-rank of X. This rank can be computed in terms of the geometric points of the p-torsion subscheme of the Jacobian, and thus is invariant under base change from $k \to k'$. We can check in this case the map is indeed an isomorphism.

Remark 14. If the morphism of $\pi_1^{\text{prof}}(Y, \bar{y}) \to \pi_1^{\text{prof}}(X, \bar{y})$ is an isomorphism then for any $k', \bar{x}$ then we say the morphism $X \to k$ is π_1-proper.

Lemma 27. *Let* $f : Y \to X$ *be a morphism of schemes which are quasi-compact and quasi-separated. Suppose that the base change functor* $\boldsymbol{FEt}(X) \to \mathbf{FEt}(Y)$ *is an equivalence of categories.*

- *The map* $\pi_0(X) \to \pi_0(Y)$ *is a homeomorphism.*
- *Suppose that one of* X *or* Y *is connected. Then so is the other, and for any geometric point* $\bar{y} \in Y$ *the map* $\pi_1^{prof}(Y, \bar{y}) \to \pi_1^{prof}(X, \bar{y})$ *is a homeomorphism.*

Proof. Refer [Stacks] tagged 0BQA □

Lemma 28. *Let* $k \to k'$ *be an extension of algebraically closed fields of characteristic* 0*. Let* X *be a* k*-scheme.*

- *The base change functor* $\boldsymbol{FEt}(X) \to \boldsymbol{FEt}(X_{k'})$ *is an equivalence of categories.*
- *If* X *is connected then for any geometric point* $\bar{x} \in X_{k'}$*, the map* $\pi_1^{prof}(X_{k'}, \bar{x}) \to \pi_1^{prof}(X, \bar{x})$ *i.e the morphism* $X \to k$ *is* π_1*-proper.*

Proof. We have trivially (a) $\implies$ (b) from previous lemma. We just have to show (a). We also may assume that X is affine. We write the co-ordinate ring A of X as finitely generated k-sub-algebras A_i and by first principles to A and to $A \otimes_k k' = \varinjlim_i (A_i \otimes_k k')$, hence we further reduce to X is of finite type over k. By forming a hypercovering of X by smooth varieties using resolution of singularities and applying first principles we may further reduce to case X is smooth. Using the Lefschetz principle, we may also assume that $k, k' \in \mathbb{C}$ we may then assume without loss of genrality $k' = \mathbb{C}$.

If X is connected then so is $X_{\mathbb{C}}$ by the previous lemma as then it follows that $\mathbf{FEt}(X) \to \mathbf{FEt}(X_{\mathbb{C}})$ is fully faithful. We are just left to prove essential surjectivity. We apply resolution of singularities to construct a compactification $\overline{X}$ of X whose boundary is a divisor of Z of simple normal crossings. Given a finite ´etale cover of $X_{\mathbb{C}}$ we obtain a corresponding $\mathbb{Z}$-local system on $X_{\mathbb{C}}^{\mathrm{an}}$ with finite global monodromy; by the Riemann–Hilbert correspondence plus GAGA, this gives rise to a vector bundle on $X_{\mathbb{C}}$ equipped with an integrable connection having regular logarithmic singularities along $Z_{\mathbb{C}}$. The moduli stack of such objects is the base extension from k to $\mathbb{C}$ of a corresponding stack of finite type over k; since the base extension must consist of discrete points, these points coincide with the connected components of the stack, which remain invariant under base extension (by previous lemma). We thus obtain a vector bundle with integrable meromorphic connection on X itself, the sheaf of sections of this bundle is the underlying $\mathcal{O}_X$-module of finite etale $\mathcal{O}_X$ algebra descending the original cover of $X_{\mathbb{C}}$. □

Lemma 29. *Let* A *be a Henselian local ring with residue field* κ*. Let* $f : X \to S := Spec(A)$ *be a proper morphism of schemes. Then the base change functor* $\boldsymbol{FEt}(X) \to \boldsymbol{FEt}(X \times_S Spec(\kappa))$ *is an equivalence of categories.*

Proof. Refer [Stacks] tag 0A48. □

An important corollary is as follows

Corollary 5. *Let* $k \to k'$ *be an extension of algebraically closed fields of any characteristic. Let* X *be a proper* k*-scheme.*

(a) The base change functor $\boldsymbol{FEt}(X) \to \boldsymbol{FEt}(X_{k'})$ *is an equivalence of categories.*
(b) If X *is connected, then for any geometric point* $\bar{x}$ *of* $X_{k'}$*, the map* $\pi_1^{prof}(X_{k'}, \bar{x}) \to \pi_1^{prof}(X, \bar{x})$ *is a homeomorphism i.e the morphism* $X \to k$ *is* π_1*-proper.*

Proof. Part (a) is just an application of previous lemma and for part (b) we have from [Stacks] tagged 0A49. □

We now turn to analogues of the homotopy exact sequence of a fiber bundle of topological spaces.

Lemma 30. *Let $X \to S$ be a qcqs morphism of schemes with connected π_1-proper geometric fibres. Assume in addition that for every geometric point $\bar{s} \in S$, every connected finite ´etale covering of $X \times_S \bar{s}$ extends to a finite etale covering of $X \times_S U$ with connected geometric fibers over some ´etale neighborhood U of $\bar{s} \in S$. Then for any finite ´etale morphism $X' \to X$, there exists a commutative diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

such that $S' \to S$ is finite etale and $X' \to S'$ has geometrically connected fibres. Additionally, this diagram is initial among all diagrams

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

where $T \to S$ is finite etale; in particular, it is unique up to unique isomorphism.

Proof. Refer [Ked+19] □

We have the following important corollary.

Corollary 6. *Suppose that for previous lemma we assume further that S is connected. Then X is connected, and for any geometric point $\bar{x} \in X$ mapping to a geometric point $\bar{s} \in S$ we get that the following sequence*

$$\pi_1^{prof}(X \times_S \bar{s}, \bar{x}) \to \pi_1^{prof}(X, \bar{x}) \to \pi_1^{prof}(S, \bar{s}) \to 1$$

is exact.

Proof. Refer [Ked+19] □

To provide a basis for Drinfeld's lemma following the motivation of Drinfeld we need a Kunneth type formulae relation product of fundamental groups to fundamental group of product.

Corollary 7. *Let k be an algebraically closed field and put $S = Spec(k)$. Let $X \to Y, Y \to S$ be morphisms such that Y is connected and $X \to S$ is qcqs and π_1-proper. Then $Z := X \times_S Y$ is also connected and we have the isomorphism*

$$\pi_1^{prof}(Z, \bar{z}) \cong \pi_1^{prof}(X, \bar{x}) \times \pi_1^{prof}(Y, \bar{y})$$

as topological groups.

Proof. Refer [Ked+19] □

Definition 69. *For X a connected scheme we say that X is $K(\pi, 1)$-scheme if for some (hence any) geometric point $\bar{x} \in X$, for every locally constant sheaf of finite abelian groups $\mathcal{F}$ on X_{et} the natural maps*

$$H^*(\pi_1^{prof}(X, \bar{x}), \mathcal{F}_{\bar{x}}) \to H^*(X_{et}, \mathcal{F})$$

are isomorphisms. This is analogous to the corresponding definition in topology, which can be formulated as the assertion that the higher homotopy groups of X all vanish.

We throw light on a following recent result of Achinger which is important in our context.

Theorem 20. *(Achinger) Let X be a connected affine scheme over $\mathbb{F}_p$. Then X is a $K(\pi, 1)$-scheme.*

We have following important corollary

Corollary 8. *Let $X := Spa(A, A^+)$ be a connected Tate adic affinoid space on which p is topologically nilpotent. Then X is a $K(\pi, 1)$ adic space.*

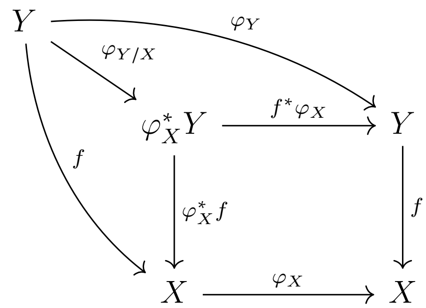
Proof. Refer [Ked+19] □

We now head towards the main parts of this thesis

5.3. **Drinfeld's Lemma(The original form).**

We next introduce a fundamental result of Drinfeld which gives a replacement for the Kunneth formula for fundamental groups for products of schemes in characteristic p. More precisely, the original result of Drinfeld gives a key special case the general case is due to E. Lau, except for a superfluous restriction to schemes of finite type.

Definition 70. *For any scheme X over $\mathbb{F}_p$, let $\varphi_X : X \to X$ be the absolute Frobenius morphism, induced by the p-th power map on rings. For $f : Y \to X$ a morphism of schemes, we define the relative Frobenius $\varphi_{Y/X} : Y \to \varphi_X^* Y$ to be the unique morphism making the diagram*



commute.

Lemma 31. *Let X be a projective scheme over $\mathbb{F}_p$.Let k be a separably closed field of characteristic p. Then pullback along $X_k \to X$ defines an equivalence of categories between coherent sheaves on X and coherent sheaves on X_k equipped with isomorphisms with their φ_k-pulbacks. Moreover, for $\mathcal{F}$ a coherent sheaf on X, the induced maps*

$$H^i(X, \mathcal{F}) \otimes_{\mathbb{F}_p} k \to H^i(X_k, \mathcal{F})$$

are φ-equivariant isomorphisms.

This is a kind of similar GAGA theorem for Serre for schemes.

Definition 71. *For X a scheme and Γ a group of automorphisms of X, let $\boldsymbol{FEt}(X/\Gamma)$ denote the category of finite ´etale coverings Y equipped with an action of Γ. That is, we must specify isomorphisms $Y \to \gamma^* Y$ for each $\gamma \in \Gamma$, subject to the condition that for $\gamma_1, \gamma_2 \in Y$ composing the γ_1-pullback of $Y \to \gamma_2^* Y$ with $Y \to \gamma_1^* Y$ yeilds the chosen map $Y \to (\gamma_1\gamma_2)^* Y$.*

Definition 72. *We say that X is Γ-connected, if $X \neq \phi$ and it is the only Γ-stable closed open set are itself and the empty set. If X is Γ-connected then for any geometric point $\bar{x} \in X$, the category $\boldsymbol{FEt}(X/\Gamma)$ equipped with the fibre functor $Y \mapsto |Y \times_X \bar{x}|$ is a Galois category.*

Lemma 32. *Let $X = Spec(A)$ be an affine scheme over $\mathbb{F}_p$. Let k be a field of characteristic p. Write A as a filtered direct limit of finitely generated $\mathbb{F}_p$-subalgebras A_i. Then the base extension functor*

$$2 - \varinjlim \boldsymbol{FEt}((A_i \otimes_{\mathbb{F}_p} k)/\varphi_k) \to \boldsymbol{FEt}((A_i \otimes_{\mathbb{F}_p} k)/\varphi_k)$$

is an equivalence of categories.

This lemma is from [Ked+19].

Lemma 33. *Let X be a scheme over $\mathbb{F}_p$. Let k be an algebraically closed field of characteristic p. Then the base extension functor*

$$\boldsymbol{FEt}(X) \to \boldsymbol{FEt}(X_k/\varphi_k)$$

is an equivalence of categories, with the quasi-inverse functor being given by taking φ_k-invariants.

Example 43. Let k be an algebraic closure of $\mathbb{F}_p$ and put $X = \operatorname{Spec}(k)$. Then X_k is highly disconnected: there is a natural homeomorphism $\pi_0(X_k) \cong \operatorname{Gal}(k/\mathbb{F}_p) \cong \widehat{\mathbb{Z}}$. However the action of φ_k on $\pi_0(X)$ is via translations by the dense subgroups $\mathbb{Z}$ of $\widehat{\mathbb{Z}}$.

An important corollary is as follows

Corollary 9. *Let X be a connected scheme over $\mathbb{F}_p$. Let k be an algebraically closed field of characteristic p.*

(a) The scheme X_k is φ_k-connected.
(b) For any geometric point $\bar{x} \in X$, the map

$$\pi_1^{prof}(X, \bar{x}) \to \pi_1^{prof}(X/\varphi_k, \bar{x})$$

is a homeomorphism of profinite groups.

Definition 73. *Let $X_1, \ldots, X_n$ be schemes over $\mathbb{F}_p$ and put $X := X_1 \times_{\mathbb{F}_p} X_2 \times_{\mathbb{F}_p} X_3 \times_{\mathbb{F}_p} \cdots \times_{\mathbb{F}_p} X_n$. We define the notation $\varphi_i = \varphi_{X_i}$. We define the category*

$$\boldsymbol{FEt}(X/\Phi) := \boldsymbol{FEt}(X/\langle \varphi_1, \varphi_2, \ldots, \varphi_n \rangle) \times_{\boldsymbol{FEt}(X/\varphi_X)} \boldsymbol{FEt}(X)$$

via the functor $\boldsymbol{FEt}(X) \to \boldsymbol{FEt}(X/\varphi_X)$. In other words, an object of $\boldsymbol{FEt}(X/\varphi)$ is a finite ´etale covering $Y \to X$ equipped with commuting isomorphisms $\beta_i : Y \cong \varphi_i^* Y$ whose composition is $\varphi_{Y/X}$. Also for any $i \in \{1, 2, \ldots, n\}$ is a canonical equivalence of categories

$$\boldsymbol{FEt}(X/\Phi) \cong \boldsymbol{FEt}(X/\langle \varphi_1, \varphi_2, \ldots, \widehat{\varphi_i}, \ldots, \varphi_n \rangle)$$

In case of $X_1, X_2, \ldots, X_n$ are connected then by previous lemma we might obtain a Galois category.

Lemma 34. *With the same setting, if $X_1, X_2, \ldots, X_n$ are connected, then X is $\langle \varphi_1, \varphi_2, \ldots, \widehat{\varphi_i}, \ldots, \varphi_n \rangle$-connected for any $i \in \{1, 2, \ldots, n\}$. We say that X is Φ-connected.*

Given this information we now proceed towards the main theorem

Theorem 21. *("Drinfeld's lemma") Let $X_1, X_2, \ldots, X_n$ be connected quasi compact and quasi seperated schemes over $\mathbb{F}_p$ and put $X := X_1 \times_{\mathbb{F}_p} X_2 \times_{\mathbb{F}_p} X_3 \times_{\mathbb{F}_p} \cdots \times_{\mathbb{F}_p} X_n$. Then for any geometric point $\bar{x} \in X$, the map*

$$\pi_1^{prof}(X/\Phi, \bar{x}) \to \prod_{i=1}^{n} \pi_1^{prof}(X_i, \bar{x})$$

is an isomorphism of topological groups.

Proof. Using the definition we may rewrite the left hand side as

$$\pi_1^{\text{prof}}(X/\Phi, \bar{x}) = \pi_1^{\text{prof}}((X_1/\varphi) \times_{\mathbb{F}_p} (X_2/\varphi) \times_{\mathbb{F}_p} (X_3/\varphi) \times_{\mathbb{F}_p} \cdots \times_{\mathbb{F}_p} (X_n/\varphi), \bar{x})$$

We proceed then by induction.

<u>Base Case</u>: $n = 1$ is trivial.

Induction Hypothesis: Let the result be true n.

Inductive Step: We prove it for $n+1$. Let $X := X_1 \times_{\mathbb{F}_p} X_2 \times_{\mathbb{F}_p} \cdots \times_{\mathbb{F}_p} X_n$. We follow in the lines of corollary 7, let $Z := X \times_{\mathbb{F}_q} X_{n+1}$ since X is qcqs over $\mathbb{F}_q$ and X_{n+1} is qcqs over $\mathbb{F}_p$. Then we have two points to note

(a) Using the lemma 30 we can get that

$$\mathbf{FEt}(X) \equiv \mathbf{FEt}(X_k/\varphi_k)$$

for an algebraically closed field k.

(b) Using the corollary 7 then, we can say

$$\pi_1^{\text{prof}}(Z, \bar{z}) \cong \pi_1^{\text{prof}}(X, \bar{z}) \times \pi_1^{\text{prof}}(X_{n+1}, \bar{z})$$

And by point above we get

$$\pi_1^{\text{prof}}(Z/\varphi, \bar{z}) \cong \pi_1^{\text{prof}}(X/\varphi, \bar{z}) \times \pi_1^{\text{prof}}(X_{n+1}/\varphi, \bar{z})$$

and from here can deduce the result again by applying the Lemma 30.

□

5.4. **Drinfeld's Lemma for Diamonds.**

We now formulae an analogue of Drinfeld's Lemma for Diamonds. This involves a re-interpretation of the Fargues-Fontaine curve in the language of diamonds. (Refer : [ANS]).

We do the setup like last section

Definition 74. *Let $X_1, X_2, \ldots, X_n$ be small v-sheaves and put $X := X_1 \times X_2 \times \cdots \times X_n$. We define the category*

$$\boldsymbol{FEt}(X/\Phi) := \boldsymbol{FEt}(X/\langle\varphi_1, \varphi_2, \ldots, \varphi_n\rangle) \times_{\boldsymbol{FEt}(X/\varphi_X)} \boldsymbol{FEt}(X))$$

where $\boldsymbol{FEt}(X) \to \boldsymbol{FEt}(X/\varphi_X)$ is the canonical section of the forgetful functor $\boldsymbol{FEt}(X) \to \boldsymbol{FEt}(X/\varphi_X)$. Again similarly for any $i \in \{1, 2, \ldots, n\}$ there is a canonical equivalence of categories

$$\boldsymbol{FEt}(X/\Phi) \cong \boldsymbol{FEt}(X/\langle\varphi_1, \varphi_2, \ldots, \widehat{\varphi_i}, \ldots, \varphi_n\rangle)$$

Definition 75. *For X a small v-sheaf, from previous definition we have*

$$\boldsymbol{FEt}((X \times Spd(\mathbb{Q}_p)/\Phi) \cong \boldsymbol{FEt}(FF_X) \cong \boldsymbol{FEt}((X \times Spd(\mathbb{Q}_p)/\varphi)$$

where FF_X is the Fargues-Fontaine curve.

We have seen previously that $\text{Spd}(\mathbb{Q}_p)/\varphi$ is a diamond and hence a result the small v-sheaf $X \times (\text{Spd}(\mathbb{Q}_p)/\varphi)$ is a diamond. We call such a thing a "mirror curve" over X.

Lemma 35. *Let X be a small v-sheaf. Let F be an algebraically closed non-archimedean field of characteristic p. Then the base extension functor*

$$\boldsymbol{FEt}(X) \to \boldsymbol{FEt}(X \times (Spd(F)/\varphi)) \cong \boldsymbol{FEt}((X/\varphi) \times Spd(F))$$

is an equivalence of categories.

Proof. Refer [Ked+19] □

The main lemma can now be done as follows

Theorem 22. *("Drinfeld's Lemma for Diamonds") Let $X_1, X_2, \dots, X_n$ be connected spatial and quasi compact and quasi seperated diamonds. Then $X := X_1 \times X_2 \times \dots \times X_n$ is Φ-connected and, for any geometric point $\bar{x} \in X$, the map*

$$\pi_1^{prof}(X/\Phi, \bar{x}) \to \prod_{i=1}^{n} \pi_1^{prof}(X_i, \bar{x})$$

is an isomorphism of profinite groups.

Proof. Same as above. □

5.5. **Shtukas in positive characteristic.**

In the introduction section of the thesis we discussed about Drinfeld shutkas and how it was used to prove the Langlands for Function Fields. The concept was originally introducted by Drinfeld as a replacement for elliptic curves in positive characteristic; that is to say, the moduli spaces of such objects constitute a replacement for modular curves and Shimura varieties as a tool for studying Galois representations of a global function field in positive characteristic (which we are now prepared to think about as representations of profinite fundamental groups).

Setting: Let C be a smooth , projective, geometrically irreducible curve over a finite field $\mathbb{F}_q$ of characteristic p.

Definition 76. *Let S be a scheme over $\mathbb{F}_q$. A shtuka over S consists of the following data*

- *A finite index set I and a morphism $(x_i)_{i \in I} : S \to C^I$.*
- *A vector bundle $\mathcal{F}$ over $C \times S$*
- *A isomorphism of bundles*

$$\Phi : (\varphi_S^* \mathcal{F})|_{(C \times S) \setminus \bigcup_{i \in I} \Gamma_{x_i}} \cong \mathcal{F}|_{(C \times S) \setminus \bigcup_{i \in I} \Gamma_{x_i}}$$

  where $\Gamma_{x_i} \subset C \times S$ denotes the graph of x_i.

The morphisms $x_i : S \to C$ are called the legs of the shtukas.

5.6. Shtukas in mixed characteristic.

Our aim was to study shtukas in mixed characteristic. We do this in this section and study the relation between Shtukas and Fargues-Fontaine curve. Our approach is mostly based on [Stacks] tag 06TF.

Definition 77. *Let $\mathcal{O} : \boldsymbol{Pfd} \to \boldsymbol{Ring}$, where $\boldsymbol{Pfd}$ denotes the category of perfectoid spaces, given by the functor taking $X \mapsto \mathcal{O}(X)$. This functor is a sheaf of rings for the v-topology. For any small v-sheaf X, we may restrict $\mathcal{O}$ to the arrow category $\boldsymbol{Pfd}_X$, the category of morphisms $S^\diamond \to X$ with $S \in \boldsymbol{Pfd}$, to obtain the structure sheaf on X.*

A vector bundle on X is a locally finite free $\mathcal{O}_X$-module; let $\boldsymbol{Vec}_X$ denote the category of such objects.

Definition 78. *Let $\mathcal{O}^\sharp : \boldsymbol{Pfd}_{Spd(\mathbb{Z}_p)} \to \boldsymbol{Ring}$ be the functor where $X \mapsto \mathcal{O}(X^\sharp)$, where we have $X^\sharp$ is the untilt of X corresponding to the structure morphism $X \to Spd(\mathbb{Z}_p)$. We can observe that this functor is again a sheaf of rings for the v-topology. For any small v-sheaf over X over $Spd(\mathbb{Z}_p)$, we may restrict $\mathcal{O}^\sharp$ to $\boldsymbol{Pfd}_X$ to obtain the untilted structure sheaf on X. An untilted vector bundle on X is a locally finite free $\mathcal{O}^\sharp_X$-module; let $Vec^\sharp_X$ be the category of such objects.*

As an immediate consequence we have

Theorem 23. *Let (A, A^+) be a perfectoid pair of characteristic p.*

(a) The pullback functor $\boldsymbol{FPMoD}_A \to \boldsymbol{Vec}_{Spd(A,A^+)}$ is an equivalence of categories.
(b) Fix a morphism $Spd(A, A^+) \to Spd(\mathbb{Z}_p)$ corresponding to an untilt $(A^\sharp, A^{\sharp+})$ of (A, A^+). Then the pullback functor $\boldsymbol{FPMoD}_{A^\sharp} \to \boldsymbol{Vec}^\sharp_{Spd(A,A^+)}$ is an equivalence of categories.

We can define shtukas over diamonds

Definition 79. *Let S be a diamond. A shtuka over S consists of the following data*

- *A finite index set I and a morphism $(x_i)_{i\in I} : S \to (Spd(\mathbb{Z}_p))^I$.*
- *An ultilted vector bundle $\mathcal{F}$ over $Spd(\mathbb{Z}_p) \times S$ with respect to the first projection which locally on S arises from a vector bundle on the underlying adic space W_S.*
- *A isomorphism of bundles*
$$\Phi : (\varphi_S^*\mathcal{F})|_{(Spd(\mathbb{Z}_p)\times S)\setminus\bigcup_{i\in I}\Gamma_{x_i}} \cong \mathcal{F}|_{(Spd(\mathbb{Z}_p)\times S)\setminus\bigcup_{i\in I}\Gamma_{x_i}}$$
where $\Gamma_{x_i} \subset Spd(\mathbb{Z}_p) \times S$ denotes the graph of x_i. We also insist that Φ be meromorphic along $\bigcup_{i\in I}\Gamma_{x_i}$, this having been implicit in the schematic case.

The morphisms $x_i : S \to Spd(\mathbb{Z}_p)$ are called the legs of the shtukas.

We state a theorem of Kedlaya.

Theorem 24. *(Kedlaya) Let (R, R^+) be a perfectoid Huber pair of characteristic p in which R is Tate, and we denote $\mathbf{A}_{inf} := \mathbf{A}_{inf}(R, R^+)$*

(a) Let $\bar{x} \in R^+$ be a topologically unit of R. Then the pullback functor from vector bundles on the scheme

$$Spec(\mathbf{A}_{inf}) \smallsetminus V(p, [\bar{x}])$$

to vector bundles on the analytic locus of $Spa(\mathbf{A}_{inf}, \mathbf{A}_{inf})$ is an equivalence of categories.

(b) Suppose that $R = F$ is a perfectoid field. Then both categories in (a) are equivalent to the category of finite free $\mathbf{A}_{inf}$-modules and the category of vector bundles on $Spa(\mathbf{A}_{inf})$.

Proof. Refer [Ked19] □

We now make things slightly complicated by considering shtukas with one leg.

Lemma 36. *Suppose that $I = \{1\}$ is a singleton set and that the morphism x_1 factors through $Spd(\mathbb{Q}_p)$. Then the following categories are canonically equivalent*

(a) shtukas over S with leg s_1

(b) data $\mathcal{F}_1 \dashrightarrow \mathcal{F}_2$, where $\mathcal{F}_1$ is a φ-equivalent bundle over $Spd(\mathbb{Z}_p) \times S$ (which locally on S descend to W_S), $\mathcal{F}_2$ is a φ-equivalent bundle over $Spd(\mathbb{Q}_p) \times S$ (which locally on S descend to Y_S), and the arrow denotes a meromorphic φ-equivariant map over Y_S which is an isomorphic away from $\bigcup_{n \in \mathbb{Z}} \varphi^n(\Gamma_{x_1})$

Proof. Refer [Ked+19]. □

We relate the notion of shtukas to p-adic Hodge Theory.

Definition 80. *Let F be a perfectoid field of characteristic p and similarly as above we write $\boldsymbol{A}_{inf} := \boldsymbol{A}_{inf}(F, \mathfrak{o}_F)$. We also fix a primitive element $z \in \boldsymbol{A}_{inf}$ corresponding to an untilt $F^\sharp$ of F of characteristic 0. A breuil-Kisin module over $\boldsymbol{A}_{inf}$ is a finite free $\boldsymbol{A}_{inf}$-module D equipped with an isomorphism with an isomorphism $\Phi : (\varphi^* D)[z^{-1}] \cong D[z^{-1}]$. Let $x_1 : Spd(F, \mathfrak{o}_F) \to Spd(\mathbb{Z}_p)$ be the morphism corresponding to the untilt $F^\sharp$ of F.*

Breuil-Kisin modules are important in analysing crystalline representations of the absolute Galois group.

Lemma 37. *Suppose that F is algebraically closed. Then restriction of φ-equivariant vector bundles along the inclusion*

$$Y_S \subset \{v \in Spa(\boldsymbol{A}_{inf}, \boldsymbol{A}_{inf}) : v(p) \neq 0\}$$

is an equivalence of categories.

Proof. Refer [Far15] □

Lemma 38. *(Fargues) Let F be algebraically closed. Then the category of Breuil-Kisin modules over $\boldsymbol{A}_{inf}$ is equivalent to the category of shtukas over Spd$(F, \mathfrak{o}_F)$ with the single leg x_1.*

We conclude this chapter henceforth by establishing this relation between p-adic Hodge Theory in the current stages.

6. Conclusion

We conclude this thesis by referring to the recent work of Scholze and Fargues titled the "Geometrization of the Local Langland's Correspondence" [FS24]. In this grand project they develop the notion of Geometric Langlands on the Fargues-Fontaine curve. Instead of getting into technical details of the paper we just sketch the proof of Drinfeld for Local Langlands in function field case (equicharacteristic) case and understand probably what improvisations can be done in the mix characteristic case.

Let C be a smooth projective and geometrically irreducible curve over a finite field $\mathbb{F}_q$ of characteristic p. Let G be a reductive group over $\mathbb{F}_q$. The moduli space of G-shtukas with one leg comes with a structure morphism to C given by the leg. This space serves as an analogue to Shimura variety over $\mathbb{Z}$.

We now consider the relative etale cohomology group $R^i f_* \overline{\mathbb{Q}}_l$ for the morphism $f :$ Sht $\to C$; this is a local system on C, which is to say a representation of the fundamental group $\pi_1(C)$. There is also an action of the adelic group $G(\mathbf{A}_F)$ given by the Hecke operators.

The group $R^i f_* \overline{\mathbb{Q}}_l$ decomposes as a direct sum of certain automorphic representations π of $G(\mathbf{A}_F)$, each tensored with a certain representation $o(\pi)$ of G_F. The idea is that the mapping $\pi \to o(\pi)$ defines the Global Langlands correspondence ! But it was later observed it was not the case in every situation. Drinfeld realised that one could find the missing representations by considering shtukas with two legs rather one.

For the complete discussion we refer to [Yun24]. We instead go towards the mixed characteristic case. As in previous setting we will also a category Sht of mixed characteristic local shtukas with one leg; the leg which defines the morphism Sht $\to$ Spd$(\mathbb{Q}_p)$. The idea is to replicate similar arguments of Drinfeld in this case.

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DECLARATION

I declare that the work in this dissertation was done purely by me with no involvement of any artificial intelligence tools or Large Language Model Artificial Intelligence tools. I did not use any Artificial intelligence engines or software to assist me in this project.

Ritoprovo Roy 27th May 2025, Budapest

Ritoprovo Roy Date