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Cyclic ordering of split matroids

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Contents

Alulírott Jánosik Áron nyilatkozom, hogy szakdolgozatom elkészítése során nem használtam MI alapú eszközöket.

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A **szakdolgozat** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

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a hallgató aláírása

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Chapter 1

Introduction

This thesis is based on an article with the same title and authors Kristóf Bérczi, Áron Jánosik and Bence Mátravölgyi [2].

Throughout the thesis, we denote a matroid by $M = (S, \mathcal{B})$, where S is a finite ground set and $\mathcal{B}$ is the *family of bases*, satisfying the so-called *basis axioms*: (B1) $\emptyset \in \mathcal{B}$, and (B2) for any $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 - B_2$, there exists $f \in B_2 - B_1$ such that $B_1 - e + f \in \mathcal{B}$. The latter property, called the *basis exchange axiom*, is one of the most fundamental tools in matroid theory. Nevertheless, it only provides a local characterization of the relationship between bases, which presents a significant stumbling block to further progress.

A rank-r matroid $M = (S, \mathcal{B})$ with $|S| = n$ is cyclically orderable if there exists an ordering $S = \{s_1, \ldots, s_n\}$ such that $\{s_i, s_{i+1}, \ldots, s_{i+r-1}\} \in \mathcal{B}$ for all $i \in [n]$, where indices are understood in a cyclic order. While studying the structure of symmetric exchanges in matroids, Gabow [15] formulated a beautiful conjecture, stating that every matroid whose ground set decomposes into two disjoint bases is cyclically orderable. This question was raised independently by Wiedemann [24] and by Cordovil and Moreira [8]. The conjecture makes a stronger claim: for a fixed partition, the cyclic ordering can be chosen such that the elements of the two bases in the partition form contiguous intervals.

Conjecture 1 (Gabow)**.** *Let $M = (S, \mathcal{B})$ be a matroid and $S = B_1 \cup B_2$ be a partition of the ground set into two disjoint bases. Then, M has a cyclic ordering in which the elements of B_1 and B_2 form intervals.*

It is not difficult to see that the statement holds for strongly base orderable matroids. The conjecture was settled for graphic matroids [8, 17, 24], sparse paving matroids [7], matroids of rank at most 4 [19] and 5 [18], split matroids [6], and regular matroids [4]. However, the existence of a cyclic ordering remains open in general, even without the constraint of the bases forming intervals.

In [17], Kajitani, Ueno, and Miyano proposed a conjecture that would provide a full characterization of cyclically orderable matroids. A matroid $M = (S, \mathcal{B})$ with *rank function* r_M is called *uniformly dense* if $|S| \cdot r_M(X) \geq r_M(S) \cdot |X|$ holds for all $X \subseteq S$. It is not difficult to see that a cyclically orderable matroid is necessarily uniformly dense as well, and the conjecture states that this condition is also sufficient.

Conjecture 2 (Kajitani, Ueno, and Miyano)**.** *A matroid is cyclically orderable if and only if it is uniformly dense.*

Despite the fact that the conjecture would provide entirely new insights into the structure of matroids, very little progress has been made so far. Van den Heuvel and Thomassé [22] showed that the conjecture is true if $|S|$ and $r(S)$ are coprimes, and Bonin's result [7] for sparse paving matroids remains true also in this more general setting.

It is worth taking a moment to consider the interpretation of the uniformly dense property. By the matroid union theorem of Edmonds and Fulkerson [10], the ground set of a matroid $M = (S, \mathcal{B})$ can be covered by k bases if and only if $k \cdot r_M(X) \geq |X|$ holds for all $X \subseteq S$. Using this, a matroid is uniformly dense if and only if its ground set can be covered by $\lceil |S|/r_M(S) \rceil$ bases. In other words, the ground set can be decomposed in 'almost' disjoint bases, where almost means that the total overlapping between distinct bases is bounded by $r_M(S) - 1$. In particular, any matroid whose ground set decomposes into pairwise disjoint bases is uniformly dense. This observation motivates the following strengthening of Gabow's conjecture.

Conjecture 3. *Let $M = (S, \mathcal{B})$ be a matroid and $S = B_1 \cup \cdots \cup B_k$ be a partition of the ground set into k pairwise disjoint bases. Then, M has a cyclic ordering in which the elements of B_i form an interval for each $i \in [k]$.*

To the best of our knowledge, Conjecture 3 has not been previously considered and remains open even for very restricted classes of matroids, such as strongly base orderable matroids. Our main contribution is proving the conjecture for the class of split matroids. Split matroids were first introduced by Joswig and Schröter [16] while studying matroid polytopes from a geometric point of view. Since then, this class of matroids has gained importance in many contexts, primarily due to the work of Ferroni and Schröter [11–14].

Theorem 1. *Conjecture 3 is true for split matroids.*

It is worth emphasizing that our proof is algorithmic, hence it provides a procedure for determining a cyclic ordering in question using a polynomial number of independence oracle calls.

Remark 2. In fact, we prove a slightly stronger statement: in the cyclic ordering obtained, the bases $B_1, \ldots, B_k$ form intervals that follow each other in this order.

The rest of the thesis is organized as follows. Basic definitions and notation are introduced in Chapter 2. We prove Conjecture 3 for split matroids in Chapter 3. In Chapter 4, we give a list of related open questions and conjectures that are subject of future research. Finally, in Chapter 5, we consider a matroid class that is a generalization of split matroids with similar structure.

Chapter 2

Preliminaries

2.1 General notation

We denote the set of *nonnegative integers* by $\mathbb{Z}_+$. For $k \in \mathbb{Z}_+$, we use $[k] = \{1, \ldots, k\}$. Given a ground set S, the *difference* of $X, Y \subseteq S$ is denoted by $X - Y$. If Y consists of a single element y, then $X - \{y\}$ and $X \cup \{y\}$ are abbreviated as $X - y$ and $X + y$, respectively. The *symmetric difference* of X and Y is denoted by $X \triangle Y \coloneqq (X - Y) \cup (Y - X)$. Given a set system $\mathcal{F}$, we use $\mathcal{F}^{\cup_2} \coloneqq \{F_1 \cup F_2 : F_1, F_2 \in \mathcal{F}\}$. We call a function $f : \mathcal{F}^{\cup_2} \to \mathbb{R}$ $\mathcal{F}$-submodular, if $\mathcal{F}$ is closed under intersection and $\forall F_1, F_2 \in \mathcal{F}$ we have $f(F_1) + f(F_2) \geq f(F_1 \cup F_2) + f(F_1 \cap F_2)$.

2.2 Split matroids

For basic definitions on matroids, we refer the reader to [21]. Let S be a ground set of size at least r, $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a (possibly empty) collection of subsets of S, and $r, r_1, \ldots, r_q$ be nonnegative integers satisfying

(H1) $$|H_i \cap H_j| \leq r_i + r_j - r \text{ for distinct } i, j \in [q],$$

(H2) $$|S - H_i| + r_i \geq r \text{ for all } i \in [q].$$

Then the corresponding *elementary split matroid* $M = (S, \mathcal{B})$ is given by $\mathcal{B} = \{X \subseteq S \mid |X| = r,\ |X \cap H_i| \leq r_i \text{ for all } i \in [q]\}$; see [3] for details. It is easy to see that the underlying hypergraph can be chosen in such a way that

(H3) $$r_i \leq r - 1 \text{ for all } i \in [q],$$

(H4) $$|H_i| \geq r_i + 1 \text{ for all } i \in [q].$$

The representation is called *non-redundant* if all of (H1)–(H4) hold. A set $F \subseteq S$ is called H_i*-tight* if $|F \cap H_i| = r_i$. Finally, a *split matroid* is the direct sum of a single elementary split matroid and some (maybe zero) uniform matroids. The connection between elementary and connected split matroids is given by the following result [3].

Lemma 3 (Bérczi, Király, Schwarcz, Yamaguchi and Yokoi)**.** *The classes of connected split matroids and connected elementary split matroids coincide.*

A nice feature of split matroids is that they generalize paving and sparse paving matroids: paving matroids correspond to the special case when $r_i = r - 1$ for all $i \in [q]$, while we get back the class of sparse paving matroids if, in addition, $|H_i| = r$ holds for all $i \in [q]$. However, unlike the class of paving matroids, split matroids are closed not only under truncation and taking minors but also under duality [16]. The following result appeared in [3].

Lemma 4 (Bérczi, Király, Schwarcz, Yamaguchi and Yokoi)**.** *Let M be a rank-r elementary split matroid with a non-redundant representation $\mathcal{H} = \{H_1, \ldots, H_q\}$ and $r, r_1, \ldots, r_q$. Let F be a set of size r.*

(a) If F is H_i-tight for some index $i \in [q]$ then F is a basis of M.

(b) If F is both H_i-tight and H_j-tight for distinct $i, j \in [q]$ then $H_i \cap H_j \subseteq F \subseteq H_i \cup H_j$.

Proof. (a) Assume F is H_i-tight, so $|F \cap H_i| = r_i$. Then we need to verify that F satisfies the independence conditions for all other H_j. We have that

$$r_i + |F \cap H_j| = |F \cap H_i| + |F \cap H_j| = |F \cap (H_i \cap H_j)| + |F \cap (H_i \cup H_j)| \leq$$

$$|H_i \cap H_j| + |F| \leq (r_i + r_j - r) + r = r_i + r_j,$$

so $|F \cap H_j| \leq r_j$ follows immediately for $i \neq j \in [q]$.

(b) Assume F is both H_i-tight and H_j-tight for $i \neq j$, then from the same inequality chain:

$$r_i + r_j = |F \cap H_i| + |F \cap H_j| = |F \cap (H_i \cap H_j)| + |F \cap (H_i \cup H_j)| \leq$$

$$|H_i \cap H_j| + |F| \leq (r_i + r_j - r) + r = r_i + r_j,$$

therefore all inequalities are satisfied with equality, that is

$$|F \cap (H_i \cap H_j)| = |H_i \cap H_j| \quad \text{and} \quad |F \cap (H_i \cup H_j)| = |F|,$$

that implies

$$H_i \cap H_j \subseteq F \quad \text{and} \quad F \subseteq H_i \cup H_j,$$

so we are done. □

By Lemma 4(a), any set of size r that is tight with respect to one of the hyperedges is a basis. We will use this observation throughout without explicitly citing the lemma, to avoid repeatedly referring to part (a).

Chapter 3

Proof of Theorem 1

Proof of Theorem 1. Throughout the proof, we use the following notation convention: given an ordered sequence $X_1, \dots, X_k$ of sets or elements $x^1, \dots, x^k$, indices are meant cyclically, meaning that $X_{k+1} = X_1$, $x^{k+1} = x^1$, $X_0 = X_k$ and $x^0 = x^k$. In addition, we interpret the set $\{x_i, \dots, x_j\}$ as empty when $i > j$.

The theorem clearly holds if $k = 1$, while the case when $k = 2$ was proved in [6]. Therefore, we assume that $k \geq 3$. Let $M = (S, \mathcal{B})$ be a split matroid and $S = B_1 \cup \dots \cup B_k$ be a partition of its ground set into k pairwise disjoint bases. First we show that it suffices to consider connected split matroids. To see this, let $M_1 = (S_1, \mathcal{B}_1), \dots, M_t = (S_t, \mathcal{B}_t)$ be the connected components of M, where $|S_j| = n_j$ and the rank of M_j is r_j for $j \in [t]$. For all $i \in [k]$ and $j \in [t]$, let $B_i^j \coloneqq B_i \cap S_j$. Then, $S_j = B_1^j \cup \dots \cup B_k^j$ is a decomposition of S_j into pairwise disjoint bases of M_j. Let $S_j = \{s_1^j, \dots, s_{n_j}^j\}$ be a cyclic ordering of M_j in which the elements of B_i^j form the interval $I_i^j \coloneqq \{s_{(i-1)\cdot r_j+1}^j, \dots, s_{i\cdot r_j}^j\}$ for each $i \in [k]$. Then,

$$S = \{I_1^1, I_1^2, \dots, I_1^t, I_2^1, I_2^2, \dots, I_2^t, \dots, I_k^1, I_k^2, \dots, I_k^t\}$$

is a cyclic ordering of M in which B_i forms an interval for each $i \in [k]$. Since Conjecture 3 clearly holds for uniform matroids, the combination of the above observation and Lemma 3 allows us to assume that M is a rank-r elementary split matroid, defined by a non-redundant representation $\mathcal{H}$.

The high-level idea of the algorithm is as follows. We build up the orderings for the bases simultaneously in phases. At the beginning of the j-th phase, the first $(j-1)$ elements in each of the bases are ordered and the goal is to find the j-th element for all of them. We denote the first $(j-1)$ elements that we have already ordered in the i-th basis by $(b_1^i, \dots, b_{j-1}^i)$. The elements that are not yet ordered will be referred to as *remaining elements* in B_i and their set is denoted by C_i, that is, $C_i = B_i - \{b^i_1, \dots, b^i_{j-1}\}$. The goal is to choose b^i_j in such a way that $(C_i - b^i_j) \cup (b^{i+1}_1, \dots, b^{i+1}_j)$ forms a basis for all $i \in [k]$; we call such a choice $(b^1_j, \dots, b^k_j)$ *valid*. Note that the condition is satisfied in the beginning as it simply requires $C_i = B_i$ to be a basis for each $i \in [k]$. If valid choices exist up to the r-th phase, then we get a cyclic ordering of the matroid with the desired properties simply by putting the ordered bases after each other. However, if the next elements cannot be chosen while satisfying the above constraints, we will slightly modify the order of the first $(j-1)$ elements to allow further steps.

Now we turn to the detailed description of the proof. For ease of discussion, we present it as an indirect proof; however, it implicitly implies an algorithm as described above. Let $j \in [r+1]$ be maximal with respect to the property that, for all $i \in [k]$, there exist $b^i_1, \dots, b^i_{j-1} \in B_i$ such that

$$(\star) \quad (b^i_\ell, \dots, b^i_{j-1}) \cup C_i \cup (b^{i+1}_1, \dots, b^{i+1}_{\ell-1}) \text{ forms a basis for all } i \in [k], \ell \in [j+1],$$

where $C_i = B_i - \{b^i_1, \dots, b^i_{j-1}\}$. If $j = r+1$ then we are done. Therefore, suppose that $j \le r$. In particular, this means that there is no valid choice of j-th elements in the bases. Let $R_i \coloneqq C_i \cup \{b^{i+1}_1, \dots, b^{i+1}_{j-1}\}$ for all $i \in [k]$. Then, R_i is a basis by applying ($\star$) for $\ell = j+1$.

Claim 5. *For all $i \in [k]$, there exist distinct elements $p_i, q_i \in C_i$ and a hyperedge H_i with value r_i satisfying the following:*

(a) $p_k \in H_k - H_1$ and $p_i \in H_i \cap H_{i+1}$ for all $i \in [k-1]$,

(b) $q_k \notin H_k$ and $q_i \notin H_i \cup H_{i+1}$ for all $i \in [k-1]$,

(c) R_{i-1} is H_i-tight for all $i \in [k]$.

Proof. Let $p_1 \in C_1$ be an arbitrary element. By the basis exchange property for R_1 and B_2, there exists an element $p_2 \in C_2$ such that $R_1 - p_1 + p_2$ forms a basis. By the repeated application of this argument we get $p_i \in C_i$ such that $R_i - p_i + p_{i+1}$ forms a basis for all $i \in [k-1]$.

If $R_k - p_k + p_1$ forms a basis, then $(p_1, \dots, p_k)$ is a valid choice, contradicting the maximality of j. Otherwise, there exists a hyperedge H_1 with value r_1 such that $|H_1 \cap (R_k - p_k + p_1)| > r_1$. Since R_k is a basis, we conclude that R_k is H_1-tight, $p_k \notin H_1$, $p_1 \in H_1$ and $|H_1 \cap (R_k - p_k + p_1)| = r_1 + 1$. By the basis exchange property, there exists an element $q_1 \in C_1 - p_1$ such that $R_k - p_k + q_1$ forms a basis, implying $q_1 \notin H_1$. As the choice $(q_1, p_2, \dots, p_k)$ cannot be valid, then there exists a hyperedge H_2 with value r_2 such that $|H_2 \cap (R_1 - q_1 + p_2)| > r_2$. Since R_1 and $R_1 - p_1 + p_2$

are both bases, we conclude that R_1 is H_2-tight, $p_1 \in H_2$, $p_2 \in H_2$, $q_1 \notin H_2$ and $|H_2 \cap (R_1 - q_1 + p_2)| = r_2 + 1$. By the basis exchange property, there exists an element $q_2 \in C_2 - p_2$ such that $R_1 - q_1 + q_2$ forms a basis, implying $q_2 \notin H_2$. Continuing this procedure, we get elements $p_1, \dots, p_k$, $q_1, \dots, q_k$ and hyperedges $H_1, \dots, H_k$ with values $r_1, \dots, r_k$ satisfying the conditions of the claim. $\square$

It is worth noting that the hyperedges $H_1, \dots, H_k$ provided by the claim are not necessarily distinct. We give an analogous claim where the roles of $p_1, \dots, p_k$ and $q_1, \dots, q_k$ are reversed. The proof follows the same reasoning as in Claim 5; however, we include it here for completeness.

Claim 6. *For all $i \in [k]$, there exist a hyperedge H'_i with value r'_i satisfying the following:*

(a) $p_k \in H'_1 - H'_k$ and $p_i \notin H'_i \cup H'_{i+1}$ for all $i \in [k-1]$,

(b) $q_k \in (H_1 \cap H'_k) - H'_1$ and $q_i \in H'_i \cap H'_{i+1}$ for all $i \in [k-1]$,

(c) R_{i-1} is H'_i-tight for all $i \in [k]$,

(d) $H_i \cap H'_i \subseteq R_{i-1} \subseteq H_i \cup H'_i$ for all $i \in [k]$.

Proof. As the choice $(q_1, \dots, q_k)$ cannot be valid, there exists a hyperedge H'_1 with value r'_1 such that $q_k \notin H'_1$, $q_1 \in H'_1$ and R_k is H'_1-tight. Note that since $q_1 \in H'_1$ and $q_1 \notin H_1$ we have that H_1 and H'_1 are distinct hyperedges. Since R_k is both H_1 and H'_1-tight, Lemma 4(b) implies that $H_1 \cap H'_1 \subseteq R_k \subseteq H_1 \cup H'_1$, thus $q_k \in H_1$, $p_k \in H'_1$ and $p_1 \notin H'_1$. As the choice $(p_1, q_2, \dots, q_k)$ cannot be valid and R_k is H_1-tight by Claim 5 and $q_k \in H_1$ therefore $R_k - q_k + p_1$ is also H_1-tight, there exists a hyperedge H'_2 with value r'_2 such that $p_1 \notin H'_2$, $q_2 \in H'_2$ and R_1 is H'_2-tight. Again as $q_2 \in H'_2$ and $q_2 \notin H_2$ we have that H_2 and H'_2 are distinct hyperedges. Since R_1 is both H_2 and H'_2-tight, Lemma 4(b) implies that $H_2 \cap H'_2 \subseteq R_1 \subseteq H_2 \cup H'_2$, thus $q_1 \in H'_2$, $p_2 \notin H'_2$. Continuing this procedure, we get hyperedges $H'_1, \dots, H'_k$ with values $r'_1, \dots, r'_k$ satisfying the conditions of the claim. $\square$

Again, let us note that the hyperedges $H'_1, \dots, H'_k$ provided by the claim are not necessarily distinct.

Claim 7. *For all $i \in [k-1]$ and $x \in C_i$, either $x \in (H_i \cap H_{i+1}) - (H'_i \cup H'_{i+1})$ or $x \in (H'_i \cap H'_{i+1}) - (H_i \cup H_{i+1})$. For $x \in C_k$, either $x \in (H_k \cap H'_1) - (H'_k \cup H_1)$ or $x \in (H'_k \cap H_1) - (H_k \cup H'_1)$.*

Proof. Consider any element $x \in C_i$ for some $i \in [k-1]$. By Claim 6, we know that $x \in H_{i+1} \cup H'_{i+1}$. We distinguish two cases based on which set x belongs to.

Assume first that $x \in H_{i+1}$. If $x \notin H_i$, then $(q_1, \ldots, q_{i-1}, x, p_{i+1}, \ldots, p_k)$ is a valid choice for the set of j-th elements, contradicting the maximality of j. This is because R_ℓ is $H_{\ell+1}$-tight for all $\ell \in [k]$ by Claim 5, so $R_{i-1} - q_{i-1} + x$ and $R_i - x + p_{i+1}$ are H_i and H_{i+1}-tight, respectively, if $i \geq 2$, while for $i = 1$, replace q_{i-1} with p_k in the reasoning and observe that $p_k \notin H_1$ by Claim 5. Thus we have $x \in H_i$. By Claim 6, $H_i \cap H'_i \subseteq R_{i-1}$. As $x \notin R_{i-1}$, we have $x \notin H'_i$. If $x \in H'_{i+1}$, then $(p_1, \ldots p_{i-1}, x, q_{i+1}, \ldots, q_k)$ is a valid choice for the set of j-th elements, contradicting the maximality of j. This is because R_ℓ is $H'_{\ell+1}$-tight for all $\ell \in [k]$ by Claim 6, so $R_{i-1} - p_{i-1} + x$ and $R_i - x + q_{i+1}$ are H'_i and H'_{i+1}-tight, respectively, if $i \geq 2$, while for $i = 1$, replace p_{i-1} by q_k in the reasoning and observe that $q_k \notin H'_1$ by Claim 6. Thus we have $x \notin H'_{i+1}$.

Consider now the case $x \in H'_{i+1}$. If $x \notin H'_i$, then $(p_1, \ldots, p_{i-1}, x, q_{i+1}, \ldots, q_k)$ is a valid choice for the set of j-th elements, contradicting the maximality of j. This is because R_ℓ is $H'_{\ell+1}$-tight for all $\ell \in [k]$ by Claim 6, so $R_{i-1} - p_{i-1} + x$ and $R_i - x + q_{i+1}$ are H'_i and H'_{i+1}-tight, respectively, if $i \geq 2$, while for $i = 1$, replace p_{i-1} by q_k in the reasoning and observe that $q_k \notin H'_1$ by Claim 6. Thus we have $x \in H'_i$. By Claim 6, $H_i \cap H'_i \subseteq R_{i-1}$. As $x \notin R_{i-1}$, we have $x \notin H_i$. If $x \in H_{i+1}$, then $(q_1, \ldots, q_{i-1}, x, p_{i+1}, \ldots, p_k)$ is a valid choice for the set of j-th elements, contradicting the maximality of j. This is because R_ℓ is $H_{\ell+1}$-tight for all $\ell \in [k]$ by Claim 5, so $R_{i-1} - q_{i-1} + x$ and $R_i - x + p_{i+1}$ are H_i and H_{i+1}-tight, respectively, if $i \geq 2$, while for $i = 1$, replace q_{i-1} by p_k in the reasoning and observe that $p_k \notin H_1$ by Claim 5. This implies $x \notin H_{i+1}$.

Finally, the statement for $x \in C_k$ follows by replacing H_{k+1} with H'_1 and H'_{k+1} with H_1 in the argument above and using the right notion of tightness everywhere. $\square$

For all $i \in [k-1]$, we define $\widehat{C}_i \coloneqq \{x \in C_i \mid x \in (H_i \cap H_{i+1}) - (H'_i \cup H'_{i+1})\}$ and $\widehat{C}'_i \coloneqq \{x \in C_i \mid x \in (H'_i \cap H'_{i+1}) - (H_i \cup H_{i+1})\}$. We further set $\widehat{C}_k \coloneqq \{x \in C_k \mid x \in (H_k \cap H'_1) - (H'_k \cup H_1)\}$ and $\widehat{C}'_k \coloneqq \{x \in C_k \mid x \in (H'_k \cap H_1) - (H_k \cup H'_1)\}$. By Claim 7, $C_i = \widehat{C}_i \cup \widehat{C}'_i$ and $\widehat{C}_i \cap \widehat{C}'_i = \emptyset$ holds for each $i \in [k]$.

Claim 8. *There exists an $s \in \mathbb{Z}_+$ such that $|\widehat{C}_i| = |\widehat{C}'_i| = s$ for each $i \in [k]$.*

Proof. As B_2 is a basis, we have $|B_2 \cap H_2| \leq r_2$. Since $|R_1 \cap H_2| = r_2$, we get $|\widehat{C}_2| = |C_2 \cap H_2| \leq |C_1 \cap H_2| = |\widehat{C}_1|$. A repeated application of the same argument leads to $|\widehat{C}_1| \geq |\widehat{C}_2| \geq \cdots \geq |\widehat{C}_k|$. Similarly, as B_1 is a basis, we have $|B_1 \cap H'_1| \leq r'_1$. Since

$|R_k \cap H'_1| = r'_1$, we get $|\widehat{C}'_1| = |C_1 \cap H'_1| \leq |C_k \cap H'_1| = |\widehat{C}_k|$. A repeated application of the same argument leads to $|\widehat{C}_k| \geq |\widehat{C}'_1| \geq |\widehat{C}'_2| \cdots \geq |\widehat{C}'_k|$. Finally, as B_1 is a basis, we have $|B_1 \cap H_1| \leq r_1$. Since $|R_k \cap H_1| = r_1$, we get $|\widehat{C}_1| = |C_1 \cap H_1| \leq |C_k \cap H_1| = |\widehat{C}'_k|$.

Concluding the above, we get $|\widehat{C}_1| \geq |\widehat{C}_2| \geq \cdots \geq |\widehat{C}_k| \geq |\widehat{C}'_1| \geq |\widehat{C}'_2| \geq \cdots \geq |\widehat{C}'_k| \geq |\widehat{C}_1|$, finishing the proof of the claim. □

Claim 9. *For each $i \in [k]$, B_i is both H_i and H'_i-tight.*

Proof. Recall that R_{i-1} is H_i-tight by Claim 5. Therefore, by Claim 8, we have

$$\begin{aligned}
|B_i \cap H_i| &= |(B_i \cap R_{i-1}) \cap H_i| + |(B_i - R_{i-1}) \cap H_i| \\
&= |(B_i \cap R_{i-1}) \cap H_i| + |\widehat{C}_i| \\
&= |(B_i \cap R_{i-1}) \cap H_i| + |\widehat{C}_{i-1}| \\
&= |R_{i-1} \cap H_i| \\
&= r_i.
\end{aligned}$$

Similarly, recall that R_{i-1} is H'_i-tight by Claim 6. Therefore, by Claim 8, we have

$$\begin{aligned}
|B_i \cap H'_i| &= |(B_i \cap R_{i-1}) \cap H'_i| + |(B_i - R_{i-1}) \cap H'_i| \\
&= |(B_i \cap R_{i-1}) \cap H'_i| + |\widehat{C}'_i| \\
&= |(B_i \cap R_{i-1}) \cap H'_i| + |\widehat{C}'_{i-1}| \\
&= |R_{i-1} \cap H'_i| \\
&= r'_i.
\end{aligned}$$

This concludes the proof of the claim. □

By Claim 5 we know that $p_k \in H_k$ and $p_k \notin H_1$ therefore $H_1 \neq H_k$. This means that there must exist consecutive indices p and $p+1$ such that $H_p \neq H_{p+1}$. By definition, we know that $|H_p \cap H_{p+1}| \geq |\widehat{C}_p| = s$. Since R_{p-1} is H_p-tight by Claim 5 and none of the elements in $\widehat{C}'_{p-1}$ is in H_p, we get $|R_{p-1}| = r \geq r_p + s$. Since R_p is H_{p+1}-tight by Claim 5 and none of the elements in $\widehat{C}'_p$ is in H_{p+1}, we get $|R_p| = r \geq r_{p+1} + s$. In the case $p = 1$ you need to replace $\widehat{C}'_{p-1}$ by $\widehat{C}_k$ and R_{p-1} by R_k in the argument above. These observations give

$$s \leq |H_p \cap H_{p+1}| \leq r_p + r_{p+1} - r \leq (r - s) + (r - s) - r = r - 2s,$$

thus $s \leq r/3$. As $r = |B_i| = j-1+|C_i| = j-1+|\widehat{C}_i|+|\widehat{C}'_i| = j-1+2s \leq j-1+2r/3$, we get $j-1 \geq r/3$. In particular, this means that at least one element is already ordered in each of $B_1, \dots, B_k$.

Now we turn our attention to the elements that have been already ordered. Consider the elements b^i_t for all $i \in [k]$, $t \in [j-1]$. Our goal is to show that the set of hyperedges containing these elements also have a specific structure.

Claim 10. *We have the following.*

(a) For all $t \in [j-1]$, $b^i_t \in (H_i \triangle H'_i) \cap (H_{i+1} \triangle H'_{i+1})$ for all $i \in [k]$.

(b) For all $t \in [j-1]$, either $\{b^i_t, b^{i+1}_t\} \subseteq H_{i+1}$ or $\{b^i_t, b^{i+1}_t\} \subseteq H'_{i+1}$.

(c) For all $t \in [j-1]$, the set $\{b^i_t, \dots, b^i_{j-1}\} \cup C_i \cup \{b^{i+1}_1, \dots, b^{i+1}_{t-1}\}$ is H_{i+1} and H'_{i+1}-tight.

Proof. Most of the proof is verifying (a) for all $t \in [j-1]$ in a decreasing order, while (b) and (c) follow easily after. Assume that the statement is true for indices greater than t and at most $j-1$; when $t = j-1$, this assumption is vacuously true since no such indices exist. We first prove that (a) holds for t. Consider an $i \in [k]$. As $b^i_t \in B_i$ and B_i is H_i and H'_i-tight by Claim 9, we get that $b^i_t \in H_i \cup H'_i$ by Lemma 4(b).

We first prove that $b^i_t \in (H_i \triangle H'_i)$. Suppose indirectly that $b^i_t \in H_i \cap H'_i$. As $b^i_t \notin R_i$, it is contained in at most one of H_{i+1} and H'_{i+1} by Claim 6 – we consider those separately.

Case 1. $b^i_t \notin H_{i+1}$.

Suppose first that $i < k$. Substitute b^i_t with q_i in the ordering of B_i. We claim that ($\star$) remains true. This is because $q_i \notin H_{i+1}$ by Claim 5 and $\{b^i_m, \dots, b^i_{j-1}\} \cup C_i \cup \{b^{i+1}_1, \dots, b^{i+1}_{m-1}\}$ is H_{i+1}-tight for all $m > t$ by assumption, thus we get that $\{b^i_m, \dots, b^i_{j-1}\} \cup C_i \cup \{b^{i+1}_1, \dots, b^{i+1}_{m-1}\} - q_i + b^i_t$ remains H_{i+1}-tight for all $m > t$. By Claim 6, $q_i \in H'_i$ and as $\{b^{i-1}_m, \dots, b^{i-1}_{j-1}\} \cup C_{i-1} \cup \{b^i_1, \dots, b^i_{m-1}\}$ is H'_i-tight for all $m > t$ by assumption, we get that $\{b^{i-1}_m, \dots, b^{i-1}_{j-1}\} \cup C_{i-1} \cup \{b^i_1, \dots, b^i_{m-1}\} - b^i_t + q_i$ remains H'_i-tight for all $m > t$. After the modification, the choice $(q_1, \dots, q_{i-1}, p_i, \dots, p_k)$ is valid for the j-th phase. This is because, by Claim 5, $q_{i-1} \notin H_i$, $p_i \in H_i$ and $q_i \notin H_i$, so $R_{i-1} - q_{i-1} - b^i_t + p_i + q_i$ is H_i-tight if $i \geq 2$, while for $i = 1$, replace q_{i-1} with p_k in the reasoning and observe that $p_k \notin H_1$ by Claim 5. Also by Claim 5, $p_{i+1} \in H_{i+1}$, $p_i \in H_{i+1}$ and $q_i \notin H_{i+1}$, so $R_i - p_i - q_i + b^i_t + p_{i+1}$ remains H_{i+1}-tight. This contradicts the maximal choice of j.

Now consider the case $i = k$ and $b_t^k \notin H_1$. Substitute b_t^k with p_k in the ordering of B_k. We claim that $(\star)$ remains true. This is because $p_k \notin H_1$ by Claim 5 and $\{b_m^k, \ldots, b_{j-1}^k\} \cup C_k \cup \{b_1^1, \ldots, b_{m-1}^1\}$ is H_1-tight for all $m > t$ by assumption, thus we get that $\{b_m^k, \ldots, b_{j-1}^k\} \cup C_k \cup \{b_1^1, \ldots, b_{m-1}^1\} - p_k + b_t^k$ remains H_1-tight for all $m > t$. By Claim 5, $p_k \in H_k$ and as $\{b_m^{k-1}, \ldots, b_{j-1}^{k-1}\} \cup C_{k-1} \cup \{b_1^k, \ldots, b_{m-1}^k\}$ is H_k-tight for all $m > t$ by assumption, we get that $\{b_m^{k-1}, \ldots, b_{j-1}^{k-1}\} \cup C_{k-1} \cup \{b_1^k, \ldots, b_{m-1}^k\} - b_t^k + p_k$ remains H_k-tight for all $m > t$. After the modification, the choice $(p_1, \ldots, p_{k-1}, q_k)$ is valid for the j-th phase. This is because, by Claim 6, $p_{k-1} \notin H'_k$, $p_k \notin H'_k$ and $q_k \in H'_k$, so $R_{k-1} - p_{k-1} - b_t^k + p_k + q_k$ is H'_k-tight. By Claims 5 and 6, $p_1 \in H_1$, $p_k \notin H_1$ and $q_k \in H_1$ so $R_k - p_k - q_k + b_t^k + p_1$ remains H_1-tight. This contradicts the maximal choice of j.

Case 2. $b_t^i \notin H'_{i+1}$.

Suppose first that $i < k$. Substitute b_t^i with p_i in the ordering of B_i. We claim that $(\star)$ remains true. This is because $p_i \notin H'_{i+1}$ by Claim 6 and $\{b_m^i, \ldots, b_{j-1}^i\} \cup C_i \cup \{b_1^{i+1}, \ldots, b_{m-1}^{i+1}\}$ is H'_{i+1}-tight for all $m > t$ by assumption, thus we get that $\{b_m^i, \ldots, b_{j-1}^i\} \cup C_i \cup \{b_1^{i+1}, \ldots, b_{m-1}^{i+1}\} - p_i + b_t^i$ remains H'_{i+1}-tight for all $m > t$. By Claim 5, $p_i \in H_i$ and as $\{b_m^{i-1}, \ldots, b_{j-1}^{i-1}\} \cup C_{i-1} \cup \{b_1^i, \ldots, b_{m-1}^i\}$ is H_i-tight for all $m > t$ by assumption, we get that $\{b_m^{i-1}, \ldots, b_{j-1}^{i-1}\} \cup C_{i-1} \cup \{b_1^i, \ldots, b_{m-1}^i\} - b_t^i + p_i$ remains H_i-tight for all $m > t$. After the modification, the choice $(p_1, \ldots, p_{i-1}, q_i, \ldots, q_k)$ is valid for the j-th phase. This is because, by Claim 6, $p_{i-1} \notin H'_i$, $p_i \notin H'_i$ and $q_i \in H'_i$ so $R_{i-1} - p_{i-1} - b_t^i + p_i + q_i$ is H'_i-tight if $i \geq 2$, while for $i = 1$, replace p_{i-1} with q_k in the reasoning and observe that $q_k \notin H'_1$ by Claim 6. Also by Claim 6, $q_{i+1} \in H'_{i+1}$, $p_i \notin H'_{i+1}$ and $q_i \in H'_{i+1}$, so $R_i - p_i - q_i + b_t^i + q_{i+1}$ remains H'_{i+1}-tight. This contradicts the maximal choice of j.

Now consider the case $i = k$ and $b_t^k \notin H'_1$. Substitute b_t^k with q_k in the ordering of B_k. We claim that $(\star)$ remains true. This is because $q_k \notin H'_1$ by Claim 6 and $\{b_m^k, \ldots, b_{j-1}^k\} \cup C_k \cup \{b_1^1, \ldots, b_{m-1}^1\}$ is H'_1-tight for all $m > t$ by assumption, thus we get that $\{b_m^k, \ldots, b_{j-1}^k\} \cup C_k \cup \{b_1^1, \ldots, b_{m-1}^1\} - q_k + b_t^k$ remains H'_1-tight for all $m > t$. By Claim 6, $q_k \in H'_k$ and as $\{b_m^{k-1}, \ldots, b_{j-1}^{k-1}\} \cup C_{k-1} \cup \{b_1^k, \ldots, b_{m-1}^k\}$ is H'_k-tight for all $m > t$ by assumption, we get that $\{b_m^{k-1}, \ldots, b_{j-1}^{k-1}\} \cup C_{k-1} \cup \{b_1^k, \ldots, b_{m-1}^k\} - b_t^k + q_k$ remains H'_k-tight for all $m > t$. After the modification, the choice $(q_1, \ldots, q_{k-1}, p_k)$ is valid for the j-th phase. This is because, by Claim 5, $q_{k-1} \notin H_k$, $p_k \in H_k$ and $q_k \notin H_k$, so $R_{k-1} - q_{k-1} - b_t^k + p_k + q_k$ is H_k-tight. Also by Claim 6, $q_1 \in H'_1$, $p_k \in H'_1$ and $q_k \notin H'_1$ so $R_k - p_k - q_k + b_t^k + q_1$ remains H'_1-tight. This contradicts the maximal choice of j.

Summarizing the above, we get $b_t^i \in (H_i \triangle H_i')$. We now prove that $b_t^i \in (H_{i+1} \triangle H_{i+1}')$. We know that $b_t^i \notin (H_{i+1} \cap H_{i+1}')$, so it suffices to show that $b_t^i \in (H_{i+1} \cup H_{i+1}')$. Suppose indirectly that $b_t^i \notin (H_{i+1} \cup H_{i+1}')$. We consider two cases based on whether $b_t^i \in H_i - (H_i' \cup H_{i+1} \cup H_{i+1}')$ or $b_t^i \in H_i' - (H_i \cup H_{i+1} \cup H_{i+1}')$.

Case 1. $b_t^i \in H_i - (H_i' \cup H_{i+1} \cup H_{i+1}')$.

Suppose first that $i < k$. Substitute b_t^i with p_i in the ordering of B_i. We claim that ($\star$) remains true. This is because $p_i \notin H_{i+1}'$ by Claim 6 and $\{b_m^i, \dots, b_{j-1}^i\} \cup C_i \cup \{b_1^{i+1}, \dots, b_{m-1}^{i+1}\}$ is H_{i+1}'-tight for all $m > t$ by assumption, thus we get that $\{b_m^i, \dots, b_{j-1}^i\} \cup C_i \cup \{b_1^{i+1}, \dots, b_{m-1}^{i+1}\} - p_i + b_t^i$ remains H_{i+1}'-tight for all $m > t$. By Claim 5, $p_i \in H_i$ and as $\{b_m^{i-1}, \dots, b_{j-1}^{i-1}\} \cup C_{i-1} \cup \{b_1^i, \dots, b_{m-1}^i\}$ is H_i-tight for all $m > t$ by assumption, we get that $\{b_m^{i-1}, \dots, b_{j-1}^{i-1}\} \cup C_{i-1} \cup \{b_1^i, \dots, b_{m-1}^i\} - b_t^i + p_i$ remains H_i-tight for all $m > t$. After the modification, the choice $(q_1, \dots, q_i, p_{i+1}, \dots, p_k)$ is valid for the j-th phase. This is because, by Claim 6, $q_{i-1} \in H_i'$, $p_i \notin H_i'$ and $q_i \in H_i'$, so $R_{i-1} - q_{i-1} - b_t^i + p_i + q_i$ is H_i'-tight if $i \geq 2$, while for $i = 1$, replace q_{i-1} with p_k in the reasoning and observe that $p_k \in H_1'$ by Claim 6. Also by Claim 5, $p_{i+1} \in H_{i+1}$, $p_i \in H_{i+1}$ and $q_i \notin H_{i+1}$, so $R_i - p_i - q_i + b_t^i + p_{i+1}$ remains H_{i+1}-tight. This contradicts the maximal choice of j.

Now consider the case $i = k$ and $b_t^k \in H_k - (H_k' \cup H_1 \cup H_1')$. Substitute b_t^k with p_k in the ordering of B_k. We claim that ($\star$) remains true. This is because $p_k \notin H_1$ by Claim 5 and $\{b_m^k, \dots, b_{j-1}^k\} \cup C_k \cup \{b_1^1, \dots, b_{m-1}^1\}$ is H_1-tight for all $m > t$ by assumption, thus we get that $\{b_m^k, \dots, b_{j-1}^k\} \cup C_k \cup \{b_1^1, \dots, b_{m-1}^1\} - p_k + b_t^k$ remains H_1-tight for all $m > t$. By Claim 5, $p_k \in H_k$ and as $\{b_m^{k-1}, \dots, b_{j-1}^{k-1}\} \cup C_{k-1} \cup \{b_1^k, \dots, b_{m-1}^k\}$ is H_k-tight for all $m > t$ by assumption, we get that $\{b_m^{k-1}, \dots, b_{j-1}^{k-1}\} \cup C_{k-1} \cup \{b_1^k, \dots, b_{m-1}^k\} - b_t^k + p_k$ remains H_k-tight for all $m > t$. After the modification, the choice $(q_1, \dots, q_k)$ is valid for the j-th phase. This is because, by Claim 5, $q_{k-1} \notin H_k$, $p_k \in H_k$ and $q_k \notin H_k$, so $R_{k-1} - q_{k-1} - b_t^k + p_k + q_k$ is H_k-tight. Also by Claim 6, $q_1 \in H_1'$, $q_k \notin H_1'$ and $p_k \in H_1'$, so $R_k - p_k - q_k + b_t^k + q_1$ remains H_1'-tight. This contradicts the maximal choice of j.

Case 2. $b_t^i \in H_i' - (H_i \cup H_{i+1} \cup H_{i+1}')$.

Suppose first that $i < k$. Substitute b_t^i with q_i in the ordering of B_i. We claim that ($\star$) remains true. This is because $q_i \notin H_{i+1}$ by Claim 5 and $\{b_m^i, \dots, b_{j-1}^i\} \cup C_i \cup \{b_1^{i+1}, \dots, b_{m-1}^{i+1}\}$ is H_{i+1}-tight for all $m > t$ by assumption, thus we get that $\{b_m^i, \dots, b_{j-1}^i\} \cup C_i \cup \{b_1^{i+1}, \dots, b_{m-1}^{i+1}\} - q_i + b_t^i$ remains H_{i+1}-tight for all $m > t$. By Claim 6, $q_i \in H_i'$ and as $\{b_m^{i-1}, \dots, b_{j-1}^{i-1}\} \cup C_{i-1} \cup \{b_1^i, \dots, b_{m-1}^i\}$ is H_i'-tight for all $m >$

t by assumption, we get that $\{b^{i-1}_m, \dots, b^{i-1}_{j-1}\} \cup C_{i-1} \cup \{b^i_1, \dots, b^i_{m-1}\} - b^i_t + q_i$ remains H'_i-tight for all $m > t$. After the modification, the choice $(p_1, \dots, p_i, q_{i+1}, \dots, q_k)$ is valid for the j-th phase. This is because, by Claim 5, $p_{i-1} \in H_i$, $p_i \in H_i$ and $q_i \notin H_i$, so $R_{i-1} - p_{i-1} - b^i_t + p_i + q_i$ is H_i-tight if $i \geq 2$, while for $i = 1$, replace p_{i-1} with q_k in the reasoning and observe that $q_k \in H_1$ by Claim 6. Also by Claim 6, $q_{i+1} \in H'_{i+1}$, $p_i \notin H'_{i+1}$ and $q_i \in H'_{i+1}$, so $R_i - p_i - q_i + b^i_t + q_{i+1}$ remains H'_{i+1}-tight. This contradicts the maximal choice of j.

Now consider the case $i = k$ and $b^k_t \in H'_k - (H_k \cup H_1 \cup H'_1)$. Substitute b^k_t with q_k in the ordering of B_k. We claim that ($\star$) remains true. This is because $q_k \notin H'_1$ by Claim 6 and $\{b^k_m, \dots, b^k_{j-1}\} \cup C_k \cup \{b^1_1, \dots, b^1_{m-1}\}$ is H'_1-tight for all $m > t$ by assumption, thus we get that $\{b^k_m, \dots, b^k_{j-1}\} \cup C_k \cup \{b^1_1, \dots, b^1_{m-1}\} - q_k + b^k_t$ remains H'_1-tight for all $m > t$. By Claim 6, $q_k \in H'_k$ and as $\{b^{k-1}_m, \dots, b^{k-1}_{j-1}\} \cup C_{k-1} \cup \{b^k_1, \dots, b^k_{m-1}\}$ is H'_k-tight for all $m > t$ by assumption, we get that $\{b^{k-1}_m, \dots, b^{k-1}_{j-1}\} \cup C_{k-1} \cup \{b^k_1, \dots, b^k_{m-1}\} - b^k_t + q_k$ remains H'_k-tight for all $m > t$. After the modification, the choice $(p_1, \dots, p_k)$ is valid for the j-th phase. This is because, by Claim 6, $p_{k-1} \notin H'_k$, $p_k \notin H'_k$ and $q_k \in H'_k$, so $R_{k-1} - p_{k-1} - b^k_t + p_k + q_k$ is H'_k-tight. Also by Claims 5 and 6, $p_1 \in H_1$, $q_k \in H_1$ and $p_k \notin H_1$, so $R_k - p_k - q_k + b^k_t + p_1$ remains H_1-tight. This contradicts the maximal choice of j.

This finishes the proof of (a), that is, $b^i_t \in (H_i \triangle H'_i) \cap (H_{i+1} \triangle H'_{i+1})$. To prove the remaining two properties, observe that $\{b^i_{t+1}, \dots, b^i_{j-1}\} \cup C_i \cup \{b^{i+1}_1, \dots, b^{i+1}_t\} - b^{i+1}_t + b^i_t$ is a basis by ($\star$). As $\{b^i_{t+1}, \dots, b^i_{j-1}\} \cup C_i \cup \{b^{i+1}_1, \dots, b^{i+1}_t\}$ is both H_{i+1} and H'_{i+1}-tight, together with $b^i_t \in (H_{i+1} \triangle H'_{i+1})$ and $b^{i+1}_t \in (H_{i+1} \triangle H'_{i+1})$, necessarily $\{b^i_t, b^{i+1}_t\} \subseteq H_{i+1}$ or $\{b^i_t, b^{i+1}_t\} \subseteq H'_{i+1}$ as otherwise the basis would have too large intersection with H_{i+1} or H'_{i+1}. This implies

$$\begin{aligned} &|(\{b^i_{t+1}, \dots, b^i_{j-1}\} \cup C_i \cup \{b^{i+1}_1, \dots, b^{i+1}_t\}) \cap H_{i+1}| = \\ &= |(\{b^i_{t+1}, \dots, b^i_{j-1}\} \cup C_i \cup \{b^{i+1}_1, \dots, b^{i+1}_t\} - b^{i+1}_t + b^i_t) \cap H_{i+1}| \end{aligned}$$

and

$$\begin{aligned} &|(\{b^i_{t+1}, \dots, b^i_{j-1}\} \cup C_i \cup \{b^{i+1}_1, \dots, b^{i+1}_t\}) \cap H'_{i+1}| = \\ &= |(\{b^i_{t+1}, \dots, b^i_{j-1}\} \cup C_i \cup \{b^{i+1}_1, \dots, b^{i+1}_t\} - b^{i+1}_t + b^i_t) \cap H'_{i+1}|, \end{aligned}$$

which means that properties (b) and (c) hold as well. $\square$

Claims 9 and 10 imply that B_i is tight with respect to H_i, H'_i, H_{i+1} and H'_{i+1}. We know that $H_m \neq H_{m+1}$ for some $m < k$. Then, $B_m \subseteq H_m \cup H_{m+1}$ which implies

$q_m \in B_m \subseteq H_m \cup H_{m+1}$. However, by Claim 5, $q_m \notin H_m \cup H_{m+1}$, a contradiction. This concludes the proof of the theorem. $\square$

Chapter 4

Further remarks and open problems

4.1 Comments on Conjecture 2

The most important result toward verifying Conjecture 2 is due to Van den Heuvel and Thomassé [22].

Theorem 11 (Van den Heuvel and Thomassé)**.** *Let $M = (S, \mathcal{B})$ be a loopless matroid with rank function $r\colon 2^S \to \mathbb{Z}_+$ and $|S| = n$, and let g denote the greatest common divisor of $r(S)$ and n. Then, there exists a partition $S = G_1 \cup \cdots \cup G_{n/g}$ into sets of size g such that $\bigcup_{t=0}^{r(S)/g-1} G_{i+t}$ is a basis for all $i \in [n/g]$ if and only if $r(S) \cdot |X| \le n \cdot r(X)$ for $X \subseteq S$.*

In particular, Theorem 11 settles Conjecture 2 in the affirmative if $r(S)$ and n are coprimes. Therefore, to prove Conjecture 2, it would be enough to verify that, when M is uniformly dense, the elements inside each G_i admit an ordering that together induces a cyclic ordering of M. Unfortunately, such an approach cannot work as shown by the following example.

Example 12. Let $S = \{a_1, \dots, a_{10}\}$ and consider the sparse paving matroid defined by the following hyperedges: $\{a_1, a_2, a_3, a_{10}\}$, $\{a_1, a_2, a_4, a_9\}$, $\{a_1, a_3, a_4, a_5\}$, $\{a_2, a_3, a_4, a_6\}$, $\{a_3, a_5, a_6, a_7\}$, $\{a_4, a_5, a_6, a_8\}$, $\{a_5, a_7, a_8, a_9\}$, $\{a_6, a_7, a_8, a_{10}\}$,$\{a_1, a_7, a_9, a_{10}\}$, $\{a_2, a_8, a_9, a_{10}\}$, with the value of r being 4; see Section 2 for the definition. If $G_i = \{a_{2i-1}, a_{2i}\}$ for all $i \in [5]$, then it is not difficult to check that $G_i \cup G_{i+1}$ is a basis for all $i \in [10]$.

However, we claim that the pairs in the sets G_i cannot be ordered in such a way that we get a cyclic ordering of the matroid M. To see this, observe that each

G_i is contained in two of the hyperedges, which excludes two of the four possible orderings of the neighboring groups G_{i-1} and G_{i+1}. Due to the exclusion of these ordering possibilities, it is not difficult to verify that no suitable ordering exists.

4.2 Exchange distance of basis sequences

Note that Gabow's conjecture can be interpreted as follows: for any two disjoint bases B_1 and B_2 of a matroid M of rank r, there is a sequence of r symmetric exchanges that transforms the pair (B_1, B_2) into (B_2, B_1). The closely related problem of transforming a sequence $(B_1, \ldots, B_k)$ of bases into another $(B'_1, \ldots, B'_k)$ was proposed by White [23]. Let $(B_1, \ldots, B_k)$ be a sequence of k bases of a matroid M, and assume that there exist $e \in B_i$, $f \in B_j$ for some $1 \leq i < j \leq k$ such that both $B_i - e + f$ and $B_j - f + e$ are bases. Then we say that the sequence $(B_1, \ldots, B_{i-1}, B_i - e + f, B_{i+1}, \ldots, B_{j-1}, B_j - f + e, B_{j+1}, \ldots, B_k)$ is obtained from the original one by a *symmetric exchange*. Accordingly, two sequences of bases are called *equivalent* if one can be obtained from the other by a composition of symmetric exchanges. White studied the following question: what is the characterization of two sequences of bases being equivalent?

There is an easy necessary condition. Namely, two sequences $(B_1, \ldots, B_k)$ and $(B'_1, \ldots, B'_k)$ are called *compatible* if the union of the B_is as a multiset coincides with the union of the B'_is as a multiset. Compatibility is obviously a necessary condition for two sequences being equivalent, and White conjectured that it is also sufficient.

Conjecture 4 (White)**.** *Two sequences of k bases are equivalent if and only if they are compatible.*

In this context, Gabow's conjecture would verify White's conjecture for two pairs of bases of the form (B_1, B_2) and (B_2, B_1). Note that, however, the conjecture says nothing on the minimum number of exchanges needed to transform one of the pairs into the other. As a common generalization of Gabow's conjecture and the special case of White's conjecture when $k = 2$, Hamidoune [8] proposed an optimization variant.

Conjecture 5 (Hamidoune)**.** *Let (B_1, B_2) and (B'_1, B'_2) be compatible basis pairs of a rank-r matroid $M = (S, \mathcal{B})$. Then, (B_1, B_2) can be transformed into (B'_1, B'_2) by using at most r symmetric exchanges.*

In [5], Bérczi, Mátravölgyi and Schwarcz formulated a weighted extension of Hamidoune's conjecture. Let $M = (S, \mathcal{B})$ be a matroid and $w\colon S \to \mathbb{R}_+$ be a weight function on the elements of the ground set S. Given a pair (B_1, B_2) of bases, we define the *weight of a symmetric exchange* $B_1 - e + f$ and $B_2 - f + e$ to be $w(e)/2 + w(f)/2$, that is, the average of the weights of the exchanged elements.

Conjecture 6 (Bérczi, Mátravölgyi, Schwarcz)**.** *Let (B_1, B_2) and (B'_1, B'_2) be compatible basis pairs of a matroid $M = (S, \mathcal{B})$, and let $w\colon S \to \mathbb{R}_+$. Then, (B_1, B_2) can be transformed into (B'_1, B'_2) by using symmetric exchanges of total weight at most $w(B_1)/2 + w(B_2)/2 = w(B'_1)/2 + w(B'_2)/2$.*

By setting the weights to be identically 1, we get back Hamidoune's conjecture. The question naturally arises: can we formulate extensions of Conjectures 5 and 6 for basis sequences of length greater than two?

Let $(B_1, \ldots, B_k)$ be a sequence of k bases of a matroid M, and assume that there exists distinct indices $\{i_1, \ldots, i_q\} \subseteq [k]$ and $e_j \in B_{i_j}$ such that $B_{i_j} - e_j + e_{j+1}$ is a basis for each $j \in [q]$. Then, we say that the sequence $(B'_1, \ldots, B'_k)$ where $B'_\ell = B_{i_j} - e_j + e_{j+1}$ if $\ell = i_j$ for some $j \in [q]$ and $B'_\ell = B_\ell$ otherwise, is obtained by a *cyclic exchange*. As a generalization of Conjecture 5, we propose the following.

Conjecture 7. *Let $(B_1, \ldots, B_k)$ and $(B'_1, \ldots, B'_k)$ be compatible sequences of k bases of a rank-r matroid. Then, $(B_1, \ldots, B_k)$ can be transformed into $(B'_1, \ldots, B'_k)$ by using at most r cyclic exchanges.*

Given a weight function $w\colon S \to \mathbb{R}_+$ on the elements of the ground set, let us define the *weight of a cyclic exchange* that moves elements $e_j \in B_{i_j}$ for $j \in [q]$ to be $\frac{1}{k}\sum_{j=1}^{q} w(e_j)$. As a generalization of Conjecture 6, the weighted counterpart is as follows.

Conjecture 8. *Let $(B_1, \ldots, B_k)$ and $(B'_1, \ldots, B'_k)$ be compatible sequences of k bases of a matroid $M = (S, \mathcal{B})$, and let $w\colon S \to \mathbb{R}_+$. Then, $(B_1, \ldots, B_k)$ can be transformed into $(B'_1, \ldots, B'_k)$ by using cyclic exchanges of total weight at most $\frac{1}{k}\sum_{i=1}^{k} w(B_i) = \frac{1}{k}\sum_{i=1}^{k} w(B'_i)$.*

Note that in both cases, the bounds are tight in the sense that r cyclic exchanges are definitely needed to transform the sequence $(B_1, \ldots, B_{k-1}, B_k)$ into $(B_2, \ldots, B_k, B_1)$.

Chapter 5

A more general matroid class

A conjecture of Crapo and Rota [9], that was made precise by Mayhew, Newman, Welsh and Whittle [20], suggests that the asymptotic fraction of matroids on n elements that are paving tends to 1 as n tends to infinity. This is based on the well-known fact that the number of paving matroids on n elements is already doubly exponential in n. The consequence of the mentioned conjecture would be that almost all matroids have a similar underlying hypergraph structure. The class of paving matroids can be generalized into the class of split matroids while keeping some of the hypergraph representation. Moreover, split matroids are closed under duality and taking minors. It raises the following question: Is there a similar, even more general underlying structure for all matroids? In this chapter, we take a look at a definition by Balcan and Harvey [1] and conclude that with a slight modification, this class provides a structure for all matroids. We define another matroid class as well as considering some variations of these classes.

Theorem 13. *Let n, k, t and r be positive integers. Let $\{A_1, \dots, A_k\}$ be a collection of subsets of a ground set S with n elements. Let $\mathcal{F}$ be a collection of subsets of $\binom{[k]}{\leq t}$, which is closed under taking intersection and $\emptyset \in \mathcal{F}$. Let $f : \mathcal{F}^{\cup_2} \to \mathbb{Z}$ be a nonnegative function such that $f(\emptyset) = r$ and f is $\mathcal{F}$-submodular. Then if $f(J) \geq |A^{\cap}(J)|$ for all $J \in \mathcal{F}^{\cup_2} - \mathcal{F}$, where $A^{\cap}(J) = \cap_{j \in J} A_j$, then*

$$\mathcal{I} = \{I \subseteq S : |I \cap A^{\cap}(J)| \leq f(J), \forall J \in \mathcal{F}\}$$

forms the independent sets of a matroid.

Proof. We will show that the set system $\mathcal{I}$ satisfies the independence axioms. Since f is nonnegative, it is clear that $\emptyset \in \mathcal{I}$.

Suppose $X \subseteq Y$ and $Y \in \mathcal{I}$. Then for all $J \in \mathcal{F}$,

$$|X \cap A^{\cap}(J)| \leq |Y \cap A^{\cap}(J)| \leq f(J),$$

thus $X \in \mathcal{I}$ as well.

We show that the size of a maximal subset of $Z \subseteq S$ that is in $\mathcal{I}$ is given by

$$r(Z) = \min \left\{ |Z|, \min_{J \in \mathcal{F}} \{f(J) + |Z - A^{\cap}(J)|\} \right\}.$$

Clearly, $r(Z) \leq |Z|$. Now, let $X \in \mathcal{I}$ be an arbitrary set contained in Z. For each $J \in \mathcal{F}$, we have

$$|X| = |X \cap A^{\cap}(J)| + |X - A^{\cap}(J)| \leq f(J) + |Z - A^{\cap}(J)|.$$

Thus,

$$r(Z) \leq \min \left\{ |Z|, \min_{J \in \mathcal{F}} \{f(J) + |Z - A^{\cap}(J)|\} \right\}.$$

For the reverse inequality, consider an arbitrary maximal subset $X \in \mathcal{I}$ of Z. Suppose that $|X| < |Z|$, as otherwise $r(Z) = |Z|$ and we are done. We will show that there exists a set $J_0 \in \mathcal{F}$ such that $X + x \notin \mathcal{I}$ for every element $x \in Z - X$ because the condition for J_0 does not hold:

$$|(X + x) \cap A^{\cap}(J_0)| > f(J_0).$$

Suppose that the addition of elements $u, v \in Z - X$ to X would violate the independence condition for sets J_u and J_v, respectively, chosen to be minimal with respect to inclusion. Then by assumption we have

$$|(X + u) \cap A^{\cap}(J_u)| > f(J_u) \quad \text{and} \quad |(X + v) \cap A^{\cap}(J_v)| > f(J_v).$$

On the other hand, X is in $\mathcal{I}$, so

$$|X \cap A^{\cap}(J_u)| \leq f(J_u) \quad \text{and} \quad |X \cap A^{\cap}(J_v)| \leq f(J_v).$$

These can only hold if

$$|X \cap A^{\cap}(J_u)| = f(J_u) \quad \text{and} \quad |X \cap A^{\cap}(J_v)| = f(J_v),$$

$u \in A^{\cap}(J_u)$ and $v \in A^{\cap}(J_v)$.

By submodularity of f, it follows that

$$f(J_u) + f(J_v) \geq f(J_u \cap J_v) + f(J_u \cup J_v).$$

On the other hand,

$$\begin{aligned} f(J_u) + f(J_v) &= |X \cap A^{\cap}(J_u)| + |X \cap A^{\cap}(J_v)| \\ &= |X \cap (A^{\cap}(J_u) \cap A^{\cap}(J_v))| + |X \cap (A^{\cap}(J_u) \cup A^{\cap}(J_v))| \\ &\leq |X \cap A^{\cap}(J_u \cup J_v)| + |X \cap A^{\cap}(J_u \cap J_v)|, \end{aligned}$$

since $A^{\cap}(J_u) \cap A^{\cap}(J_v) = A^{\cap}(J_u \cup J_v)$ and $A^{\cap}(J_u) \cup A^{\cap}(J_v) \subseteq A^{\cap}(J_u \cap J_v)$ because if an element is in $A^{\cap}(J_u) \cup A^{\cap}(J_v)$, then it is in all sets corresponding to J_u or J_v, so it is also in all sets corresponding to $J_u \cap J_v$.

Thus, we obtain

$$f(J_u \cap J_v) + f(J_u \cup J_v) \leq |X \cap A^{\cap}(J_u \cap J_v)| + |X \cap A^{\cap}(J_u \cup J_v)|.$$

Since $\mathcal{F}$ is closed under intersection, $J_u \cap J_v \in \mathcal{F}$, implying

$$f(J_u \cap J_v) \geq |X \cap A^{\cap}(J_u \cap J_v)|.$$

We now distinguish two cases:

- If $J_u \cup J_v \in \mathcal{F}$, then

  $$f(J_u \cup J_v) \geq |X \cap A^{\cap}(J_u \cup J_v)|,$$

  and thus all inequalities become equalities. So we have

  $$f(J_u \cap J_v) = |X \cap A^{\cap}(J_u \cap J_v)|,$$

  and since $u \in A^{\cap}(J_u)$ and $v \in A^{\cap}(J_v)$, we get $u, v \in A^{\cap}(J_u \cap J_v)$, implying

  $$|(X+u) \cap A^{\cap}(J_u \cap J_v)| > f(J_u \cap J_v) \quad \text{and} \quad |(X+v) \cap A^{\cap}(J_u \cap J_v)| > f(J_u \cap J_v).$$

  Hence, the addition of elements u and v to X would also violate the independence condition for $J_u \cap J_v$, contradicting the minimality of J_u and J_v, unless $J_u = J_v$.

- If $J_u \cup J_v \notin \mathcal{F}$, then by the definition of f, we have

  $$f(J_u \cup J_v) \geq |A^{\cap}(J_u \cup J_v)| \geq |X \cap A^{\cap}(J_u \cup J_v)|,$$

  leading again to equality in all inequalities. Similarly, this implies that the addition of u and v to X would also violate the independence condition for $J_u \cap J_v$, contradicting minimality unless $J_u = J_v$.

Hence, all elements of $Z - X$ violate the independence condition for the same minimal set J_0. Thus, we have

$$Z - X \subseteq A^{\cap}(J_0) \quad \text{and} \quad |X \cap A^{\cap}(J_0)| = f(J_0),$$

which implies

$$|X| = |X \cap A^{\cap}(J_0)| + |X - A^{\cap}(J_0)| = f(J_0) + |Z - A^{\cap}(J_0)|.$$

□

Definition 14. We call these matroids submodular $\mathcal{F}$-cap matroids.

Remark 15. The rank function of a submodular $\mathcal{F}$-cap matroid is $r(Z) = \min\{|Z|, \min_{J\in\mathcal{F}}\{f(J) + |Z - A^{\cap}(J)|\}\}$. If $f(J) + |S - A^{\cap}(J)| \geq f(\emptyset) = r$ for all $J \in \mathcal{F}$, then the rank of the matroid is $r(S) = f(\emptyset) + |S - A^{\cap}(\emptyset)| = f(\emptyset) = r$. We may assume this as we can decrease the value of a submodular function on the empty set and it still remains submodular. The independent sets of the matroid do not change either as we only strengthen the condition for the empty set that gives $|X| \leq r$, and every independent set satisfies this if r is the rank of the matroid.

Definition 16. If f is modular, we call these matroids modular $\mathcal{F}$-cap matroids.

Definition 17. If $\mathcal{F} = \binom{n}{\leq t}$, then we call these matroids submodular/modular t-cap matroids accordingly.

Theorem 18. *Let n, k, t and r be positive integers. Let $\{A_1, \dots, A_k\}$ be a collection of subsets of a ground set S with n elements. Let $\mathcal{F}$ be a collection of subsets of $\binom{[k]}{\leq t}$, which is closed under taking intersection and $\emptyset \in \mathcal{F}$. Let $g : \mathcal{F}^{\cup_2} \to \mathbb{Z}$ be a function such that $g(\emptyset) = 0$, g is $\mathcal{F}$-submodular, and $g(J) + |A^{\cup}(J)| \geq 0$ for all $J \in \mathcal{F}$. Then if $g(J) + |A^{\cup}(J)| \geq r$ for all $J \in \mathcal{F}^{\cup_2} - \mathcal{F}$, where $A^{\cup}(J) = \cup_{j\in J} A_j$, then*

$$\mathcal{I} = \{I \subseteq S : |I \cap A^{\cup}(J)| \leq g(J) + |A^{\cup}(J)|, \forall J \in \mathcal{F}\}$$

forms the independent sets of a matroid.

Proof. We will show that the independence axioms hold for the set system $\mathcal{I}$. Since for all $J \in \mathcal{F}$, $g(J) + |A^{\cup}(J)| \geq 0$, clearly $\emptyset \in \mathcal{I}$.

Suppose that $X \subseteq Y$ and that $Y \in \mathcal{I}$. Then for all $J \in \mathcal{F}$,

$$|X \cap A^{\cup}(J)| \leq |Y \cap A^{\cup}(J)| \leq g(J) + |A^{\cup}(J)|,$$

therefore $X \in \mathcal{I}$ as well.

For the last axiom we prove that the size of a maximal set from $\mathcal{I}$ in any set $Z \subseteq S$ is given by

$$r(Z) = \min\{|Z|, \min_{J \in \mathcal{F}}\{g(J) + |A^{\cup}(J)| + |Z - A^{\cup}(J)|\}\}.$$

Clearly, $r(Z) \leq Z$. Now, consider an arbitrary set $X \in \mathcal{I}$ contained in Z. Then for all $J \in \mathcal{F}$, we have

$$|X| = |X \cap A^{\cup}(J)| + |X - A^{\cup}(J)| \leq g(J) + |A^{\cup}(J)| + |Z - A^{\cup}(J)|.$$

Thus,

$$r(Z) \leq \min\{|Z|, \min_{J \in \mathcal{F}}\{g(J) + |A^{\cup}(J)| + |Z - A^{\cup}(J)|\}\}$$

.

For the other direction, consider an arbitrary maximal subset $X \in \mathcal{I}$ of Z. Assume $|X| < |Z|$, as otherwise $r(Z) = |Z|$ and we are done. Now we prove that there exists a set $J_0 \in \mathcal{F}$ such that $X + x \notin \mathcal{I}$ for every element $x \in Z - X$ because the condition for J_0 does not hold:

$$|(X + x) \cap A^{\cup}(J_0)| > g(J_0) + |A^{\cup}(J_0)|.$$

Suppose that the addition of elements $u, v \in Z - X$ to X would violate the independence condition for sets J_u and J_v, respectively, chosen to be minimal with respect to inclusion. By assumption we have

$$|(X+u) \cap A^{\cup}(J_u)| > g(J_u) + |A^{\cup}(J_u)| \quad \text{and} \quad |(X+v) \cap A^{\cup}(J_v)| > g(J_v) + |A^{\cup}(J_v)|.$$

On the other hand, X is in $\mathcal{I}$, so

$$|X \cap A^{\cup}(J_u)| \leq g(J_u) + |A^{\cup}(J_u)| \quad \text{and} \quad |X \cap A^{\cup}(J_v)| \leq g(J_v) + |A^{\cup}(J_v)|.$$

These can only hold if

$$|X \cap A^{\cup}(J_u)| = g(J_u) + |A^{\cup}(J_u)| \quad \text{and} \quad |X \cap A^{\cup}(J_v)| = g(J_v) + |A^{\cup}(J_v)|,$$

$u \in A^{\cup}(J_u)$ and $v \in A^{\cup}(J_v)$.

By definition and submodularity,

$$g(J_u) + |A^{\cup}(J_u)| + g(J_v) + |A^{\cup}(J_v)| \geq g(J_u \cap J_v) +$$
$$+ |A^{\cup}(J_u \cap J_v)| + g(J_u \cup J_v) + |A^{\cup}(J_u \cup J_v)| + |(A^{\cup}(J_u) \cap A^{\cup}(J_v)) - A^{\cup}(J_u \cap J_v)|.$$

On the other hand,

$$g(J_u) + |A^{\cup}(J_u)| + g(J_v) + |A^{\cup}(J_v)| = |X \cap A^{\cup}(J_u)| + |X \cap A^{\cup}(J_v)| =$$
$$= |X \cap A^{\cup}(J_u \cap J_v)| + |X \cap (A^{\cup}(J_u \cup J_v))| + |X \cap (A^{\cup}(J_u) \cap A^{\cup}(J_v) - A^{\cup}(J_u \cap J_v)).$$

Thus, we obtain

$$g(J_u \cap J_v) + |A^{\cup}(J_u \cap J_v)| + g(J_u \cup J_v) + |A^{\cup}(J_u \cup J_v)| + |A^{\cup}(J_u) \cap A^{\cup}(J_v) - A^{\cup}(J_u \cap J_v)|$$
$$\leq |X \cap A^{\cup}(J_u \cap J_v)| + |X \cap (A^{\cup}(J_u \cup J_v))| + |X \cap (A^{\cup}(J_u) \cap A^{\cup}(J_v) - A^{\cup}(J_u \cap J_v))|$$

Since $\mathcal{F}$ is closed under intersection, we have $J_u \cap J_v \in \mathcal{F}$, implying

$$g(J_u \cap J_v) + |A^{\cup}(J_u \cap J_v)| \geq |X \cap A^{\cup}(J_u \cap J_v)|.$$

For the last terms,

$$|A^{\cup}(J_u) \cap A^{\cup}(J_v) - A^{\cup}(J_u \cap J_v)| \geq |X \cap (A^{\cup}(J_u) \cap A^{\cup}(J_v) - A^{\cup}(J_u \cap J_v))|.$$

Now we distinguish two cases:

- If $(J_u \cup J_v) \in \mathcal{F}$, then

  $$g(J_u \cup J_v) + |A^{\cup}(J_u \cup J_v)| \geq |X \cap A^{\cup}(J_u \cup J_v)|,$$

  implying that all inequalities are satisfied with equality. But then u and v also violates the condition for $J_u \cap J_v$, from which by the minimality $J_u = J_v$ follows.

- If $(J_u \cup J_v) \notin \mathcal{F}$, then since $J_u \cup J_v \in \mathcal{F}^{\cup_2} - \mathcal{F}$, we obtain

  $$g(J_u \cup J_v) + |A^{\cup}(J_u \cup J_v)| \geq r.$$

  However,

  $$|X \cap A(J_u \cup J_v)| \leq |X| < |Z| \leq r,$$

  thus

  $$g(J_u \cup J_v) + |A^{\cup}(J_u \cup J_v)| > |X \cap A(J_u \cup J_v)|,$$

  which contradicts the previous inequality.

Therefore, all elements of $Z - X$ violate the independence condition for the same minimal set J_0. This means

$$Z - X \subseteq A^{\cup}(J_0) \quad \text{and} \quad |X \cap A^{\cup}(J_0)| = g(J_0) + |A^{\cup}(J_0)|,$$

so $|X| = g(J_0) + |A^{\cup}(J_0)| + |Z - A^{\cup}(J_0)|$. $\square$

Definition 19. We call these matroids submodular $\mathcal{F}$-cup matroids.

Remark 20. The rank function of a submodular $\mathcal{F}$-cup matroid is $r(Z) = \min\{|Z|, \min_{J\in\mathcal{F}}\{g(J) + |Z - A^{\cup}(J)| + |A^{\cup}(J)|\}\}$. If $g(J) + |S - A^{\cup}(J)| + |A^{\cup}(J)| = g(J) + n \geq r$ for all $J \in \mathcal{F}$, then the rank of the matroid is r. We may assume this, because choosing a smaller r only weakens the conditions in the definition of $\mathcal{F}$-cup matroids.

Definition 21. If f is modular, we call these matroids modular $\mathcal{F}$-cup matroids.

Definition 22. If $\mathcal{F} = \binom{n}{\leq t}$, then we call these matroids submodular/modular t-cup matroids accordingly.

Modular t-cup matroids were already defined by Balcan and Harvey [1] as it gave a solution to their structural question of how much the rank function of a matroid can vary. In fact, along partition matroids and paving matroids as a special case of their construction, they noted the class of elementary split matroids in the subsection E.2 Pairwise intersections way before they were introduced in [3].

Theorem 23. *The dual of an $\mathcal{F}$-cap matroid is an $\mathcal{F}$-cup matroid with the same $\mathcal{F}$, and vice versa.*

Proof. Let us take an $\mathcal{F}$-cap matroid with a representation where the rank is r. We know that for an independent set $X \in \mathcal{I}$,

$$|X \cap A^{\cap}(J)| \leq f(J)$$

for all $J \in \mathcal{F}$. So, for the dual matroid, if we take $Y \in \mathcal{I}^*$, we have

$$|Y \cap \overline{A^{\cap}(J)}| \leq n - |A^{\cap}(J)| - (r - f(J))$$

Let us define $A_i^* = \overline{A_i}$ for all i in the collection of subsets in the definition of the $\mathcal{F}$-cup matroid. Then $A^{*\cup}(J) = \overline{A^{\cap}(J)}$. So we have

$$|Y \cap A^{*\cup}(J)| \leq |A^{*\cup}(J)| + f(J) - (n - r^*).$$

Then if we take $g(J) = f(J) - (n - r^*)$, we get the condition for an $\mathcal{F}$-cup matroid. We need to check the conditions for g.
Since f was $\mathcal{F}$-submodular, and $\mathcal{F}$ is the same, $g = f - (n - r^*)$ is also $\mathcal{F}$-submodular.
$f(\emptyset) = r$, so $g(\emptyset) = 0$.
The rank of the original matroid was r, so $f(J) + |S - A^{\cap}(J)| \geq r$ for all $J \in \mathcal{F}$. This implies

$$g(J) + (n - r^*) + |A^{*\cup}(J)| \geq (n - r^*),$$

therefore

$$g(J) + |A^{*\cup}(J)| \geq 0$$

for all $J \in \mathcal{F}$.
We had $f(J) \geq |A^{\cap}(J)|$ for all $j \in \mathcal{F}^{\cup_2} - \mathcal{F}$, so

$$g(J) + (n - r^*) \geq n - |A^{*\cup}(J)|,$$

and

$$g(J) + |A^{*\cup}(J)| \geq r^*$$

follows for all $J \in \mathcal{F}^{\cup_2} - \mathcal{F}$. So the conditions for g are satisfied, the dual matroid is indeed an $\mathcal{F}$-cup matroid.

For the other direction, notice that all of these conditions were equivalent to the original, except the one that followed from the rank being r.
So we only need to verify that backwards. We use the same notations since the dual of the dual is the original matroid.
Take the representation for the dual matroid where the rank is r^*. So we have $g(J) + n \geq r^*$ for all $J \in \mathcal{F}$, from which

$$f(J) - (n - r^*) + n \geq r^*,$$

and the nonnegativity of f on $\mathcal{F}$ follows. On $\mathcal{F}^{\cup_2} - \mathcal{F}$ it follows as well from the other condition. □

Remark 24. The above proof works for each subclass as well, which implies that the dual of a modular $\mathcal{F}$-cap matroid is a modular $\mathcal{F}$-cup matroid, and the dual of a t-cap matroid is a t-cup matroid.

Theorem 25. *Every matroid is a submodular $\mathcal{F}$-cap matroid.*

Proof. Take an arbitrary matroid with a ground set S and identify it with $S := [n]$. Let us define $A_i = S - i$ for all $i \in S$, therefore $k = n$. Take $\mathcal{F} = \binom{[n]}{\leq n-1}$. Define $f(J) = r(A^{\cap}(J)) = r(S - J)$ for all $J \subseteq [n]$. Then f is submodular (therefore $\mathcal{F}$-submodular), since

$$f(J_1)+f(J_2) = r(S-J_1)+r(S-J_2) \geq r((S-J_1)\cap(S-J_2))+r((S-J_1)\cup(S-J_2))$$

$$= r(S - (J_1 \cup J_2)) + r(S - J_1 \cap J_2) = f(J_1 \cup J_2) + f(J_1 \cap J_2)$$

by the submodularity of the rank function.
f is nonnegative by definition and $\mathcal{F}^{\cup_2} - \mathcal{F} = [n]$, so the condition for $\mathcal{F}^{\cup_2} - \mathcal{F}$ is satisfied because

$$f([n]) = r(S - [n]) = r(\emptyset) = 0 = |\emptyset| = |A^{\cap}([n])|$$

$f(\emptyset) = r(S) = r$ is satisfied too.
The condition for the independent sets gives

$$|I \cap A^{\cap}(J)| \leq f(J),$$

that is

$$|I \cap (S - J)| \leq r(S - J)$$

for all $J \subsetneq S$, or in other form

$$|I \cap H| \leq r(H)$$

for all $\emptyset \neq H \subseteq S$, which clearly defines the original matroid. $\square$

Remark 26. Since each matroid is the dual of its dual matroid, combining the previous theorems we get that every matroid is a submodular $\mathcal{F}$-cup matroid.

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